

# Derivations and differential operators on rings and fields

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## Abstract

Let  $R$  be an integral domain of characteristic zero. We prove that a function  $D: R \rightarrow R$  is a derivation of order  $n$  if and only if  $D$  belongs to the closure of the set of differential operators of degree  $n$  in the product topology of  $R^R$ , where the image space is endowed with the discrete topology. In other words,  $f$  is a derivation of order  $n$  if and only if, for every finite set  $F \subset R$ , there is a differential operator  $D$  of degree  $n$  such that  $f = D$  on  $F$ . We also prove that if  $d_1, \dots, d_n$  are nonzero derivations on  $R$ , then  $d_1 \circ \dots \circ d_n$  is a derivation of exact order  $n$ .

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## 1 Introduction and main results

By a ring we mean a commutative ring with unit. An integral domain is a ring with no zero-divisors other than 0. The ring  $R$  has characteristic zero if  $n \cdot x \neq 0$  for every  $x \in R \setminus \{0\}$  and for every positive integer  $n$ .

A *derivation* on a ring  $R$  is a map  $d: R \rightarrow R$  such that

$$d(x + y) = d(x) + d(y) \quad \text{and} \quad d(xy) = d(x)y + d(y)x \quad (1)$$

for every  $x, y \in R$ . Derivations of higher order are defined by induction as follows (cf. [5]).

Let  $R$  be a ring. The identically 0 function defined on  $R$  is called the derivation of order 0. Let  $n > 0$ , and suppose we have defined the derivations of order at most  $n - 1$ . A function  $D: R \rightarrow R$  is called a *derivation of order at most  $n$* , if  $D$  is additive and satisfies

$$D(xy) - D(x)y - D(y)x = B(x, y) \quad (2)$$

for every  $x, y \in R$ , where  $B(x, y)$  is a derivation of order at most  $n - 1$  in each of its variables. We denote by  $\mathcal{D}^n(R)$  the set of derivations of order at most  $n$  defined on  $R$ . We may write  $\mathcal{D}^n$  instead of  $\mathcal{D}^n(R)$  if the ring  $R$  is clear from the context. We say that the order of a derivation  $D$  is  $n$  if  $D \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}$ . (We have  $\mathcal{D}^{-1} = \emptyset$  by definition).

Clearly, a function  $d: R \rightarrow R$  is a derivation if and only if  $d \in \mathcal{D}_1$ .

Now we define differential operators on a ring  $R$ . We say that the map  $D: R \rightarrow R$  is a *differential operator of degree at most  $n$*  if  $D$  is the linear combination, with coefficients from  $R$ , of finitely many maps of the form  $d_1 \circ \dots \circ d_k$ , where  $d_1, \dots, d_k$  are derivations on  $R$  and  $k \leq n$ . If  $k = 0$  then we interpret  $d_1 \circ \dots \circ d_k$  as the identity function on  $R$ . We denote by  $\mathcal{O}^n(R)$  the set of differential operators of degree at most  $n$  defined on  $R$ . We may write  $\mathcal{O}^n$  instead of  $\mathcal{O}^n(R)$  if the ring  $R$  is clear from the context. We say that the degree of a differential operator  $D$  is  $n$  if  $D \in \mathcal{O}^n \setminus \mathcal{O}^{n-1}$  (where  $\mathcal{O}^{-1} = \emptyset$  by definition).

The term ‘‘differential operator’’ is justified by the following fact. Let  $K = \mathbb{Q}(t_1, \dots, t_k)$ , where  $t_1, \dots, t_k$  are algebraically independent over  $\mathbb{Q}$ . Then  $K$  is the field of all rational functions of  $t_1, \dots, t_k$  with rational coefficients. It is clear that  $d_i = \frac{\partial}{\partial t_i}$  is a derivation on  $K$  for every  $i = 1, \dots, k$ . Therefore, every differential operator

$$D = \sum_{i_1 + \dots + i_k \leq n} c_{i_1, \dots, i_k} \cdot \frac{\partial^{i_1 + \dots + i_k}}{\partial t_1^{i_1} \dots \partial t_k^{i_k}}, \quad (3)$$

where the coefficients  $c_{i_1, \dots, i_k}$  belong to  $K$ , is a differential operator of degree at most  $n$ . The converse is also true: if  $D$  is a differential operator of degree at most  $n$  on the field  $K = \mathbb{Q}(t_1, \dots, t_k)$ , then  $D$  is of the form (3) (see [3, Proposition 3.2] and the proof of Lemma 2.6 below).

**Remark 1.1.** If  $d$  is a derivation on  $R$ , then  $c \cdot d$  is also a derivation for every  $c \in R$ . Thus every differential operator is the sum of terms of the form  $d_1 \circ \dots \circ d_k$ , where  $k \geq 1$  and  $d_1, \dots, d_k$  are derivations, and of a term  $c \cdot j$ , where  $c \in R$  and  $j$  is the identity function. Since  $d(1) = 0$  for every derivation  $d$ , it follows that a differential operator  $D$  satisfies  $D(1) = 0$  if and only if the term  $c \cdot j$  is missing; that is, if  $D$  is the sum of terms of the form  $d_1 \circ \dots \circ d_k$ , where  $k \geq 1$  and  $d_1, \dots, d_k$  are derivations. We denote by  $\mathcal{O}_0^n$  the set of all differential operators  $D$  of degree at most  $n$  satisfying  $D(1) = 0$ .

