Characterizations of nondecreasing semilattice operations on chains

AAA96

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Motivation

Let X be a nonempty set

Definition

$F\colon X^2 \to X$ is said to be

- *idempotent* if

$$F(x,x) \;=\; x \qquad x \in X$$

- *quasitrivial* if

$$F(x,y) \in \{x,y\} \qquad x,y \in X$$

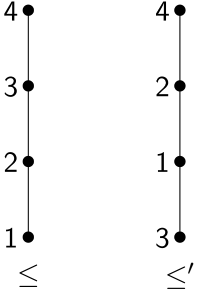
- *$\leq$-preserving* for some total order $\leq$ on X if

$$F(x,y) \leq F(x',y') \quad \text{whenever} \quad x \leq x' \text{ and } y \leq y'$$

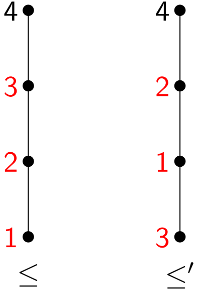
Motivation

Fact. F is associative, quasitrivial, and commutative iff there exists a total order $\leq$ on X such that $F = \vee$.

Example. On $X = \{1, 2, 3, 4\}$, consider $\leq$ and $\leq'$



Motivation



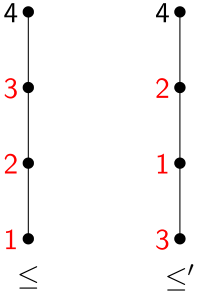
$\vee'(1,2) = 2$ and $\vee'(1,3) = 1 \quad \Rightarrow \quad \vee'$ is not $\leq$-preserving

What are the $\leq'$ for which $\vee'$ are $\leq$-preserving?

Single-peakedness

Definition. (Black, 1948) $\leq'$ is said to be *single-peaked for $\leq$* if for all $a, b, c \in X$,

$$a \leq b \leq c \implies b \leq' a \vee' c \in \{a, c\}$$

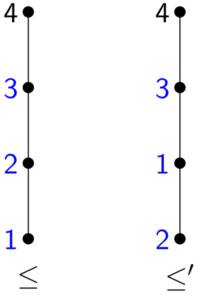


$\leq'$ is not single-peaked for $\leq$

Single-peakedness

Definition. (Black, 1948) $\leq'$ is said to be *single-peaked for $\leq$* if for all $a, b, c \in X$,

$$a \leq b \leq c \implies b \leq' a \vee' c \in \{a, c\}$$



$\leq'$ is single-peaked for $\leq$ and $\vee'$ is $\leq$-preserving

Single-peakedness

Definition. (Black, 1948) $\leq'$ is said to be *single-peaked for $\leq$* if for all $a, b, c \in X$,

$$a \leq b \leq c \implies b \leq' a \vee' c \in \{a, c\}$$

F is associative, quasitrivial, and commutative iff $F = \vee$

Theorem (Devillet et al., 2017)

For any $F\colon X^2 \to X$, the following are equivalent.

(i) F is associative, quasitrivial, commutative, and $\leq$-preserving

(ii) $F = \vee'$ for some $\leq'$ that is single-peaked for $\leq$

How can we generalize this result by relaxing quasitriviality into idempotency?

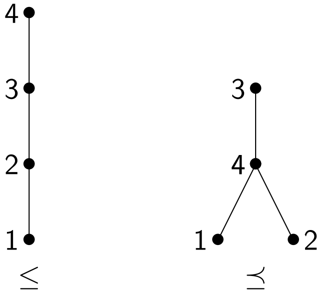
Towards a generalization

$\leq$ will denote a total order on X

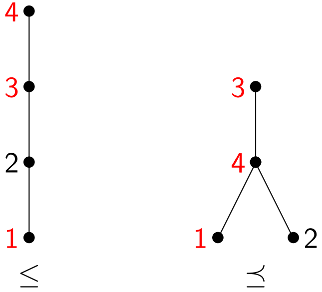
$\preceq$ will denote a join-semilattice order on X

F is associative, idempotent, and commutative iff there exists $\preceq$ such that $F = \curlyvee$.

Example. On $X = \{1, 2, 3, 4\}$, consider $\leq$ and $\preceq$



Towards a generalization



$\curlyvee(1,4) = 4$ and $\curlyvee(3,4) = 3$ $\Rightarrow$ $\curlyvee$ is not $\leq$-preserving

What are the $\preceq$ for which $\curlyvee$ are $\leq$-preserving?

CI-property

Definition. We say that $\preceq$ has the *convex-ideal property* (*CI-property* for short) *for* $\leq$ if for all $a, b, c \in X$,

$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c$$

Proposition

The following are equivalent.

