LOCAL CURVATURE ESTIMATES FOR THE LAPLACIAN FLOW

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ABSTRACT. In this paper we give local curvature estimates for the Laplacian flow on closed G_2-structures under the condition that the Ricci curvature is bounded along the flow. The main ingredient consists of the idea of Kotschwar-Munteanu-Wang [24] who gave local curvature estimates for the Ricci flow on complete manifolds and then provided a new elementary proof of Sesum's result [36], and the particular structure of the Laplacian flow on closed G_2-structures. As an immediate consequence, this estimates give a new proof of Lotay-Wei's [33] result which is an analogue of Sesum's theorem.

The second result is about an interesting evolution equation for the scalar curvature of the Laplacian flow of closed G_2-structures. Roughly speaking, we can prove that the time derivative of the scalar curvature R_t is equal to the Laplacian of R_t, plus an extra term which can be written as the difference of two nonnegative quantities.

1. INTRODUCTION

Let $\mathcal{M}$ be a smooth 7-manifold. The Laplacian flow for closed G_2-structures on $\mathcal{M}$ introduced by Bryant [1] is to study the torsion-free G_2-structures

$$\partial_t \varphi_t = \Delta_{\varphi_t} \varphi_t, \quad \varphi_0 = \varphi, \tag{1.1}$$

where $\Delta_{\varphi_t}\varphi_t = dd^*_{\varphi_t}\varphi_t + d^*_{\varphi_t} d\varphi_t$ is the Hodge Laplacian of g_{φ_t} and φ is an initial closed G_2-structure. Since $d\partial_t\varphi_t = \partial_t d\Delta_{\varphi_t}\varphi_t = 0$, we see that the flow (1.1) preserves the closedness of φ_t. For more background on G_2-structures, see Section 2. When $\mathcal{M}$ is compact, the flow (1.1) can be viewed as the gradient flow for the Hitchin functional introduced by Hitchin [17]

$$\mathscr{H} : [\overline{\varphi}]_+ \longrightarrow \mathbb{R}^+, \quad \varphi \longmapsto \frac{1}{7}\int_{\mathcal{M}} \varphi \wedge \psi = \int_{\mathcal{M}} *_\varphi 1. \tag{1.2}$$

Here $\overline{\varphi}$ is a closed G_2-structure on $\mathcal{M}$ and $[\overline{\varphi}]_+$ is the open subset of the cohomology class $[\overline{\varphi}]$ consisting of G_2-structures. Any critical point of $\mathscr{H}$ gives a torsion-free G_2-structure.

The study of Laplacian flows on some special 7-manifolds, Laplacian solitons, and other flows on G_2-structures can be found in [12, 13, 14, 15, 18, 23, 28, 34, 35, 37, 38].

Recently, Donaldson [6, 7, 8, 9] studied the co-associative Kovalev-Lefschetz fibrations G_2-manifolds and G_2-manifolds with boundary.

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1.1. **Notions and conventions.** To state the main results, we fix our notions used throughout this paper. Let $\mathcal{M}$ be as before a smooth 7-manifold. The space of smooth functions and the space of smooth vector fields are denoted respectively by $C^\infty(\mathcal{M})$ and $\mathfrak{X}(\mathcal{M})$. The space of k-tenors (i.e., $(0,k)$-covariant tensor fields) and k-forms on $\mathcal{M}$ are denoted, respectively, by $\otimes^k(\mathcal{M}) = C^\infty(\otimes^k(T^*\mathcal{M}))$ and $\wedge^k(\mathcal{M}) = C^\infty(\wedge^k(T^*\mathcal{M}))$. For any k-tensor field $\boldsymbol{T} \in \otimes^k(\mathcal{M})$, we locally have the expression $\boldsymbol{T} = \boldsymbol{T}_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} =: \boldsymbol{T}_{i_1\cdots i_k} dx^{i_1 \otimes \cdots \otimes i_k}$. A k-form α on $\mathcal{M}$ can be written in the *standard form* as $\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} =: \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1\wedge\cdots\wedge i_k}$, where $\alpha_{i_1\cdots i_k}$ is fully skew-symmetric in its indices. Using the standard forms, if we take the interior product $X \lrcorner \alpha$ of a k-form $\alpha \in \wedge^k(\mathcal{M})$ with a vector field $X \in \mathfrak{X}(\mathcal{M})$, we obtain the $(k-1)$-form $X \lrcorner \alpha = \frac{1}{(k-1)!} X^m \alpha_{m i_1 \cdots i_{k-1}} dx^{i_1\wedge\cdots\wedge i_{k-1}}$ which is also in the standard form. In particular, consider the vector space $\otimes^2(\mathcal{M})$ of 2-tensors. For any 2-tensor $\boldsymbol{A} = \boldsymbol{A}_{ij} dx^{i\otimes j}$, define $\boldsymbol{A}^{\odot} := \frac{1}{2}(\boldsymbol{A}_{ij} + \boldsymbol{A}_{ji}) dx^{i\otimes j} \equiv \boldsymbol{A}^{\odot}_{ij} dx^{i\otimes j}$ and $\boldsymbol{A}^{\wedge} := \frac{1}{2}(\boldsymbol{A}_{ij} - \boldsymbol{A}_{ji}) dx^{i\otimes j} \equiv \boldsymbol{A}^{\wedge}_{ij} dx^{i\otimes j}$. Then $\boldsymbol{A}^{\odot}$ is an element of $\odot^2(\mathcal{M})$, the space of symmetric 2-tensors. Since[1] $dx^{i\wedge j} = dx^{i\otimes j} - dx^{j\otimes i}$, it follows that $\boldsymbol{A}^{\wedge} = \frac{1}{2}\boldsymbol{A}_{ij} dx^{i\wedge j}$. Define $\alpha^{\boldsymbol{A}} := \frac{1}{2}\alpha^{\boldsymbol{A}}_{ij} dx^{i\wedge j}$ with $\alpha^{\boldsymbol{A}}_{ij} := \boldsymbol{A}_{ij}$. Then we see that $\alpha^{\boldsymbol{A}} = \boldsymbol{A}^{\wedge} \in \wedge^2(\mathcal{M})$ and $\otimes^2(\mathcal{M}) = \odot^2(\mathcal{M}) \oplus \wedge^2(\mathcal{M})$.

A given Riemannian metric g on $\mathcal{M}$ determines two isomorphisms between vector fields and 1-forms: $\flat_g : \mathfrak{X}(\mathcal{M}) \longrightarrow \wedge^1(\mathcal{M})$ and $\sharp_g : \wedge^1(\mathcal{M}) \longrightarrow \mathfrak{X}(\mathcal{M})$, where, for every vector field $X = X^i \frac{\partial}{\partial x^i}$ and 1-form $\alpha = \alpha_i dx^i$, $\flat_g(X) = X^i g_{ij} dx^j \equiv X_j dx^j$ and $\sharp_g(\alpha) = \alpha_i g^{ij} \frac{\partial}{\partial x^j} \equiv \alpha^j \frac{\partial}{\partial x^j}$. Using these two natural maps, we can frequently raise or lower indices on tensors. The metric g also induces a metric on k-forms $g(dx^{i_1\wedge\cdots\wedge i_k}, dx^{j_1\wedge\cdots\wedge j_k}) = \det(g(dx^{i_a}, dx^{j_b})) = \sum_{\sigma\in\mathfrak{S}_7} \mathrm{sgn}(\sigma) g^{i_1 j_{\sigma(1)}} \cdots g^{i_k j_{\sigma(k)}}$ where $\mathfrak{S}_7$ is the group of permutations of seven letters and $\mathrm{sgn}(\sigma)$ denotes the sign (± 1) of an element σ of $\mathfrak{S}_7$. The inner product $\langle\cdot,\cdot\rangle_g$ of two k-forms $\alpha, \beta \in \wedge^k(\mathcal{M})$ now is given by $\langle \alpha, \beta\rangle_g = \frac{1}{k!}\alpha_{i_1\cdots i_k}\beta^{i_1\cdots i_k} = \frac{1}{k!}\alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_k} g^{i_1 j_1}\cdots g^{i_k j_k}$.

Given two 2-tensors $\boldsymbol{A}, \boldsymbol{B} \in \otimes^2(\mathcal{M})$, with the forms $\boldsymbol{A} = \boldsymbol{A}_{ij} dx^{i\otimes j}$ and $\boldsymbol{B} = \boldsymbol{B}_{ij} dx^{i\otimes j}$. Define $\langle\langle \boldsymbol{A}, \boldsymbol{B}\rangle\rangle_g := \boldsymbol{A}_{ij}\boldsymbol{B}^{ij}$. There are two special cases which will be used later:

(1) $\alpha = \frac{1}{2}\alpha_{ij} dx^{i\wedge j} \in \wedge^2(\mathcal{M})$ and $\boldsymbol{B} = \boldsymbol{B}_{ij} dx^{i\otimes j} \in \otimes^2(\mathcal{M})$. In this case, α can be written as a 2-tensor $\boldsymbol{A}^\alpha = \boldsymbol{A}^\alpha_{ij} dx^{i\otimes j}$ with $\boldsymbol{A}^\alpha_{ij} = \alpha_{ij}$. Then $\langle\langle \alpha, \boldsymbol{B}\rangle\rangle_g := \langle\langle \boldsymbol{A}^\alpha, \boldsymbol{B}\rangle\rangle_g = \alpha_{ij}\boldsymbol{B}^{ij}$.

(2) $\alpha = \frac{1}{2}\alpha_{ij} dx^{i\wedge j}$ and $\beta = \frac{1}{2}\beta_{ij} dx^{i\wedge j} \in \wedge^2(\mathcal{M})$. In this case, α, β can be both written as 2-tensors $\boldsymbol{A}^\alpha = \boldsymbol{A}^\alpha_{ij} dx^{i\otimes j}$ and $\boldsymbol{B}^\beta = \boldsymbol{B}^\beta_{ij} dx^{i\otimes j}$ with $\boldsymbol{A}^\alpha_{ij} = \alpha_{ij}$ and $\boldsymbol{B}^\beta_{ij} = \beta_{ij}$. Then $\langle\langle \alpha, \beta\rangle\rangle_g := \langle\langle \boldsymbol{A}^\alpha, \boldsymbol{B}^\beta\rangle\rangle_g = \alpha_{ij}\beta^{ij} = 2\langle\alpha,\beta\rangle_g$.

[1] In our convention, for any 2-form $\alpha = \frac{1}{2}\alpha_{ij} dx^{ij}$, we have

$$\alpha\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right) = \frac{1}{2}\alpha_{ij}\left(dx^{i\otimes j} - dx^{j\otimes i}\right)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right) = \frac{1}{2}\alpha_{ij}\left(\delta^i_k\delta^j_\ell - \delta^j_k\delta^i_\ell\right) = \frac{1}{2}\left(\alpha_{k\ell} - \alpha_{\ell k}\right) = \alpha_{k\ell}$$

which justifies the notion $\alpha_{k\ell}$ as $\alpha(\partial/\partial x^k, \partial/\partial x^\ell)$. In general, for any k-form $\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1\wedge\cdots\wedge i_k}$ we have $\alpha_{i_1\cdots i_k} = \alpha(\partial/\partial x^{i_1}, \cdots, \partial/\partial x^{i_k})$, because $dx^{i_1\wedge\cdots\wedge i_k} = \sum_{\sigma\in\mathfrak{S}_k}\mathrm{sgn}(\sigma) dx^{i_{\sigma(1)}\otimes\cdots\otimes i_{\sigma(k)}}$.

The norm of $\boldsymbol{A} \in \otimes^2(\mathcal{M})$ is defined by $||\boldsymbol{A}||_g^2 := \langle\langle \boldsymbol{A}, \boldsymbol{A}\rangle\rangle_g = \boldsymbol{A}_{ij}\boldsymbol{A}^{ij}$, while the norm of $\alpha \in \wedge^k(\mathcal{M})$ is $|\alpha|_g^2 := \langle \alpha, \alpha\rangle_g = \frac{1}{k!}\alpha_{i_1\cdots i_k}\alpha^{i_1\cdots i_k}$. In particular, $||X||_g^2 = X_i X^i = |\flat_g(X)|_g^2$ and $||\alpha||_g^2 = 2|\alpha|_g^2$, for any vector field $X \in \mathfrak{X}(\mathcal{M})$ and 2-form α.

The Levi-Civita connection associated to a given Riemannian metric g is denoted by ${}^g\nabla$ or simply ∇. Our convention on Riemann curvature tensor is $R_{ijk}^m \frac{\partial}{\partial x^m} := \mathrm{Rm}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} = (\nabla_i \nabla_j - \nabla_j \nabla_i)\frac{\partial}{\partial x^k}$ and $R_{ijk\ell} := R_{ijk}^m g_{m\ell}$. The Ricci curvature of g is given by $R_{jk} := R_{ijk\ell}g^{i\ell}$. We use dV_g and $*_g$ to denote the volume form and Hodge star operator, respectively, on $\mathcal{M}$ associated to a metric g and an orientation.

We use the standard notion $A * B$ to denote some linear combination of contractions of the tensor product $A \otimes B$ relative to the metric g_t associated the φ_t. In Theorem 1.4 and its proof, all universal constants c, C below depend only on the given real number p.

1.2. **Main results.** Applying De Turck's trick and Hamilton's Nash-Moser inverse function theorem, Bryant and Xu [2] proved the following local time existence for (1.1).

Theorem 1.1. (Bryant-Xu [2]**)** *For a compact 7-manifold $\mathcal{M}$, the initial value problem (1.1) has a unique solution for a short time interval $[0, T_{\max})$ with the maximal time $T_{\max} \in (0, \infty]$ depending on φ.*

As in the Ricci flow, we can prove following results on the long time existence for the Laplacian flow (1.1).

Theorem 1.2. (Lotay-Wei [33]**)** *Let $\mathcal{M}$ be a compact 7-manifold and φ_t, $t \in [0, T)$, where $T < \infty$, be a solution to the flow (1.1) for closed G_2-structures with associated metric $g_t = g_{\varphi_t}$ for each t.*

(a) *If the velocity of the flow satisfies*
$$\sup_{\mathcal{M}\times[0,T)} ||\Delta_t \varphi_t||_t < \infty,$$
then the solution φ_t can be extended past time T.

(b) *If $T = T_{\max}$, then*
$$\lim_{t\to T_{\max}} \sup_{\mathcal{M}} \left(||\mathrm{Rm}_t||_t^2 + ||\nabla_t \boldsymbol{T}_t||_t^2\right) = \infty.$$
Here $\boldsymbol{T}_t$ is the torsion of φ_t (see (2.14)).

In this paper, we give a new elementary proof of Theorem 1.2, based on the idea of [24] and the structure of the equation (1.1).

Theorem 1.3. *Let $\mathcal{M}$ be a compact 7-manifold and φ_t, $t \in [0, T)$, where $T < \infty$, be a solution to the flow (1.1) for closed G_2-structures with associated metric $g_t = g_{\varphi_t}$ for each t. Suppose that*
$$K := \sup_{\mathcal{M}\times[0,T)} ||\mathrm{Ric}_t||_t < \infty, \quad \Lambda := \sup_{\mathcal{M}} ||\mathrm{Rm}_0||_0.$$

Then

$$\sup_{\mathcal{M}\times[0,T)} ||\mathrm{Rm}_t||_t < \infty,$$

where the bound depends only on n, K, T *and* Λ.

When $\mathcal{M}$ is compact, the theorem immediately implies the part (a) in Theorem 1.2. Indeed, we shall show that (see (3.18) and (3.37))

$$\sup_{\mathcal{M}\times[0,T)} ||\Delta_t\varphi_t||_t < \infty \Longleftrightarrow \sup_{\mathcal{M}\times[0,T)} ||\mathrm{Ric}_t||_t < \infty.$$

In the compact case, Theorem 1.3 shows that, if the conclusion in part (a) does not hold, then $T = T_{\max}$ and $\sup_{\mathcal{M}\times[0,T_{\max})} ||\mathrm{Rm}_t||_t < \infty$ which implies $\sup_{\mathcal{M}\times[0,T_{\max})}$ $(||\mathrm{Rm}_t||_t^2 + ||\nabla_t \boldsymbol{T}_t||_t^2) < \infty$, since the norm $||\nabla_t\boldsymbol{T}_t||_t^2$ can be controlled by $||\mathrm{Rm}_t||_t^2$ (see (3.63)). However, by part (b) in Theorem 1.2, it is impossible. Therefore, the conclusion in part (a) is true.

As remarked in [24], to prove Theorem 1.3, it suffices to establish the following integral estimate.