Let  $G$  be an Abelian semigroup, and let  $H$  be an Abelian group. The *difference operator*  $\Delta_g$  ( $g \in G$ ) is defined by  $\Delta_g f(x) = f(x + g) - f(x)$  for every  $f: G \rightarrow H$  and  $x \in G$ . A function  $f: G \rightarrow H$  is a *generalized polynomial*, if there is a  $k$  such that  $\Delta_{g_1} \dots \Delta_{g_{k+1}} f = 0$  for every  $g_1, \dots, g_{k+1} \in G$ . The smallest  $k$  for which this holds for every  $g_1, \dots, g_{k+1} \in G$  is the *degree* of the generalized

polynomial  $f$ , denoted by  $\deg f$ . The degree of the identically zero function is  $-1$  by definition. It is clear that the nonzero constant functions are generalized polynomials of degree 0, and the nonconstant additive functions; that is, the nonzero homomorphism from  $G$  to  $H$ , are generalized polynomials of degree 1.

If  $X, Y$  are nonempty sets, then  $Y^X$  denotes the set of all maps  $f: X \rightarrow Y$ . We endow the space  $Y$  with the discrete topology, and  $Y^X$  with the product topology. The closure of a set  $\mathcal{A} \subset Y^X$  with respect to the product topology is denoted by  $\text{cl } \mathcal{A}$ . Clearly, a function  $f: X \rightarrow Y$  belongs to  $\text{cl } \mathcal{A}$  if and only if, for every finite set  $F \subset X$  there is a function  $g \in \mathcal{A}$  such that  $f(x) = g(x)$  for every  $x \in F$ .

It is clear that a function  $f: G \rightarrow H$  is a generalized polynomial of degree at most  $n$  if and only if, for every finite set  $F \subset G$ , there is a generalized polynomial  $h$  of degree at most  $n$  such that  $f = h$  on  $F$ . This means that *the set of generalized polynomials of degree at most  $n$  is closed in  $H^G$* .

If  $R$  is a ring, then we denote by  $R^*$  the Abelian semigroup  $R \setminus \{0\}$  under multiplication. We denote by  $j$  the identity function on  $R$ .

In this note our aim is to prove that, for every integral domain of characteristic zero and for every positive integer  $n$ , we have  $\mathcal{D}^n = \text{cl } \mathcal{O}_0^n$ . That is, *a map  $D: R \rightarrow R$  is a derivation of order at most  $n$  if and only if  $D$  belongs to the closure of the set of all differential operators of degree at most  $n$  satisfying  $D(1) = 0$* . More precisely, we prove the following result.

**Theorem 1.1.** *Let  $R$  be an integral domain of characteristic zero,  $K$  its field of fractions, and let  $n$  be a positive integer. Then, for every function  $D: R \rightarrow R$ , the following are equivalent.*

- (i)  $D \in \mathcal{D}^n(R)$ .
- (ii)  $D \in \text{cl } (\mathcal{O}_0^n(R))$ .
- (iii)  $D$  is additive on  $R$ ,  $D(1) = 0$ , and  $D/j$ , as a map from the semigroup  $R^*$  to  $K$ , is a generalized polynomial of degree at most  $n$ .

As an immediate consequence of the theorem above we find the following corollary.

**Corollary 1.1.** *Let  $R$  be an integral domain of characteristic zero,  $K$  its field of fractions, and let  $n$  be a positive integer. Then, for every function  $D: R \rightarrow R$ , the following are equivalent.*

- (i)  $D \in \mathcal{D}^n(R) \setminus \mathcal{D}^{n-1}(R)$ .
- (ii)  $D \in (\text{cl } \mathcal{O}_0^n(R)) \setminus \text{cl } (\mathcal{O}_0^{n-1}(R))$ .

(iii)  $D$  is additive on  $R$ ,  $D(1) = 0$ , and  $D/j$ , as a map from the semigroup  $R^*$  to  $K$ , is a generalized polynomial of degree  $n$ .

Indeed, suppose  $D \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}$ . Then, by Theorem 1.1, we have  $D \in \text{cl } \mathcal{O}_0^n$ . If  $D \notin \text{cl } (\mathcal{O}_0^n) \setminus \text{cl } (\mathcal{O}_0^{n-1})$ , then  $D \in \text{cl } \mathcal{O}_0^{n-1}$ . This implies  $D \in \mathcal{D}^{n-1}$ , which is impossible. Therefore, (i) of Corollary 1.1 implies (ii) of Corollary 1.1. The other implications can be shown similarly.

**Remark 1.2.** Theorem 1.1 and Corollary 1.1 do not hold without assuming that  $R$  is of characteristic zero. Consider the following example.

Let  $F_2$  denote the field having two elements, and let  $R = F_2[x]$  be the ring of polynomials with coefficients from  $F_2$ . We put

$$D \left( \sum_{i=0}^n a_i \cdot x^i \right) = \sum_{i=2}^n \frac{i(i-1)}{2} \cdot a_i \cdot x^{i-2}$$

for every  $n \geq 0$  and  $a_0, \dots, a_n \in F_2$ . It is easy to check that  $D$  is a derivation of order at most two on  $R$ . Since  $D(x) = 0$  and  $D(x^2) = 1$ , it follows that  $D$  is not a derivation, and thus  $D \in \mathcal{D}^2 \setminus \mathcal{D}^1$ .

