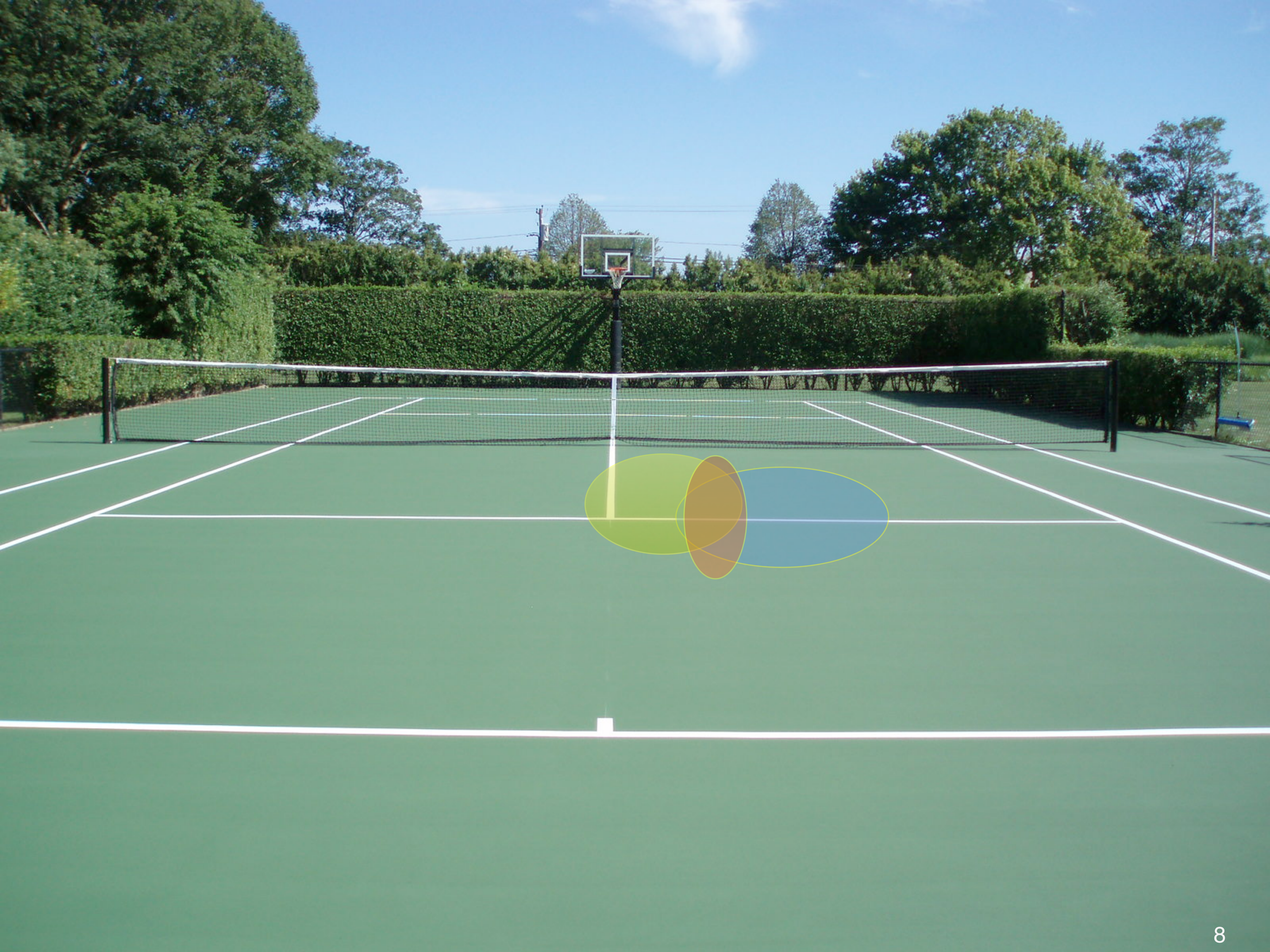
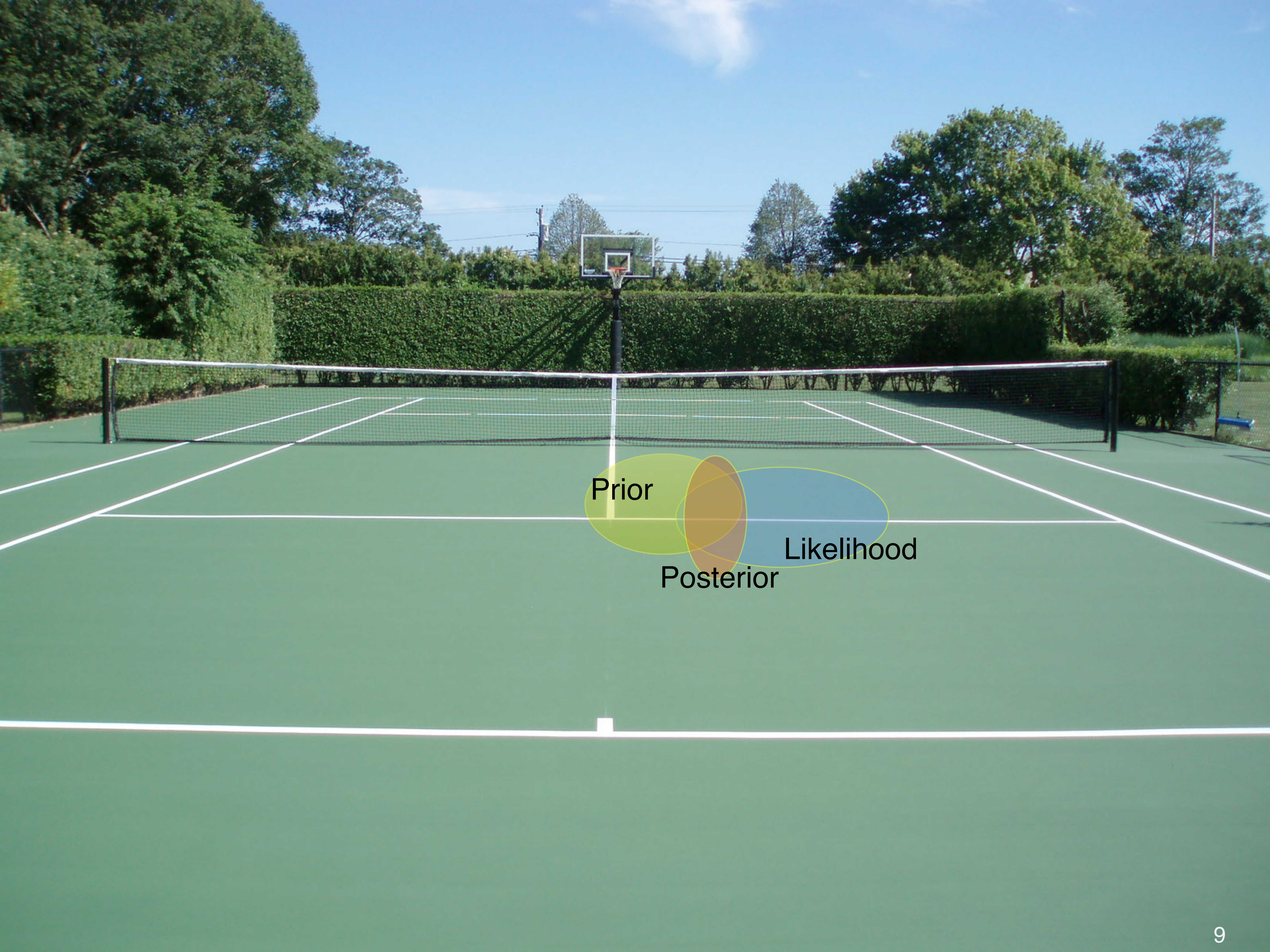
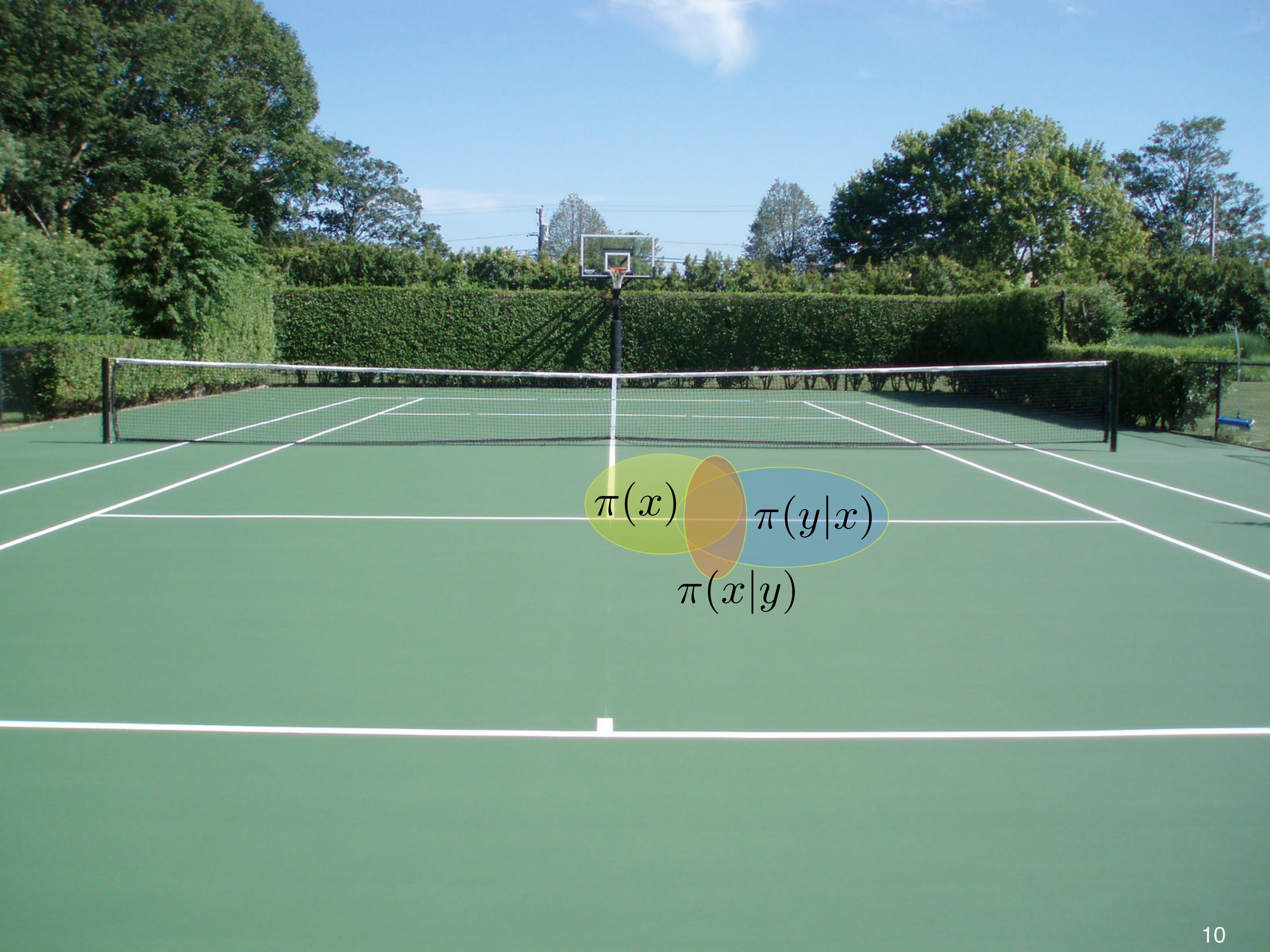
Bayesian inference