Theorem 1.4. *Let* $\mathcal{M}$ *be a smooth 7-manifold and* φ_t, $t \in [0,T)$, *where* $T < \infty$, *be a solution to the flow (1.1) for closed* G_2*-structures with associated metric* $g_t = g_{\varphi_t}$ *for each* t. *Assume that there exist constants* $A, K > 0$ *and a point* $x_0 \in \mathcal{M}$ *such that the geodesic ball* $B_{g_0}(x_0, A/\sqrt{K})$ *is compactly contained in* $\mathcal{M}$ *and that*

$$|\mathrm{Ric}_t|_t \leq K \quad \text{on } B_{g_0}\left(x_0, \frac{A}{\sqrt{K}}\right) \times [0,T].$$

Then, for any $p \geq 5$, *there exists* $c = c(p) > 0$ *so that*

$$\begin{aligned}\int_{B_{g_0}(x_0,A/2\sqrt{K})} ||\mathrm{Rm}_t||_t^p dV_t \quad &\leq \quad c(1+K)e^{cKT}\int_{B_{g_0}(x_0,A/\sqrt{K})} ||\mathrm{Rm}_0||_0^p dV_0 \\ &\quad + cK^p\left(1 + A^{-2p}\right)e^{cKT}\mathrm{vol}_t\left(B_{g_0}\left(x_0, \frac{A}{\sqrt{K}}\right)\right) \qquad (1.3)\end{aligned}$$

for all $t \in [0,T]$.

Now by the standard De Giorgi-Nash-Moser iteration (our manifold is compact and the Ricci curvature is uniformly bounded), under the condition in Theorem 1.4, we can prove

$$||\mathrm{Rm}_T||_T(x_0) \leq d_1(d_2 + \Lambda_0), \qquad (1.4)$$

where d_1, d_2 are constants depending on K, T, A, and

$$\Lambda_0 := \sup_{B_{g_0}(x_0,A/\sqrt{K})} ||\mathrm{Rm}_0||_0.$$

Actually, this follows from the same argument in [24] by noting that

$$(\Delta_t - \partial_t)||\mathrm{Rm}_t||_t \geq -c||\mathrm{Rm}_t||_t^2. \qquad (1.5)$$

To verify (1.5), we use (2.26), (3.61) and (3.65) to deduce that $||\nabla_t\boldsymbol{T}_t|| \leq c||\mathrm{Rm}_t||_t$ and

$$||\nabla_t^2\boldsymbol{T}_t||_t \leq c||\nabla_t\mathrm{Rm}_t||_t + c||\mathrm{Rm}_t||_t^{3/2}.$$

Then, by (3.31) and the Cauchy inequality

$$\begin{aligned}||\nabla_t \mathrm{Rm}_t||_t^2 &\leq -\frac{1}{2}(\partial_t - \Delta_t)||\mathrm{Rm}_t||_t^2 + c||\mathrm{Rm}_t||_t^3 + c||\mathrm{Rm}_t||_t^{3/2}||\nabla_t \mathrm{Rm}_t||_t \\ &\leq -\frac{1}{2}(\partial_t - \Delta_t)||\mathrm{Rm}_t||_t^2 + c||\mathrm{Rm}_t||_t^3 + ||\nabla_t \mathrm{Rm}_t||_t^2\end{aligned}$$

which implies (1.5). Now the estimate (1.4) yields Theorem 1.3.

The analogue of Theorem 1.2 in the Ricci flow was proved by Hamilton [16] (for part (b)) and Sesum [36] (for part (a)). It is an open question (due to Hamilton, see [3]) that the Ricci flow will exist as long as the scalar curvature remains bounded. For the Kähler-Ricci flow [39] or type-I Ricci flow [10], this question was settled. For the general case, some partial result on Hamilton's conjecture was carried out in [3].

For the Ricci-harmonic flow introduce by List [29, 30] (see also, [31, 32]), the analogue of Theorem 1.2 was proved in [29, 30] (see also, [31, 32]) and [4] (see [27] for another proof). The author [25, 26] extended Cao's result [3] to the Ricci-harmonic flow. The same Hamilton's conjecture was asked by the author in [25, 26].

We can ask the same question for the Laplacian flow on closed G_2-structures. In [33] (see Page 171, line -6 to -3, or Open Problem (3) in Page 230), Lotay and Wei asked that whether the Laplacian flow on closed G_2-structures will exist as long as the torsion tensor or scalar curvature remains bounded. Let g_t be the associated metric of φ_t. Then the evolution equation for g_t is given by

$$\partial_t g_{ij} = -2R_{ij} - \frac{4}{3}|\boldsymbol{T}_t|_t^2 g_{ij} - 4\boldsymbol{T}_i{}^k \boldsymbol{T}_{kj}. \tag{1.6}$$

For the Laplacian flow on closed G_2-structures, the torsion $\boldsymbol{T}_t$ is actually a 2-form for each t, hence we use the norm $|\cdot|_t$ in (1.6). The standard formula for the scalar curvature R_t gives (see (3.23))

$$\partial_t R_t = \Delta_t R_t + 2||\mathrm{Ric}_t||_t^2 - \frac{2}{3}R_t^2 + 4R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + 4(\nabla^j \boldsymbol{T}^{ik})(\nabla_i \boldsymbol{T}_{jk}). \tag{1.7}$$

Now the above mentioned open problem states that

$$\text{Is it ture that } \lim_{t\to T_{\max}} R_t = -\infty?$$

The "minus infinity" comes from the fact that along the Laplacian flow on closed G_2-structures the scalar curvature is always nonpositive (see (2.26)). The following Proposition 1.5 is motivate to solve this problem, and starts from the basic evolution equation (1.7) where the last two terms on the right-hand side do not have good signature. However, using the closedness of φ_t (in particular, the identity (3.23)), we can prove the following interesting evolution equation for R_t.

Proposition 1.5. *Let $\mathcal{M}$ be a smooth 7-manifold and φ_t, $t \in [0, T)$, where $T \in (0, \infty]$, be a solution to the flow (1.1) for closed G_2-structures with associated metric $g_t = g_{\varphi_t}$ for* *each t. Then the scalar curvature R_t satisfies*

$$
\begin{aligned}
\partial_t R_t &= \Delta_t R_t + \left\{ 2\left|\left| R_{ij} + \frac{2}{3}|\boldsymbol{T}_t|^2_t g_{ij} \right|\right|^2_t + \frac{1}{2}\left|\left| R_{ijab}R^{ij}{}_{mn} - \psi_{abmn} \right|\right|^2_t \right. \\
&\quad + \frac{1}{2}\left|\left| 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn} - \psi_{abmn} \right|\right|^2_t + \frac{1}{2}\left|\left| 2\widehat{\boldsymbol{T}}_{am}\widehat{\boldsymbol{T}}_{bn} - \psi_{abmn} \right|\right|^2_t \\
&\quad \left. + 2||\widehat{\boldsymbol{T}}_t||^2_t + 4||\nabla_t \boldsymbol{T}_t||^2_t \right\} - \left\{ ||\mathrm{Rm}_t||^2_t + \frac{26}{9}R^2_t + \frac{1}{2}\left|\left| R_{ijab}R^{ij}{}_{mn} \right|\right|^2_t \right. \\
&\quad \left. + 2\left|\left| \boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn} \right|\right|^2_t + 2||\widehat{\boldsymbol{T}}_t||^4_t + 210 \right\}.
\end{aligned} \tag{1.8}
$$

Here $\widehat{\boldsymbol{T}}_{ij} = \boldsymbol{T}_i{}^k \boldsymbol{T}_{kj}$.

Observe that the above well-arranged evolution equation can give us a weakly lower bound for R_t, which can not prove or disprove the conjecture of Lotay and Wei.

We give an outline of the current paper. We review the basic theory in Section 2 about G_2-structures, G_2-decompositions of 2-forms and 3-forms, and general flows on G_2-structures. In Section 3, we rewrite results in Section 2 for closed G_2-structures, and the local curvature estimates will be given in the last subsection.

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2. BASIC THEORY OF G_2-STRUCTURES

In this section, we view some basic theory of G_2-structures, following [1, 19, 20, 21, 22, 33]. Let $\{e_1, \cdots, e_7\}$ denote the standard basis of $\mathbb{R}^7$ and let $\{e^1, \cdots, e^7\}$ be its dual basis. Define the 3-form

$$\phi := e^{1\wedge 2\wedge 3} + e^{1\wedge 4\wedge 5} + e^{1\wedge 6\wedge 7} + e^{2\wedge 4\wedge 6} - e^{2\wedge 5\wedge 7} - e^{3\wedge 4\wedge 7} - e^{3\wedge 5\wedge 6},$$

where $e^{i\wedge j\wedge k} := e^i \wedge e^j \wedge e^k$. The subgroup G_2, which fixes ϕ, of $\mathbf{GL}(7, \mathbb{R})$ is the 14-dimensional Lie subgroup of $\mathbf{SO}(7)$, acts irreducibly on $\mathbb{R}^7$, and preserves the metric and orientation for which $\{e_1, \cdots, e_7\}$ is an oriented orthonormal basis. Note that G_2 also preserves the 4-form

$$*_\phi \phi = e^{4\wedge 5\wedge 6\wedge 7} + e^{2\wedge 3\wedge 6\wedge 7} + e^{2\wedge 3\wedge 4\wedge 5} + e^{1\wedge 3\wedge 5\wedge 7} - e^{1\wedge 3\wedge 4\wedge 6} - e^{1\wedge 2\wedge 5\wedge 6} - e^{1\wedge 2\wedge 4\wedge 7}.$$

where the Hodge star operator $*_\phi$ is determined by the metric and orientation.

For a smooth 7-manifold $\mathcal{M}$ and a point $x \in \mathcal{M}$, define as in [33]

$$\wedge^3_+(T^*_x\mathcal{M}) := \left\{ \varphi_x \in \wedge^3(T^*_x\mathcal{M}) : \begin{array}{c} \mathrm{u}^*\phi = \varphi_x \text{ for some invertible} \\ \text{map } \mathrm{u} \in \mathrm{Hom}_{\mathbb{R}}(T_x\mathcal{M}, \mathbb{R}^7) \end{array} \right\}$$

and the bundle

$$\wedge^3_+(T^*\mathcal{M}) := \bigsqcup_{x\in\mathcal{M}} \wedge^3_+(T^*_x\mathcal{M}).$$

We call a section φ of $\wedge^3_+(T^*\mathcal{M})$ a *positive* 3*-form* on $\mathcal{M}$ or a G_2*-structure* on $\mathcal{M}$, and denote the space of positive 3-forms by $\wedge^3_+(\mathcal{M})$. The existence of G_2-structures is equivalent to the property that $\mathcal{M}$ is oriented and spin, which is equivalent to the vanishing of the first and second Stiefel-Whitney classes. From the definition of G_2-structures, we see that any $\varphi \in \wedge^3_+(\mathcal{M})$ uniquely determines a Riemannian metric g_φ and an orientation dV_φ, hence the Hodge star operator $*_\varphi$ and the associated 4-form

$$\psi := *_\varphi \varphi. \tag{2.1}$$

We also have the isomorphisms $\flat_\varphi := \flat_{g_\varphi}$ and $\sharp_\varphi := \sharp_{g_\varphi}$. For a given G_2-structure $\varphi \in \wedge^3_+(\mathcal{M})$, we denote by $\langle\cdot,\cdot\rangle_\varphi$, $\langle\langle\cdot,\cdot\rangle\rangle$, $|\cdot|_\varphi$, $||\cdot||_\varphi$, the corresponding inner products $\langle\cdot,\cdot\rangle_{g_\varphi}$, $\langle\langle\cdot,\cdot\rangle\rangle_{g_\varphi}$ and norms $|\cdot|_{g_\varphi}$, $||\cdot||_{g_\varphi}$.

Given a G_2-structure $\varphi \in \wedge^3_+(\mathcal{M})$. We say that φ is *torsion-free* if φ is parallel with respect to the metric g_φ. Equivalently, φ is torsion-free if and only if ${}^\varphi\nabla\varphi = 0$, where ${}^\varphi\nabla$ is the Levi-Civita connection of g_φ.

Theorem 2.1. (Fernández-Gray [11]**)** *The G_2-structure φ is torsion-free if and only if φ is both closed (i.e., $d\varphi = 0$) and co-closed (i.e., $d *_\varphi \varphi = d\psi = 0$).*

When $\mathcal{M}$ is compact, the above theorem says that a G_2-structure φ is torsion-free if and only if φ is harmonic with respect to the induces metric g_φ.

We say that a G_2-structure φ is *closed* (resp., *co-closed*) if $d\varphi = 0$ (resp., $d\psi = 0$). Theorem 2.1 can be restated as that a G_2-structure is torsion-free if and only if it is both closed and co-closed.

2.1. **G_2-decompositions of $\wedge^2(\mathcal{M})$ and $\wedge^3(\mathcal{M})$.** A G_2-structure φ induces splittings of the bundles $\wedge^k(T^*\mathcal{M}), 2 \le k \le 5$, into direct summands, which we denote by $\wedge^k_\ell(T^*\mathcal{M}, \varphi)$ with ℓ being the rank of the bundle. We let the space of sections of $\wedge^k_\ell(T^*\mathcal{M}, \varphi)$ by $\wedge^k_\ell(\mathcal{M}, \varphi)$. Define the natural projections

$$\pi^k_\ell : \wedge^k(\mathcal{M}) \longrightarrow \wedge^k_\ell(\mathcal{M}, \varphi), \quad \alpha \longmapsto \pi^k_\ell(\alpha). \tag{2.2}$$

We mainly focus on the G_2–decompositions of $\wedge^2(\mathcal{M})$ and $\wedge^3(\mathcal{M})$. Recall that

$$\wedge^2(\mathcal{M}) = \wedge^2_7(\mathcal{M}, \varphi) \oplus \wedge^2_{14}(\mathcal{M}, \varphi), \tag{2.3}$$

$$\wedge^3(\mathcal{M}) = \wedge^3_1(\mathcal{M}, \varphi) \oplus \wedge^3_7(\mathcal{M}, \varphi) \oplus \wedge^3_{27}(\mathcal{M}, \varphi). \tag{2.4}$$

Here each component is determined by

$$\begin{aligned}\wedge^2_7(\mathcal{M},\varphi) &= \{X\lrcorner\varphi : X\in\mathfrak{X}(\mathcal{M})\} = \{\beta\in\wedge^2(\mathcal{M}) : *_\varphi(\varphi\wedge\beta)=2\beta\},\\ \wedge^2_{14}(\mathcal{M},\varphi) &= \{\beta\in\wedge^2(\mathcal{M}) : \psi\wedge\beta=0\} = \{\beta\in\wedge^2(\mathcal{M}) : *_\varphi(\varphi\wedge\beta)=-\beta\},\\ \wedge^3_1(\mathcal{M},\varphi) &= \{f\varphi : f\in C^\infty(\mathcal{M})\},\\ \wedge^3_7(\mathcal{M},\varphi) &= \left\{*_\varphi(\varphi\wedge\alpha) : \alpha\in\wedge^1(\mathcal{M})\right\} = \{X\lrcorner\psi : X\in\mathfrak{X}(\mathcal{M})\},\\ \wedge^3_{27}(\mathcal{M},\varphi) &= \{\eta\in\wedge^3(\mathcal{M}) : \eta\wedge\varphi=\eta\wedge\psi=0\}.\end{aligned}$$

For any 2-form $\beta=\frac{1}{2}\beta_{ij}dx^{i\wedge j}\in\wedge^2(\mathcal{M})$, its two components $\pi^2_7(\beta)$ and $\pi^2_{14}(\beta)$ are determined by

$$\pi^2_7(\beta) = \frac{\beta+*_\varphi(\varphi\wedge\beta)}{3} = \frac{1}{2}\left(\frac{1}{3}\beta_{ab}+\frac{1}{6}\beta^{\ell m}\psi_{\ell mab}\right)dx^{ab}, \tag{2.5}$$

$$\pi^2_{14}(\beta) = \frac{2\beta-*_\varphi(\varphi\wedge\beta)}{3} = \frac{1}{2}\left(\frac{2}{3}\beta_{ab}-\frac{1}{6}\beta^{\ell m}\psi_{\ell mab}\right)dx^{ab}. \tag{2.6}$$

To decompose 3-forms, recall two maps introduce by Bryant [1]

$$\mathrm{i}_\varphi : \odot^2(\mathcal{M})\longrightarrow\wedge^3(\mathcal{M}),\quad \mathrm{j}_\varphi : \wedge^3(\mathcal{M})\longrightarrow\odot^2(\mathcal{M}), \tag{2.7}$$

where

$$\begin{aligned}\mathrm{i}_\varphi(h) &:= h_{ij}g^{j\ell}dx^i\wedge\left(\frac{\partial}{\partial x^\ell}\lrcorner\varphi\right) = \frac{1}{2}h_{i\ell}\varphi^\ell{}_{jk}dx^{ijk}\\ &= \frac{1}{6}\left(h_{i\ell}\varphi^\ell{}_{jk}+h_{j\ell}\varphi_i{}^\ell{}_k+h_{k\ell}\varphi_{ij}{}^\ell\right)dx^{ijk},\quad h=h_{ij}dx^{ij}\in\odot^2(\mathcal{M}),\end{aligned} \tag{2.8}$$

and

$$\left(\mathrm{j}_\varphi(\eta)\right)(X,Y) := *_\varphi\left((X\lrcorner\varphi)\wedge(Y\lrcorner\varphi)\wedge\eta\right). \tag{2.9}$$