On the other hand, if  $d_1$  and  $d_2$  are arbitrary derivations on  $R$ , then  $d_1 \circ d_2$  is also a derivation. Indeed,

$$d_1(d_2(x^k)) = d_1(k \cdot x^{k-1} \cdot d_2(x)) = k(k-1) \cdot x^{k-2} \cdot d_1(x) \cdot d_2(x) + k \cdot x^{k-1} \cdot d_1(d_2(x))$$

for every  $k \geq 2$ . Since  $k(k-1)$  is even, we find that

$$(d_1 \circ d_2)(x^k) = k \cdot x^{k-1} \cdot a \tag{4}$$

for every  $k \geq 2$ , where  $a = d_1(d_2(x)) \in R$ . It is easy to check that (4) is true for  $k = 0$  and  $k = 1$  as well. Since derivations are additive, (4) gives  $d_1(d_2(p)) = a \cdot \frac{\partial p}{\partial x}$  for every  $p \in R$ , and thus  $d_1 \circ d_2 \in \mathcal{O}_0^1$ . This implies that  $\mathcal{O}_0^2 = \mathcal{O}_0^1$ , and thus  $\mathcal{D}^2$  is strictly larger than  $\mathcal{O}_0^2$ .

**Remark 1.3.** In the proof of Theorem 1.1 the crucial step is to show that if  $R$  is of characteristic zero and the transcendence degree of the field of fractions  $K$  of  $R$  over  $\mathbb{Q}$  is finite, then  $\mathcal{D}^n = \mathcal{O}_0^n$  (see Lemma 2.7). Comparing to Theorem 1.1 we find that under these conditions, for every function  $f: R \rightarrow R$  we have

$(f \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}) \iff (f \in \mathcal{O}_0^n \setminus \mathcal{O}_0^{n-1}) \iff D$  is additive on  $K$ ,  $D(1) = 0$ , and  $D/j$ , defined on the group  $K^*$ , is a generalized polynomial of degree  $n$ .

We also prove that for every integral domain  $R$  of characteristic zero, if there are nonzero derivation on  $R$ , then the sets  $\mathcal{D}^n \setminus \mathcal{D}^{n-1}$  are nonempty; that is, there are derivations of any given order. More precisely, we prove the following <sup>1</sup>.

<sup>1</sup>Added in proof: it came to our notice recently that the statement of Theorem 1.2 was also proved, using different methods, by Bruce Ebanks in his submitted paper "Derivations and Leibniz differences on rings".

**Theorem 1.2.** *Let  $R$  be an integral domain of characteristic zero, and let  $n$  be a positive integer. If  $d_1, \dots, d_n$  are nonzero derivations on  $K$ , then  $d_1 \circ \dots \circ d_n \in \mathcal{D}^n \setminus \mathcal{D}^{n-1}$ .*

(For integral domains of characteristic zero this generalizes [2, Remark 3], where the case  $d_1 = \dots = d_n$  is considered.)

**Remark 1.4.** The statement of the theorem above does not hold without assuming that  $R$  is of characteristic zero. Consider the example described in Remark 1.2. Clearly,  $d(p) = \frac{\partial p}{\partial x}$  ( $p \in R$ ) defines a nonzero derivation on  $R$ . However, as we saw in Remark 1.2,  $d \circ d$  is a derivation of order 1.

The statement of the theorem is not true for rings in general; not even for rings of characteristic zero. Let  $R = \mathbb{Q}[x] \times \mathbb{Q}[x]$ , and put  $d_1(p, q) = (\frac{\partial p}{\partial x}, 0)$  and  $d_2(p, q) = (0, \frac{\partial q}{\partial x})$  for every  $(p, q) \in R$ . Then  $d_1$  and  $d_2$  are nonzero derivations on  $R$ , but  $d_1 \circ d_2 = 0$ .

## 2 Lemmas

**Lemma 2.1.** *For every ring  $R$  and for every nonnegative integer  $n$ , the set  $\mathcal{D}^n$  is closed in  $R^R$ .*

*Proof.* We prove by induction on  $n$ . If  $n = 0$ , then  $\mathcal{D}^0 = \{0\}$  is closed. Let  $n > 0$ , and suppose that  $\mathcal{D}^{n-1}$  is closed. Let  $f \in \text{cl } \mathcal{D}^n$  be arbitrary. We have to prove that  $f \in \mathcal{D}^n$ ; that is, for every fixed  $y \in R$ , the map  $x \mapsto g(x) = f(xy) - yf(x) - xf(y)$  belongs to  $\mathcal{D}^{n-1}$ . By the induction hypothesis, it is enough to show that  $g \in \text{cl } \mathcal{D}^{n-1}$ ; that is, for every finite set  $F \subset R$  there is a function  $h \in \mathcal{D}^{n-1}$  such that  $g(x) = h(x)$  for every  $x \in F$ .