Primer



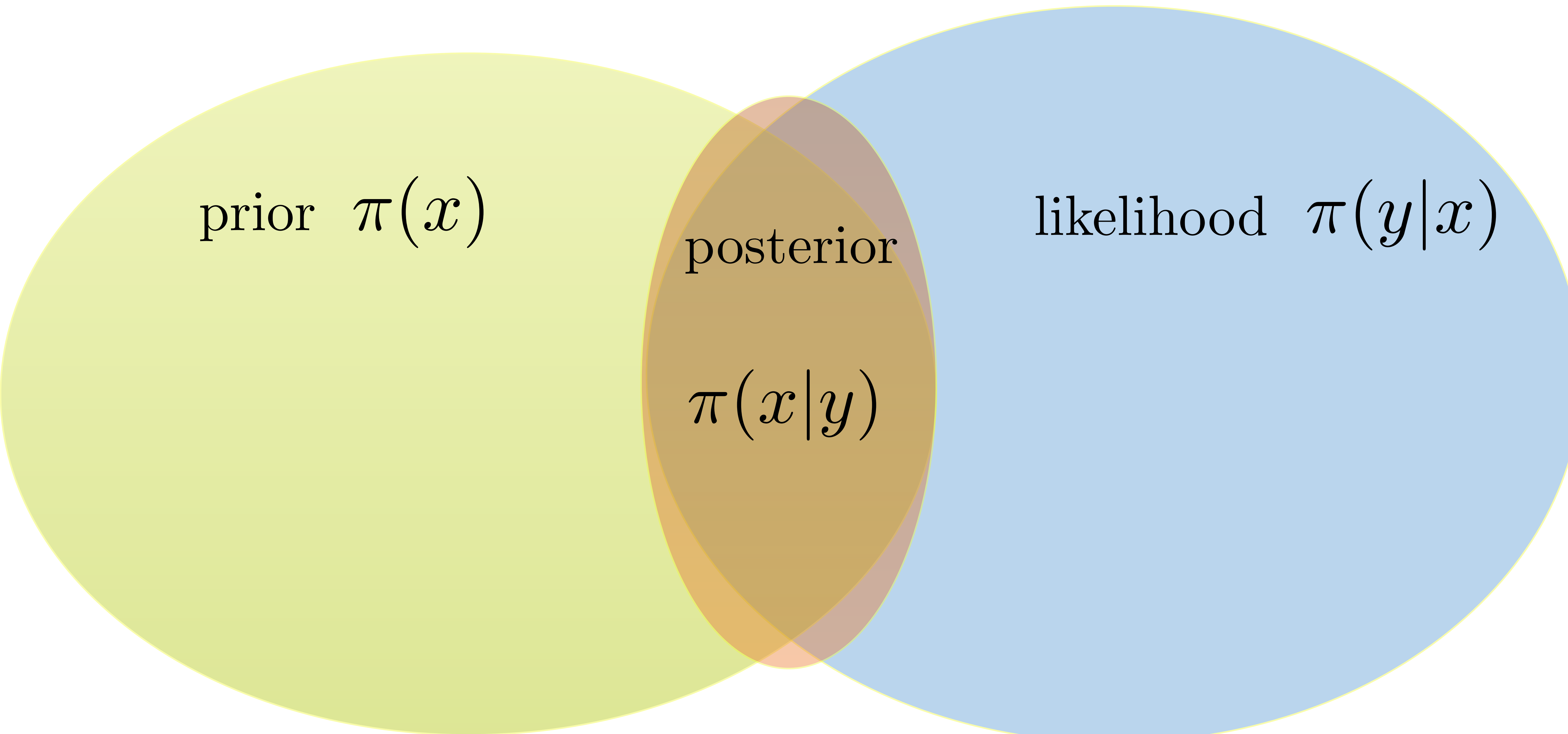

Prior
Likelihood
Posterior


$\pi(x)$
$\pi(y|x)$
$\pi(x|y)$

Bayes' theorem

$$\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{evidence}}$$

$$\pi(x|y) = \frac{\pi(x)\pi(y|x)}{\int \pi(x)\pi(y|x)dx}$$



Parameter identification: Bayesian approach

Bayes' theorem

$$\pi(x|y) = \frac{\pi(x)\pi(y|x)}{\int \pi(x)\pi(y|x)dx}$$

$\pi(.)$: probability distribution function

$\pi(.|.)$: conditional probability distribution function

x : material parameter

y : observations

Parameter identification: Bayesian approach

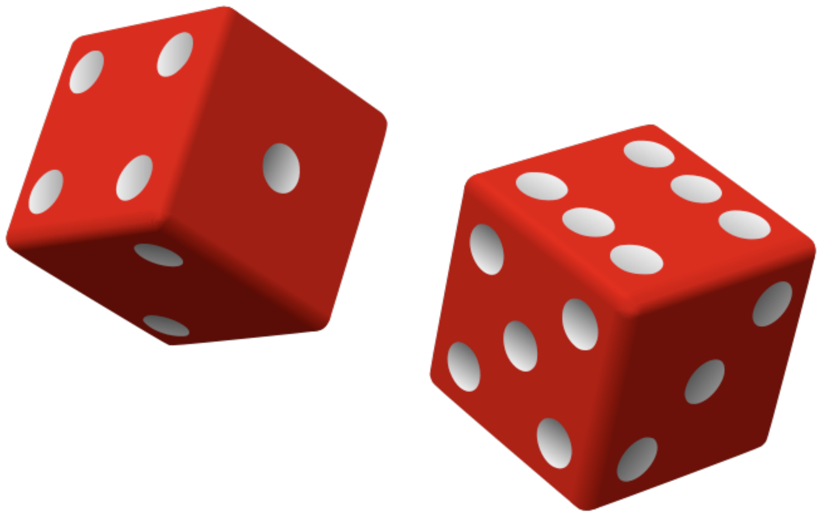
Bayes' theorem

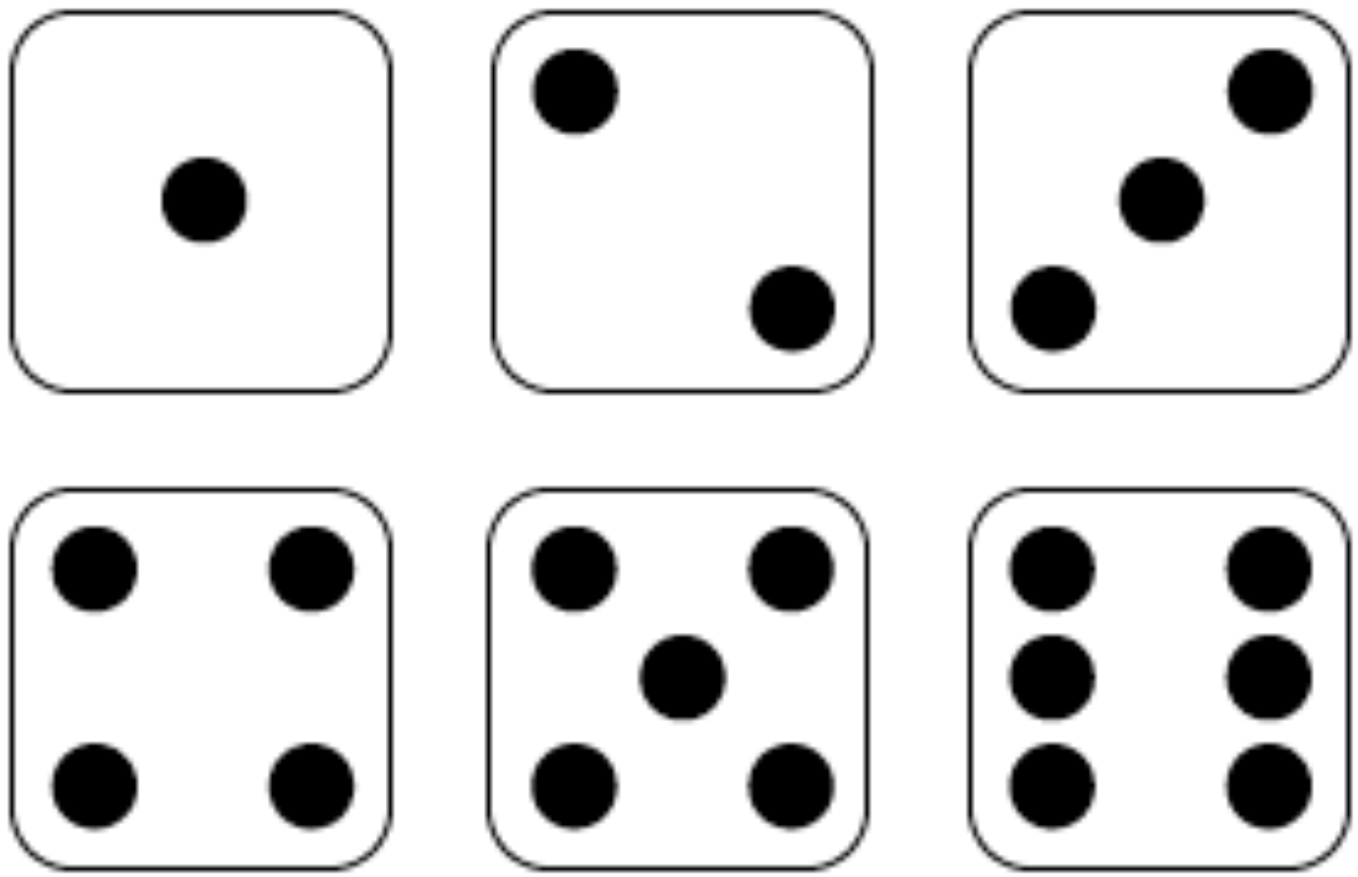
$$\pi(x|y) = \frac{\pi(x)\pi(y|x)}{\int \pi(x)\pi(y|x)dx}$$

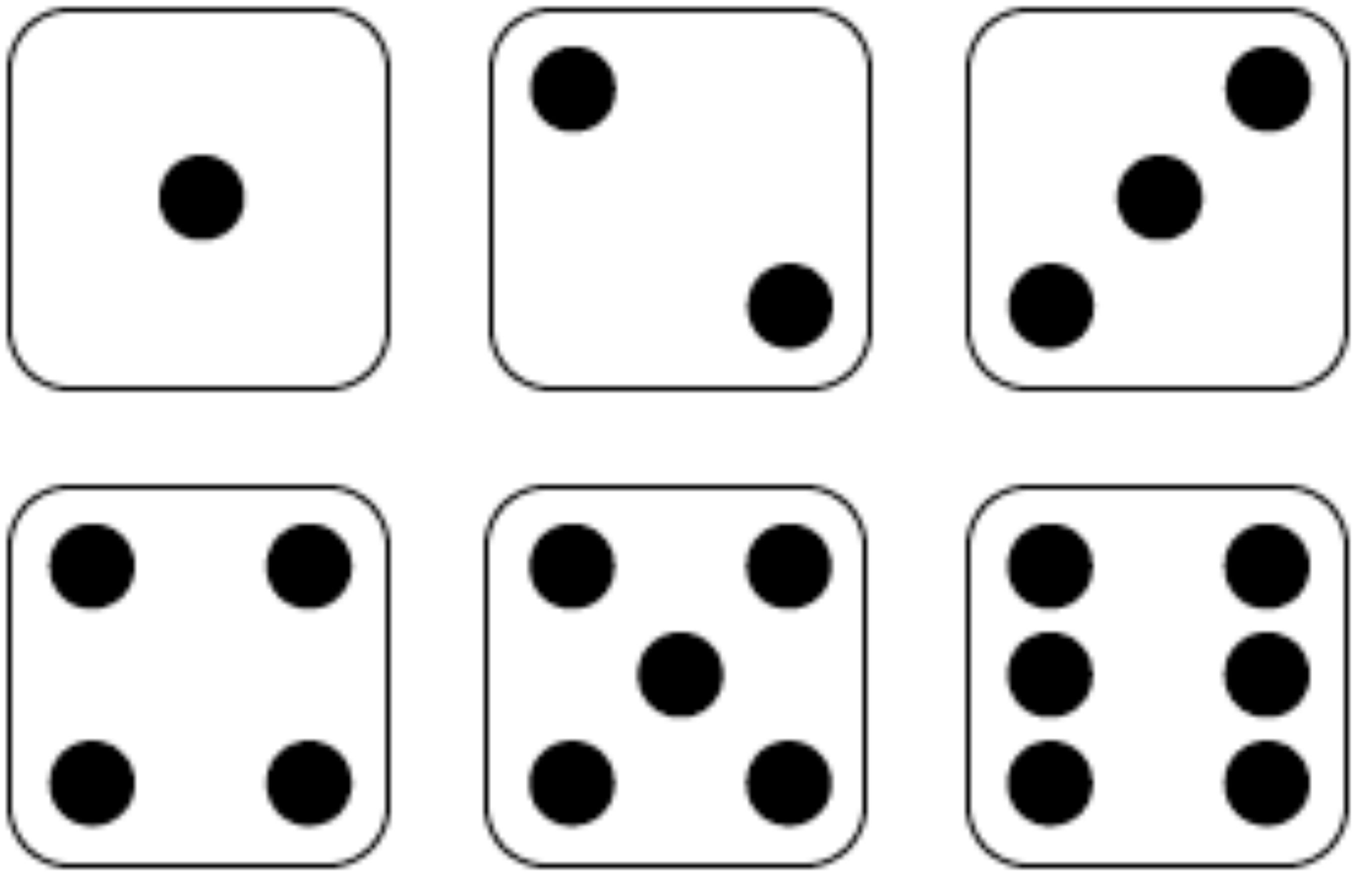
Descriptive formula

$$\text{Posterior} = \frac{\text{Prior} \times \text{Likelihood}}{\text{Evidence}}$$