Then i_φ is injective and is isomorphic onto $\wedge^3_1(\mathcal{M},\varphi)\oplus\wedge^3_{27}(\mathcal{M},\varphi)$, and j_φ is an isomorphism between $\wedge^3_1(\mathcal{M},\varphi)\oplus\wedge^3_{27}(\mathcal{M},\varphi)$ and $\odot^2(\mathcal{M})$. Moreover, for any 3-form $\eta\in\wedge^3(\mathcal{M})$, we have

$$\eta = \mathrm{i}_\varphi(h)+X\lrcorner\psi \tag{2.10}$$

for some symmetric 2-tensor $h\in\odot^2(\mathcal{M})$ and vector field $X\in\mathfrak{X}(\mathcal{M})$. Then

$$\begin{aligned}\eta &= h_i{}^\ell dx^i\wedge\left(\frac{\partial}{\partial x^\ell}\lrcorner\varphi\right)+X^\ell\left(\frac{\partial}{\partial x^\ell}\lrcorner\psi\right) = \frac{1}{2}h_i{}^\ell\varphi_{\ell jk}dx^{ijk}+\frac{1}{6}X^\ell\psi_{\ell ijk}dx^{ijk}\\ &= \frac{1}{6}\left(3h_i{}^\ell\varphi_{\ell jk}+X^\ell\psi_{\ell ijk}\right)dx^{ijk} = \frac{1}{6}\eta_{ijk}dx^{ijk}.\end{aligned}$$

Write h as $h_{ij}=\mathring{h}_{ij}+\frac{1}{7}\mathrm{tr}_\varphi(h)g_\varphi$, where $\mathring{h}\in\odot^2_0(\mathcal{M})$ is the trace-free part of h, one has

$$\eta = \underbrace{\frac{3}{7}\left(\mathrm{tr}_\varphi(h)\right)\varphi}_{\pi^3_1(\eta)}+\underbrace{\frac{1}{2}\mathring{h}_i{}^\ell\varphi_{\ell jk}dx^{ijk}}_{\pi^3_{27}(\eta)}+\underbrace{\frac{1}{6}X^\ell\psi_{\ell ijk}dx^{ijk}}_{\pi^3_7(\eta)}. \tag{2.11}$$

2.2. **The torsion tensors of a G_2-structure.** By Hodge duality we obtain the G_2-decompositions of 4-forms $\wedge^4(\mathcal{M}) = \wedge^4_1(\mathcal{M},\varphi) \oplus \wedge^4_7(\mathcal{M},\varphi) \oplus \wedge^4_{27}(\mathcal{M},\varphi)$ and 5-forms $\wedge^5(\mathcal{M}) = \wedge^5_7(\mathcal{M},\varphi) \oplus \wedge^5_{14}(\mathcal{M},\varphi)$, respectively. By definition, we can find forms $\tau_0 \in C^\infty(\mathcal{M})$, $\tau_1, \widetilde{\tau}_1 \in \wedge^1(\mathcal{M})$, $\tau_2 \in \wedge^2_{14}(\mathcal{M},\varphi)$, and $\tau_3 \in \wedge^3_{27}(\mathcal{M},\varphi)$ such that

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *_\varphi\tau_3, \quad d\psi = 4\widetilde{\tau}_1 \wedge \psi - *_\varphi\tau_2. \tag{2.12}$$

Since $\tau_2 \in \wedge^2_{14}(\mathcal{M},\varphi)$, it follows that $\tau_2 \wedge \varphi = - *_\varphi \tau_2$. Then (2.12) can be written as in the sense of Bryant [1]

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *_\varphi\tau_3, \quad d\psi = 4\widetilde{\tau}_1 \wedge \psi + \tau_2 \wedge \varphi. \tag{2.13}$$

It can be proved that $\tau_1 = \widetilde{\tau}_1$ (see [22]). We call τ_0 the *scalar torsion*, τ_1 the *vector torsion*, τ_2 the *Lie algebra torsion*, and τ_3 the *symmetric traceless torsion*. We also call $\boldsymbol{\tau}_\varphi := \{\tau_0, \tau_1, \tau_2, \tau_3\}$ the *intrinsic torsion forms* of the G_2-structure φ.

Recall that a G_2-structure φ is torsion-free if and only if $d\varphi = d\psi = 0$ by Theorem 2.1. From (2.12) we see that φ is torsion-free if and only if the intrinsic torsion forms $\boldsymbol{\tau}_\varphi \equiv= 0$; that is, $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$.

Lemma 2.2. (Fernández-Gray, [11]**)** *For any $X \in \mathfrak{X}(\mathcal{M})$, the 3-form $\nabla_X\varphi$ lines in the space $\wedge^3_7(\mathcal{M},\varphi)$. Therefore the covariant derivative $\nabla\varphi \in \wedge^1(\mathcal{M}) \otimes \wedge^3_7(\mathcal{M})$.*

Consequently, there exists a 2-tensor $\boldsymbol{T} = \boldsymbol{T}_{ij}dx^{i\otimes j}$, called the *full torsion tensor*, such that

$$\nabla_\ell\varphi = \boldsymbol{T}_\ell{}^n\psi_{nabc}. \tag{2.14}$$

Equivalently,

$$\boldsymbol{T}_{\ell m} = \frac{1}{24}(\nabla_\ell\varphi_{abc})\psi_m{}^{abc}. \tag{2.15}$$

Write

$$\tau_1 = (\tau_1)_i dx^i \in \wedge^1(\mathcal{M}), \tag{2.16}$$

$$\tau_2 = \frac{1}{2}(\tau_2)_{ab}dx^{ab} \in \wedge^2_{14}(\mathcal{M}), \tag{2.17}$$

$$\tau_3 = \frac{1}{2}(\tau_3)_i{}^\ell\varphi_{\ell ij}dx^{ijk} \in \wedge^3_{27}(\mathcal{M},\varphi). \tag{2.18}$$

The associated 2-tensor $\boldsymbol{\tau}_3 := (\tau_3)_{ij}dx^{i\otimes j}$ of τ_3 lies in the space $\odot^2_0(\mathcal{M})$. With this convenience, the full torsion tensor $\boldsymbol{T}_{\ell m}$ is determined by

$$\boldsymbol{T}_{\ell m} = \frac{\tau_0}{4}g_{\ell m} - (\boldsymbol{\tau}_3)_{\ell m} - \left(\sharp_\varphi(\tau_1)\lrcorner\varphi\right)_{\ell m} - \frac{1}{2}(\tau_2)_{\ell m} \tag{2.19}$$

or as 2-tensors,

$$\boldsymbol{T} = \frac{\tau_0}{4}g_\varphi - \boldsymbol{\tau}_3 - \sharp_\varphi(\tau_1)\lrcorner\varphi - \frac{1}{2}\tau_2. \tag{2.20}$$

Here the 2-form $\sharp_\varphi(\tau_1)\lrcorner\varphi$ is defined by

$$\sharp_\varphi(\tau_1)\lrcorner\varphi = \frac{1}{2}\left(\sharp_\varphi(\tau_1)\lrcorner\varphi\right)dx^{a\wedge b} = \frac{1}{2}\left((\tau_1)_k\varphi^k{}_{ab}\right)dx^{a\wedge b}.$$

As an application, this gives another proof of Theorem 2.1.

For fixed indices i and j, set

$$R_{ij|k\ell} := R_{ijk\ell} \text{ is skew-symmetric in } k \text{ and } \ell, \tag{2.21}$$

where

$$R_{ij|\bullet\bullet} := \frac{1}{2}R_{ij|k\ell}dx^{k\ell} = \frac{1}{2}R_{ijk\ell}dx^{k\ell} \in \wedge^2(\mathcal{M}). \tag{2.22}$$

Then, according to (2.5) and (2.6)

$$R_{ijk\ell} = R_{ij|k\ell} = \left(\pi^2_7(R_{ij|\bullet\bullet})\right)_{k\ell} + \left(\pi^2_{14}(R_{ij|\bullet\bullet})\right)_{k\ell},$$

where

$$\begin{aligned} \left(\pi^2_7(R_{ij|\bullet\bullet})\right)_{k\ell} &= \frac{1}{3}R_{ij|k\ell} + \frac{1}{6}R_{ij|ab}\psi^{ab}{}_{k\ell} &= \frac{1}{3}R_{ijk\ell} + \frac{1}{6}R_{ijab}\psi^{ab}{}_{k\ell}, \\ \left(\pi^2_{14}(R_{ij|\bullet\bullet})\right)_{k\ell} &= \frac{2}{3}R_{ij|k\ell} - \frac{1}{6}R_{ij|ab}\psi^{ab}{}_{k\ell} &= \frac{1}{3}R_{ijk\ell} - \frac{1}{6}R_{ijab}\psi^{ab}{}_{k\ell}. \end{aligned}$$

Karigiannis [22] (see also the equivalent formula obtained by Bryant in [1]) proved that the Ricci curvature is given by

$$\begin{aligned} R_{jk} &= R_{ijk\ell}g^{i\ell} = 3\left(\pi^2_7(R_{ij|\bullet\bullet})\right)_{k\ell}g^{i\ell} = \frac{3}{2}\left(\pi^2_{14}(R_{ij|\bullet\bullet})\right)_{k\ell}g^{i\ell} \\ &= -\left(\nabla_i \boldsymbol{T}_{jm} - \nabla_j \boldsymbol{T}_{im}\right)\varphi^m{}_k{}^i - \boldsymbol{T}_j{}^i\boldsymbol{T}_{ik} + \left(\mathrm{tr}_\varphi \boldsymbol{T}\right)\boldsymbol{T}_{jk} + \boldsymbol{T}_{jb}\boldsymbol{T}_{ia}\psi^{iab}{}_k, \\ &= -\nabla_i\left(\boldsymbol{T}_j{}^n\varphi_{nk}{}^i\right) + \nabla_j\left(\boldsymbol{T}_i{}^n\varphi_{nk}{}^i\right) - \boldsymbol{T}_j{}^i\boldsymbol{T}_{ik} + \left(\mathrm{tr}_\varphi\boldsymbol{T}\right)\boldsymbol{T}_{jk} - \boldsymbol{T}_{jb}\boldsymbol{T}_{ia}\psi^{iab}{}_k. \end{aligned} \tag{2.23}$$

Cleyton and Ivanov [5] also derived a formula for the Ricci tensor for closed G_2-structures in terms of $d^*_\varphi\varphi$. Taking the trace of (2.23), we obtain Btyant's formula [1] for the scalar curvature

$$\begin{aligned} R &= -12\nabla^\ell(\tau_1)_\ell + \frac{21}{8}\tau_0^2 - ||\boldsymbol{\tau}_3||^2_\varphi + 5||\sharp_\varphi(\tau_1)\lrcorner\varphi||^2_\varphi - \frac{1}{4}||\tau_2||^2_\varphi, \\ &= -12\nabla^\ell(\tau_1)_\ell + \frac{21}{8}\tau_0^2 - ||\boldsymbol{\tau}_3||^2_\varphi + 30|\tau_1|^2_\varphi - \frac{1}{2}|\tau_2|^2_\varphi, \end{aligned} \tag{2.24}$$

For a closed G_2-structure, we have $\tau_0 = \tau_1 = \tau_3 = 0$ and then $R = -\frac{1}{4}||\tau_2||^2_\varphi \leq 0$. On the other hand, we have $(\tau_2)_{ij} = -2\boldsymbol{T}_{ij}$ by (2.20). Thus the full torsion tensor $\boldsymbol{T}$ is actually a 2-form

$$\boldsymbol{T} = \frac{1}{2}\boldsymbol{T}_{ij}dx^{ij} \in \wedge^2(\mathcal{M}) \tag{2.25}$$

and the scalar curvature can be written in terms of T

$$R = -||\boldsymbol{T}||^2_\varphi = -2|\boldsymbol{T}|^2_\varphi \leq 0. \tag{2.26}$$

Hence, for closed G_2-structures, scalar curvatures are always non-positive.

Finally, we mention a Bianchi type identity

$$\nabla_i\boldsymbol{T}_{j\ell} - \nabla_j\boldsymbol{T}_{i\ell} = -\frac{1}{2}R_{ijab}\varphi^{ab}{}_\ell - \boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\varphi^{ab}{}_\ell = -\left(\frac{1}{2}R_{ijab} + \boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\varphi^{ab}{}_\ell. \tag{2.27}$$

The proof can be found in [22].

2.3. **General flows on G_2-structures.** For any family $(\varphi_t)_t$ of G_2-structures, according to the decomposition (2.10), we can consider the general flow

$$\partial_t \varphi_t = \mathrm{i}_{\varphi_t}(h_t) + X_t \lrcorner \psi_t \tag{2.28}$$

where $h_t \in \odot^2(\mathcal{M})$ and $X_t \in \mathfrak{X}(\mathcal{M})$. The general flow (2.28) locally can be written as

$$\partial_t \varphi_{ijk} = h_i{}^\ell \varphi_{\ell jk} + h_j{}^\ell \varphi_{i\ell k} + h_k{}^\ell \varphi_{ij\ell} + X^\ell \psi_{\ell ijk}. \tag{2.29}$$

We write for g_t and dV_t the metric and volume form associated to φ_t, respectively.

Theorem 2.3. *Under the general flow (2.28), we have*

$$\partial_t g_{ij} = 2h_{ij}, \tag{2.30}$$

$$\partial_t g^{ij} = -2h^{ij}, \tag{2.31}$$

$$\partial_t dV_t = (\mathrm{tr}_t h_t)\, dV_t, \tag{2.32}$$

$$\partial_t \boldsymbol{T}_{pq} = \boldsymbol{T}_p{}^m h_{mq} - \boldsymbol{T}_p{}^m X^k \varphi_{kmq} - (\nabla_k h_{ip})\varphi^{ki}{}_q + \nabla_p X_q. \tag{2.33}$$

These evolution equations can be found in [22].

3. Laplacian flows on closed G_2-structures

We now consider the Laplacian flow for closed G_2-structures

$$\partial_t \varphi_t = \Delta_{\varphi_t} \varphi_t = \Delta_t \varphi_t, \quad \varphi_0 = \varphi, \tag{3.1}$$

where $\Delta_{\varphi_t}\varphi_t = dd^*_{\varphi_t}\varphi_t + d^*_{\varphi_t} d\varphi_t$ is the Hodge Laplacian of g_{φ_t} and φ is an initial closed G_2-structure. The short time existence for (3.1) was proved by Bryant and Xu [2], see also Theorem 1.1.

A criterion for the long time existence for the Lapalcian flow on compact manifolds was given in Theorem 1.2. In this section, we give a new elementary proof of Lotay-Wei's result in compact case.

3.1. **Basic theory of closed G_2-structures.** Let $\wedge^3_{+,\bullet}(\mathcal{M}) \subset \wedge^3_+(\mathcal{M}, \varphi)$ be the set of all closed G_2-structures on $\mathcal{M}$. If $\varphi \in \wedge^3_{+,\bullet}(\mathcal{M})$ is closed, i.e., $d\varphi = 0$, then τ_0, τ_1, τ_3 are all zero, so the only nonzero torsion form is

$$\boldsymbol{\tau} \equiv \tau_2 = \frac{1}{2}(\tau_2)_{ij} dx^{ij} = \frac{1}{2}\boldsymbol{\tau}_{ij} dx^{ij}. \tag{3.2}$$

According to (2.20) and (2.25), we have $\boldsymbol{T}_{ij} = -\frac{1}{2}\boldsymbol{\tau}_{ij}$ so that

$$\boldsymbol{T} \equiv \frac{1}{2}\boldsymbol{T}_{ij} dx^{ij} \quad \text{or equivalently} \quad \boldsymbol{T} = -\frac{1}{2}\boldsymbol{\tau}, \tag{3.3}$$

is a 2-form. Since $d\psi = \boldsymbol{\tau} \wedge \varphi = - *_\varphi \boldsymbol{\tau}$, we get $d^*_\varphi \boldsymbol{\tau} = *_\varphi d *_\varphi \boldsymbol{\tau} = - *_\varphi d^2\psi = 0$ which is given in local coordinates by

$$\nabla^i \boldsymbol{\tau}_{ij} = 0 \tag{3.4}$$

For a closed G_2-structure φ, according to (2.23), the Ricci curvature is given by (in this case $\boldsymbol{T}_{ij}$ is a 2-form)

$$R_{jk} = \left(\nabla_j \boldsymbol{T}_{im} - \nabla_i \boldsymbol{T}_{jm}\right) \varphi^m{}_k{}^i - \boldsymbol{T}_j{}^i \boldsymbol{T}_{ik} + \boldsymbol{T}_{jb}\boldsymbol{T}_{ia}\psi^{iab}{}_k.$$

Since $\boldsymbol{\tau} \in \wedge^2_{14}(\mathcal{M}, \varphi)$ and $\boldsymbol{T}_{ij} = -\frac{1}{2}\boldsymbol{\tau}_{ij}$, it follows from [33] (see page 179 – 180) that

$$(\nabla_j \boldsymbol{T}_{im})\varphi^m{}_k{}^i = 2\boldsymbol{T}_j{}^\ell \boldsymbol{T}_{\ell k}. \tag{3.5}$$

and therefore, for a closed G_2-structure φ, the Ricci curvature is given by

$$R_{jk} = -(\nabla_i \boldsymbol{T}_{jm})\varphi_k{}^{im} - \boldsymbol{T}_j{}^i \boldsymbol{T}_{ik}. \tag{3.6}$$

Taking the trace of (3.6) yields (2.26). Moreover, the factor $\nabla_i \boldsymbol{T}_{jm}$ in (3.6) can be expressed as (see Proposition 2.4 in [33])

$$\begin{aligned} \nabla_i \boldsymbol{T}_{jk} &= -\frac{1}{4}R_{ijmn}\varphi_k{}^{mn} - \frac{1}{4}R_{kjmn}\varphi_i{}^{mn} + \frac{1}{4}R_{ikmn}\varphi_j{}^{mn} \\ &\quad - \frac{1}{2}\boldsymbol{T}_{im}\boldsymbol{T}_{jn}\varphi_k{}^{mn} - \frac{1}{2}\boldsymbol{T}_{km}\boldsymbol{T}_{jn}\varphi_i{}^{mn} + \frac{1}{2}\boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}. \end{aligned} \tag{3.7}$$

If φ is a closed G_2-structure, Section 2.2 in [33] shows that $\pi^3_7(\Delta_\varphi \varphi) = 0$ and hence, according to (2.10),

$$\Delta_\varphi \varphi = \mathrm{i}_\varphi(h) \in \wedge^3_1(\mathcal{M}, \varphi) \oplus \wedge^3_{27}(\mathcal{M}, \varphi), \tag{3.8}$$

where

$$h_{ij} = \frac{1}{2}\nabla_m \boldsymbol{\tau}_{ni}\varphi_j{}^{mn} - \frac{1}{6}|\boldsymbol{\tau}|^2_\varphi g_{ij} - \frac{1}{4}\boldsymbol{\tau}_i{}^\ell \boldsymbol{\tau}_{\ell j} = -R_{ij} - \frac{2}{3}|\boldsymbol{T}|^2_\varphi g_{ij} - 2\boldsymbol{T}_i{}^k \boldsymbol{T}_{kj}. \tag{3.9}$$

Here $|\boldsymbol{T}|^2_\varphi = \frac{1}{2}\boldsymbol{T}_{k\ell}\boldsymbol{T}^{k\ell} = \frac{1}{2}||\boldsymbol{T}||^2_\varphi$.