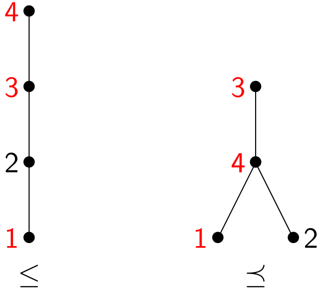
(i) $\preceq$ has the CI-property for $\leq$

(ii) Every ideal of $(X, \preceq)$ is a convex subset of $(X, \leq)$

CI-property

Definition. We say that $\preceq$ has the *CI-property for $\leq$* if for all $a, b, c \in X$,

$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c$$

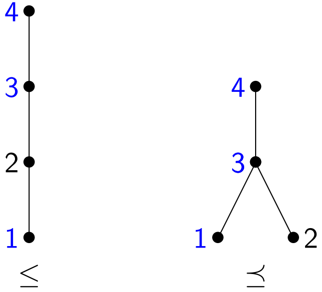


$\preceq$ does not have the CI-property for $\leq$

CI-property

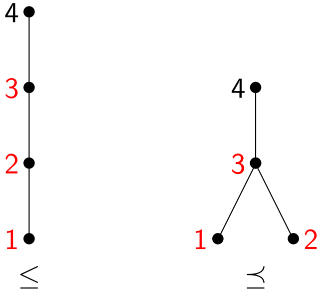
Definition. We say that $\preceq$ has the *CI-property for $\leq$* if for all $a, b, c \in X$,

$$a \leq b \leq c \quad \Longrightarrow \quad b \preceq a \curlyvee c$$



$\preceq$ has the CI-property for $\leq$

CI-property



$\Upsilon(1,2) = 3$ and $\Upsilon(2,2) = 2$ $\quad\Longrightarrow\quad$ Υ is not $\leq$-preserving

Internality

Definition. $F\colon X^2 \to X$ is said to be *internal* if $x \leq F(x, y) \leq y$ for every $x, y \in X$ with $x \leq y$

Definition. We say that $\preceq$ is *internal for* $\leq$ if for all $a, b, c \in X$,

$$a < b < c \quad \Longrightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b)$$

Proposition

The following are equivalent.

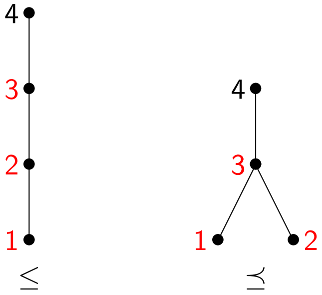
(i) $\preceq$ is internal for $\leq$

(ii) The join operation $\curlyvee$ of $\preceq$ is internal

Internality

Definition. We say that $\preceq$ is *internal for* $\leq$ if for all $a, b, c \in X$,

$$a < b < c \quad \Longrightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b)$$

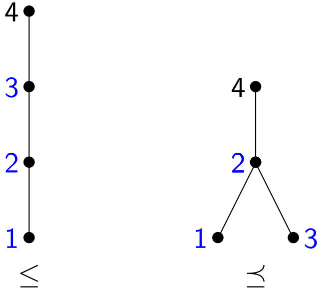


$\preceq$ has the CI-property but is not internal for $\leq$

Internality

Definition. We say that $\preceq$ is *internal for* $\leq$ if for all $a, b, c \in X$,

$$a < b < c \quad \implies \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b)$$



$\preceq$ has the CI-property and is internal for $\leq$

Also, $\curlyvee$ is $\leq$-preserving

Nondecreasingness

Definition. We say that $\preceq$ is *nondecreasing for $\leq$* if

- CI-property for $\leq$
- internal for $\leq$.

F is associative, idempotent, and commutative iff $F = \curlyvee$

Theorem

For any $F\colon X^2 \to X$, the following are equivalent.

(i) F is associative, idempotent, commutative, and $\leq$-preserving

(ii) $F = \curlyvee$ for some $\preceq$ that is nondecreasing for $\leq$

Finite case

Assume that $X = \{1, \ldots, n\}$, is endowed with the usual total order

$$1 < \ldots < n$$

Proposition

The number of nondecreasing join-semilattice orders on X is the n^{th} Catalan number.

Finite case

By a *binary tree* we mean an unordered rooted tree in which every vertex has at most two children.

Proposition

The following are equivalent.

(i) $\preceq$ is nondecreasing for $\leq$

(ii) The Hasse diagram of $(X, \preceq)$ is a binary tree satisfying $(*)$

Finite case

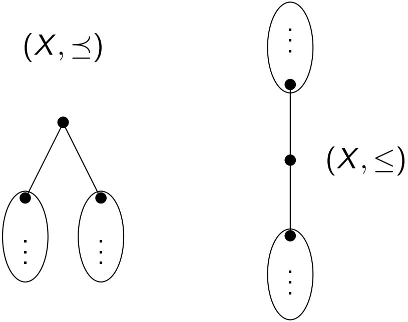
Proposition

The following are equivalent.

(i) $\preceq$ is nondecreasing for $\leq$

(ii) The Hasse diagram of $(X, \preceq)$ is a binary tree satisfying $(*)$

$(*)$:



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