If  $F$  is finite, then so is  $A = F \cup \{xy : x \in F\} \cup \{y\}$ . Since  $f \in \text{cl } \mathcal{D}^n$ , there is a function  $D \in \mathcal{D}^n$  such that  $f(z) = D(z)$  for every  $z \in A$ . If  $x \in F$ , then  $x, y, xy \in A$ , and thus

$$g(x) = f(xy) - yf(x) - xf(y) = D(xy) - yD(x) - xD(y).$$

The function  $x \mapsto h(x) = D(xy) - yD(x) - xD(y)$  belongs to  $\mathcal{D}^{n-1}$ , as  $D \in \mathcal{D}^n$ . Since  $g(x) = h(x)$  for every  $x \in F$ , the lemma is proved.  $\square$

**Lemma 2.2.** *For every ring  $R$  we have  $\text{cl } \mathcal{O}_0^n \subset \mathcal{D}^n$ .*

*Proof.* Since  $\mathcal{D}^n$  is closed by Lemma 2.1, it is enough to show that  $\mathcal{O}_0^n \subset \mathcal{D}^n$ . Let  $D$  be a differential operator of degree at most  $n$  satisfying  $D(1) = 0$ . According to Remark 1.1,  $D$  is the sum of terms of the form  $d_1 \circ \dots \circ d_k$ , where  $1 \leq k \leq n$  and  $d_1, \dots, d_k$  are derivations. Since  $\mathcal{D}^n$  is a linear space, it is enough to show

that  $d_1 \circ \dots \circ d_k \in \mathcal{D}^k$  whenever  $k \geq 1$  and  $d_1, \dots, d_k$  are derivations. This, in turn, is easy to prove by induction on  $k$ .  $\square$

The statement of the following lemma is probably known. In order to make these notes as self-contained as possible, we provide the proof.

**Lemma 2.3.** *Let  $G$  be an Abelian semigroup, and let  $K$  be a field. If  $p: G \rightarrow K$  is a generalized polynomial of degree  $n \geq 0$  and  $a: G \rightarrow K$  is a nonzero additive function, then  $p \cdot a$  is a generalized polynomial of degree at most  $n + 1$ .*

*If  $K$  is of characteristic zero, then  $\deg(p \cdot a) = n + 1$ .*

*Proof.* We prove by induction on  $n$ . If  $n = 0$ , then  $p$  is a nonzero constant, and  $p \cdot a$  is a nonzero additive function, hence a generalized polynomial of degree 1.

Let  $n > 0$ , and suppose that the statement is true for  $n - 1$ . Let  $p$  be a generalized polynomial of degree  $n$ . We have

$$\Delta_g(p \cdot a)(x) = a(x) \cdot \Delta_g p(x) + a(g) \cdot p(x + g) \quad (5)$$

for every  $x, g \in G$ . Since  $\deg \Delta_g p(x) \leq n - 1$ , it follows from the induction hypothesis that  $\deg(a(x) \cdot \Delta_g p(x)) \leq n$ . Therefore, by (5), we have  $\deg \Delta_g(p \cdot a) \leq n$  for every  $g \in G$ , and thus  $\deg(p \cdot a) \leq n + 1$ . We have to prove that if  $K$  is characteristic zero, then  $\deg(p \cdot a) \geq n + 1$ .

Since the image space  $K$  is a torsion free and divisible Abelian group, it follows from Djoković's theorem [1] that  $p = P_n + \dots + P_1 + P_0$ , where  $P_i$  is a monomial of degree  $i$  for every  $i = 1, \dots, n$ , and  $P_0$  is constant. Then there is a symmetric function  $A(x_1, \dots, x_n)$ , additive in each of its variables, such that  $P_n(x) = A(x, \dots, x)$  ( $x \in G$ ). Since  $q = p - P_n$  is a generalized polynomial of degree  $\leq n - 1$ , it follows from the induction hypothesis that  $\deg(q \cdot a) \leq n$ . Therefore, in order to prove  $\deg(p \cdot a) \geq n + 1$ , it is enough to show that  $\deg(P_n \cdot a) = n + 1$ .

First we show that there exists an element  $g \in G$  such that  $P_n(g) \neq 0$  and  $a(g) \neq 0$ . By assumption, there is an  $x \in G$  such that  $a(x) \neq 0$ . Since  $\deg P_n = n \geq 0$ , it follows that  $P_n$  is nonzero. Let  $y \in G$  be such that  $P_n(y) \neq 0$ . Now  $a(kx + y) = k \cdot a(x) + a(y)$  for every positive integer  $k$ . Since  $a(x), a(y) \in K$  and  $a(x) \neq 0$ , we have  $a(kx + y) \neq 0$  for every  $k$  with at most one exception.

Using the fact that  $A(x_1, \dots, x_n)$  is symmetric and additive in each of its variables, we find

$$P_n(kx + y) = \sum_{i=0}^n \binom{n}{i} A_i(kx, y) \quad (6)$$

for every positive integer  $k$ , where

$$A_i(kx, y) = A(\underbrace{kx, \dots, kx}_i, \underbrace{y, \dots, y}_{k-i}) = k^i \cdot A(\underbrace{x, \dots, x}_i, \underbrace{y, \dots, y}_{k-i}).$$

Therefore, by (6),  $Q(kx + y)$  is a polynomial of  $k$  with coefficients from  $K$ . Since the constant term of this polynomial is  $A(y, \dots, y) \neq 0$ ,  $Q(kx + y)$  is not the identically zero polynomial, and thus  $P_n(kx + y) \neq 0$  for all but finitely many  $k$ . Therefore, we may choose a  $k$  such that  $P_n(g) \neq 0$  and  $a(g) \neq 0$ , where  $g = kx + y$ .