A discrete example of Bayes' theorem

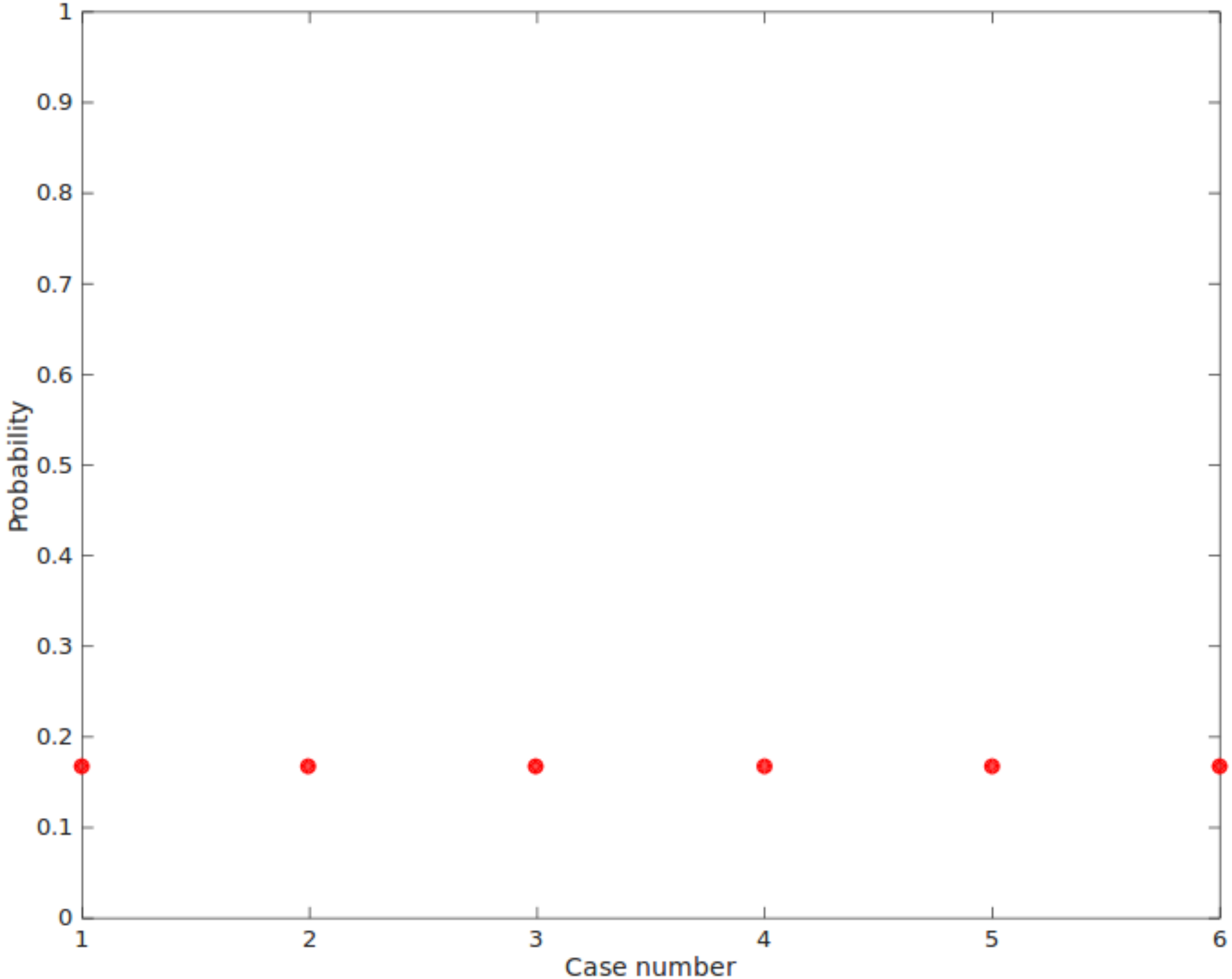


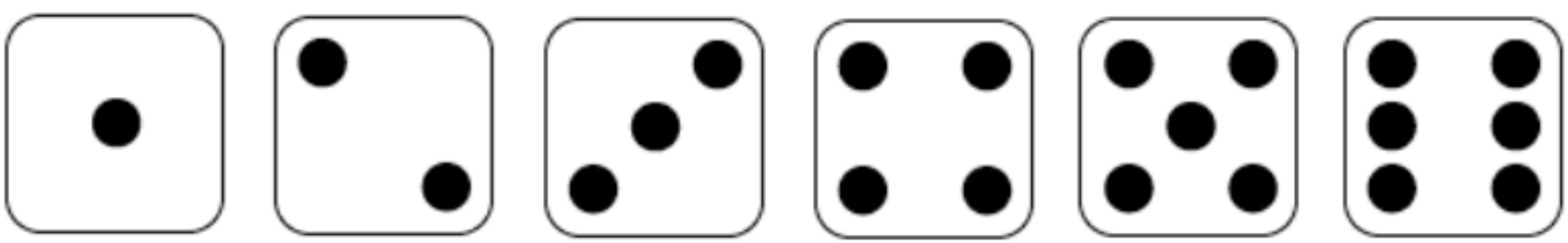




This is our prior information for the probability of each face: 1/6

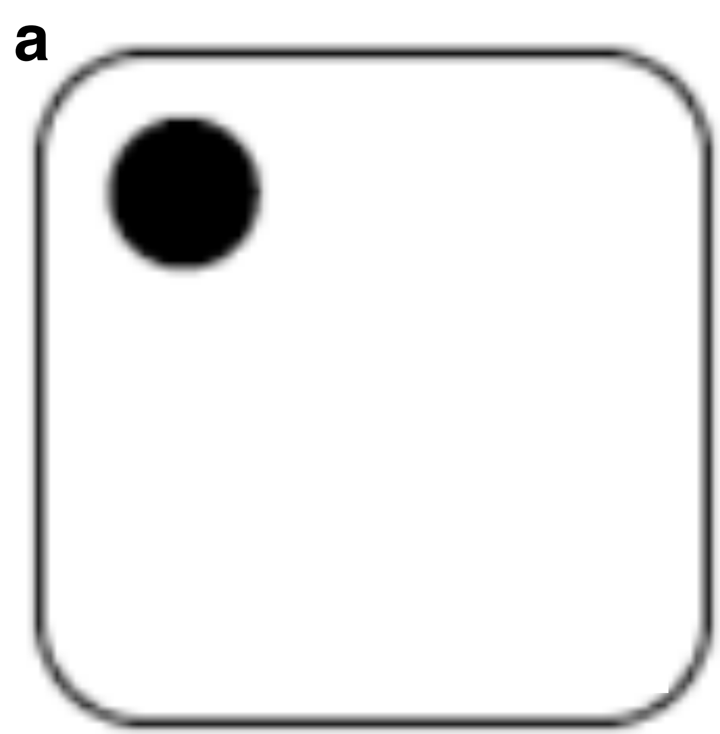
P=1/6

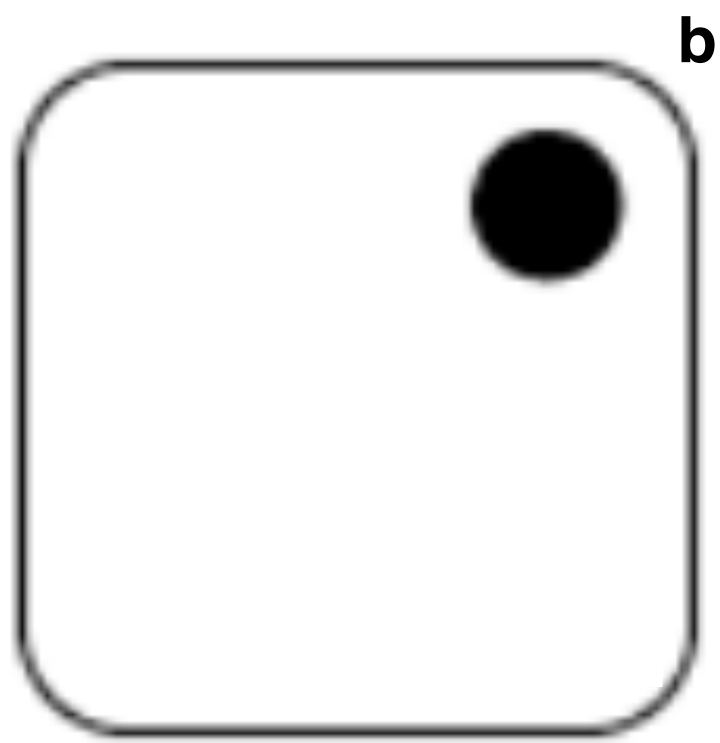


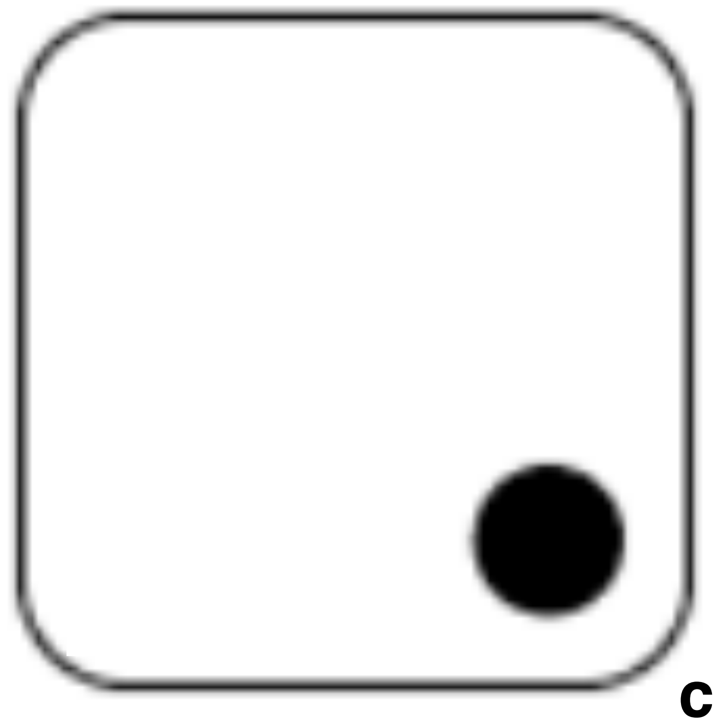


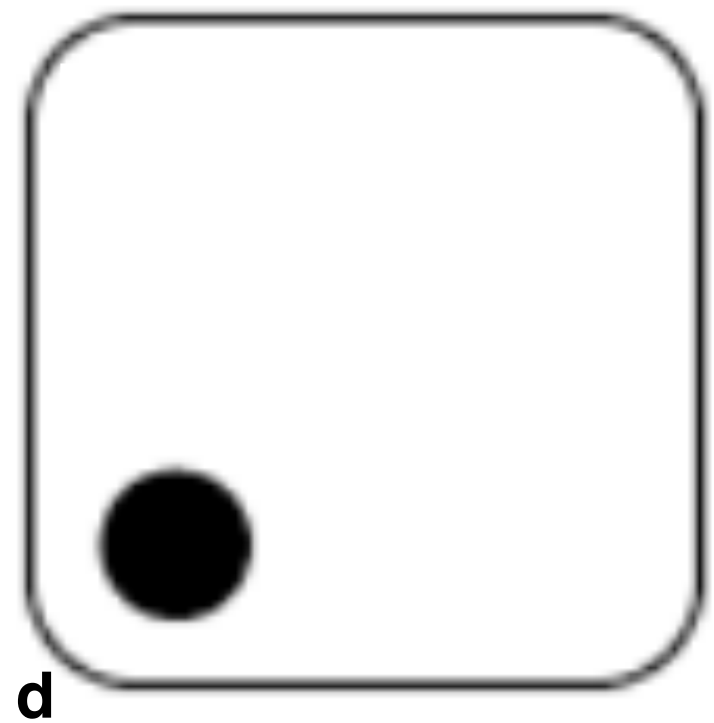
Assume that after throwing the dice, you see the above evidence

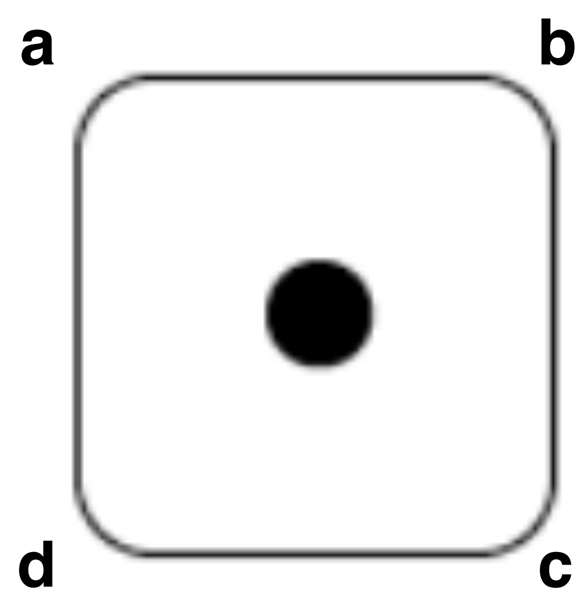
Goal: determine the probability of this evidence for each face of the dice

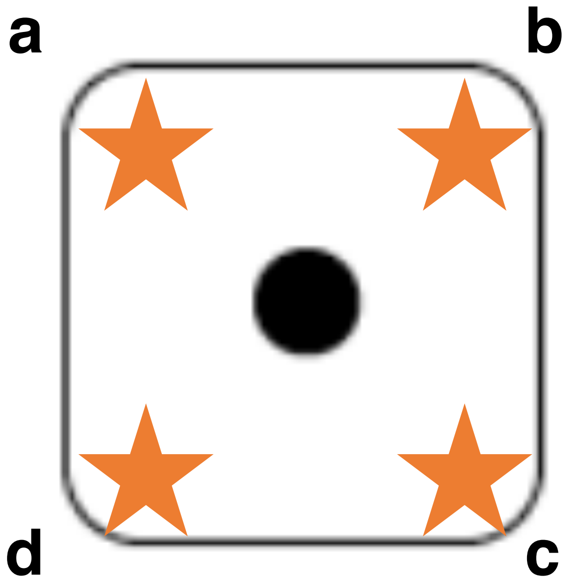

a

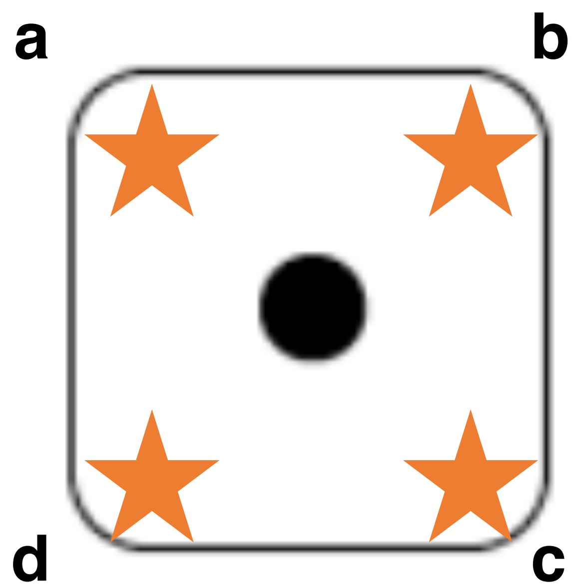

b


c

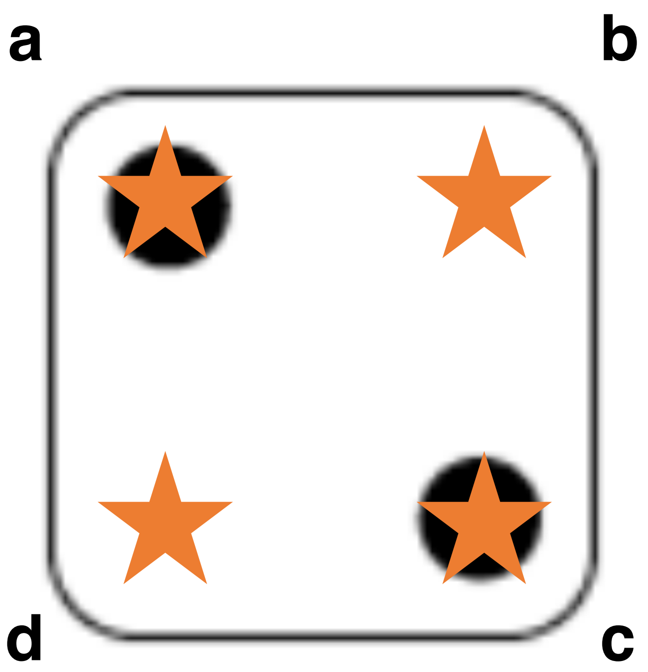

d

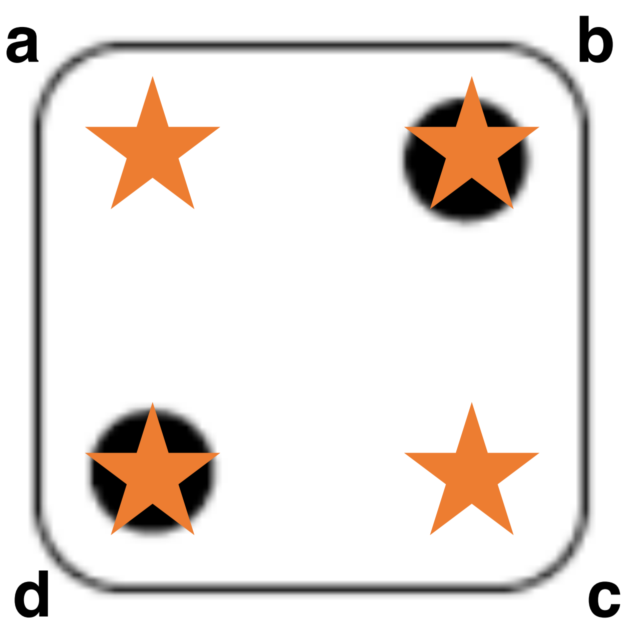

a
b
d
c


a
b
d
c

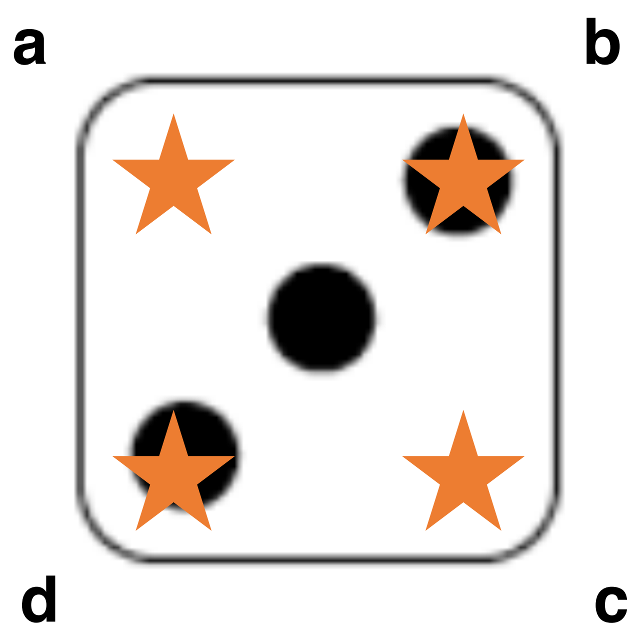


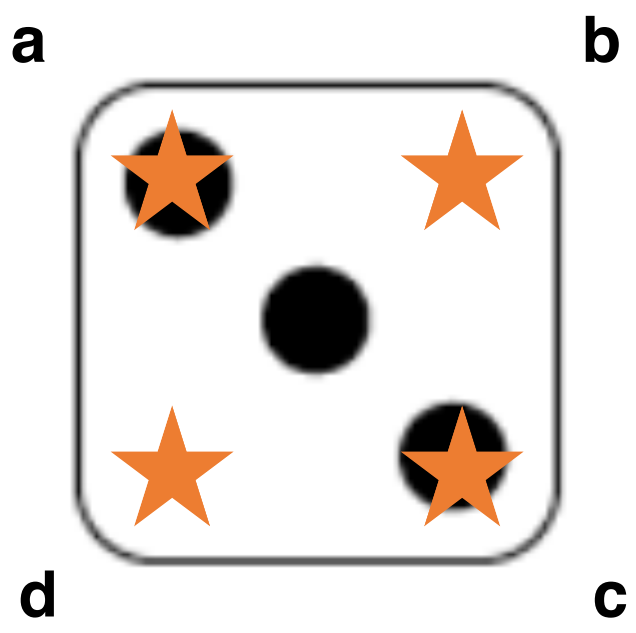
One would never see a dot at the star positions for this face
The probability of the evidence is *zero*



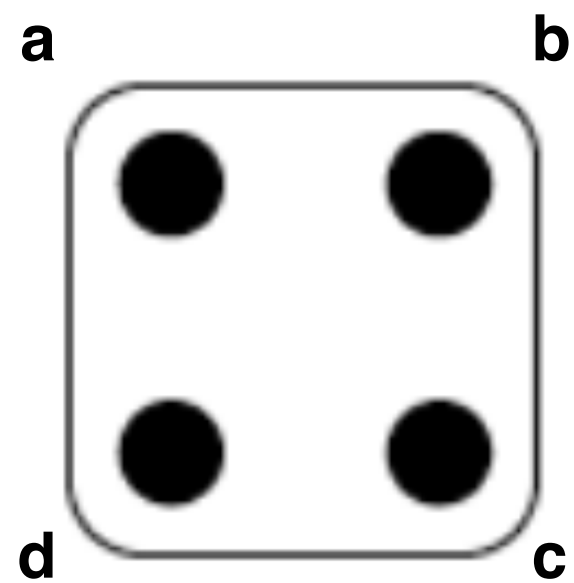


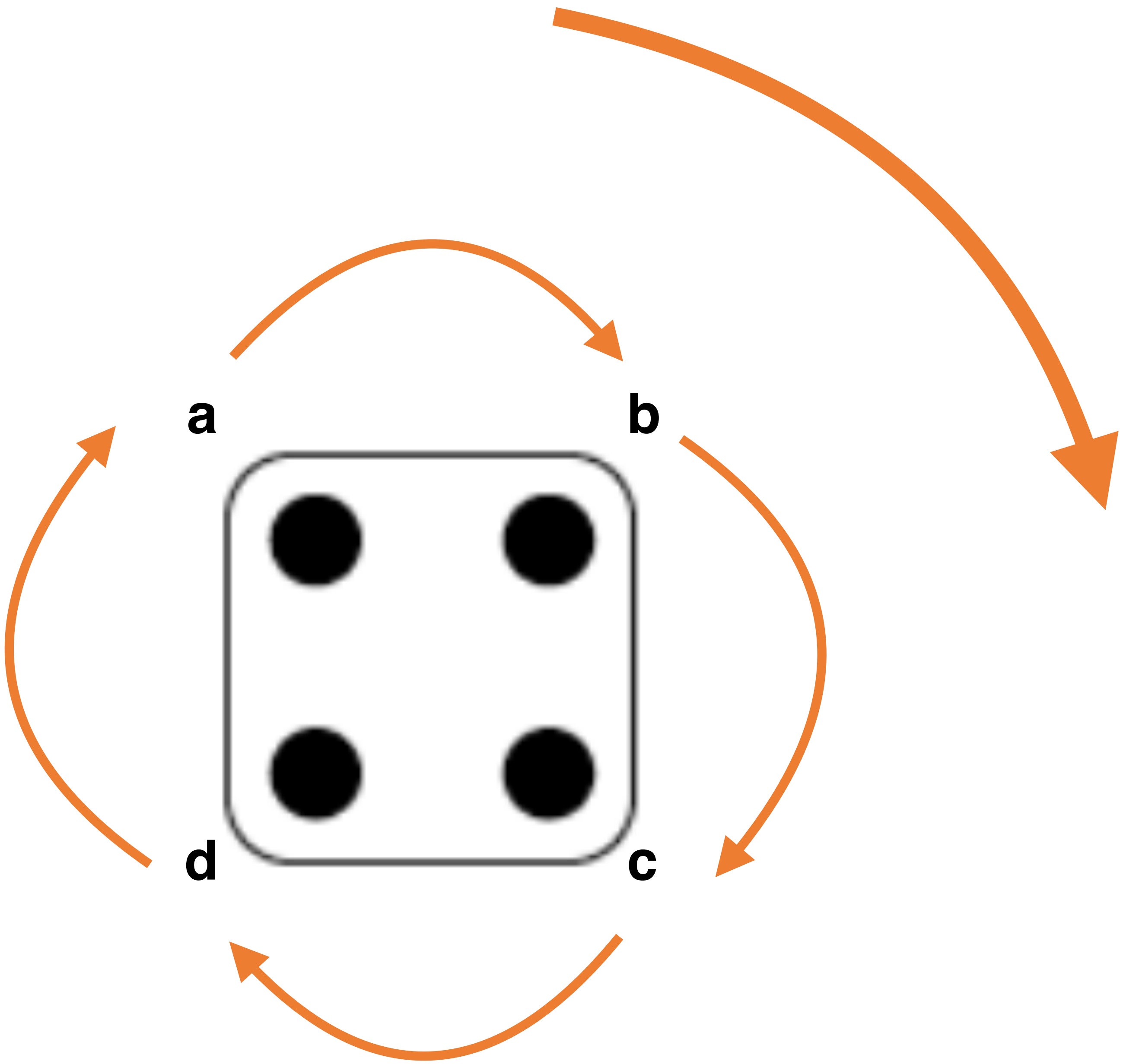
Two possibilities (a,c) and (b,d)