3.2. **Evolution equations for closed G_2-structures.** Since the Laplacian flow (3.1) preserves the closedness of φ_t, it follows from (3.10) that we have

$$\Delta_{\varphi_t}\varphi_t = \mathrm{i}_{\varphi_t}(h_t) \in \wedge^3_1(\mathcal{M}, \varphi_t) \oplus \wedge^3_{27}(\mathcal{M}, \varphi_t), \tag{3.10}$$

where

$$h_{ij} = -R_{ij} - \frac{2}{3}|\boldsymbol{T}_t|^2_t g_{ij} - 2\boldsymbol{T}_i{}^k \boldsymbol{T}_{kj}. \tag{3.11}$$

From Theorem 2.3, we see that the associated metric tensor g_t evolves by

$$\partial_t g_{ij} = 2h_{ij} = -2R_{ij} - \frac{4}{3}|\boldsymbol{T}_t|^2_t g_{ij} - 4\boldsymbol{T}_i{}^k \boldsymbol{T}_{kj}. \tag{3.12}$$

and the volume form dV_t evolves by

$$\partial_t dV_t = (\mathrm{tr}_t h_t)dV_t = \left(-R - \frac{14}{3}|\boldsymbol{T}_t|^2_t + 4|\boldsymbol{T}_t|^2_t\right)dV_t = \frac{4}{3}|\boldsymbol{T}_t|^2_t dV_t. \tag{3.13}$$

Hence, along the flow (3.1), the volume of g_t is nondecreasing.

Introduce the following notions

$$\blacksquare_t := \partial_t - \blacktriangle_t, \quad |\cdot|_t := |\cdot|_{\varphi_t}, \quad \Delta_t := \Delta_{\varphi_t}, \tag{3.14}$$

where $\blacktriangle_t := g^{ij}\nabla_i\nabla_j$ is the usual Laplacian of g_t and Δ_t is the Hodge Laplacian of g_t, and also the 2-tenor Sic_t with components

$$S_{ij} := R_{ij} + \frac{2}{3}|\boldsymbol{T}_t|^2_t g_{ij} + 2\boldsymbol{T}_i{}^k \boldsymbol{T}_{kj} = -h_{ij}. \tag{3.15}$$

Then the evolution equation (3.12) can be written as

$$\partial_t g_{ij} = -2S_{ij}. \tag{3.16}$$

Moreover, the trace of Sic_t is exactly the scalar curvature, up to a multiplying constant,

$$S_t := \mathrm{tr}_t \mathrm{Sic}_t = R_t + \frac{14}{3}|\boldsymbol{T}_t|_t^2 - 4|\boldsymbol{T}_t|_t^2 = -\frac{4}{3}|\boldsymbol{T}_t|_t^2 = \frac{2}{3}R_t. \tag{3.17}$$

It was proved in [33] that

$$|\Delta_t \varphi_t|_t^2 = (\mathrm{tr}_t h_t)^2 + 2||h_t||_t^2 = \frac{16}{9}|\boldsymbol{T}_t|_t^4 + 2||\mathrm{Sic}_t||_t^2. \tag{3.18}$$

This identity together with (2.26) shows that the boundedness of $\Delta_t\varphi_t$ is equivalent to the boundedness of Ric_t.

The evolution equation (2.33) implies that for the Laplacian flow on closed G_2-structures, the torsion T_{ij} evolves by evolves

$$\partial_t \boldsymbol{T}_{ij} = \boldsymbol{T}_i{}^k h_{kj} - (\nabla_m h_{ni})\varphi_j{}^{mn}. \tag{3.19}$$

Furthermore, we can prove

Proposition 3.1. *Under the flow (3.1), we have*

$$\begin{aligned}\blacksquare_t \boldsymbol{T}_{ij} \quad = \quad & 3R_j{}^k\boldsymbol{T}_{ki} - R_i{}^k\boldsymbol{T}_{kj} - \frac{1}{2}R_{ijmk}\boldsymbol{T}^{mk} - \frac{1}{2}R_{mpi}{}^k R_{qk}\psi_j{}^{pqm} - \frac{2}{3}\varphi_{ji}{}^m\nabla_m|\boldsymbol{T}_t|_t^2 \\ & + \nabla_p\boldsymbol{T}_{qi}\left(\boldsymbol{T}^{pk}\varphi_{kj}{}^q - 2\boldsymbol{T}^{qk}\varphi_{kj}{}^p\right) - \frac{2}{3}|\boldsymbol{T}_t|_t^2\boldsymbol{T}_{ij} - 4T_i{}^k\boldsymbol{T}_k{}^m\boldsymbol{T}_{mj}. \end{aligned} \tag{3.20}$$

Proof. See [33]. □

For a geometric flow $\partial_t g_{ij} = \eta_{ij}$, for some symmetric 2-tensor η_{ij}, we have

$$\begin{aligned}\partial R_{ijk}^\ell &= \frac{1}{2}g^{\ell p}\Big(\nabla_i\nabla_j\eta_{kp} + \nabla_i\nabla_k\eta_{jp} - \nabla_i\nabla_p\eta_{jk} \\ &\quad - \nabla_j\nabla_i\eta_{kp} - \nabla_j\nabla_k\eta_{ip} + \nabla_j\nabla_p\eta_{ik}\Big), \\ \partial_t R_{jk} &= \frac{1}{2}g^{pq}\left(\nabla_q\nabla_j\eta_{kp} + \nabla_q\nabla_k\eta_{jp} - \nabla_q\nabla_p\eta_{jk} - \nabla_j\nabla_k\eta_{qp}\right), \\ \partial_t R_t &= -\blacktriangle_t \mathrm{tr}_t\eta_t + \mathrm{div}_t(\mathrm{div}_t\eta_t) - R_{ij}h^{ij},\end{aligned}$$

where $(\mathrm{div}_t\eta_t)_j = \nabla^i\eta_{ij}$. Applying those evolution equations to $\eta_{ij} = -2R_{ij} - \frac{4}{3}|\boldsymbol{T}_t|_t^2 g_{ij} - 4\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj} = -2S_{ij}$ we have

$$\begin{aligned}\mathrm{tr}_t\eta_t &= -2R_t - \frac{28}{3}|\boldsymbol{T}_t|_t^2 + 8|\boldsymbol{T}_t|_t^2 \ = \ \frac{8}{3}|\boldsymbol{T}_t|_t^2, \\ (\mathrm{div}_t\eta_t)_j &= -2\nabla^i R_{ij} - \frac{4}{3}\nabla_j|\boldsymbol{T}_t|_t^2 - 4\nabla^i\widehat{\boldsymbol{T}}_{ij} \\ &= -\nabla_j R_t - \frac{4}{3}\nabla_j|\boldsymbol{T}_t|_t^2 - 4\nabla^i\widehat{\boldsymbol{T}}_{ij}, \\ \mathrm{div}_t(\mathrm{div}_t\eta_t) &= \nabla^j(\mathrm{div}_t\eta_t)_j \ = \ -\blacktriangle_t R_t - \frac{4}{3}\blacktriangle_t|\boldsymbol{T}_t|_t^2 - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij},\end{aligned}$$

where the symmetric 2-tensor $\widehat{\boldsymbol{T}}$ is given by

$$\widehat{\boldsymbol{T}}_{ij} := \boldsymbol{T}_{ik}\boldsymbol{T}^k{}_j. \tag{3.21}$$

Plugging those identities into the above evolution equation for R_t, we get

$$\begin{aligned}\partial_t R_t &= -4\blacktriangle_t|\boldsymbol{T}_t|_t^2 - \blacktriangle_t R_t - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij} - R^{ij}\left(-2R_{ij} - \frac{4}{3}|\boldsymbol{T}_t|_t^2 g_{ij} - 4\widehat{\boldsymbol{T}}_{ij}\right)\\ &= \blacktriangle_t R_t - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij} + 2||\mathrm{Ric}_t||_t^2 + \frac{4}{3}|\boldsymbol{T}_t|_t^2 R_t + 4R^{ij}\widehat{\boldsymbol{T}}_{ij}\end{aligned}$$

which implies

$$\blacksquare_t R_t = 2||\mathrm{Ric}_t||_t^2 - \frac{2}{3}R_t^2 - 4\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij} + 4\langle\langle \mathrm{Ric}_t, \widehat{\boldsymbol{T}}\rangle\rangle_t. \tag{3.22}$$

Observe that the last two terms on the right-hand side of (3.22) are not determined of their signs. In the following, we shall use the identity

$$\nabla^i\boldsymbol{T}_{ij} = 0 \tag{3.23}$$

follows from from (3.3) and (3.4), to simplify those two terms. Using the identity (3.23), the term $\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij}$ can be simplified as follows.

$$\begin{aligned}\nabla^j\nabla^i\widehat{\boldsymbol{T}}_{ij} &= \nabla^j\nabla^i\left(\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}\right) = \nabla^j\left[(\nabla^i\boldsymbol{T}_i{}^k)\boldsymbol{T}_{kj} + \boldsymbol{T}_i{}^k(\nabla^i\boldsymbol{T}_{kj})\right]\\ &= \boldsymbol{T}^{ik}(\nabla_j\nabla_i\boldsymbol{T}_k{}^j) - (\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}).\end{aligned}$$

On the other hand, from the Ricci identity

$$\nabla_j\nabla_i\boldsymbol{T}_k{}^j = \nabla_i\nabla_j\boldsymbol{T}_k{}^j - R_{jik\ell}\boldsymbol{T}^{\ell j} - R_{ji}{}^{j\ell}\boldsymbol{T}_{k\ell} = R_{ijk\ell}\boldsymbol{T}^{\ell j} + R_{i\ell}\boldsymbol{T}_k{}^\ell,$$

we see that the evolution equation (3.22) is equivalent to

$$\blacksquare_t R_t = 2||\mathrm{Ric}_t||_t^2 - \frac{2}{3}R_t^2 + 4R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + 4(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}). \tag{3.24}$$

From (3.15) and (3.21) we can rewrite the term $||\mathrm{Ric}_t||_t^2$ in (3.24) in terms of Sic_t according to the following relation:

$$\begin{aligned}||\mathrm{Sic}_t||_t^2 &= \left(R_{ij} + \frac{2}{3}|\boldsymbol{T}_t|_t^2 g_{ij} + 2\widehat{\boldsymbol{T}}_{ij}\right)\left(R^{ij} + \frac{2}{3}|\boldsymbol{T}_t|_t^2 g^{ij} + 2\widehat{\boldsymbol{T}}^{ij}\right)\\ &= ||\mathrm{Ric}_t||_t^2 + \frac{4}{3}|\boldsymbol{T}_t|_t^2 R_t + 4\langle\langle \mathrm{Ric}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle_t + \frac{28}{9}|\boldsymbol{T}_t|_t^4 + \frac{8}{3}|\boldsymbol{T}_t|_t^2\mathrm{tr}_t\widehat{\boldsymbol{T}}_t + 4||\widehat{\boldsymbol{T}}_t||_t^2\\ &= ||\mathrm{Ric}_t||_t^2 - \frac{2}{3}R_t^2 + 4\langle\langle \mathrm{Ric}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle_t + \frac{7}{9}R_t^2 - \frac{4}{3}R_t^2 + 4||\widehat{\boldsymbol{T}}_t||_t^2\\ &= ||\mathrm{Ric}_t||_t^2 + 4||\widehat{\boldsymbol{T}}_t||_t^2 + 4\langle\langle \mathrm{Ric}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle_t - \frac{11}{9}R_t^2,\end{aligned}$$

where we used $\mathrm{tr}_t\widehat{\boldsymbol{T}}_t = g^{ij}\boldsymbol{T}_{ik}\boldsymbol{T}^k{}_j = \boldsymbol{T}_{ik}\boldsymbol{T}^{ki} = -2|\boldsymbol{T}_t|_t^2$ and $R_t = -2|\boldsymbol{T}_t|_t^2$. Replacing R_t by S_t according to the identity (3.17), we can rewrite (3.24) as

$$\begin{aligned}\blacksquare_t S_t &= \frac{4}{3}||\mathrm{Sic}_t||_t^2 - \frac{16}{3}||\widehat{\boldsymbol{T}}||_t^2 - \frac{16}{3}\langle\langle \mathrm{Ric}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle_t + \frac{32}{27}R_t^2\\ &\quad + \frac{8}{3}R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + \frac{8}{3}(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}).\end{aligned}$$

Similarly, replacing $\langle\langle \mathrm{Ric}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle_t$ by $\langle\langle \mathrm{Sic}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle_t$ with respect to the identity

$$\langle\langle \mathrm{Sic}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle = \left(R_{ij} + \frac{2}{3}|\boldsymbol{T}_t|_t^2 g_{ij} + 2\widehat{\boldsymbol{T}}_{ij}\right)\widehat{\boldsymbol{T}}^{ij} = \langle\langle \mathrm{Ric}_t, \widehat{\boldsymbol{T}}_t\rangle\rangle_t - \frac{1}{3}R_t^2 + 2||\widehat{\boldsymbol{T}}_t||_t^2,$$

we obtain the following evolution equation for S_t,

$$\blacksquare_t S_t = \frac{4}{3}\left[\left|\left|\mathrm{Sic}_t - 2\widehat{\boldsymbol{T}}_t\right|\right|_t^2 - S_t^2\right] + \frac{8}{3}\left[R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell} + (\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk})\right]. \tag{3.25}$$

Next, we try to deal with the last bracket in (3.25), which contains two terms $R_{ijk\ell}\boldsymbol{T}^{ik}\boldsymbol{T}^{j\ell}$ and $(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk})$. Using (2.27) and (3.7), the term $(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk})$ is equal to

$$\begin{aligned}
(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}) &= \left[\nabla^i\boldsymbol{T}^{jk} + \left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\varphi^{kab}\right]\nabla_i\boldsymbol{T}_{jk} \\
&= ||\nabla_t\boldsymbol{T}_t||_t^2 + \frac{1}{2}\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left[-\frac{1}{2}R_{ijmn}\varphi^{mn}{}_k\varphi^{kab} - \frac{1}{2}R_{kjmn}\varphi_i{}^{mn}\varphi^{kab}\right. \\
&\quad \left. + \frac{1}{2}R_{ikmn}\varphi_j{}^{mn}\varphi^{kab} - \boldsymbol{T}_{im}\boldsymbol{T}_{jn}\varphi^{mn}{}_k\varphi^{kab} - \boldsymbol{T}_{km}\boldsymbol{T}_{jn}\varphi_i{}^{mn}\varphi^{kab} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right].
\end{aligned}$$