Let  $Q = P_n \cdot a$ , and suppose that  $\deg Q \leq n$ . Then  $Q = Q_n + \dots + Q_1 + Q_0$ , where  $Q_i$  is a monomial of degree  $i$  for every  $i = 1, \dots, n$ , and  $Q_0$  is constant. For every  $i = 1, \dots, n$ , there is a symmetric function  $B_i(x_1, \dots, x_i)$ , additive in each of its variables, such that  $Q_i(x) = B_i(x, \dots, x)$  ( $x \in G$ ). Then

$$Q(k \cdot g) = Q_0 + \sum_{i=1}^n B_i(kg, \dots, kg) = Q_0 + \sum_{i=1}^n k^i \cdot B_i(g, \dots, g)$$

for every positive integer  $k$ . Therefore, the map  $k \mapsto Q(k \cdot g)$  is a polynomial of degree  $\leq n$  with coefficients from  $K$ . However,

$$Q(k \cdot g) = k^n \cdot A(g, \dots, g) \cdot k \cdot a(g) = k^{n+1} \cdot A(g, \dots, g) \cdot a(g)$$

is a polynomial of degree  $n + 1$ . This is a contradiction, proving  $\deg Q = n + 1$ .  $\square$

**Lemma 2.4.** *Let  $R$  be an integral domain, and let  $K$  be its field of fractions. If  $d_1, \dots, d_n$  are nonzero derivations on  $R$  and  $D = d_1 \circ \dots \circ d_n$ , then  $D/j$ , as a map from the semigroup  $R^*$  to  $K$ , is a generalized polynomial of degree at most  $n$ .*

*If  $R$  is of characteristic zero, then  $\deg D/j = n$ .*

*Proof.* We prove by induction on  $n$ . If  $n = 1$ , then  $D$  is a nonzero derivation. It is clear that in this case  $D/j$  is additive, hence a generalized polynomial of degree at most 1 on the semigroup  $R^*$ . Suppose  $\deg D/j \leq 0$ . Then  $D/j$  is constant on  $R^*$ , and thus  $D = c \cdot j$  on  $R$ , where  $c \in R$  is a constant. Since  $D$  is a derivation, we have  $c = D(1) = 0$  and  $d = 0$ , a contradiction. Thus  $\deg D/j = 1$ .

Suppose that  $n > 1$ , and the statement is true for  $n - 1$ . Let  $d_1, \dots, d_n$  be nonzero derivations on  $R$ . By the induction hypothesis,  $(d_2 \circ \dots \circ d_n)/j = p$  is a generalized polynomial of degree at most  $n - 1$ . Since  $d_1$  is a derivation, we have

$$D(x) = (d_1 \circ \dots \circ d_n)(x) = d_1(p(x) \cdot x) = d_1(p(x)) \cdot x + p(x) \cdot d_1(x)$$

for every  $x \in R^*$ . Thus

$$D/j = (d_1 \circ p) + p \cdot (d_1/j) \tag{7}$$

on  $R^*$ . Since  $p: R^* \rightarrow K$  is a generalized polynomial of degree  $\leq n - 1$  and  $d_1: R \rightarrow R$  is additive, it follows that  $d_1 \circ p$  is a generalized polynomial of degree

$\leq n - 1$  on  $R^*$ . (This is because, if  $G$  is an Abelian semigroup,  $H$  is an Abelian group,  $p: G \rightarrow H$  is a generalized polynomial of degree  $k$ , and  $d: H \rightarrow H$  is additive, then  $d \circ p$  is a generalized polynomial of degree at most  $k$ .)

If  $R$  is of characteristic zero, then so is  $K$ . In this case  $p \cdot (d_1/j)$  is a generalized polynomial of degree  $n$  by Lemma 2.3, since  $d_1/j$  is nonzero and additive on  $R^*$ . Therefore,  $D/j$  is a generalized polynomial of degree  $n$ .  $\square$

**Lemma 2.5.** *Let  $R$  be an integral domain, and let  $K$  be its field of fractions. If  $D \in \text{cl } \mathcal{O}_0^n(R)$ , then  $D/j$ , as a map from the semigroup  $R^*$  to  $K$ , is a generalized polynomial of degree at most  $n$ .*

*Proof.* Let  $D \in \text{cl } \mathcal{O}_0^n$  be given. As the set of generalized polynomials of degree  $\leq n$  is closed, it is enough to show that for every finite set  $F \subset R^*$  there is a generalized polynomial  $p: R^* \rightarrow K$  such that  $\deg p \leq n$  and  $D/j = p$  on  $F$ . Since  $D \in \text{cl } \mathcal{O}_0^n$ , there is an  $f \in \mathcal{O}_0^n$  such that  $D = f$  on  $F$ . It is clear from Remark 1.1 and Lemma 2.4 that  $f/j$  is a generalized polynomial of degree at most  $n$ . Now we have  $D/j = f/j$  on  $F$ , completing the proof.  $\square$

The statement of the following lemma is proved, in a different context, in Lemma 3.3 of [3]. We give the proof adjusted to our purposes.