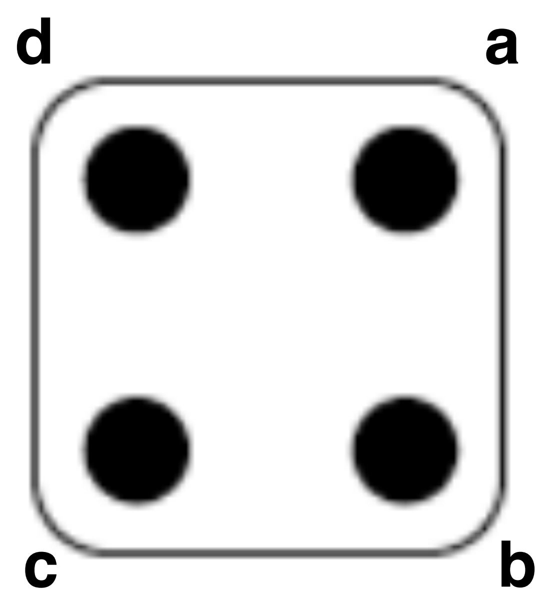


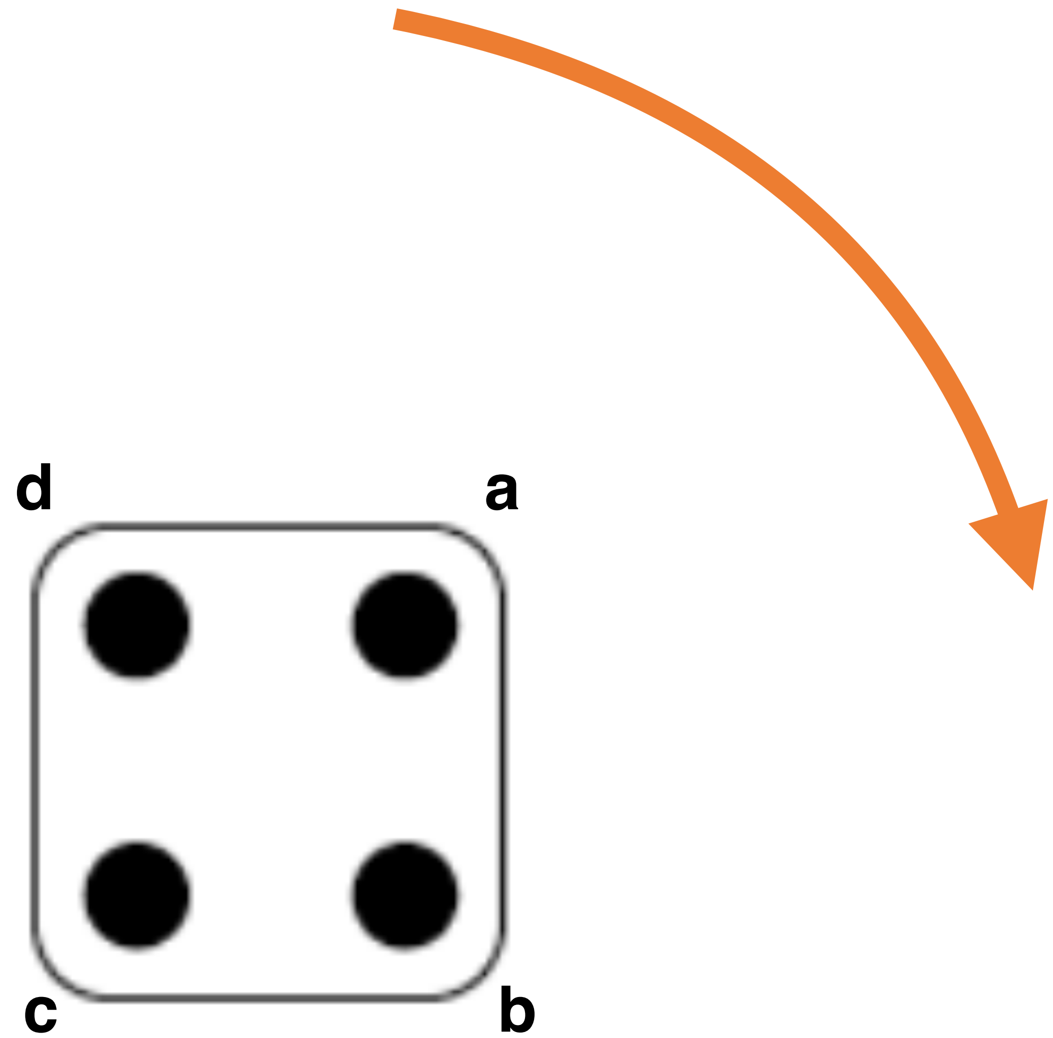


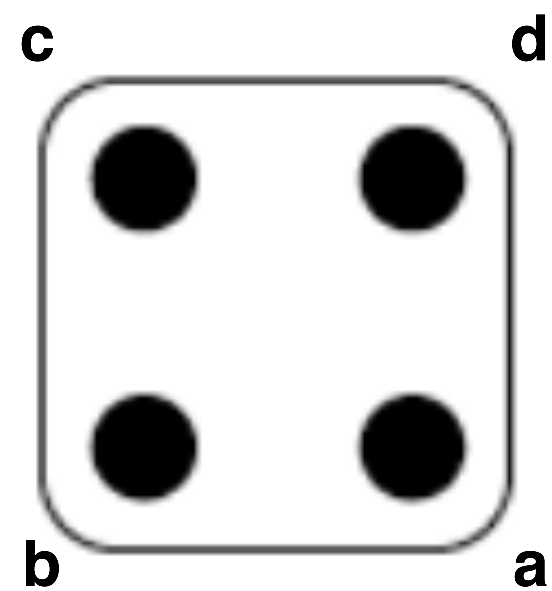
Also two possibilities (a,c) and (b,d)