By symmetry the term

$$\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\frac{1}{2}R_{kjmn}\varphi_i{}^{mn}\varphi^{kab} + \frac{1}{2}R_{ikmn}\varphi_j{}^{mn}\varphi^{kab}\right)$$

is equal to, interchanging $i \leftrightarrow j$ and $a \leftrightarrow b$ in the second term,

$$\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\frac{1}{2}R_{kjmn}\varphi_i{}^{mn}\varphi^{kab}\right) + \left(\frac{1}{2}R^{ji}{}_{ba} + \boldsymbol{T}^j{}_b\boldsymbol{T}^i{}_a\right)\left(\frac{1}{2}R_{jkmn}\varphi_i{}^{mn}\varphi^{kba}\right)$$

which is zero. Similarly, we have, by interchanging $m \leftrightarrow n$ and then $i \leftrightarrow j$, $a \leftrightarrow b$ in the first term,

$$\begin{aligned}
&\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\boldsymbol{T}_{km}\boldsymbol{T}_{jn}\varphi_i{}^{mn}\varphi^{kab} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right) \\
&= \left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\boldsymbol{T}_{kn}\boldsymbol{T}_{jm}\varphi_i{}^{nm}\varphi^{kab} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right) \\
&= \left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(-\boldsymbol{T}_{kn}\boldsymbol{T}_{im}\varphi_j{}^{nm}\varphi^{kba} + \boldsymbol{T}_{im}\boldsymbol{T}_{kn}\varphi_j{}^{mn}\varphi^{kab}\right) = 0.
\end{aligned}$$

Therefore, using the identity $\varphi_{ijk}\varphi^k{}_{ab} = g_{ia}g_{jb} - g_{ib}g_{ja} + \psi_{ijab}$ (see [22]), we arrive at

$$\begin{aligned}
(\nabla^j\boldsymbol{T}^{ik})(\nabla_i\boldsymbol{T}_{jk}) &= ||\nabla_t\boldsymbol{T}_t||_t^2 - \frac{1}{2}\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(\frac{1}{2}R_{ij}{}^{mn} + \boldsymbol{T}_i{}^m\boldsymbol{T}_j{}^n\right)\varphi_{mnk}\varphi^{kab} \\
&= ||\nabla_t\boldsymbol{T}_t||_t^2 - \frac{1}{2}\left(\frac{1}{2}R^{ij}{}_{ab} + \boldsymbol{T}^i{}_a\boldsymbol{T}^j{}_b\right)\left(\frac{1}{2}R_{ij}{}^{mn} + \boldsymbol{T}_i{}^m\boldsymbol{T}_j{}^n\right)\left(\delta^a_m\delta^b_n - \delta^b_m\delta^a_n + \psi_{mn}{}^{ab}\right) \\
&= ||\nabla_t\boldsymbol{T}_t||_t^2 - \frac{1}{8}\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left[\left(R^{ijab} + 2\boldsymbol{T}^{ia}\boldsymbol{T}^{jb}\right)\right. \\
&\quad \left. - \left(R^{ijba} + 2\boldsymbol{T}^{ib}\boldsymbol{T}^{ja}\right) + \left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab}\right].
\end{aligned}$$

Since, by our convention,

$$\begin{aligned}
\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijab} + 2\boldsymbol{T}^{ia}\boldsymbol{T}^{jb}\right) &= ||\mathrm{Rm}_t||_t^2 + 4R_{ijab}\boldsymbol{T}^{ia}\boldsymbol{T}^{jb} + 4||\boldsymbol{T}_t||_t^4, \\
\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijba} + 2\boldsymbol{T}^{ib}\boldsymbol{T}^{ja}\right) &= -||\mathrm{Rm}_t||_t^2 - 4R_{ijab}\boldsymbol{T}^{ia}\boldsymbol{T}^{jb} + 4||\widehat{\boldsymbol{T}}_t||_t^2,
\end{aligned}$$

it follows that

$$(\nabla^j \boldsymbol{T}^{ik})(\nabla_i \boldsymbol{T}_{jk}) = ||\nabla_t \boldsymbol{T}_t||_t^2 + \frac{1}{8}\Big[-2||\mathrm{Rm}_t||_t^2 - 8R_{ijab}\boldsymbol{T}^{ia}\boldsymbol{T}^{jb} - 4||\boldsymbol{T}_t||_t^4 + 4||\widehat{\boldsymbol{T}}_t||_t^2 - \left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab}\Big]$$

and (3.25) can be written as

$$\begin{aligned} \blacksquare_t S_t &= \frac{4}{3}\left|\left|\mathrm{Sic}_t - 2\widehat{\boldsymbol{T}}_t\right|\right|_t^2 + \frac{8}{3}||\nabla_t \boldsymbol{T}_t||_t^2 + \frac{4}{3}||\widehat{\boldsymbol{T}}_t||_t^2 - \frac{2}{3}||\mathrm{Rm}_t||_t^2 - \frac{13}{3}S_t^2 \\ &\quad - \frac{1}{3}\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab}. \end{aligned} \tag{3.26}$$

Finally, we deal with the last term J on the right-hand side of (3.26). From the identity $\psi_{ijk\ell}\psi^{ijk\ell} = 168$, we find that

$$\begin{aligned} J &:= -\frac{1}{3}\left(R_{ijab} + 2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}\right)\left(R^{ijmn} + 2\boldsymbol{T}^{im}\boldsymbol{T}^{jn}\right)\psi_{mn}{}^{ab} \\ &= \frac{1}{3}\left(-R_{ij}{}^{ab}R^{ijmn}\psi_{mnab} - 4\boldsymbol{T}_i{}^a\boldsymbol{T}_j{}^b R^{ijmn}\psi_{mnab} - 4\boldsymbol{T}^a{}_i\boldsymbol{T}^{im}\boldsymbol{T}^b{}_j\boldsymbol{T}^{jn}\psi_{mnab}\right) \\ &= \frac{1}{3}\Big[\left|\left|R_{ij}{}^{ab}R^{ijmn} - \frac{1}{2}\psi^{abmn}\right|\right|_t^2 - \left|\left|R_{ij}{}^{ab}R^{ijmn}\right|\right|_t^2 - \frac{168}{4} \\ &\quad + \left|\left|2\boldsymbol{T}_i{}^a\boldsymbol{T}_j{}^b R^{ijmn} - \psi^{abmn}\right|\right|_t^2 - 4\left|\left|\boldsymbol{T}_i{}^a\boldsymbol{T}_j{}^b R^{ijmn}\right|\right|_t^2 - 168 \\ &\quad + \left|\left|2\widehat{\boldsymbol{T}}^{am}\widehat{\boldsymbol{T}}^{bn} - \psi^{mnab}\right|\right|_t^2 - 4||\widehat{\boldsymbol{T}}_t||_t^4 - 168\Big]. \end{aligned}$$

Plugging the expression for J into (3.26), we obtain

Proposition 3.2. *The scalar curvature R_t or S_t evolves by*

$$\begin{aligned} \blacksquare_t S_t &= \frac{4}{3}\left|\left|\mathrm{Sic}_t - 2\widehat{\boldsymbol{T}}_t\right|\right|_t^2 + \frac{8}{3}||\nabla_t \boldsymbol{T}||_t^2 + \frac{1}{3}\left|\left|R_{ijab}R^{ij}{}_{mn} - \psi_{abmn}\right|\right|_t^2 + \frac{4}{3}||\widehat{\boldsymbol{T}}_t||_t^2 \\ &\quad + \frac{1}{3}\left|\left|2\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn} - \psi_{abmn}\right|\right|_t^2 + \frac{1}{3}\left|\left|2\widehat{\boldsymbol{T}}_{am}\widehat{\boldsymbol{T}}_{bn} - \psi_{abmn}\right|\right|_t^2 - \frac{4}{3}||\widehat{\boldsymbol{T}}_t||_t^4 \\ &\quad - \frac{2}{3}||\mathrm{Rm}_t||_t^2 - \frac{13}{3}S_t^2 - \frac{1}{3}\left|\left|R_{ijab}R^{ij}{}_{mn}\right|\right|_t^2 - \frac{4}{3}\left|\left|\boldsymbol{T}_{ia}\boldsymbol{T}_{jb}R^{ij}{}_{mn}\right|\right|_t^2 - 126. \end{aligned} \tag{3.27}$$

Since $S_t = \frac{2}{3}R_t$, it follows from the above theorem that (1.6) holds true.

Before giving local curvature estimates for Laplacian flow in the next subsection, we derive evolution equations for Ric_t, Rm_t, and $\boldsymbol{T}_t$ in different forms. Using the Lichnerowicz Laplacian

$$\blacktriangle_{L,t}\eta_{jk} := \blacktriangle_t \eta_{jk} - R_j{}^p\eta_{pk} - R_k{}^p\eta_{jp} + 2R_{pjkq}h^{qp},$$

we see that the evolution equation for R_{ij} can be written as

$$\partial_t R_{jk} = -\frac{1}{2}\left[\blacktriangle_{L,t}\eta_{jk} + \nabla_j\nabla_k \mathrm{tr}_t\eta_t + \nabla_j(d_t^*\eta_t)_k + \nabla_k(d_t^*\eta_t)_j\right],$$

where $(d_t^*\eta_t)_k := -\nabla^j\eta_{jk}$. For $\eta_{ij} = -2R_{ij} - \frac{4}{3}||\boldsymbol{T}_t||_t^2 g_{ij} - 4\boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}$ we have proved $\mathrm{tr}_t\eta_t = \frac{8}{3}||\boldsymbol{T}_t||_t^2$ and $(d_t^*\eta_t)_j = \nabla_j R_t + \frac{4}{3}\nabla_j||\boldsymbol{T}_t||_t^2 + 4\nabla^i\widehat{\boldsymbol{T}}_{ij}$ with $\widehat{\boldsymbol{T}}_{ij} = \boldsymbol{T}_i{}^k\boldsymbol{T}_{kj}$. Then

$$\begin{aligned}
\partial_t R_{jk} &= \blacktriangle_{L,t}\left(R_{jk} + \frac{2}{3}||\boldsymbol{T}_t||_t^2 g_{jk} + 2\widehat{\boldsymbol{T}}_{jk}\right) - \frac{1}{2}\nabla_j\left(\nabla_k R_t + \frac{4}{3}\nabla_k||\boldsymbol{T}_t||_t^2 + 4\nabla^i\widehat{\boldsymbol{T}}_{ik}\right)\\
&\quad - \frac{4}{3}\nabla_j\nabla_k||\boldsymbol{T}_t||_t^2 - \frac{1}{2}\nabla_k\left(\nabla_j R_t + \frac{4}{3}\nabla_j||\boldsymbol{T}_t||_t^2 + 4\nabla^i\widehat{\boldsymbol{T}}_{ij}\right)\\
&= \blacktriangle_{L,t}\left(R_{jk} + \frac{2}{3}||\boldsymbol{T}_t||_t^2 g_{jk} + 2\widehat{\boldsymbol{T}}_{jk}\right) - 2\nabla_j\nabla^i\widehat{\boldsymbol{T}}_{ik} - 2\nabla_k\nabla^i\widehat{\boldsymbol{T}}_{ij} - \frac{2}{3}\nabla_j\nabla_k||\boldsymbol{T}_t||_t^2.
\end{aligned}$$

But the first term is equal to

$$\begin{aligned}
\blacktriangle_{L,t}\left(R_{jk} + \frac{2}{3}||\boldsymbol{T}_t||_t^2 g_{jk} + 2\widehat{\boldsymbol{T}}_{jk}\right) &= \blacktriangle_t R_{jk} - 2R_j{}^pR_{pk} + 2R_{pjkq}R^{pq}\\
&\quad + \left[\frac{2}{3}\left(\blacktriangle_t||\boldsymbol{T}_t||_t^2\right)g_{jk} + 2\blacktriangle_t\widehat{\boldsymbol{T}}_{jk} - 2R_j{}^p\widehat{\boldsymbol{T}}_{pk} - 2\widehat{\boldsymbol{T}}_j{}^pR^p{}_k + 4R_{pjkq}\widehat{\boldsymbol{T}}^{pq}\right],
\end{aligned}$$

we have

$$\begin{aligned}
\blacksquare_t R_{ij} &= -2R_i{}^pR_{pj} + 2R_{pijq}R^{pq} + \left[\frac{2}{3}\left(\blacktriangle_t||\boldsymbol{T}_t||_t^2\right)g_{ij} + 2\blacktriangle_t\widehat{\boldsymbol{T}}_{ij} - 2R_i{}^p\widehat{\boldsymbol{T}}_{pj}\right.\\
&\quad \left. - 2\widehat{\boldsymbol{T}}_i{}^pR_{pj} + 4R_{pijq}\widehat{T}^{pq} - 2\nabla_i\nabla^p\widehat{\boldsymbol{T}}_{pj} - 2\nabla_j\nabla^p\widehat{\boldsymbol{T}}_{pi} - \frac{2}{3}\nabla_i\nabla_j||\boldsymbol{T}_t||_t^2\right].
\end{aligned}\tag{3.28}$$

Consequently, the norm of Ric_t satisfies

$$\begin{aligned}
\blacksquare_t||\mathrm{Ric}_t||_t^2 &= -2||\nabla_t\mathrm{Ric}_t||_t^2 + 4R_{kij\ell}R^{k\ell}R^{ij} + \left[\frac{4}{3}R_t\blacktriangle_t||\boldsymbol{T}_t||_t^2 + 8R^k{}_{ij}{}^\ell\widehat{\boldsymbol{T}}_{k\ell}R^{ij} + \right.\\
&\quad \left.\frac{8}{3}||\mathrm{Ric}_t||_t^2||\boldsymbol{T}_t||_t^2 + 4R^{ij}\blacktriangle_t\widehat{\boldsymbol{T}}_{ij} - 8R^{ij}\nabla_i\nabla^k\widehat{\boldsymbol{T}}_{kj} - \frac{4}{3}R^{ij}\nabla_i\nabla_j||\boldsymbol{T}_t||_t^2\right].
\end{aligned}\tag{3.29}$$

The general formula for R_{ijk}^ℓ gives

$$\begin{aligned}
\partial_t R_{ijk}^\ell &= -\nabla_i\nabla_k R_j{}^\ell - \nabla_j\nabla^\ell R_{ik} + \nabla_i\nabla^\ell R_{jk} + \nabla_j\nabla_k R_i{}^\ell + R_{ijk}{}^qR_q{}^\ell + R_{ij}{}^{\ell q}R_{kp}\\
&\quad + 2R_{ijk}{}^q\widehat{\boldsymbol{T}}_q{}^\ell + 2R_{ij}{}^{\ell q}\widehat{\boldsymbol{T}}_{kp} - \frac{2}{3}\left(\nabla_i\nabla_k||\boldsymbol{T}_t||_t^2\right)g_j{}^\ell - \frac{2}{3}\left(\nabla_j\nabla^\ell||\boldsymbol{T}_t||_t^2\right)g_{ik}\\
&\quad + \frac{2}{3}\left(\nabla_i\nabla^\ell||\boldsymbol{T}_t||_t^2\right)g_{jk} + \frac{2}{3}\left(\nabla_j\nabla_k||\boldsymbol{T}_t||_t^2\right)g_i{}^\ell\\
&\quad - 2\nabla_i\nabla_k\widehat{\boldsymbol{T}}_j{}^\ell - 2\nabla_j\nabla^\ell\widehat{\boldsymbol{T}}_{ik} + 2\nabla_i\nabla^\ell\widehat{\boldsymbol{T}}_{jk} + 2\nabla_j\nabla_k\widehat{\boldsymbol{T}}_i{}^\ell.
\end{aligned}\tag{3.30}$$

Hence, the evolution equation for $||\mathrm{Rm}_t||_t^2$ is given by

$$\begin{aligned}
\partial_t||\mathrm{Rm}_t||_t^2 &= \nabla_t^2\mathrm{Ric}_t * \mathrm{Rm}_t + \mathrm{Ric}_t * \mathrm{Rm}_t * \mathrm{Rm}_t + \mathrm{Rm}_t * \mathrm{Rm}_t * \widehat{\boldsymbol{T}}_t\\
&\quad + \mathrm{Ric}_t * \nabla_t^2||\boldsymbol{T}_t||_t^2 + \mathrm{Rm}_t * \nabla_t^2\widehat{\boldsymbol{T}}_t + \frac{8}{3}|\boldsymbol{T}_t|_t^2||\mathrm{Rm}_t||_t^2.
\end{aligned}\tag{3.31}$$

Moreover, it was proved in [33] that

$$\begin{aligned}
||\nabla_t\mathrm{Rm}_t||_t^2 &\leq -\frac{1}{2}\blacksquare_t||\mathrm{Rm}_t||_t^2 + C_1||\mathrm{Rm}_t||_t^3\\
&\quad + C_1||\mathrm{Rm}_t||_t^{3/2}||\nabla_t^2\boldsymbol{T}_t||_t + C_1||\mathrm{Rm}_t||_t||\nabla_t\boldsymbol{T}_t||_t^2
\end{aligned}\tag{3.32}$$

where C_1 is some universal constant, and

$$\blacksquare_t \boldsymbol{T}_t = \mathrm{Rm}_t * \boldsymbol{T}_t + \mathrm{Rm}_t * \boldsymbol{T}_t * \psi_t + \nabla_t \boldsymbol{T}_t * \boldsymbol{T}_t * \varphi_t + \boldsymbol{T}_t * \boldsymbol{T}_t * \boldsymbol{T}_t. \tag{3.33}$$

Squaring (3.33) gives

$$\begin{aligned} ||\nabla_t \boldsymbol{T}_t||_t^2 \quad &\leq \quad -\frac{1}{2}\blacksquare_t ||\boldsymbol{T}_t||_t^2 + C_2 ||\mathrm{Rm}_t||_t ||\boldsymbol{T}_t||_t^2 \\ &\quad + C_2 ||\nabla_t \boldsymbol{T}_t||_t ||\boldsymbol{T}_t||_t^2 + C_2 ||\boldsymbol{T}_t||_t^4 \end{aligned} \tag{3.34}$$

for another universal constant C_2 which may differs from C_1. The Cauchy-Schwartz inequality shows $2C_2||\nabla_t \boldsymbol{T}_t||_t ||\boldsymbol{T}_t||_t^2 \leq ||\nabla_t \boldsymbol{T}_t||^2 + C_2^2 ||\boldsymbol{T}_t||_t^4$, so that the evolution inequality (3.34) becomes

$$||\nabla_t \boldsymbol{T}_t||_t^2 \leq -\blacksquare_t ||\boldsymbol{T}_t||_t^2 + C_3 ||\mathrm{Rm}_t||_t ||\boldsymbol{T}_t||_t^2 + C_3 ||\boldsymbol{T}_t||_t^4. \tag{3.35}$$

Here C_3 is a universal constant.