**Lemma 2.6.** *Let  $R$  be a subring of  $\mathbb{C}$ , let  $K \subset \mathbb{C}$  be its field of fractions, and suppose that the transcendence degree of  $K$  over  $\mathbb{Q}$  is finite. Let the map  $D: R \rightarrow R$  be additive. If  $D/j$ , as a map from the semigroup  $R^*$  to  $\mathbb{C}$  is a generalized polynomial of degree at most  $n$ , then  $D \in \mathcal{O}^n$ .*

*Proof.* Let  $k$  be the transcendence degree of  $K$  over  $\mathbb{Q}$ , and let the elements  $u_1, \dots, u_k \in K$  be algebraically independent over  $\mathbb{Q}$ . Let  $u_i = a_i/b_i$ , where  $a_i, b_i \in R$  for every  $i = 1, \dots, k$ . Then the field  $\mathbb{Q}(a_1, b_1, \dots, a_k, b_k)$  has transcendence degree  $k$  over  $\mathbb{Q}$ , and thus we can choose elements  $t_1, \dots, t_k \in \{a_1, b_1, \dots, a_k, b_k\} \subset R^*$  such that  $t_1, \dots, t_k$  are algebraically independent over  $\mathbb{Q}$ .

By assumption, the function  $p = D/j$  is a generalized polynomial of degree  $\leq n$  on  $R^*$ . By Djoković's theorem, we have  $p = P_n + \dots + P_1 + P_0$ , where  $P_j$  is a monomial of degree  $j$  for every  $j = 1, \dots, n$ , and  $P_0$  is constant. Using the fact that  $P_j(x) = A_j(x, \dots, x)$ , where  $A_j(x_1, \dots, x_j)$  is symmetric and additive in each of its variables, it is easy to see that for every  $j = 1, \dots, n$  there is a homogeneous polynomial  $\bar{p}_j \in K[x_1, \dots, x_k]$  of degree  $j$  such that

$$P_j(t_1^{i_1} \cdots t_k^{i_k}) = \bar{p}_j(i_1, \dots, i_k)$$

whenever  $i_1, \dots, i_k$  are nonnegative integers. (Note that the semigroup operation in  $R^*$  is multiplication.) Putting  $\bar{p} = P_0 + \sum_{j=1}^n \bar{p}_j$  we find that  $\bar{p} \in K[x_1, \dots, x_k]$ , and

$$\bar{p}(t_1^{i_1} \cdots t_k^{i_k}) = q(i_1, \dots, i_k)$$



for every  $i_1, \dots, i_k \geq 0$ . We shall use the notation  $x^{[0]} = 1$  and  $x^{[j]} = x(x-1)\cdots(x-j+1)$  for every  $j = 1, 2, \dots$  and  $x \in \mathbb{Z}$ . It is easy to see that every polynomial belonging to  $K[x_1, \dots, x_k]$  and of degree  $\leq n$  can be written in the form  $\sum c_j \cdot x_1^{[j_1]} \cdots x_k^{[j_k]}$ , where  $j = (j_1, \dots, j_k)$  runs through the set of  $k$ -tuples of nonnegative integers with  $j_1 + \dots + j_k \leq n$ , and in each term the coefficient  $c_j$  belongs to  $K$ . Therefore, the polynomial  $\bar{p}$  also has such a representation. Then we have

$$\begin{aligned} D(t_1^{i_1} \cdots t_k^{i_k}) &= p(t_1^{i_1} \cdots t_k^{i_k}) \cdot t_1^{i_1} \cdots t_k^{i_k} = \\ &= \sum c_j \cdot i_1^{[j_1]} \cdots i_k^{[j_k]} \cdot t_1^{i_1} \cdots t_k^{i_k} = \\ &= \sum c_j \cdot t_1^{j_1} \cdots t_k^{j_k} \cdot i_1^{[j_1]} \cdots i_k^{[j_k]} \cdot t_1^{i_1-j_1} \cdots t_k^{i_k-j_k} = \\ &= E(t_1^{i_1} \cdots t_k^{i_k}) \end{aligned} \tag{8}$$

for every  $i_1, \dots, i_k \geq 0$ , where  $E$  is the differential operator

$$\sum c_j \cdot t_1^{j_1} \cdots t_k^{j_k} \cdot \frac{\partial^{j_1+\dots+j_k}}{\partial t_1^{j_1} \cdots \partial t_k^{j_k}}.$$

By extending the derivations  $\partial/\partial t_i$  to  $K$ , we can extend  $E$  to  $K$  as a differential operator  $\bar{E}$  of degree at most  $n$ . Then  $\bar{E}$  is additive on  $K$ , and  $\bar{E}/j$  is a generalized polynomial on  $K^*$  by Lemma 2.4. Let  $q(0) = 0$ , and let  $q(x) = p(x) - \bar{E}(x)/x$  for every  $x \in R^*$ . Then  $q \cdot j = D - \bar{E}$  is additive on  $R$ , and  $q$  is a generalized polynomial on  $R^*$ . Let  $G$  denote the semigroup generated by the elements  $t_1, \dots, t_k$ . Then  $q$  vanishes on  $G$  by (8). From these conditions it follows that  $q = 0$  on  $R$ . This is proved in [3, Lemma 3.6] under the stronger condition that  $G$  is the group (and not the semigroup) generated by  $t_1, \dots, t_k$ . One can see that the same argument works in our more general case as well; however, for the sake of completeness we give the proof in the appendix. Thus we have  $q = 0$ ; that is,  $D = \bar{E}$  on  $R$ , which completes the proof.  $\square$