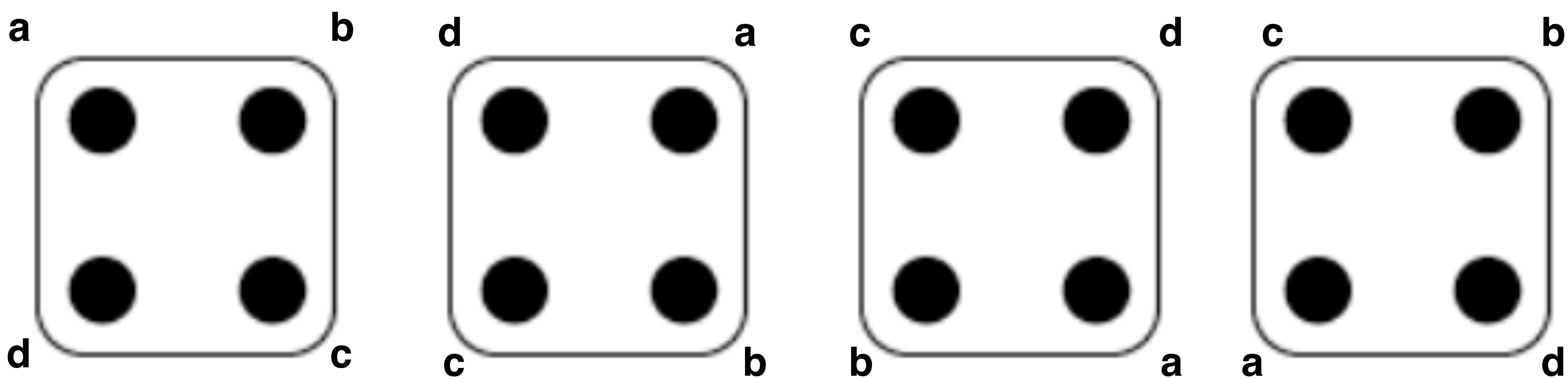

a
b
d
c

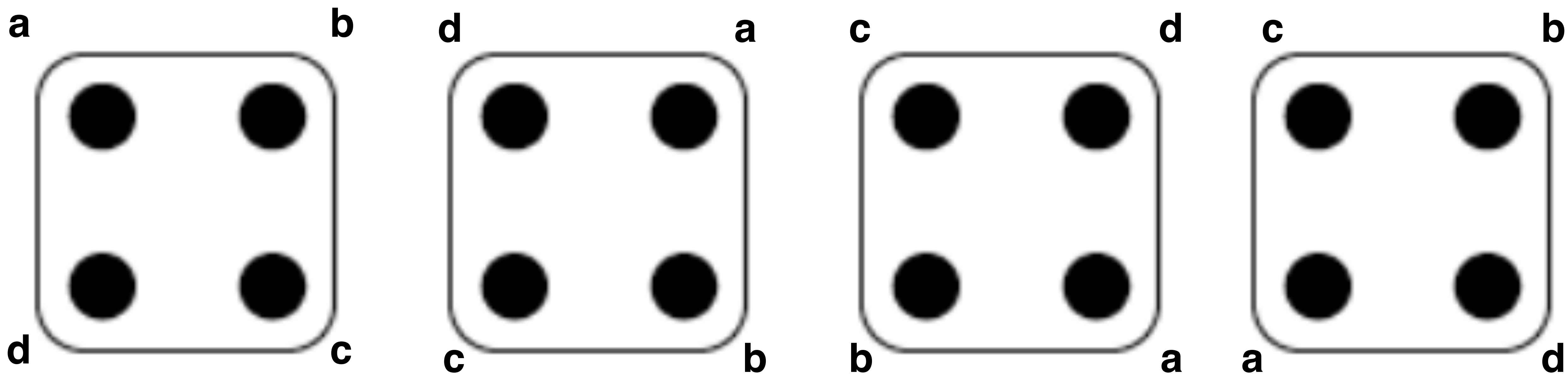

a
b
c
d


d
a
c
b

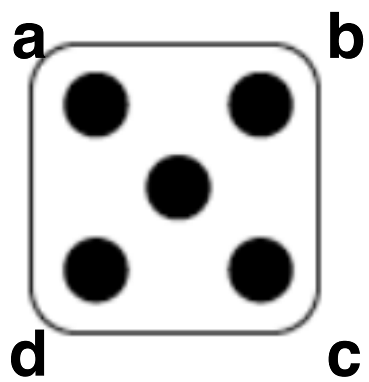

d
a
c
b


c
d
b
a

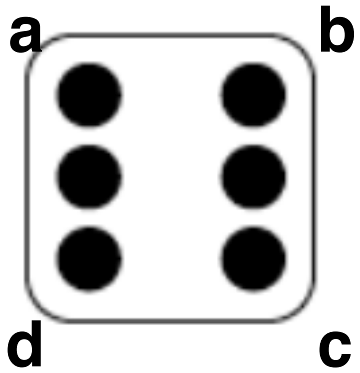

a b
d c
d a
c b
c d
b a
c b
a d



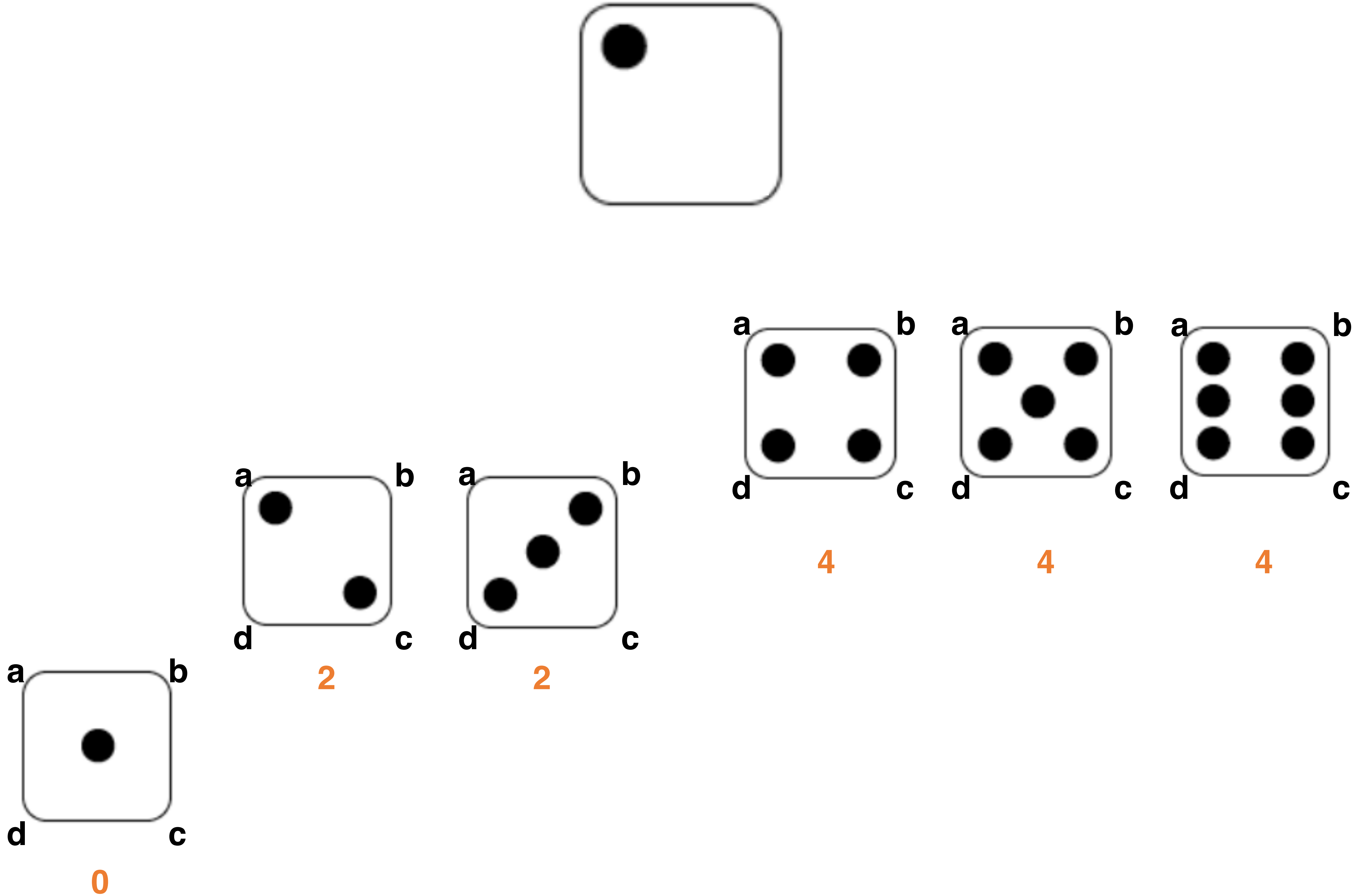
Four possibilities



Four possibilities



Four possibilities


a
b
d
c
0
a
b
d
c
2
a
b
d
c
2
a
b
d
c
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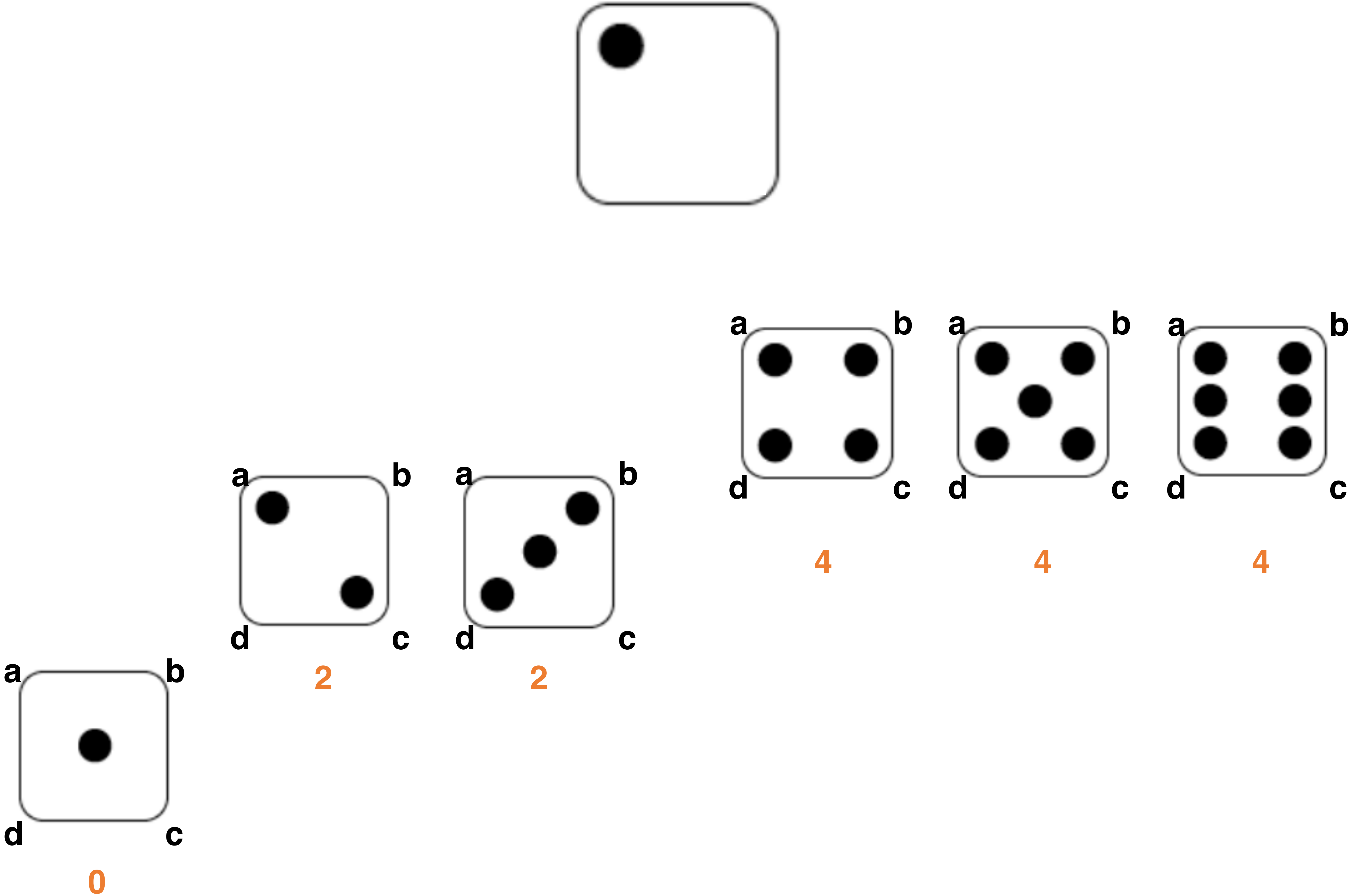
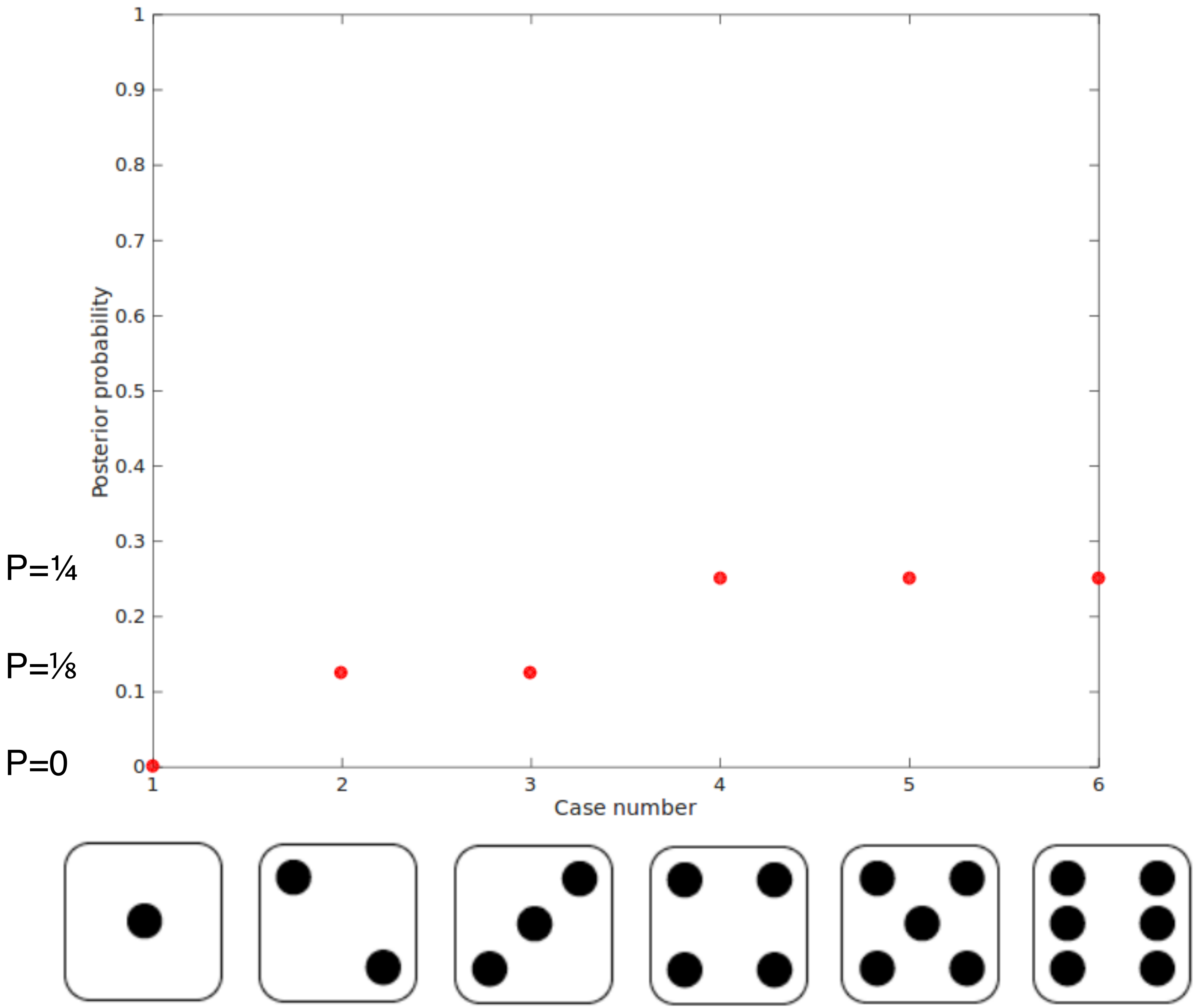
a
b
d
c
4
a
b
d
c
4

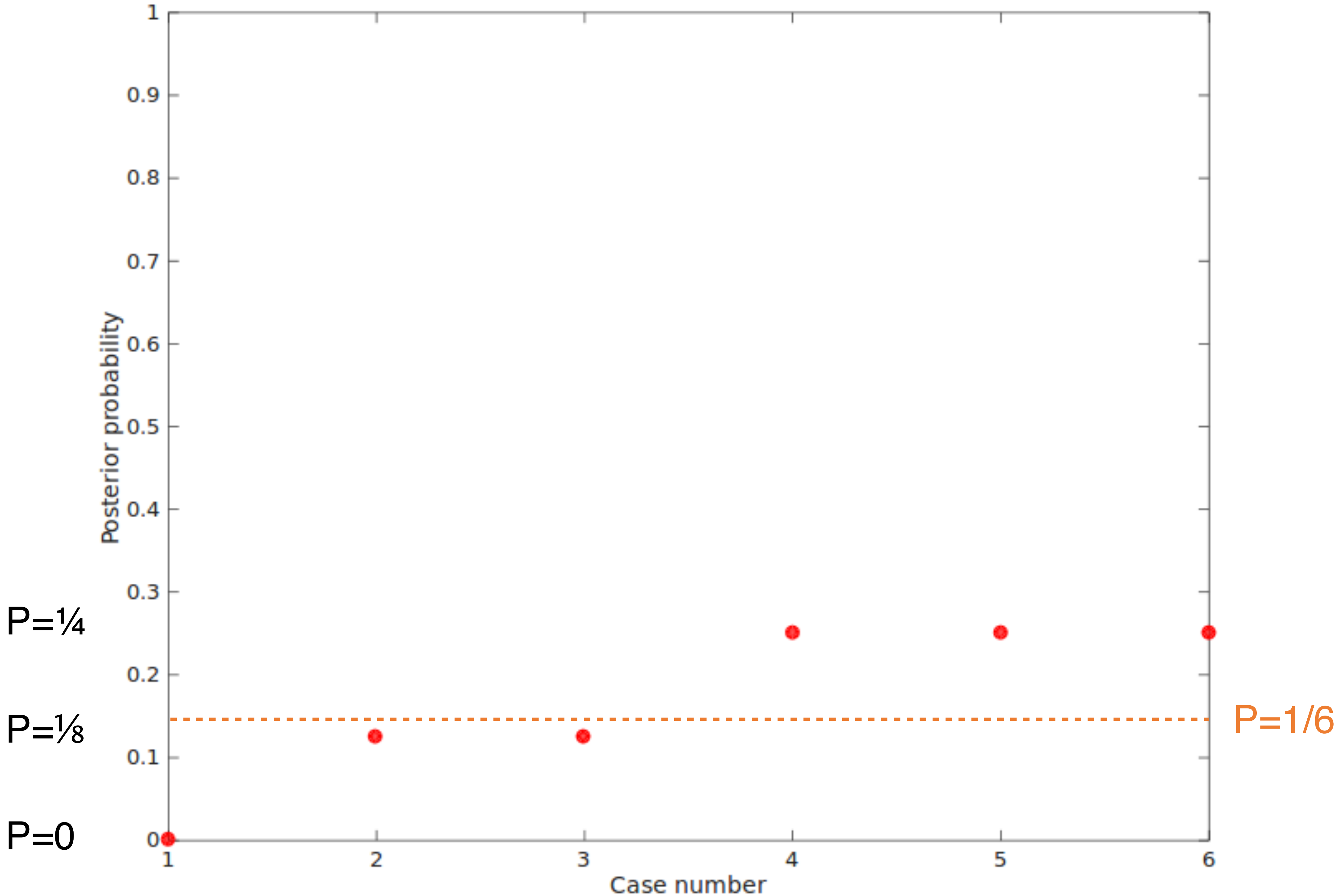


$$\pi(y) = \frac{0+2+2+4+4+4}{6 \times 4} = \frac{16}{24}$$

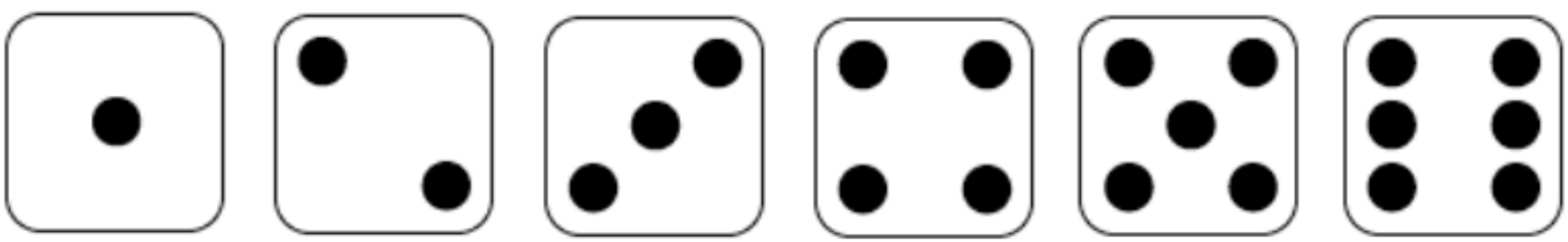
$$\pi(x|y) = \frac{\text{Prior} \times \text{Likelihood}}{\text{Evidence}} = \frac{\frac{1}{6} \times \frac{1}{2}}{\frac{16}{24}} = 0.125$$

Probability that  **was the face of the dice knowing** 

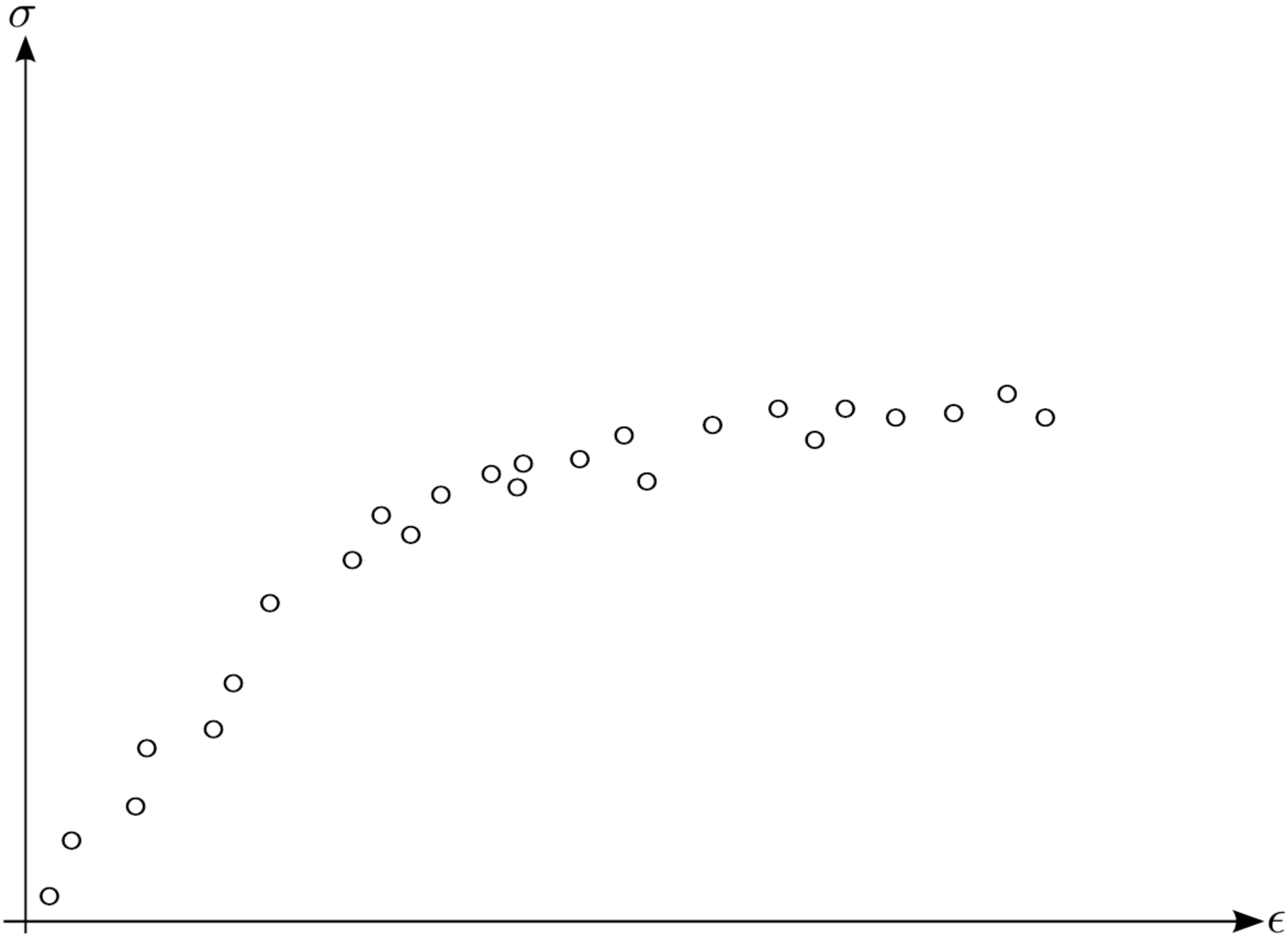

P=¼
P=⅛
P=0
Posterior probability
Case number
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1
2
3
4
5
6



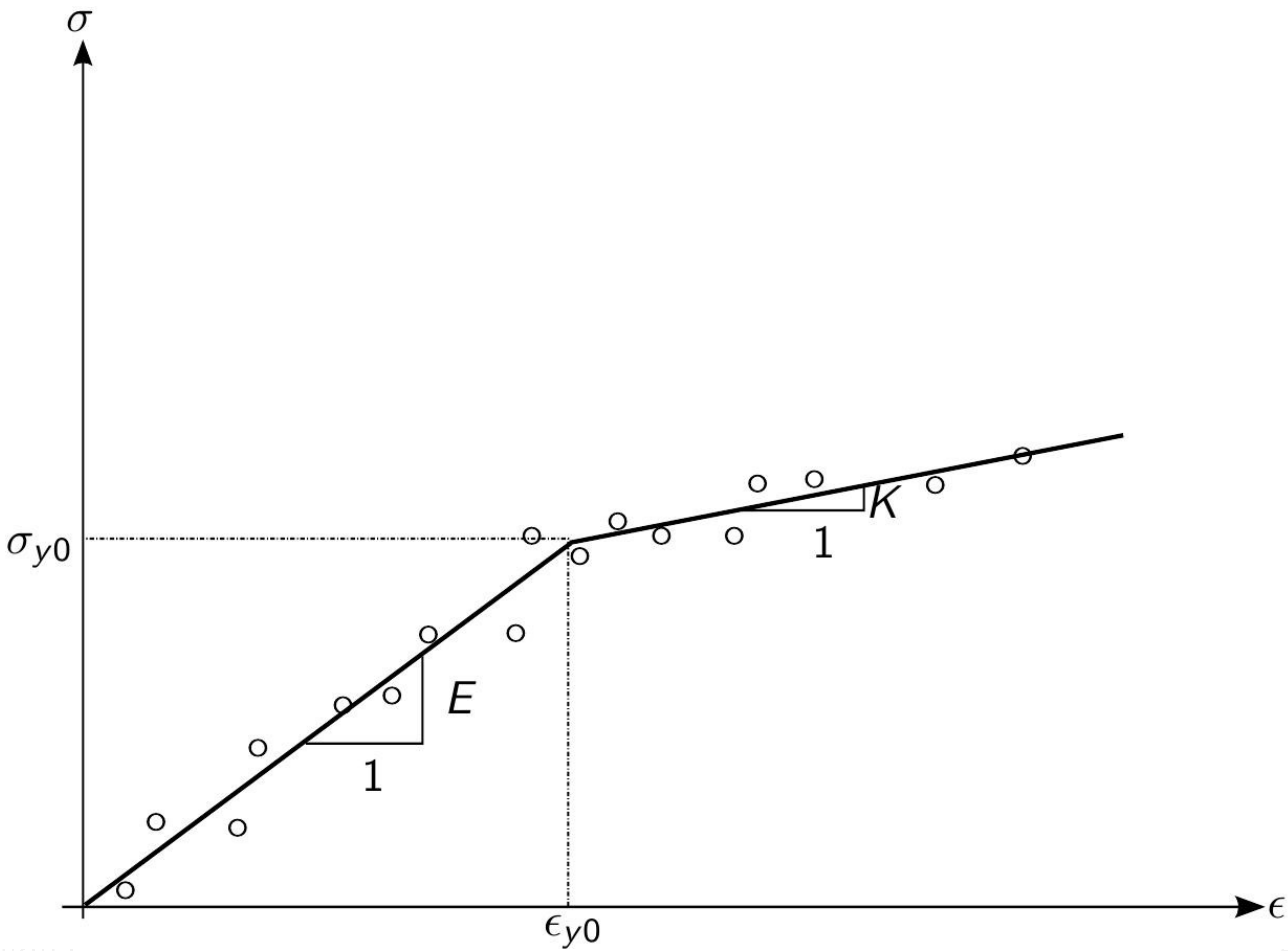
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Posterior probability
1
2
3
4
5
6
Case number
P=¼
P=⅛
P=0
P=1/6