3.3. **Local curvature estimates.** In this section, we consider the Laplacian flow (3.1) on $\mathcal{M} \times [0, T]$, where $T \in (0, T_{\max})$. From now on we always omit the time subscripts from all considered quantities. From (3.15), (3.29), (3.31), (3.32), and (3.35) we have

$$\begin{aligned} ||\nabla \mathrm{Ric}||^2 \quad &= \quad -\frac{1}{2}\blacksquare ||\mathrm{Ric}||^2 + \mathrm{Ric} * \mathrm{Ric} * \mathrm{Rm} - \frac{1}{3}(\blacktriangle R) R - \frac{2}{3}||\mathrm{Ric}||^2 R \\ &\quad + 2\langle\langle \mathrm{Ric}, \blacktriangle \widehat{\boldsymbol{T}} \rangle\rangle + \frac{1}{3}\langle\langle \mathrm{Ric}, \nabla^2 R\rangle\rangle + \mathrm{Ric} * \widehat{\boldsymbol{T}} * \mathrm{Rm} + \mathrm{Ric} * \nabla^2 \widehat{\boldsymbol{T}}, \\ ||\nabla \mathrm{Rm}||^2 \quad &\leq \quad -\frac{1}{2}\blacksquare ||\mathrm{Rm}||^2 + C||\mathrm{Rm}||^3 + C||\mathrm{Rm}||^{3/2} ||\nabla^2 \boldsymbol{T}|| + C||\mathrm{Rm}|| ||\nabla \boldsymbol{T}||^2, \\ \partial_t ||\mathrm{Rm}||^2 \quad &= \quad \nabla^2 \mathrm{Ric} * \mathrm{Rm} + \mathrm{Ric} * \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} * \widehat{\boldsymbol{T}} \\ &\quad + \mathrm{Ric} * \nabla^2 ||\boldsymbol{T}||^2 + \mathrm{Rm} * \nabla^2 \widehat{\boldsymbol{T}} + \frac{4}{3}||\boldsymbol{T}||^2 ||\mathrm{Rm}||^2, \\ ||\nabla \boldsymbol{T}||^2 \quad &\leq \quad -\blacksquare ||\boldsymbol{T}||^2 + C||\mathrm{Rm}|| ||\boldsymbol{T}||^2 + C||\boldsymbol{T}||^4, \\ \partial_t dV \quad &= \quad \frac{2}{3}||\boldsymbol{T}||^2 dV, \quad R \quad = \quad -||\boldsymbol{T}||^2. \end{aligned}$$

Choose an open domain Ω of $\mathcal{M}$ and assume that

$$||\mathrm{Ric}|| \leq K \tag{3.36}$$

on $\Omega \times [0, T]$, Then the torsion $\boldsymbol{T}$ satisfies $||\boldsymbol{T}|| \lesssim K^{1/2}$ and metrics g_t are all equivalent to g_0. We also observe from (2.25) and (3.19) that

$$||\mathrm{Ric}|| \lesssim 1 \Longleftrightarrow |\Delta\varphi| \lesssim 1 \tag{3.37}$$

and the following simple fact

$$\partial_t ||A||^2 = \frac{p}{2}||A||^{p-2} \partial_t ||A||^2 \tag{3.38}$$

for any tensor A.

Choose a Lipschitz function η with support in Ω and consider the quantity

$$\frac{d}{dt}\int ||\mathrm{Rm}||^p \eta^{2p} dV, \quad \int := \int_{\mathcal{M}},$$

where $p \geq 5$. As in [27], we introduce the following "good" quantities

$$A_1 := \int ||\mathrm{Rm}||^p \eta^{2p} dV, \quad A_2 := \int ||\mathrm{Rm}||^{p-1}\eta^{2p} dV,$$
$$A_3 := \int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-1} dV, \quad A_4 := \int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2} dV$$

and also "bad" quantities

$$B_1 := \frac{1}{K}\int ||\nabla \mathrm{Ric}||^2 ||\mathrm{Rm}||^{p-1}\eta^{2p} dV, \quad B_2 := \int ||\nabla \mathrm{Rm}||^2 ||\mathrm{Rm}||^{p-3}\eta^{2p} dV.$$

We split the proof of Theorem 1.4 into four steps.

(a) In the first step, we can show that, see Lemma 3.3,

$$\begin{aligned}\frac{d}{dt}A_1 \ &\leq \ B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2A_2 \\ &+ c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p} dV.\end{aligned}$$

(b) In the second step, we can prove that the term

$$c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p} dV$$

is bounded from above by (see (3.47))

$$B_1 + cKB_2 + cK^2A_2 + cKA_1 - \frac{d}{dt}\left[\int c(-R)||\mathrm{Rm}||^{p-1}\eta^{2p} dV\right].$$

Observe that the above integral is nonnegative, since the scalar curvature R is nonpositive along the Laplacian flow on closed G_2-structures. Hence we obtain from the first step that, see Lemma 3.4,

$$\begin{aligned}\frac{d}{dt}A_1 \ &\leq \ 2B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2A_2 \\ &- \frac{d}{dt}\left[\int c(-R)||\mathrm{Rm}||^{p-1}\eta^{2p} dV\right].\end{aligned}$$

(c) In the next two steps, we estimate the bad terms B_1 and B_2. In the third step, B_1 is estimated by (see (3.57))

$$\begin{aligned}B_1 \ &\leq \ cKB_2 + cKA_4 + cKA_1 + cK^2A_2 \\ &- \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p} dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p} dV\right].\end{aligned}$$

Then the second step can be simplified as, see Lemma 3.5,

$$\begin{aligned}\frac{d}{dt}A_1 \ &\leq \ cKB_2 + cKA_4 + cKA_1 + cK^2A_2 \\ &- \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p} dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p} dV\right].\end{aligned}$$

(d) Finally, we estimate the term B_2. In this step we shall use the assumption that $p \geq 5$. Using the inequality $||\nabla \boldsymbol{T}|| \lesssim ||\mathrm{Rm}||$ and $||\nabla^2\boldsymbol{T}|| \lesssim ||\nabla\mathrm{Rm}|| + ||\mathrm{Rm}||||\boldsymbol{T}|| + ||\nabla\boldsymbol{T}||||\boldsymbol{T}|| + ||\boldsymbol{T}||^3$, we can prove (see (3.67))

$$B_2 \leq cA_4 + cA_1 - \frac{d}{dt}\left[\frac{1}{p-1}\int ||\mathrm{Rm}||^{p-1}\eta^{2p} dV\right].$$

Plugging it into the third step, we arrive at, see Lemma 3.6,

$$\begin{aligned}\frac{d}{dt}(A_1+cKA_2) &\leq cK(A_1+cKA_2)+cKA_4\\ &\quad -\frac{d}{dt}\left[\frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right.\\ &\quad \left.+c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].\end{aligned}$$

If we choose a geodesic ball $\Omega := B_{g_0}(x_0,\rho/\sqrt{K})$ and a cut-off function η so that $||\nabla\phi|| \leq \sqrt{K}e^{cKT}/\rho$, then the above inequality gives a proof of Theorem 1.4.

We are going to carry out the above mentioned four steps. From (3.39) and the above evolution equations, we have

$$\begin{aligned}\frac{d}{dt}\int ||\mathrm{Rm}||^p\eta^{2p}dV &= \int\left(\partial_t||\mathrm{Rm}||^p\right)\eta^{2p}dV+\int ||\mathrm{Rm}||^p\eta^{2p}\partial_t dV\\ &= \int \frac{p}{2}||\mathrm{Rm}||^{p-2}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV+\int ||\mathrm{Rm}||^p\eta^{2p}\left(-\frac{2}{3}R\right)dV\\ &= \int\frac{p}{2}||\mathrm{Rm}||^{p-2}\left[\begin{array}{c}\nabla^2\mathrm{Ric}*\mathrm{Rm}+\mathrm{Ric}*\mathrm{Rm}*\mathrm{Rm}\\ +\mathrm{Rm}*\mathrm{Rm}*\widehat{\boldsymbol{T}}+\mathrm{Ric}*\nabla^2||\boldsymbol{T}||^2\\ +\mathrm{Rm}*\nabla^2\widehat{\boldsymbol{T}}+\frac{4}{3}||\boldsymbol{T}||^2||\mathrm{Rm}||^2\end{array}\right]\eta^{2p}dV\\ &\quad -\frac{2}{3}\int R||\mathrm{Rm}||^p\eta^{2p}dV \qquad (3.39)\\ &\leq c\int ||\mathrm{Rm}||^{p-2}\Big[\nabla^2\mathrm{Ric}*\mathrm{Rm}+K||\mathrm{Rm}||^2+K||\mathrm{Rm}||^2+\nabla^2||\boldsymbol{T}||^2*\mathrm{Ric}\\ &\quad +\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\Big]\eta^{2p}dV+cK\int ||\mathrm{Rm}||^p\eta^{2p}dV\\ &\leq c\int ||\mathrm{Rm}||^{p-2}\left[\nabla^2\mathrm{Ric}*\mathrm{Rm}+\nabla^2||\boldsymbol{T}||^2*\mathrm{Ric}+\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right]\eta^{2p}dV\\ &\quad +cK\int ||\mathrm{Rm}||^p\eta^{2p}dV.\end{aligned}$$

It was proved in [24] that the first integral in (3.39) is bounded by

$$\begin{aligned}c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2\mathrm{Ric}*\mathrm{Rm}\right)\eta^{2p}dV &\leq \frac{1}{K}\int ||\nabla\mathrm{Ric}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\ &\quad +cK\int ||\nabla\mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV+cK\int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2}dV. \qquad (3.40)\end{aligned}$$

Since $||\boldsymbol{T}||^2=-R$, the same inequality holds for the integral

$$c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2||\boldsymbol{T}||^2*\mathrm{Ric}\right)\eta^{2p}dV.$$

To deal with the last term in the bracket of (3.39), we use the same argument of [24] to conclude

$$\begin{aligned}c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)\eta^{2p}dV &= c\int\left(\nabla||\mathrm{Rm}||^{p-2}*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)\eta^{2p}dV\\ &\quad +c\int\left(||\mathrm{Rm}||^{p-2}*\nabla\widehat{\boldsymbol{T}}*\nabla\mathrm{Rm}\right)\eta^{2p}dV\end{aligned}$$

$$
\begin{aligned}
&\quad + c\int \left(||\mathrm{Rm}||^{p-2} * \nabla\widehat{\boldsymbol{T}} * \mathrm{Rm} * \nabla\eta\right)\eta^{2p-1}dV \\
&\leq c\int ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||||\nabla\widehat{\boldsymbol{T}}||\eta^{2p}dV + c\int ||\mathrm{Rm}||^{p-2}||\nabla\widehat{\boldsymbol{T}}||||\nabla\mathrm{Rm}||\eta^{2p}dV \\
&\quad + c\int ||\mathrm{Rm}||^{p-1}||\nabla\widehat{\boldsymbol{T}}||||\nabla\eta||\eta^{2p-1}dV \\
&\leq c\int ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||||\nabla\widehat{\boldsymbol{T}}||\eta^{2p}dV + c\int ||\mathrm{Rm}||^{p-1}||\nabla\widehat{\boldsymbol{T}}||||\nabla\eta||\eta^{2p-1}dV.
\end{aligned}
$$

According to the Cauchy-Schwartz inequality, the first and second integrals are bounded by

$$
\begin{aligned}
&\int ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||||\nabla\widehat{\boldsymbol{T}}||\eta^{2p}dV \\
&= \int \left(||\mathrm{Rm}||^{\frac{p-3}{2}}||\nabla\mathrm{Rm}||\eta^{p}\right)\left(||\mathrm{Rm}||^{\frac{p-1}{2}}||\nabla\widehat{\boldsymbol{T}}||\eta^{p}\right)dV \\
&\leq cK\int ||\nabla\mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV + \frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV
\end{aligned}
$$

and

$$
\begin{aligned}
&\int ||\mathrm{Rm}||^{p-1}||\nabla\widehat{\boldsymbol{T}}||||\nabla\eta||\eta^{2p-1}dV \\
&= \int \left(||\mathrm{Rm}^{\frac{p-1}{2}}||\nabla\widehat{\boldsymbol{T}}||\eta^{p}\right)\left(||\mathrm{Rm}||^{\frac{p-1}{2}}||\nabla\eta||\eta^{p-1}\right)dV \\
&\leq \frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + cK\int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2}dV.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
c\int ||\mathrm{Rm}||^{p-2}\left(\nabla^2\widehat{\boldsymbol{T}} * \mathrm{Rm}\right)\eta^{2p}dV &\leq \frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&\quad + cK\int ||\nabla\mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV + cK\int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2}dV. \qquad (3.41)
\end{aligned}
$$

Using $\widehat{\boldsymbol{T}} = \boldsymbol{T} * \boldsymbol{T}$ and $R = -||\boldsymbol{T}||^2$ yields

$$
\begin{aligned}
&\frac{1}{K}\int ||\nabla\widehat{\boldsymbol{T}}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&\leq \frac{c}{K}\int ||\nabla\boldsymbol{T}||^2||\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \leq c\int ||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \qquad (3.42) \\
&\leq c\int \left(-\frac{1}{4}\blacksquare||\boldsymbol{T}||^2 + c||\mathrm{Rm}||||\boldsymbol{T}||^2 + c||\boldsymbol{T}||^4\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&= c\int \left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&\quad + cK\int ||\mathrm{Rm}||^{p}\eta^{2p}dV + cK^2\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV.
\end{aligned}
$$

Hence, using (3.40), (3.41), and (3.42), we arrive at

Lemma 3.3. *One has*

$$
\begin{aligned}
A_1' \equiv \frac{d}{dt}A_1 &\leq B_1 + cKB_2 + cKA_4 + cKA_1 + cK^2A_2 \\
&\quad + c\int \left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV. \qquad (3.43)
\end{aligned}
$$

In the following computations, we are mainly going to estimate or simplify the bad terms B_1, B_2, and also the term involving $-\blacksquare||\boldsymbol{T}||^2$. Integration by parts on the last integral in (3.43) and using $R = -||\boldsymbol{T}||^2$, we obtain

$$
\begin{aligned}
c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV &= c\int\left((\partial_t-\Delta)R\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\\
&= c\int(\partial_t R)\,||\mathrm{Rm}||^{p-1}\eta^{2p}dV + c\int\left\langle\nabla R,\nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)\right\rangle dV\\
&= \frac{d}{dt}\left(c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right) - c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV\\
&\quad - c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}\partial_t dV + c\int\left\langle\nabla R, ||\mathrm{Rm}||^{p-3}\mathrm{Rm}*\nabla\mathrm{Rm}\right\rangle\eta^{2p}dV\\
&\quad + c\int\left\langle\nabla R, ||\mathrm{Rm}||^{p-1}\eta^{2p-1}\nabla\eta\right\rangle dV\\
&\leq c\int||\mathrm{Rm}||^{p-2}\langle\nabla R,\nabla\mathrm{Rm}\rangle\eta^{2p}dV + c\int||\mathrm{Rm}||^{p-1}||\nabla R||\,||\nabla\eta||\eta^{2p-1}dV\\
&\quad + c\int R^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV - c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV\\
&\quad + \frac{d}{dt}\left(c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right).
\end{aligned}
$$