**Lemma 2.7.** *Let  $R$  be a subring of  $\mathbb{C}$ , let  $K \subset \mathbb{C}$  be its field of fractions, and suppose that the transcendence degree of  $K$  over  $\mathbb{Q}$  is finite. Then  $\mathcal{D}^n(R) = \mathcal{O}_0^n(R)$ .*

*Proof.* By Lemma 2.2, we only have to show that  $\mathcal{D}^n \subset \mathcal{O}_0^n$ . It is easy to prove, by induction on  $n$  that if  $D \in \mathcal{D}^n$ , then  $D(1) = 0$ . Therefore, it is enough to show that if  $D \in \mathcal{D}^n$ , then  $D$  is a differential operator of degree at most  $n$ . We prove by induction on  $n$ .

The statement is obvious if  $n = 0$ . Let  $n > 0$ , and suppose that the statement is true for  $n - 1$ . Let  $D$  be a derivation of order at most  $n$ . By Lemma 2.6, it is enough to show that  $p = D/j$ , defined on the semigroup  $R^*$ , is a generalized

polynomial of degree at most  $n$ . Let  $y \in R^*$  be fixed. Dividing (2) by  $xy$  we obtain

$$\frac{D(xy)}{xy} - \frac{D(x)}{x} - \frac{D(y)}{y} = \frac{B(x, y)}{xy},$$

and thus  $p(xy) - p(x) - p(y) = B(x, y)/xy$  for every  $x \in K^*$ . Therefore we have

$$\Delta_y p(x) = p(y) + \frac{1}{y} \cdot \frac{B(x, y)}{x} \quad (9)$$

on  $R^*$ . The map  $x \mapsto B(x, y)$  is a derivation of order at most  $n - 1$ . We also have  $B(1, y) = 0$  by  $D(1) = 0$ . Therefore, by Lemma 2.4, the map  $x \mapsto B(x, y)/x$  is a generalized polynomial of degree at most  $n$ . Then so is  $\Delta_y p$  by (9). Since this is true for every  $y \in K^*$ , it follows that  $p$  is a generalized polynomial of degree at most  $n$ .  $\square$

### 3 Proof of Theorems 1.1 and 1.2.

First we prove Theorem 1.1. The implication (ii) $\implies$ (iii) is proved in Lemma 2.5.

(iii) $\implies$ (ii): Suppose that  $D$  is additive,  $D(1) = 0$ , and  $D/j$  is a generalized polynomial of degree at most  $n$ . In order to prove  $D \in \text{cl } \mathcal{O}_0^n$ , we have to show that for every finite set  $F \subset K$  there is a function  $f \in \mathcal{O}_0^n$  such that  $D = f$  on  $F$ . Let  $F \subset K$  be finite, and let  $L$  denote the subfield of  $K$  generated by  $F$ . Obviously, the transcendence degree of  $L$  over  $\mathbb{Q}$  is finite. It is well-known that every field of characteristic zero and having finite transcendence degree over  $\mathbb{Q}$  is isomorphic to a subfield of  $\mathbb{C}$ . Therefore, we may assume that  $L \subset \mathbb{C}$ . Thus, by Lemma 2.6, the restriction  $D|_L$  of  $D$  to the field  $L$  is a derivation of order at most  $n$ . Since  $D(1) = 0$ , we also have  $D|_L \in \mathcal{O}_0^n(L)$ . It is well-known that every derivation on  $L$  can be extended to  $K$  as a derivation (see [4, pp. 351-352]). This implies that every differential operator on  $L$  of degree at most  $n$  can be extended to  $K$  as a differential operator of degree at most  $n$ . If  $f$  is such an extension of  $D|_L$ , then, obviously,  $D(x) = f(x)$  for every  $x \in F$ . This proves (iii) $\implies$ (ii).

(ii) $\implies$ (i): This is Lemma 2.2.

(i) $\implies$ (ii): Let  $D \in \mathcal{D}^n$ . In order to prove  $f \in \text{cl } \mathcal{O}_0^n$  we have to show that for every finite set  $F \subset K$  there is a function  $f \in \mathcal{O}_0^n$  such that  $D = f$  on  $F$ . Let  $L$  denote the field generated by  $F$ . Obviously, the transcendence degree of  $L$  over  $\mathbb{Q}$  is finite. Thus, by Lemma 2.7, the restriction  $D|_L$  of  $D$  to the field  $L$  is a derivation of order at most  $n$ , vanishing at 1. Let  $f$  be an extension of  $D|_L$  to  $K$  as a function  $f \in \mathcal{O}_0^n$ . Then, obviously,  $D(x) = f(x)$  for every  $x \in F$ . This proves (i) $\implies$ (ii).  $\square$

The statement of Theorem 1.2 is an immediate consequence of Corollary 1.1 and Lemma 2.4.  $\square$