Stress-strain data



Identify the parameters



Construct the likelihood function

Model

$Y = f(X, \Omega)$ observations=f(parameters, error)

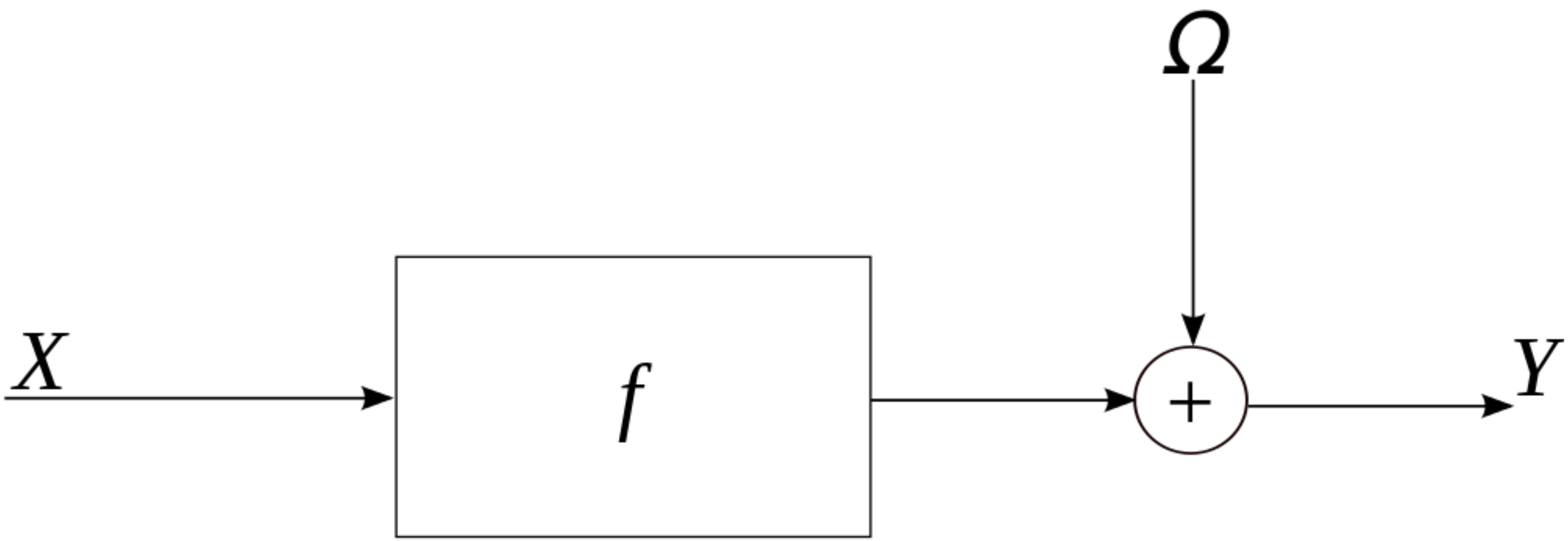
Ω : Error

X : Material parameter

Noise model

Additive noise model

$$Y = f(X) + \Omega$$



Likelihood function

Likelihood function for additive model

$$\pi(y|x) = \pi(\omega) = \pi(y - f(x))$$

$$Y = f(X) + \Omega$$

Constitutive law: linear elasticity

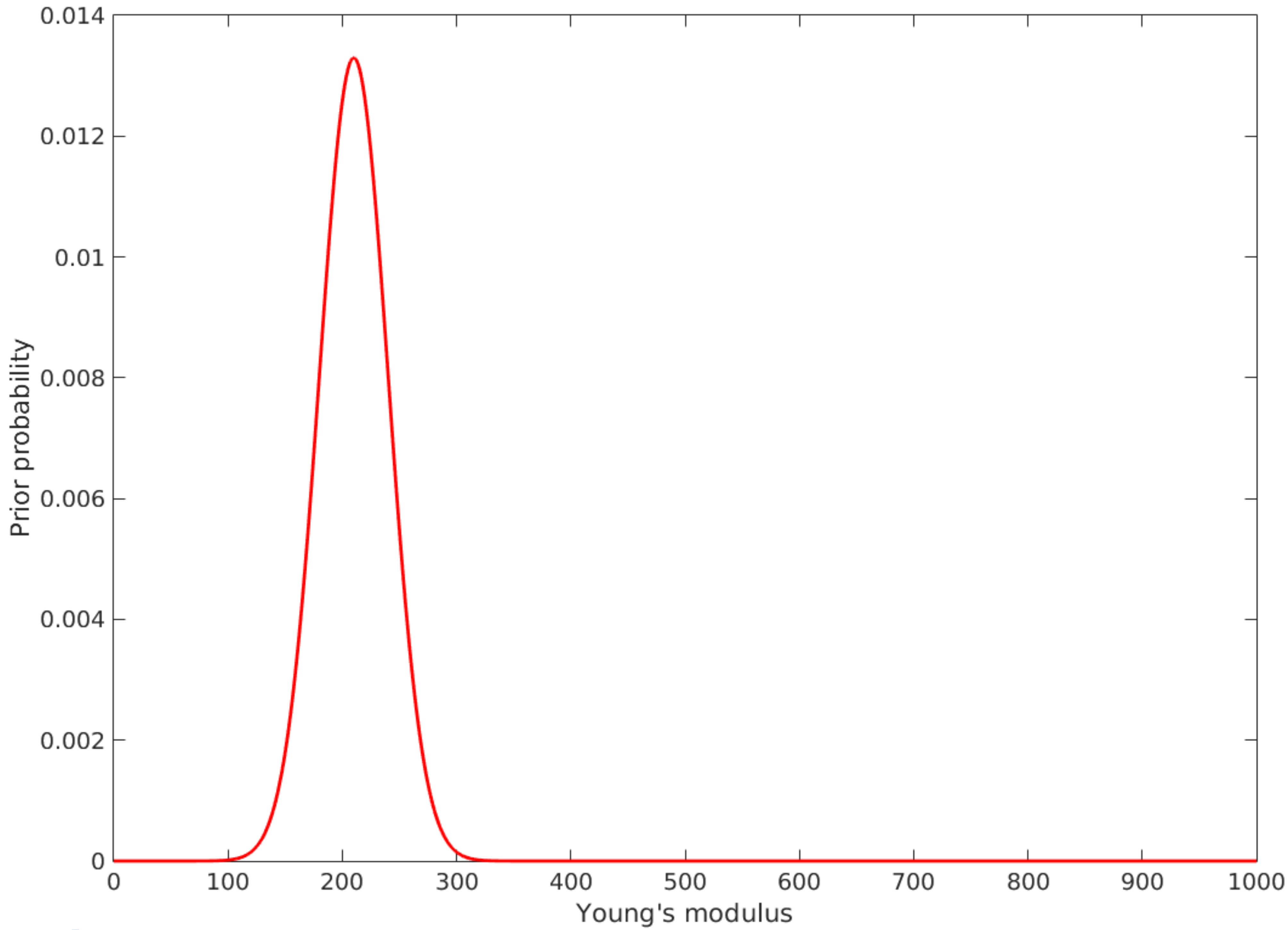
Constitutive model

$$\sigma = E\epsilon \text{ or } \sigma = x\epsilon$$

Observed data

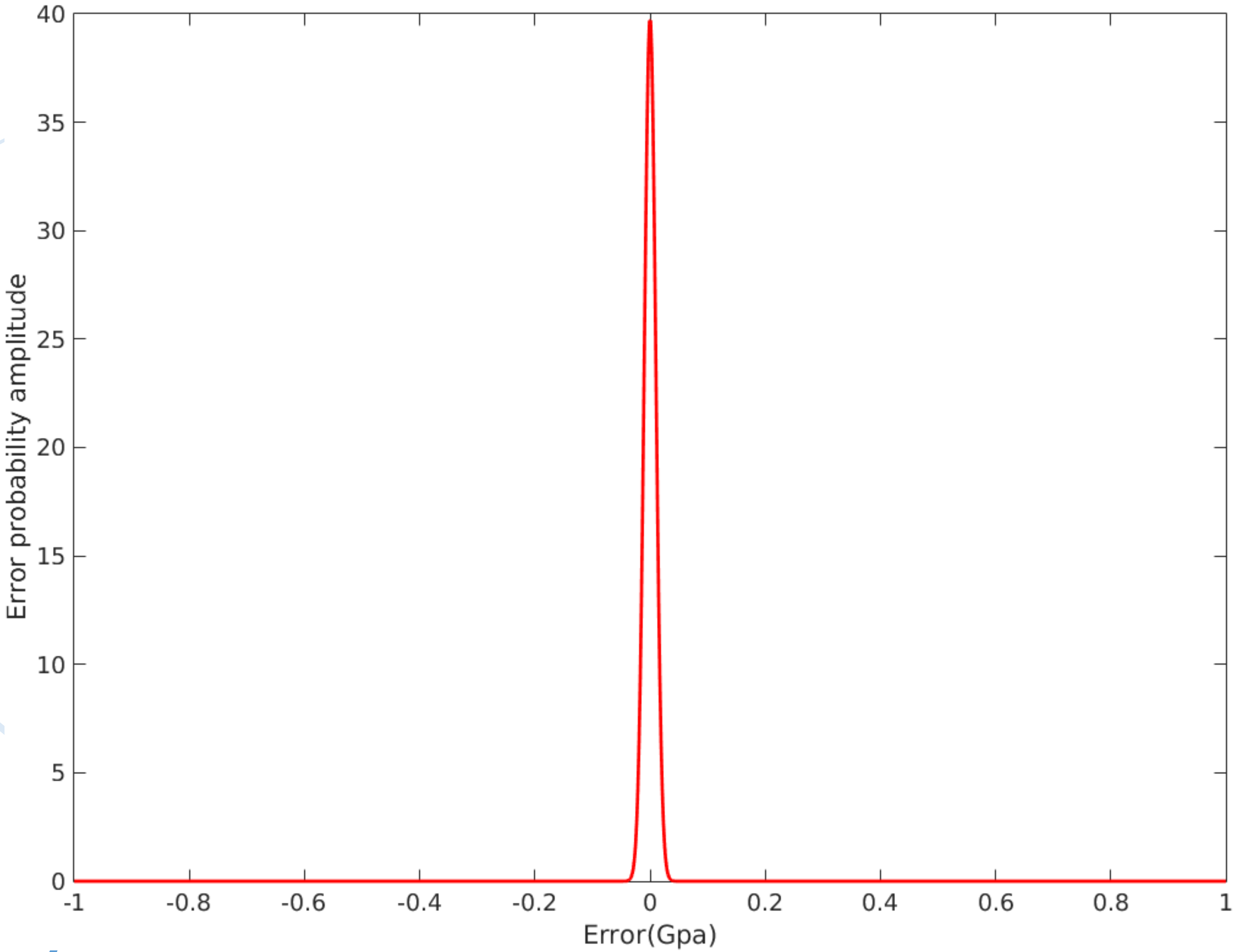
$$Y = X\epsilon + \Omega$$

Prior information on Young's modulus



$$\pi_{prior}(x) = N(210, 900)$$

Error model (noise)



$$\pi(e)_{error} = N(0, 0.0001)$$

Likelihood function

Likelihood function

$$\pi(y|x) = N(y - x\epsilon, 0.0001)$$

$$\pi(y|x) = \pi(\omega) = \pi(y - f(x))$$

Bayes' theorem: calculate the posterior

$$\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{evidence}}$$

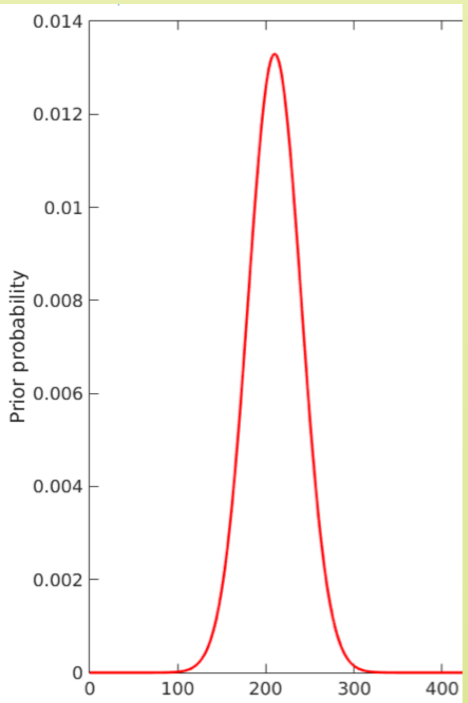
$$\pi(x|y) = \frac{\pi(x)\pi(y|x)}{\int \pi(x)\pi(y|x)dx}$$

Bayes' theorem: calculate the posterior

$$\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{evidence}}$$

$$\pi(x|y) = \frac{\pi(x)\pi(y|x)}{\int \pi(x)\pi(y|x)dx}$$

prior $\pi(x)$

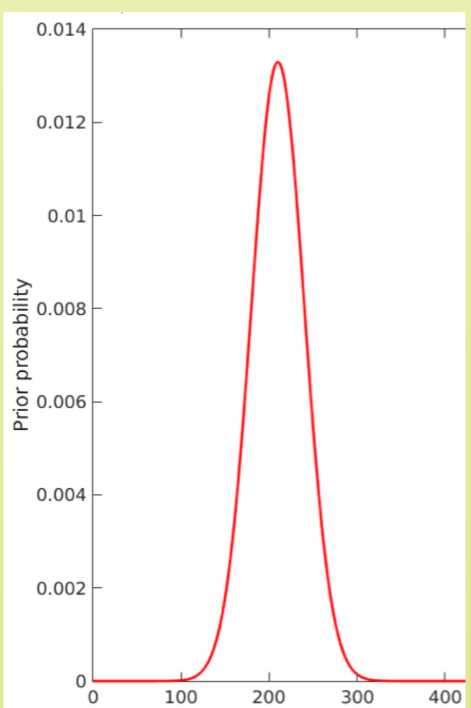


Bayes' theorem: calculate the posterior

$$\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{evidence}}$$

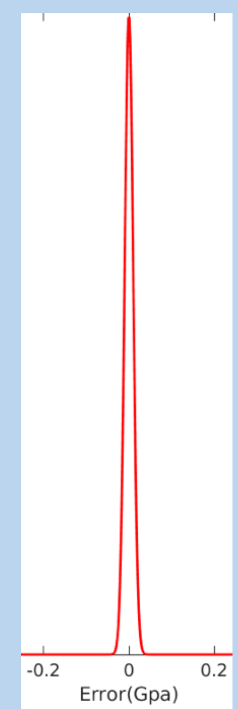
$$\pi(x|y) = \frac{\pi(x)\pi(y|x)}{\int \pi(x)\pi(y|x)dx}$$

prior $\pi(x)$



likelihood $\pi(y|x)$

$$\pi(y|x) = N(y - x\epsilon, 0.0001)$$
$$\pi(y|x) = \pi(\omega) = \pi(y - f(x))$$