The first two integrals can be simplified by using the Cauchy-Schwarz inequality as follows:

$$
\begin{aligned}
c\int||\mathrm{Rm}||^{p-2}\langle\nabla R,\nabla\mathrm{Rm}\rangle\eta^{2p}dV &\leq c\int||\nabla\mathrm{Ric}||\,||\nabla\mathrm{Rm}||\,||\mathrm{Rm}||^{p-2}\eta^{2p}dV\\
\leq c\int\left(||\nabla\mathrm{Rm}||\,||\mathrm{Rm}||^{\frac{p-3}{2}}\eta^p\right)\left(||\nabla\mathrm{Ric}||\,||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^p\right)dV &\leq \frac{1}{50}B_1 + cKB_2
\end{aligned}
$$

and

$$
\begin{aligned}
c\int||\mathrm{Rm}||^{p-1}||\nabla R||\,||\nabla\eta||\eta^{2p-1}dV &\leq c\int||\mathrm{Rm}||^{p-1}||\nabla\mathrm{Ric}||\,||\nabla\eta||\eta^{2p-1}dV\\
\leq c\int\left(||\mathrm{Rm}||^{\frac{p-1}{2}}||\nabla\eta||\eta^{p-1}\right)\left(||\mathrm{Rm}||^{\frac{p-1}{2}}||\nabla\mathrm{Ric}||\eta^p\right)dV &\leq \frac{1}{50}B_1 + cKA_4.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c\int\left(-\blacksquare||\boldsymbol{T}||^2\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV &\leq \frac{2}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2\\
&\quad + \frac{d}{dt}\left(c\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right) - c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV. \qquad (3.44)
\end{aligned}
$$

Now, the second integral in (3.44) is equal to

$$
\begin{aligned}
-c\int R\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV &= c\int(-R)\,||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV\\
&= c\int(-R)||\mathrm{Rm}||^{p-3}\Big[\nabla^2\mathrm{Ric}*\mathrm{Rm} + \mathrm{Ric}*\mathrm{Rm}*\mathrm{Rm} + \mathrm{Rm}*\mathrm{Rm}*\widehat{\boldsymbol{T}}\\
&\quad + \mathrm{Ric}*\nabla^2||\boldsymbol{T}||^2 + \mathrm{Rm}*\nabla^2\widehat{\boldsymbol{T}} + \frac{4}{3}||\boldsymbol{T}||^2||\mathrm{Rm}||^2\Big]\eta^{2p}dV\\
&\leq c\int(-R)||\mathrm{Rm}||^{p-3}\left[\nabla^2\mathrm{Ric}*\mathrm{Rm} - \mathrm{Ric}*\nabla^2 R + \nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right]\eta^{2p}dV + cK^2A_2.
\end{aligned}
$$

Using the identity, where $p \geq 5$,

$$\nabla||\mathrm{Rm}||^{p-3} = \frac{p-3}{2}\left(||\mathrm{Rm}||^2\right)^{\frac{p-3}{2}-1}\nabla||\mathrm{Rm}||^2 = ||\mathrm{Rm}||^{p-5}\mathrm{Rm}*\nabla\mathrm{Rm}$$

we obtain

$$\begin{aligned}
c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla^2\mathrm{Ric}*\mathrm{Rm})dV &= c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\mathrm{Ric}*\nabla\mathrm{Rm})dV \\
&+ c\int\left\{\nabla\left[(-R)||\mathrm{Rm}||^{p-3}\phi^{2p}\right]*\nabla\mathrm{Ric}*\mathrm{Rm}\right\}dV \\
= c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\mathrm{Ric}*\nabla\mathrm{Rm})dV &+ c\int||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla R*\nabla\mathrm{Ric}*\mathrm{Rm})dV \\
&+ c\int(-R)\eta^{2p}\left(\nabla||\mathrm{Rm}||^{p-3}*\nabla\mathrm{Ric}*\mathrm{Rm}\right)dV \\
&+ c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p-1}\left(\nabla\phi*\nabla\mathrm{Ric}*\mathrm{Rm}\right)dV \\
\leq\ c\int||\mathrm{Rm}||^{p-2}\eta^{2p}||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||dV &+ c\int||\nabla\mathrm{Ric}||||\nabla R||||\mathrm{Rm}||^{p-2}\eta^{2p}dV \\
+ c\int||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||\eta^{2p}dV &+ c\int||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||||\nabla\mathrm{Ric}||dV \\
&\leq\ c\int\left(||\nabla\mathrm{Ric}||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^p\right)\left(||\nabla\mathrm{Rm}||||\mathrm{Rm}||^{\frac{p-3}{2}}\eta^p\right)dV \\
+ c\int\left(||\nabla\mathrm{Ric}||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^p\right)\left(||\nabla\phi||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^{p-1}\right)dV &\ \leq\ \frac{1}{50}B_1+cKB_2+cKA_4.
\end{aligned}$$

Similarly, we can prove

$$c\int(-R)||\mathrm{Rm}||^{p-3}\left(-\mathrm{Ric}*\nabla^2R\right)\eta^{2p}dV \leq \frac{1}{50}B_1+cKB_2+cKA_4.$$

Using $\nabla\widehat{\boldsymbol{T}} = \nabla\boldsymbol{T}*\boldsymbol{T} \leq c||\nabla\boldsymbol{T}||||\boldsymbol{T}|| \leq cK^{1/2}||\nabla\boldsymbol{T}||$ yields

$$\begin{aligned}
c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla^2\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)dV\ &=\ c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\widehat{\boldsymbol{T}}*\nabla\mathrm{Rm})dV \\
&+ c\int\left\{\nabla\left[(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}\right]*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm}\right\}dV \\
=\ c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla\widehat{T}*\nabla\mathrm{Rm})dV &+ c\int||\mathrm{Rm}||^{p-3}\eta^{2p}(\nabla R*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm})dV \\
&+ c\int(-R)\eta^{2p}\left(\nabla||\mathrm{Rm}||^{p-3}*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)dV \\
&+ c\int(-R)||\mathrm{Rm}||^{p-3}\eta^{2p-1}\left(\nabla\eta*\nabla\widehat{\boldsymbol{T}}*\mathrm{Rm}\right)dV \\
\leq\ c\int\left(||\mathrm{Rm}||^{p-2}\eta^{2p}||\nabla\mathrm{Rm}||\right. &+ \left.||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||\right)\left(K^{1/2}||\nabla\boldsymbol{T}||\right)dV \\
&\leq\ c\int\left(||\nabla\mathrm{Rm}||||\mathrm{Rm}||^{\frac{p-3}{2}}\eta\right)\left(||\nabla\boldsymbol{T}||K^{1/2}||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^p\right)dV \\
&+\ \int\left(||\nabla\eta||||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^{p-1}\right)\left(||\nabla\boldsymbol{T}||K^{1/2}||\mathrm{Rm}||^{\frac{p-1}{2}}\eta^p\right)dV \\
&\leq\ \epsilon c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + \frac{cK}{\epsilon}B_2 + \frac{cK}{\epsilon}A_4.
\end{aligned}$$

According to (3.44) we get

$$c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV$$

$$\leq \; c \int \left(-\blacksquare ||\boldsymbol{T}||^2 \right) ||\mathrm{Rm}||^{p-1} \eta^{2p} dV + cKA_1 + cK^2 A_2$$

$$\leq \; \frac{2}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1$$

$$+ \frac{d}{dt} \left(c \int R ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right) - c \int R \left(\partial_t ||\mathrm{Rm}||^{p-1} \right) \eta^{2p} dV$$

$$\leq \; \frac{2}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1$$

$$+ \frac{d}{dt} \left(\int cR ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right) + c \int (-R) ||\mathrm{Rm}||^{p-3} \left(\partial_t ||\mathrm{Rm}||^2 \right) \eta^{2p} dV.$$

Hence

$$c \int (-R) ||\mathrm{Rm}||^{p-3} \left(\partial_t ||\mathrm{Rm}||^2 \right) \eta^{2p} dV \;\; \leq \;\; \frac{2}{50} B_1 + cKB_2 + cKA_4 + \frac{cK}{\epsilon} B_2 + \frac{cK}{\epsilon} A_4$$

$$+ \epsilon \left[\frac{2}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left(\int cR ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right) \right]$$

$$+ \epsilon c \int (-R) ||\mathrm{Rm}||^{p-3} \left(\partial_t ||\mathrm{Rm}||^2 \right) \eta^{2p} dV.$$

Choosing $\epsilon = \frac{1}{2}$ yields

$$\frac{c}{2} \int (-R) ||\mathrm{Rm}||^{p-3} \left(\partial_t ||\mathrm{Rm}||^2 \right) \eta^{2p} dV$$

$$\leq \; \frac{3}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left(\int cR ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right)$$

and

$$c \int ||\nabla \boldsymbol{T}||^2 ||\mathrm{Rm}||^{p-1} \eta^{2p} dV$$

$$\leq \; \frac{8}{50} B_1 + cKB_2 + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left(\int 2cR ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right).$$

Thus

$$\begin{aligned} c \int (-R) ||\mathrm{Rm}||^{p-3} \left(\partial_t ||\mathrm{Rm}||^2 \right) \eta^{2p} dV \;\; &\leq \;\; \frac{3}{50} B_1 + cKB_2 \\ + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left(\int cR ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right) & \qquad (3.45) \end{aligned}$$

and

$$\begin{aligned} c \int ||\nabla \boldsymbol{T}||^2 ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \;\; &\leq \;\; \frac{8}{50} B_1 + cKB_2 \\ + cKA_4 + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left(\int cR ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right) & \qquad (3.46) \end{aligned}$$

and

$$\begin{aligned} c \int \left(-\blacksquare ||\boldsymbol{T}||^2 \right) ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \;\; &\leq \;\; \frac{5}{50} B_1 + cKB_2 \\ + cK^2 A_2 + cKA_1 + \frac{d}{dt} \left(\int cR ||\mathrm{Rm}||^{p-1} \eta^{2p} dV \right). & \qquad (3.47) \end{aligned}$$

From (3.43) and (3.47) we arrive at

Lemma 3.4. *One has*

$$\begin{aligned} A_1' &\le 2B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\ &\quad + \frac{d}{dt}\left(\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right). \end{aligned} \tag{3.48}$$

We next estimate B_1 and B_2. Actually, we shall see that B_1 can be estimated in terms of B_2. Hence the key step is to estimate B_2. For B_1, using

$$\begin{aligned} ||\nabla \mathrm{Ric}||^2 &= -\frac{1}{2}\blacksquare||\mathrm{Ric}||^2 + \mathrm{Ric} * \mathrm{Ric} * \mathrm{Rm} - \frac{1}{3}(\blacktriangle R)\boldsymbol{T} - \frac{2}{3}R||\mathrm{Ric}||^2 \\ &\quad + 2\langle\langle \mathrm{Ric}, \blacktriangle\widehat{\boldsymbol{T}}\rangle\rangle + \frac{1}{3}\langle\langle \mathrm{Ric}, \nabla^2 R\rangle\rangle + \mathrm{Ric} * \widehat{\boldsymbol{T}} * \mathrm{Rm} + \mathrm{Ric} * \nabla^2\widehat{\boldsymbol{T}}. \end{aligned}$$

we obtain

$$\begin{aligned} B_1 &\le \frac{1}{2K}\int ||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\blacktriangle - \partial_t\right)||\mathrm{Ric}||^2 dV + cKA_1 \\ &\quad + \frac{1}{3K}\int (-R)||\mathrm{Rm}||^{p-1}\eta^{2p}\Delta R\, dV + \frac{2}{K}\int \langle\langle \mathrm{Ric}, \blacktriangle\widehat{\boldsymbol{T}}\rangle\rangle ||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &\quad + \frac{1}{3K}\int \langle\langle \mathrm{Ric}, \nabla^2 R\rangle\rangle ||\mathrm{Rm}||^{p-1}\eta^{2p}dV + \frac{1}{K}\int ||\mathrm{Rm}||^{p-1}\left(\mathrm{Ric} * \nabla^2\widehat{\boldsymbol{T}}\right)\eta^{2p}dV. \end{aligned} \tag{3.49}$$

From the estimates $\nabla||\mathrm{Ric}||^2 \lesssim ||\mathrm{Ric}||||\nabla\mathrm{Ric}||$, $\nabla||\mathrm{Rm}||^{p-1} \lesssim ||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||$, and $\partial_t||\mathrm{Rm}||^{p-1} = \frac{p-1}{2}||\mathrm{Rm}||^{p-3}\partial_t||\mathrm{Rm}||^2$, we have

$$\begin{aligned} &\int ||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\blacktriangle - \partial_t\right)||\mathrm{Ric}||^2 dV \\ &= \int \nabla||\mathrm{Ric}||^2 * \nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)dV - \int ||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\partial_t||\mathrm{Ric}||^2\right)dV \\ &= \int\left(\nabla||\mathrm{Ric}||^2 * \nabla||\mathrm{Rm}||^{p-1}\right)\eta^{2p}dV + \int\left(\nabla||\mathrm{Ric}||^2 * \nabla\eta\right)||\mathrm{Rm}||^{p-1}\eta^{2p-1}dV \\ &\quad - \frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}\eta^{2p}||\mathrm{Ric}||^2 dV\right] + \int\left(\partial_t||\mathrm{Rm}||^{p-1}\right)\eta^{2p}||\mathrm{Ric}||^2 dV \\ &\quad + \int ||\mathrm{Rm}||^{p-1}\eta^{2p}||\mathrm{Ric}||^2(\partial_t dV) \\ &\le cK\int ||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||||\mathrm{Rm}||^{p-2}\eta^{2p}dV + cK\int ||\nabla\mathrm{Ric}||||\nabla\eta||||\mathrm{Rm}||^{p-1}\eta^{2p-1}dV \\ &\quad + c\int ||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}||\mathrm{Ric}||^2 dV + cK^2A_1 \\ &\quad - \frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right] \\ &\le cK\left(\frac{1}{50c}B_1 + cKB_2\right) + cK\left(\frac{1}{50c}B_1 + cKA_4\right) + cK^2A_1 \\ &\quad + c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV - \frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right] \\ &\le \frac{2}{50}KB_1 + cK^2B_2 + cK^2A_4 + cK^2A_1 \\ &\quad + c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV - \frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right]. \end{aligned}$$

Thus

$$\int ||\mathrm{Rm}||^{p-1}\eta^{2p}\blacksquare||\mathrm{Ric}||^2 dV \le \frac{2}{50}KB_1 + cK^2B_2 + cK^2A_4 + cK^2A_1 \tag{3.50}$$

$$+ c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV - \frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right].$$

Consider the term

$$c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV = c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}$$
$$\Big[\nabla^2\mathrm{Ric} * \mathrm{Rm} + \mathrm{Ric} * \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} * \widehat{\boldsymbol{T}} + \mathrm{Ric} * \nabla^2||\boldsymbol{T}||^2 + \mathrm{Rm} * \nabla^2\widehat{\boldsymbol{T}}$$
$$+ \frac{4}{3}||\boldsymbol{T}||^2||\mathrm{Rm}||^2\Big]dV \le c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\Big[\nabla^2\mathrm{Ric} * \mathrm{Rm} - \nabla^2 R * \mathrm{Ric}$$
$$+ \nabla^2\widehat{\boldsymbol{T}} * \mathrm{Rm}\Big]dV + cK^2A_2.$$

The three terms in the bracket can be estimated as follows. Firstly

$$\begin{aligned}
& c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla^2\mathrm{Ric} * \mathrm{Rm}\right)dV \\
&= c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla\mathrm{Ric} * \nabla\mathrm{Rm}\right)dV \\
&+ c\int \left\{\nabla\left[||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\right] * \nabla\mathrm{Ric} * \mathrm{Rm}\right\}dV \\
&= c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla\mathrm{Ric} * \nabla\mathrm{Rm}\right)dV \\
&+ c\int ||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla||\mathrm{Ric}||^2 * \nabla\mathrm{Ric} * \mathrm{Rm}\right)dV \\
&+ c\int ||\mathrm{Ric}||^2\eta^{2p}\left(\nabla||\mathrm{Rm}||^{p-3} * \nabla\mathrm{Ric} * \mathrm{Rm}\right)dV \\
&+ c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p-1}\left(\nabla\eta * \nabla\mathrm{Ric} * \mathrm{Rm}\right)dV
\end{aligned}$$

$$\le cK\int ||\mathrm{Rm}||^{p-2}\eta^{2p}||\nabla\mathrm{Ric}||||\nabla\mathrm{Rm}||dV + cK\int ||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\mathrm{Ric}||||\nabla\eta||dV$$
$$\le cK\left(\epsilon B_1 + \frac{K}{\epsilon}B_2\right) + cK\left(\epsilon B_1 + \frac{K}{\epsilon}A_4\right) \le \frac{1}{50}KB_1 + cK^2B_2 + cK^2A_4.$$

The same estimate holds for

$$c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(-\nabla^2 R * \mathrm{Ric}\right)dV.$$