## 4 Appendix

**Lemma 4.1.** *Let  $R$  be a subring of  $\mathbb{C}$ , and let  $K \subset \mathbb{C}$  be its field of fractions. Suppose that the transcendence degree of  $K$  over  $\mathbb{Q}$  is  $k < \infty$ , and let the elements  $t_1, \dots, t_k \in R$  be algebraically independent over  $\mathbb{Q}$ . Let  $f: R \rightarrow \mathbb{C}$  be additive on  $R$  (with respect to addition) and such that  $q = f/j$ , as a map from the semigroup  $R^*$  to  $\mathbb{C}$  is a generalized polynomial. If  $f = 0$  on the semigroup  $G$  generated by  $t_1, \dots, t_k$ , then  $f = 0$  on  $R$ .*

*Proof.* We prove by induction on  $\deg q$ . If  $\deg q = 0$ , then  $q$  is constant. Since  $f = 0$  on  $G$ , we have  $q = 0$  on  $G$ , and thus  $q = 0$  on  $R$ .

Suppose  $m = \deg q > 0$ , and that the statement is true for degrees less than  $m$ . Let  $g \in G$  be fixed, and put  $f_1(x) = g^{-1}f(gx) - f(x)$  ( $x \in R$ ). Then  $f_1$  is additive on  $R$ . Also,  $f_1/j$  is a generalized polynomial on  $R^*$ , since

$$\frac{f_1(x)}{x} = \frac{g^{-1}f(gx) - f(x)}{x} = \frac{f(gx)}{gx} - \frac{f(x)}{x} = q(gx) - q(x) = \Delta_g q(x)$$

for every  $x \in R^*$ . Since  $\deg(f_1/j) = \deg \Delta_g q \leq m - 1$  and  $f_1 = 0$  on  $G$ , it follows from the induction hypothesis that  $f_1 = 0$  on  $R$ . Thus  $f(gx) = g \cdot f(x)$  for every  $g \in G$  and  $x \in R$ . By the additivity of  $f$  we obtain

$$f(cx) = c \cdot f(x) \quad (c \in \mathbb{Q}[t_1, \dots, t_k], x \in R). \quad (10)$$

Since the transcendence degree of  $K$  over  $\mathbb{Q}$  is  $k$  and  $t_1, \dots, t_k$  are algebraically independent over  $\mathbb{Q}$ , it follows that every element of  $K$  is algebraic over  $\mathbb{Q}(t_1, \dots, t_k)$ . Let  $\alpha \in R$  be arbitrary. Then  $\alpha$  is algebraic over the field  $\mathbb{Q}(t_1, \dots, t_k)$ , and there are elements  $c_0, \dots, c_N \in \mathbb{Q}[t_1, \dots, t_k]$  such that

$$c_N \alpha^N + \dots + c_1 \alpha + c_0 = 0, \quad (11)$$

where  $c_N \neq 0$  and  $N$  is minimal. Let  $f(\alpha^i) = a_i$  ( $i = 0, 1, \dots$ ). Multiplying (11) by  $\alpha^{n-N}$  for every  $n \geq N$  we obtain

$$c_N \alpha^n + \dots + c_1 \alpha^{n-N+1} + c_0 \alpha^{n-N} = 0.$$

By (10) and by the additivity of  $f$ , this implies

$$c_N a_n + \dots + c_1 a_{n-N+1} + c_0 a_{n-N} = 0$$

for every  $n \geq N$ . Therefore, the sequence  $(a_n)$  satisfies a linear recurrence relation. It is well-known that  $a_n$  can be uniquely represented in the form  $a_n = \sum_{\lambda \in \Lambda} p_\lambda(n) \cdot \lambda^n$ , where  $\lambda$  runs through  $\Lambda$ , the set of roots of the characteristic

polynomial  $\chi(x) = c_N x^N + \dots + c_0$ , and for every root  $\lambda \in \Lambda$ ,  $p_\lambda \in \mathbb{C}[x]$  is a polynomial of the degree less than the multiplicity of  $\lambda$ .

Since  $N$  is minimal, the polynomial  $\chi$  is irreducible over  $\mathbb{Q}(t_1, \dots, t_k)$ . Therefore, every  $\lambda$  is a simple root of  $\chi$ , and thus

$$a_n = \sum_{\lambda \in \Lambda} d_\lambda \cdot \lambda^n \quad (12)$$

for every  $n$ , where  $d_\lambda$  is a constant for every  $\lambda \in \Lambda$ .

Since  $q$  is a generalized polynomial on  $R^*$  it follows that the map  $n \mapsto q(\alpha^n)$  is a polynomial on  $\{0, 1, \dots\}$ . Now, we have  $a_n = f(\alpha^n) = q(\alpha^n) \cdot \alpha^n$  for every  $n$ . The uniqueness of the representation (12) implies that  $\alpha \in \Lambda$ , and the function  $n \mapsto q(\alpha^n)$  ( $n = 0, 1, \dots$ ) is constant. Since  $q(1) = f(1) = 0$  by  $1 \in G$ , it follows that  $q(\alpha^n) = 0$  for every  $n$ . In particular,  $q(\alpha) = 0$  and  $f(\alpha) = 0$ . Since this is true for every  $\alpha \in R$ , we obtain  $f = 0$  on  $R$ .  $\square$

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