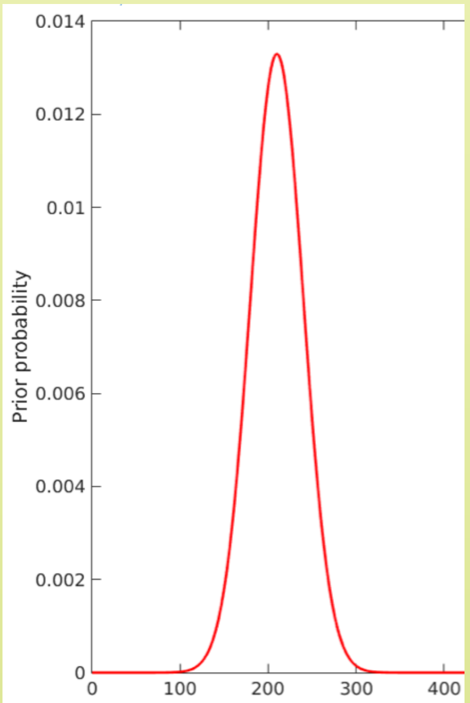


Bayes' theorem: calculate the posterior

$$\text{posterior} = \frac{\text{prior} \times \text{likelihood}}{\text{evidence}}$$

$$\pi(x|y) = \frac{\pi(x)\pi(y|x)}{\int \pi(x)\pi(y|x)dx}$$

prior $\pi(x)$

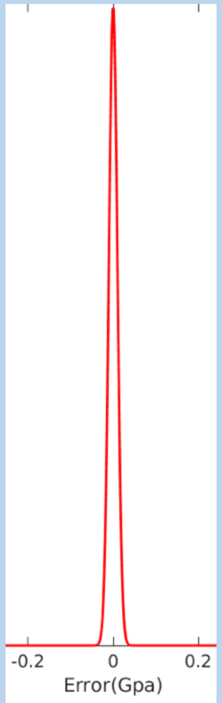


posterior

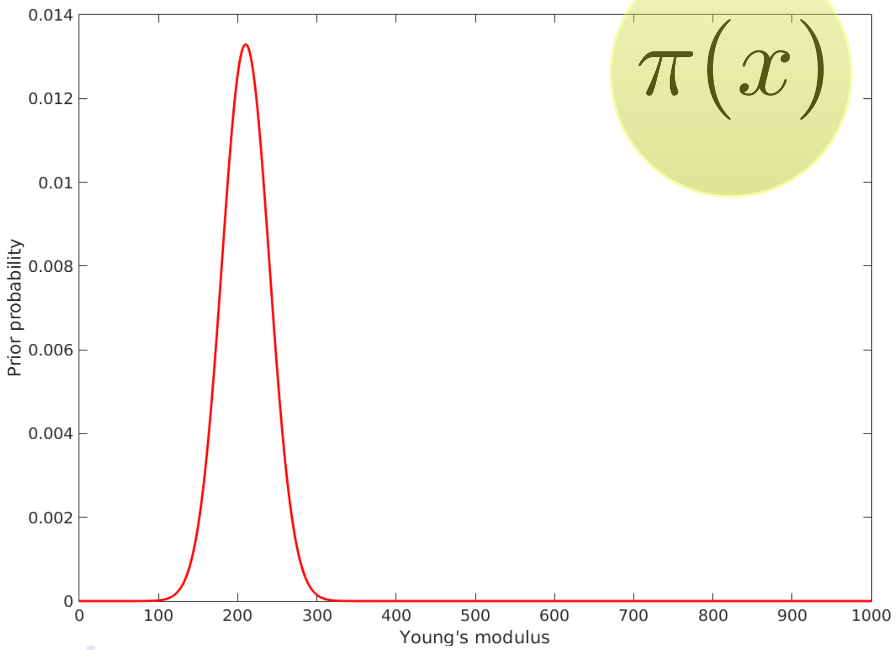
$\pi(x|y)$

likelihood $\pi(y|x)$

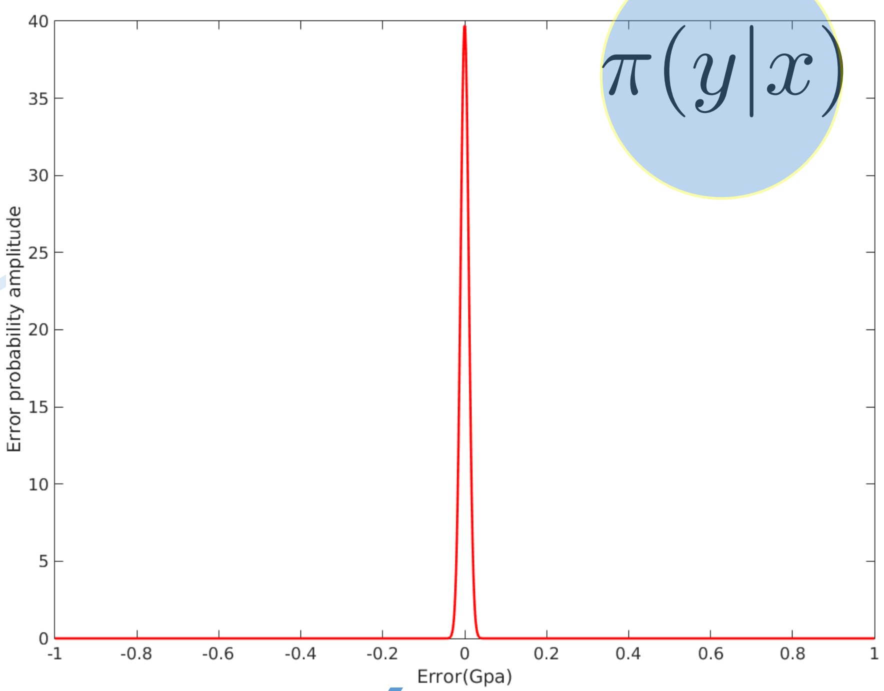
$$\pi(y|x) = N(y - x\epsilon, 0.0001)$$
$$\pi(y|x) = \pi(\omega) = \pi(y - f(x))$$