Finally,

$$c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\nabla^2\widehat{\boldsymbol{T}} * \mathrm{Rm}\right)dV = c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}$$
$$\left(\nabla\widehat{\boldsymbol{T}} * \nabla\mathrm{Rm}\right)dV + c\int\left\{\nabla\left(||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\right) * \nabla\widehat{\boldsymbol{T}} * \mathrm{Rm}\right\}dV$$
$$\begin{aligned}
&\le c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(K^{1/2}||\nabla\boldsymbol{T}||||\nabla\mathrm{Rm}||\right)dV \\
&+ c\int\left(\nabla||\mathrm{Ric}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||dV \\
&+ c\int ||\mathrm{Rm}||^2\left(\nabla||\mathrm{Rm}||^{p-3}\right)\eta^{2p}||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||dV
\end{aligned}$$

$$
\begin{aligned}
&+ c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p-1}||\nabla\eta||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||dV \\
&\leq\ cK\int ||\mathrm{Rm}||^{p-2}\eta^{2p}\left(K^{1/2}||\nabla \boldsymbol{T}||||\nabla\mathrm{Rm}||\right)dV \\
&+ cK\int ||\mathrm{Rm}||^{p-1}\eta^{2p-1}\left(K^{1/2}||\nabla\eta||||\nabla \boldsymbol{T}||\right)dV \\
&\leq\ K\left[cKB_2+\frac{cK}{\epsilon}A_4+\epsilon c\int ||\nabla \boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\
&\leq\ \frac{8}{50}KB_1+cK^2B_2+cK^2A_4+cK^3A_2+cK^2A_1+\frac{d}{dt}\left[cK\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c\int ||\mathrm{Ric}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}\left(\partial_t||\mathrm{Rm}||^2\right)dV\ &\leq\ \frac{10}{50}KB_1+cK^2B_2+cK^2A_4+cK^3A_2 \\
&+ cK^2A_1+cK\frac{d}{dt}\left[\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \qquad (3.51)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2K}\int ||\mathrm{Rm}||^{p-1}\eta^{2p}(\blacktriangle-\partial_t)||\mathrm{Ric}||^2dV\ &\leq\ \frac{6}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1 \\
&- \frac{1}{K}\frac{d}{dt}\left[\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV\right]+c\frac{d}{dt}\left[\int R||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\
&\leq\ \frac{6}{50}B_1+cKB_2+cKA_4+cK^2A_2+cKA_1 \qquad (3.52) \\
&- \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV+c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
\end{aligned}
$$

In the following, we estimate the left four terms in (3.49). We start from terms involving the scalar curvature.

$$
\begin{aligned}
\frac{1}{3K}\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}\Delta R\,dV\ &=\ -\frac{1}{3K}\int\nabla R\cdot\nabla\left[(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}\right]dV \\
&=\ -\frac{1}{3K}\int\nabla R\cdot\Big[-\nabla R||\mathrm{Rm}||^{p-1}\eta^{2p}+(-R)\nabla||\mathrm{Rm}||^{p-1}\eta^{2p} \\
&+2p(-R)||\mathrm{Rm}||^{p-1}\eta^{2p-1}\nabla\eta\Big]dV\ \leq\ \frac{1}{3K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \qquad (3.53) \\
&+\frac{c}{K}\int(-R)||\mathrm{Rm}||^{p-2}||\nabla R||||\nabla\mathrm{Rm}||\eta^{2p}dV \\
&+\frac{c}{K}\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla R||||\nabla\eta||dV \\
&\leq\ \frac{1}{3K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+\frac{1}{3K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKB_2 \\
&+\frac{1}{3K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKA_4 \\
&\leq\ \frac{1}{K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV+cKB_2+cKA_4.
\end{aligned}
$$

The another term involving the scalar curvature can be estimated by

$$\begin{aligned}
\frac{1}{3K}\int\langle\langle \mathrm{Ric}, \nabla^2 R\rangle\rangle||\mathrm{Rm}||^{p-1}\eta^{2p}dV &= -\frac{1}{3K}\int \nabla^j R\nabla^i\left[R_{ij}||\mathrm{Rm}||^{p-1}\eta^{2p}\right]dV \\
&= -\frac{1}{3K}\int\nabla^j R\bigg[\frac{1}{2}\nabla_j R||\mathrm{Rm}||^{p-1}\eta^{2p}+R_{ij}\nabla^i||\mathrm{Rm}||^{p-1}\eta^{2p} \\
+R_{ij}||\mathrm{Rm}||^{p-1}2p\eta^{2p-1}\nabla^i\eta\bigg]dV &\leq -\frac{1}{6K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&+\frac{c}{K}\int||\mathrm{Ric}||||\nabla R||||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||\eta^{2p}dV \\
&+\frac{c}{K}\int||\nabla R||||\mathrm{Ric}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV \\
\leq -\frac{1}{6K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV &+ \frac{1}{18K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + cKB_2 \\
+\frac{1}{18K}\int||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV &+ cKA_4 \leq cKB_2 + cKA_4.
\end{aligned} \tag{3.54}$$

Using (3.46) we obtain

$$\begin{aligned}
\frac{2}{K}\int\langle\langle\mathrm{Ric}, \blacktriangle\widehat{\boldsymbol{T}}\rangle\rangle||\mathrm{Rm}||^{p-1}\eta^{2p}dV &= \frac{1}{K}\int\left(\mathrm{Ric}*\blacktriangle\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
= \frac{1}{K}\int\left(\nabla\mathrm{Ric}*\nabla\widehat{T}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV &+ \frac{1}{K}\int\mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}*\nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)dV \\
\leq\frac{c}{K}\int||\nabla\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p}dV &+ \frac{c}{K}\int||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||\eta^{2p}dV \\
&+\frac{c}{K}\int||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV \\
&\leq \frac{1}{50}B_1 + c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + cKB_2 \\
+c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV &+ cKA_4 + c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&\leq \frac{1}{50}B_1 + cKB_2 + cKA_4 + c\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
\leq \frac{9}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 &+ cKA_1 + \frac{d}{dt}\left[\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
\end{aligned} \tag{3.55}$$

Similarly, we can prove

$$\begin{aligned}
\frac{1}{K}\int\left(\mathrm{Ric}*\nabla^2\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV &= \frac{1}{K}\int\left(\nabla\mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
+\frac{1}{K}\int\mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}*\nabla\left(||\mathrm{Rm}||^{p-1}\eta^{2p}\right)dV &\leq \frac{1}{K}\int\left(\nabla\mathrm{Ric}*\nabla\widehat{\boldsymbol{T}}\right)||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\
&+\frac{c}{K}\int||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||\eta^{2p}dV \\
&+\frac{c}{K}\int||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV \\
\leq\frac{c}{K}\int||\nabla\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p}dV &+ \frac{c}{K}\int||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||\eta^{2p}dV \\
&+\frac{c}{K}\int||\mathrm{Ric}||||\nabla\widehat{\boldsymbol{T}}||||\mathrm{Rm}||^{p-1}\eta^{2p-1}||\nabla\eta||dV
\end{aligned} \tag{3.56}$$

$$\leq \frac{9}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 + \frac{d}{dt}\left[\int cR||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].$$

Plugging (3.50) and (3.53) – (3.56) into (3.49), and using (3.46) and $||\nabla R||^2 \leq cK||\nabla \boldsymbol{T}||^2$, we obtain

$$\begin{aligned} B_1 &\leq \frac{6}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\ &- \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\ &+ \frac{1}{K}\int ||\nabla R||^2||\mathrm{Rm}||^{p-1}\eta^{2p}dV + \frac{18}{50}B_1 - \frac{d}{dt}\left[c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\ &\leq \frac{32}{50}B_1 + cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\ &- \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]. \end{aligned}$$

Thus

$$\begin{aligned} B_1 &\leq cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\ &- \frac{d}{dt}\left[\frac{1}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \end{aligned} \tag{3.57}$$

From (3.48) and (3.57), we can conclude that

Lemma 3.5. *One has*

$$\begin{aligned} A_1' &\leq cKB_2 + cKA_4 + cK^2A_2 + cKA_1 \\ &- \frac{d}{dt}\left[\frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]. \end{aligned} \tag{3.58}$$

Observe that two terms in the bracket are both nonnegative, since $R = -||\boldsymbol{T}||^2 \leq 0$.

Finally, we estimate the term B_2. Using the evolution inequality

$$||\nabla \mathrm{Rm}||^2 \leq -\frac{1}{2}\blacksquare||\mathrm{Rm}||^2 + c||\mathrm{Rm}||^3 + c||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{3/2} + c||\mathrm{Rm}||||\nabla\boldsymbol{T}||^2$$

we obtain

$$\begin{aligned} B_2 &= \int ||\nabla \mathrm{Rm}||^2||\mathrm{Rm}||^{p-3}\eta^{2p}dV \leq \int\Big[-\frac{1}{2}\blacksquare||\mathrm{Rm}||^2 + c||\mathrm{Rm}||^3 \\ &\quad + c||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{3/2} + c||\mathrm{Rm}||||\nabla\boldsymbol{T}||^2\Big]||\mathrm{Rm}||^{p-3}\eta^{2p}dV \\ &\leq -\frac{1}{2}\int\left(\blacksquare||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV + cA_1 \\ &\quad + c\int ||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{p-3/2}\eta^{2p}dV + c\int||\nabla^2\boldsymbol{T}||^2||\mathrm{Rm}||^{p-2}\eta^{2p}dV. \end{aligned} \tag{3.59}$$

For the first integral one has

$$-\frac{1}{2}\int\left(\blacksquare||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV = \frac{1}{2}\int\left(\blacktriangle||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV$$

$$
\begin{aligned}
-\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV &= -\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV\\
&-\frac{1}{2}\int\nabla||\mathrm{Rm}||^2\left[\left(\nabla||\mathrm{Rm}||^{p-3}\right)\eta^{2p}+||\mathrm{Rm}||^{p-3}\left(\nabla\eta^{2p}\right)\right]dV\\
&= -\frac{p-3}{4}\int\left(\nabla||\mathrm{Rm}||^2\right)^2||\mathrm{Rm}||^{p-5}\eta^{2p}dV\\
+c\int||\mathrm{Rm}||^{p-2}||\nabla\mathrm{Rm}||&|\nabla\eta||\eta^{2p-1}dV-\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV\\
&\leq \frac{1}{50}B_2+cA_4-\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV.
\end{aligned}
$$

Here we used the assumption that $p\geq 5$. On the other hand,

$$
\begin{aligned}
-\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV &= -\frac{1}{2}\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]\\
+\frac{1}{2}\int||\mathrm{Rm}||^2\left(\partial_t||\mathrm{Rm}||^{p-3}\right)\eta^{2p}dV&+\frac{1}{2}\int||\mathrm{Rm}||^{p-1}\eta^{2p}\left(\partial_t dV\right)\\
\leq \frac{p-3}{4}\int||\mathrm{Rm}||^{p-3}\left(\partial_t||\mathrm{Rm}||^2\right)\eta^{2p}dV&+cA_1-\frac{1}{2}\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]
\end{aligned}
$$

so that

$$
-\frac{1}{2}\int\left(\partial_t||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV\leq cA_1-\frac{1}{p-1}\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
$$

Therefore

$$
\begin{aligned}
-\frac{1}{2}\int\left(\blacksquare||\mathrm{Rm}||^2\right)||\mathrm{Rm}||^{p-3}\eta^{2p}dV &\leq \frac{1}{50}B_2+cA_4+cA_1\\
&-\frac{1}{p-1}\frac{d}{dt}\left[\int||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].
\end{aligned}\tag{3.60}
$$

To estimate the remainder two integrals, we recall from (3.9) that

$$
\nabla\boldsymbol{T}=\mathrm{Rm}*\varphi+\boldsymbol{T}*\boldsymbol{T}*\varphi\tag{3.61}
$$

and from (2.14) that

$$
\nabla\varphi=\boldsymbol{T}*\psi.\tag{3.62}
$$

From (3.61) we get

$$
||\nabla\boldsymbol{T}||\leq c||\mathrm{Rm}||+c||\boldsymbol{T}||^2\leq c||\mathrm{Rm}||.\tag{3.63}
$$

In particular, the inequality (3.63) yields

$$
\int||\nabla\boldsymbol{T}||^2||\mathrm{Rm}||^{p-2}\eta^{2p}dV\leq c\int||\mathrm{Rm}||^p\eta^{2p}dV\leq cA_1.\tag{3.64}
$$

Taking the derivative of (3.61) and using (3.62) we obtain

$$
\nabla^2\boldsymbol{T}=\nabla\mathrm{Rm}*\varphi+\mathrm{Rm}*\boldsymbol{T}*\psi+\nabla\boldsymbol{T}*\boldsymbol{T}*\varphi+\boldsymbol{T}*\boldsymbol{T}*\boldsymbol{T}*\psi.\tag{3.65}
$$

The particular case $||\nabla^2\boldsymbol{T}||\leq c||\nabla\mathrm{Rm}||+c||\mathrm{Rm}||||\boldsymbol{T}||+c||\nabla\boldsymbol{T}||||\boldsymbol{T}||+c||\boldsymbol{T}||^3$ leads to

$$
\begin{aligned}
c\int||\nabla^2\boldsymbol{T}||||\mathrm{Rm}||^{p-3/2}\eta^{2p}dV &\leq c\int\Big[||\nabla\mathrm{Rm}||+||\mathrm{Rm}||||\boldsymbol{T}||+||\nabla\boldsymbol{T}||||\boldsymbol{T}||\\
+||\boldsymbol{T}||^3\Big]||\mathrm{Rm}||^{p-3/2}\phi^{2p}dV &\leq c\int\left(||\nabla\mathrm{Rm}||||\mathrm{Rm}||^{p-3/2}\eta^p\right)\left(||\mathrm{Rm}||^{p/2}\eta^p\right)dV
\end{aligned}
$$

$$+ c\int ||\mathrm{Rm}||^p\eta^{2p}dV \ \le\ \frac{1}{50}B_2 + cA_1. \tag{3.66}$$

Plugging (3.60), (3.64), and (3.66) into (3.59) we arrive at

$$B_2 \le cA_4 + cA_1 - \frac{d}{dt}\left[\frac{1}{p-1}\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right]. \tag{3.67}$$

Together with (3.58) and (3.67) we finally obtain

$$\begin{aligned}(A_1 + cKA_2)' \ &\le\ cK(A_1 + cKA_2) + cKA_4 \\ &- \frac{d}{dt}\left[\frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right].\end{aligned} \tag{3.68}$$

Equivalently,

Lemma 3.6. *If* $||\mathrm{Ric}|| \le K$ *and* $p \ge 5$*, one has*

$$\begin{aligned}&\frac{d}{dt}\left[A_1 + cKA_2 + \frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV\right] \\ &\le\ cK(A_1 + cKA_2) + cKA_4.\end{aligned} \tag{3.69}$$

As in [24, 27], we choose the domain $\Omega := B_{g_0}(x_0, \rho/\sqrt{K})$ and the function

$$\eta = \left(\frac{\rho/\sqrt{K} - d_{g(0)}(x_0, \cdot)}{\rho/\sqrt{K}}\right)_+.$$

Then, for all $t \in [0, T]$,

$$e^{-cKt}g_0 \le g(t) \le e^{cKt}g_0, \quad ||\nabla_{g(t)}\phi||_{g(t)} \le e^{cKT}||\nabla_{g_0}\phi||_{g_0} \le \frac{\sqrt{K}e^{cKT}}{\rho}.$$

The proof of Theorem 1.4. Define

$$\begin{aligned}U \ &:=\ \int ||\mathrm{Rm}||^p\phi^{2p}dV + cK\int ||\mathrm{Rm}||^{p-1}\eta^{2p}dV \\ &+ \frac{c}{K}\int ||\mathrm{Rm}||^{p-1}||\mathrm{Ric}||^2\eta^{2p}dV + c\int(-R)||\mathrm{Rm}||^{p-1}\eta^{2p}dV.\end{aligned} \tag{3.70}$$

Then (3.69) yields

$$U' \le cKU + cKA_4. \tag{3.71}$$

For A_4, using the Young inequality, we have

$$\begin{aligned}A_4 \ &=\ \int ||\mathrm{Rm}||^{p-1}||\nabla\eta||^2\eta^{2p-2}dV \ \le\ \int_{B_{g_0}(x_0,\rho/\sqrt{K})} ||\mathrm{Rm}||^{p-1}\eta^{2p-2}K\rho^{-2}e^{cKT}dV \\ &\le\ \int_{B_{g_0}(x_0,\rho/\sqrt{K})}\left[\frac{(||\mathrm{Rm}||^{p-1}\eta^{2p-2})^{p/(p-1)}}{\frac{p}{p-1}} + \frac{(K\rho^{-2}e^{cKT})^p}{p}\right]dV \\ &\le\ A_1 + K^p\rho^{-2p}pe^{cKT}\mathrm{vol}_{g(t)}\left(B_{g_0}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right) \\ &\le\ U + cK^pe^{cKT}\rho^{-2p}\mathrm{vol}_{g(t)}\left(B_{g_0}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right).\end{aligned}$$

Thus

$$U' \le cKU + cK^{p+1}e^{cKT}\rho^{-2-p}\mathrm{vol}_{g(t)}\left(B_{g_0}\left(x_0, \frac{\rho}{\sqrt{K}}\right)\right).$$

As in the proof of [24], one can easily deduce from above that

$$\int_{B_{g_0}(x_0,\frac{\rho}{2\sqrt{K}})}||\mathrm{Rm}_{g(t)}||^p_{g(t)}dV_{g(t)} \leq c(1+K)e^{cKT}\int_{B_{g_0}(x_0,\frac{\rho}{\sqrt{K}})}||\mathrm{Rm}_{g_0}||^p_{g_0}dV_{g_0}$$

$$+ cK^p\left(1+\rho^{-2p}\right)e^{cKT}\mathrm{vol}_{g(t)}\left(B_{g_0}\left(x_0,\frac{\rho}{\sqrt{K}}\right)\right). \tag{3.72}$$

As an immediate consequence of the inequality (3.72) we give another proof of the part (a) in Theorem 1.2.

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