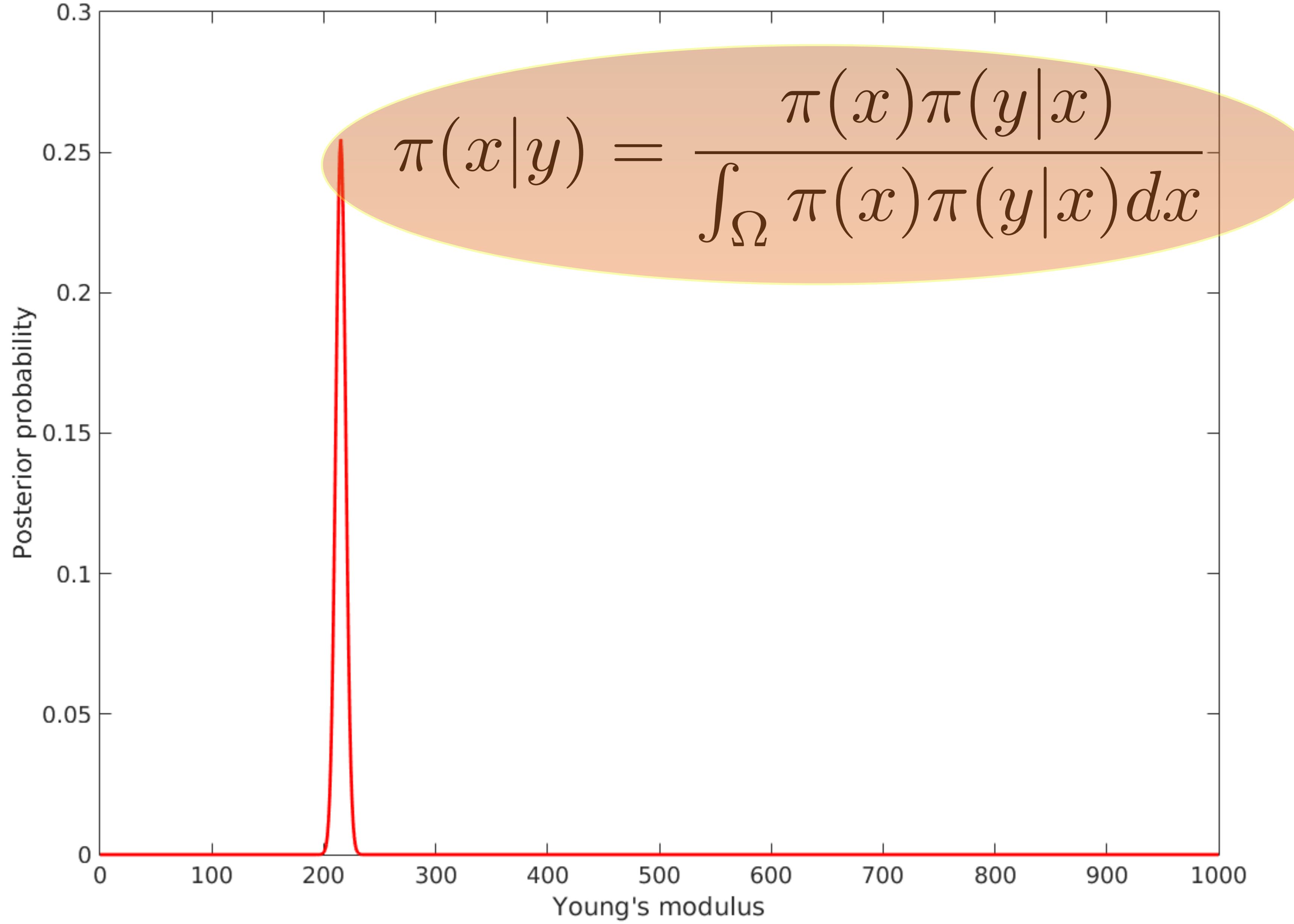
Posterior probability



$$\pi_{prior}(x) = N(210, 900)$$

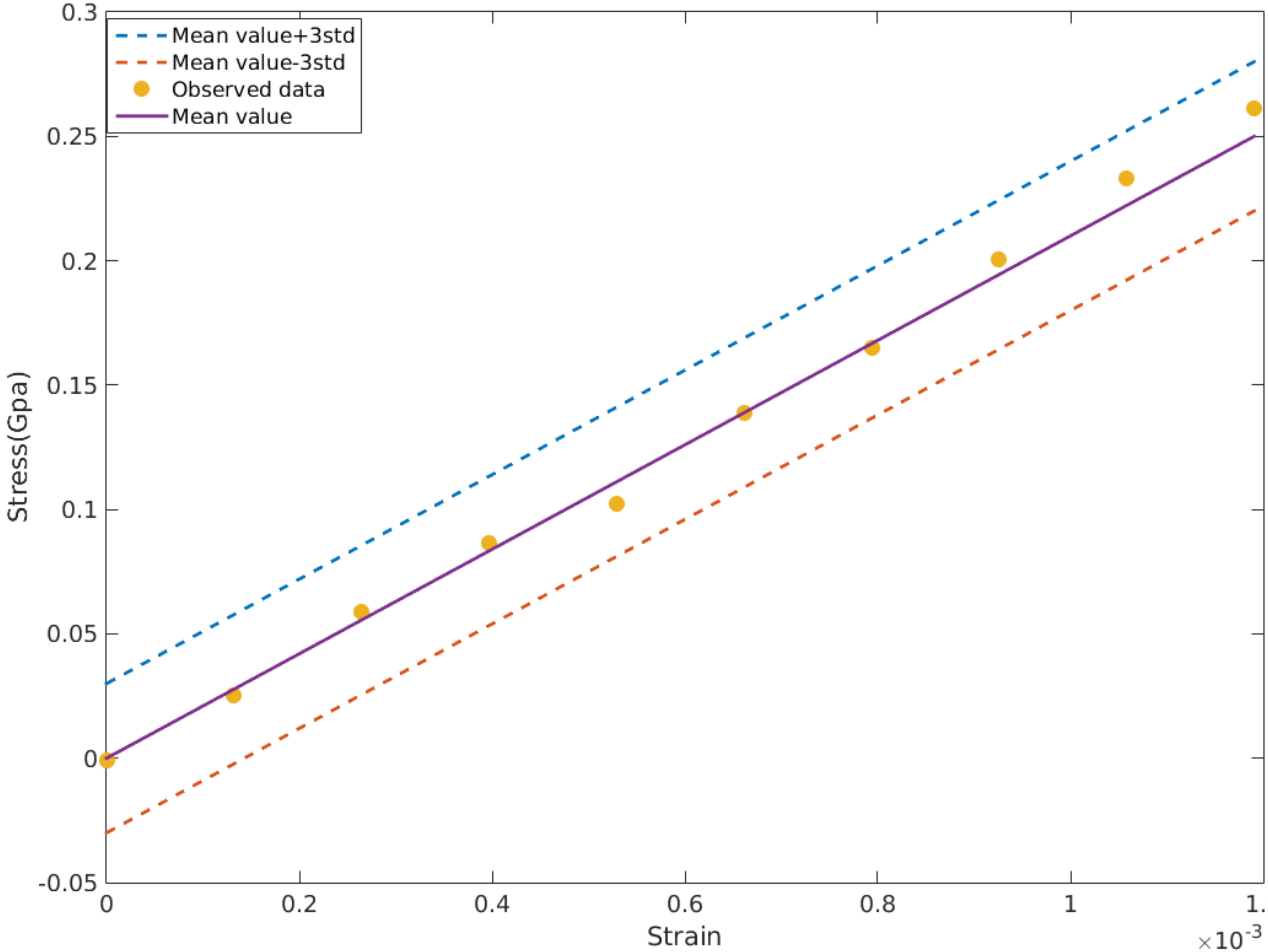


$$\pi(e)_{error} = N(0, 0.0001)$$

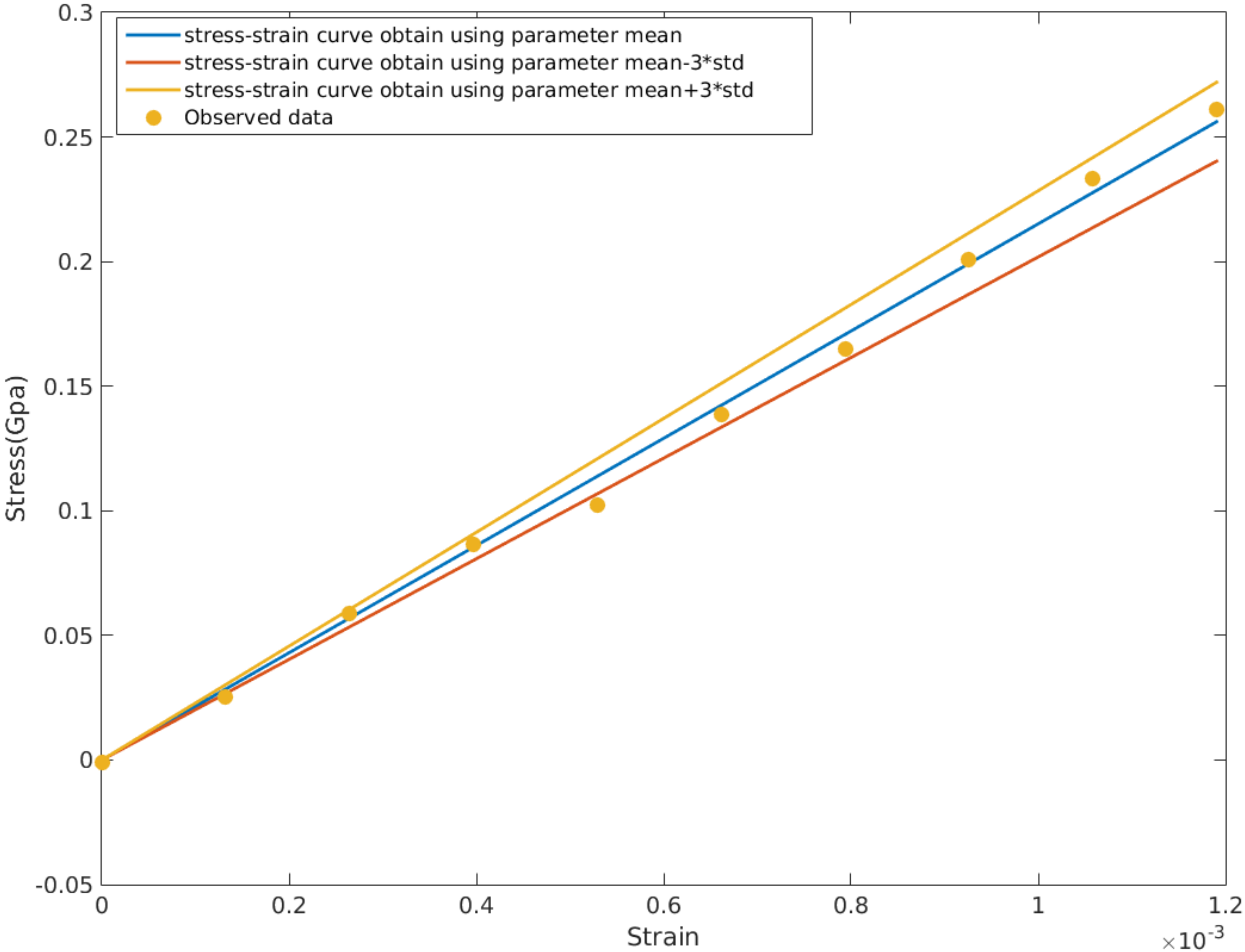


$$\pi_{posterior} = N(215.1533, 19.6168)$$
$$N_{sample} = 10$$

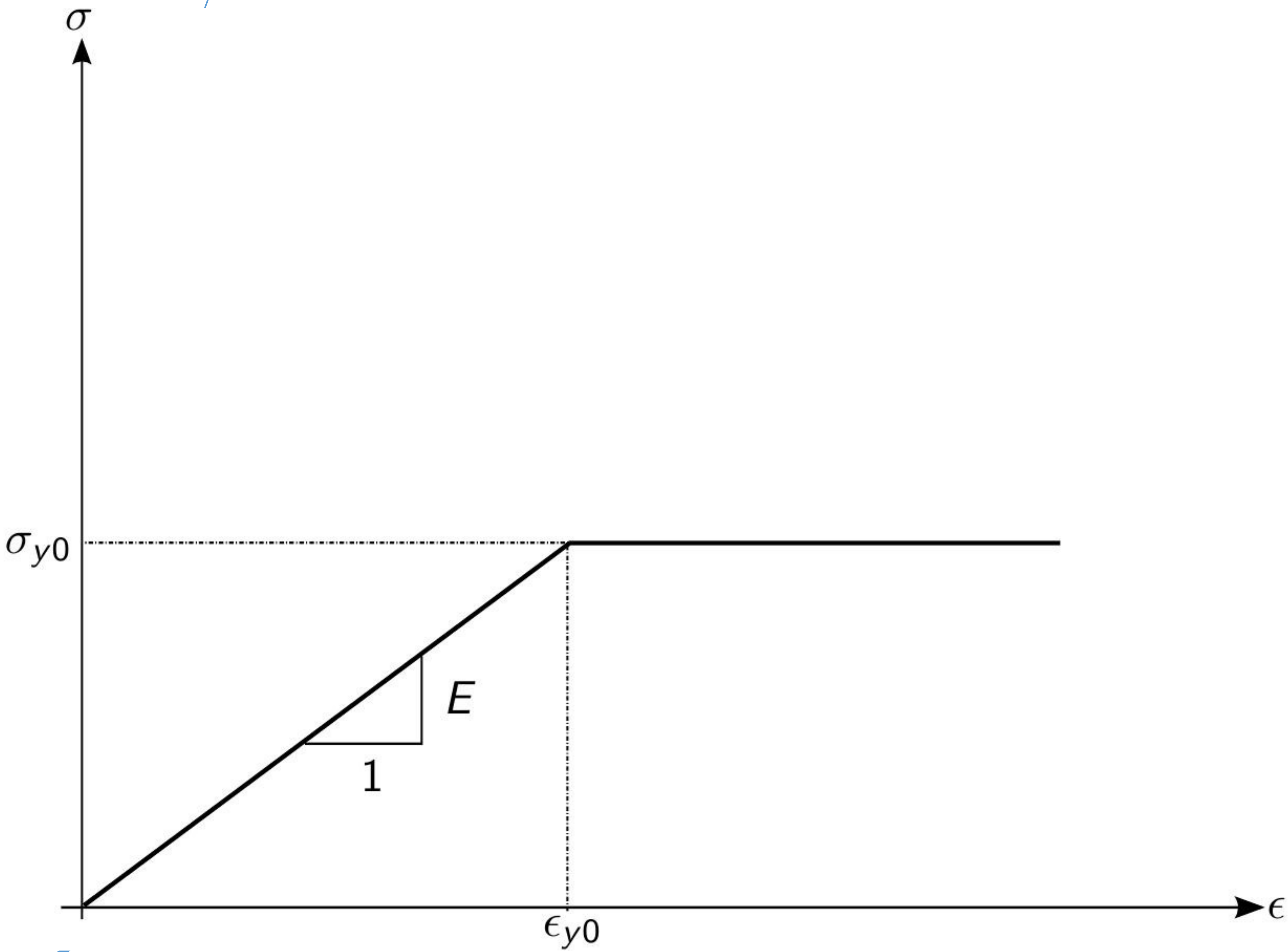
The 99.73% rule: observations



Propagation of the uncertainty to the constitutive model



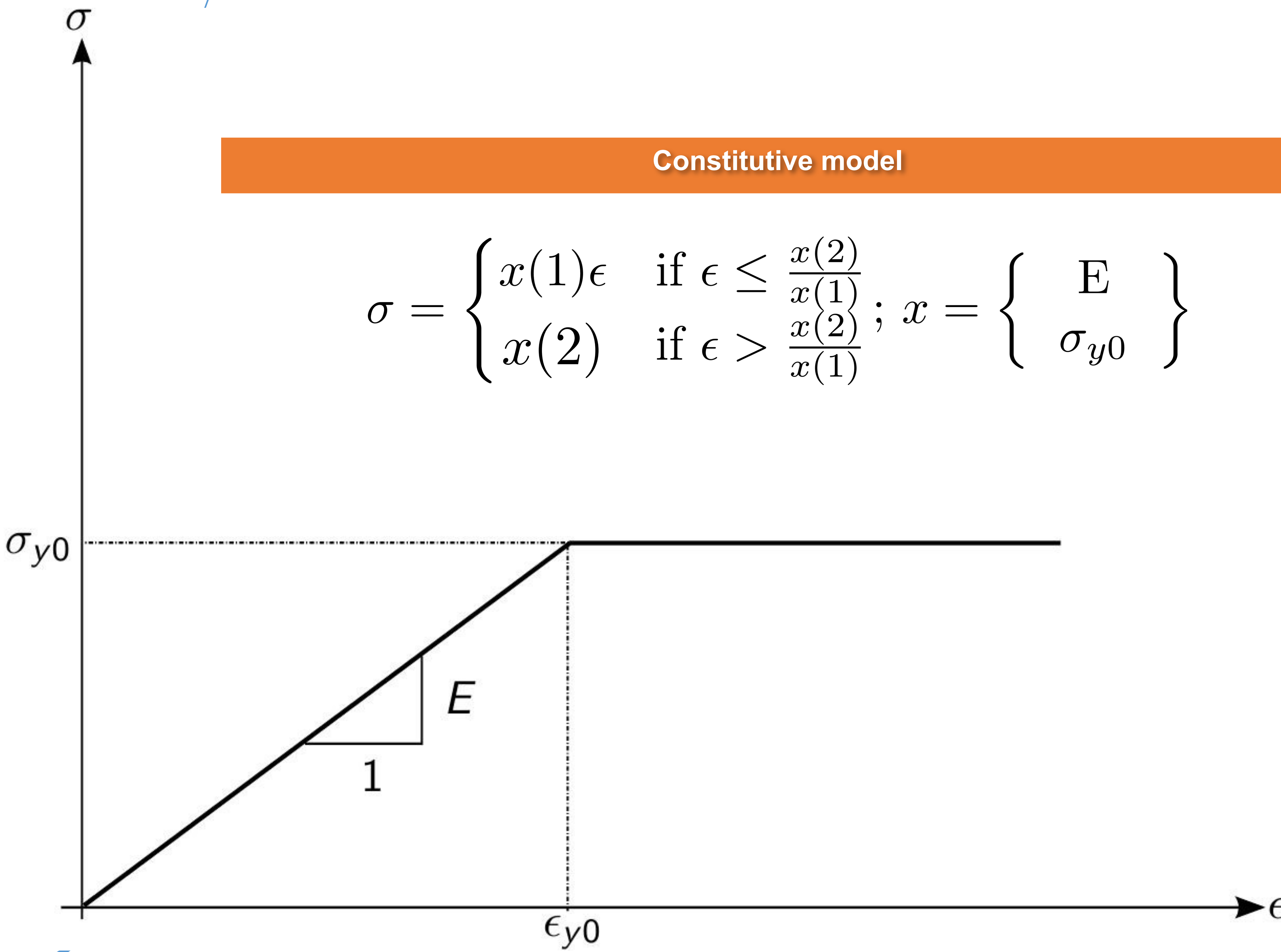
Perfect plasticity



Perfect plasticity

Constitutive model

$$\sigma = \begin{cases} x(1)\epsilon & \text{if } \epsilon \leq \frac{x(2)}{x(1)} \\ x(2) & \text{if } \epsilon > \frac{x(2)}{x(1)} \end{cases} ; \; x = \left\{ \begin{array}{c} \mathrm{E} \\ \sigma_{y0} \end{array} \right\}$$



Perfect plasticity

Modified form for constitutive model

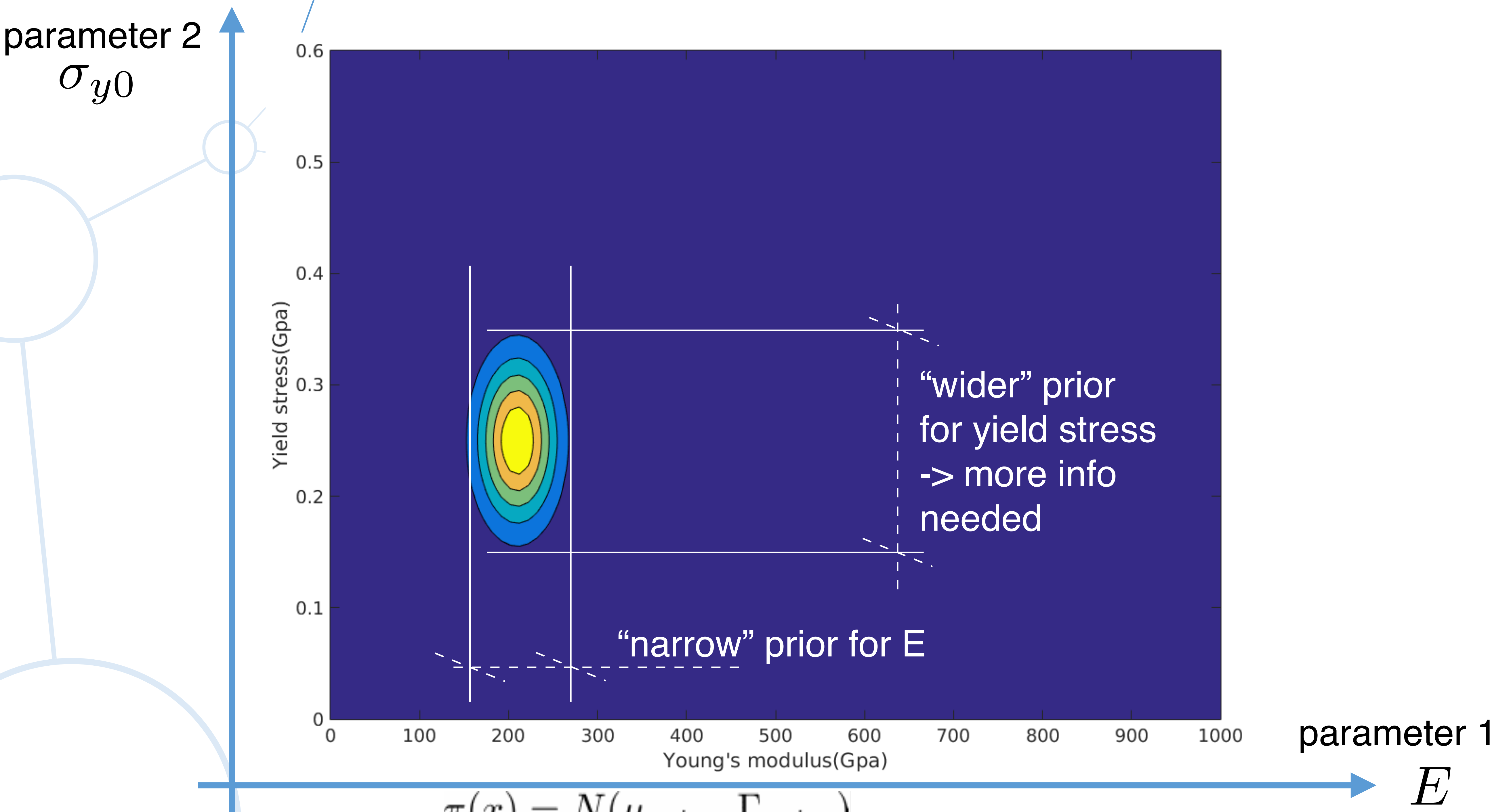
$$\sigma = x(1)\epsilon(1 - h(\sigma - x(2))) + x(2)h(\sigma - x(2))$$

$$h : \text{heaviside function}$$

Observed data

$$Y = \sigma + \Omega$$

Perfect plasticity: contour plot of prior in parameter space



$$\pi(x) = N(\mu_{prior}, \Gamma_{prior})$$

$$\mu_{prior} = \begin{bmatrix} 210 \\ 0.25 \end{bmatrix}; \Gamma_{prior} = \begin{bmatrix} 900 & 0 \\ 0 & 0.0025 \end{bmatrix}$$

Perfect plasticity

Posterior probability

$$\pi(x|y_{N_{obs}}) \propto exp\left(-\frac{1}{2}\left((x-\mu_{prior})^T\Gamma^{-1}_{prior}(x-\mu_{prior}) + \frac{\sum_{i=1}^{N_{obs}}(y_i - F_i)^2}{\sigma_\Omega^2}\right)\right)$$

σ_Ω : Error standard deviation

Perfect plasticity

Posterior probability

$$(x-\mu)^2$$

$$\pi(x|y_{N_{obs}}) \propto exp\left(-\frac{1}{2}\left((x-\mu_{prior})^T \Gamma^{-1}_{prior} (x-\mu_{prior}) + \frac{\sum_{i=1}^{N_{obs}} (y_i - F_i)^2}{\sigma_\Omega^2} \right) \right)$$

σ_Ω : Error standard deviation

$$1/\sigma^2$$

likelihood for each observation

Perfect plasticity

Posterior probability

stress measurement stress model

$$(x-\mu)^2$$

$$\pi(x|y_{N_{obs}}) \propto exp\left(-\frac{1}{2}\left((x-\mu_{prior})^T\Gamma_{prior}^{-1}(x-\mu_{prior}) + \frac{\sum_{i=1}^{N_{obs}}(y_i - F_i)^2}{\sigma_\Omega^2}\right)\right)$$

σ_Ω : Error standard deviation

$$1/\sigma^2$$

likelihood for each observation

Perfect plasticity

Posterior probability

stress measurement stress model

$$(x-\mu)^2$$

$$\pi(x|y_{N_{obs}}) \propto exp\left(-\frac{1}{2}\left((x-\mu_{prior})^T\Gamma^{-1}_{prior}(x-\mu_{prior}) + \frac{\sum_{i=1}^{N_{obs}}(y_i - F_i)^2}{\sigma_\Omega^2}\right)\right)$$

σ_Ω : Error standard deviation

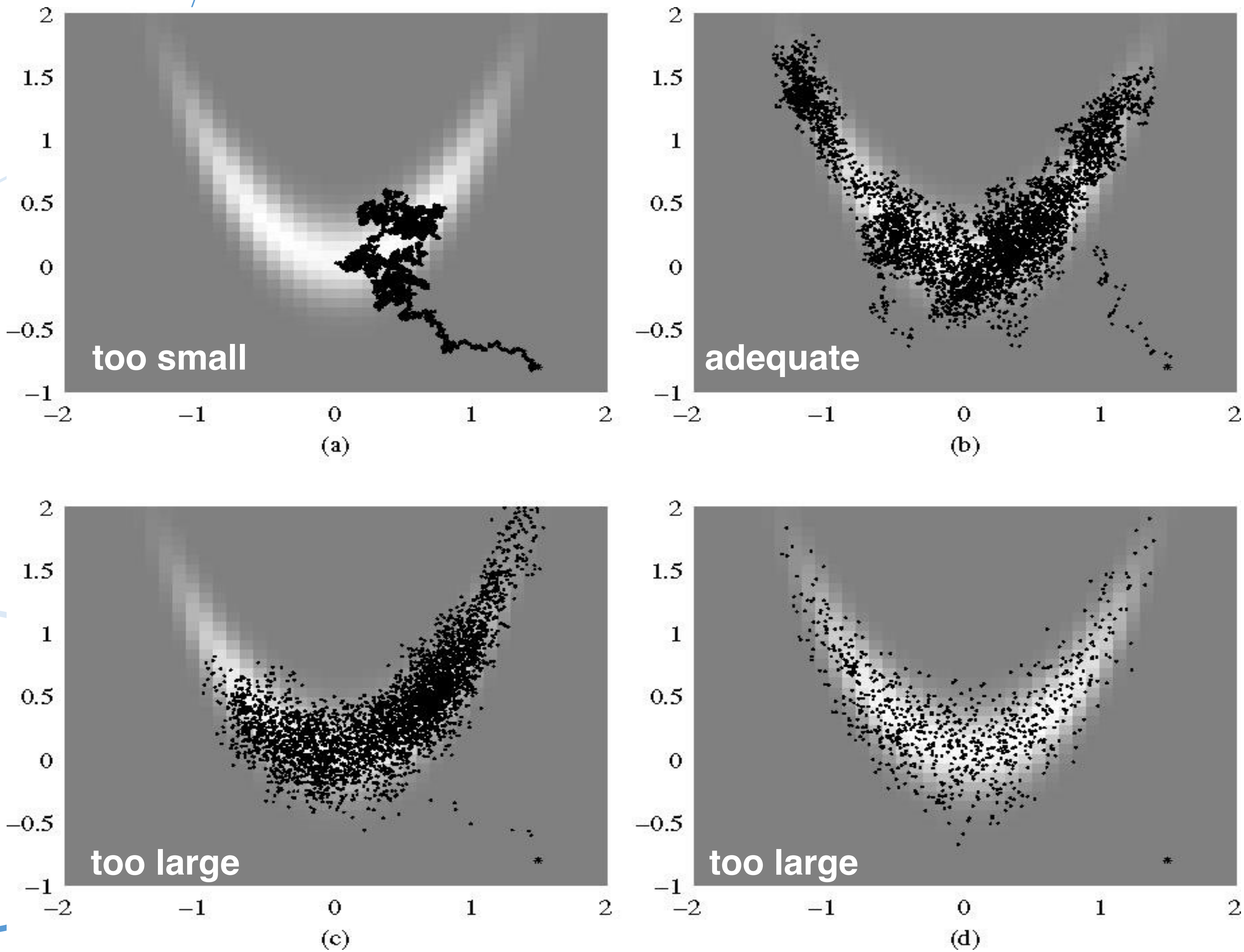
likelihood for each observation

$$1/\sigma^2$$

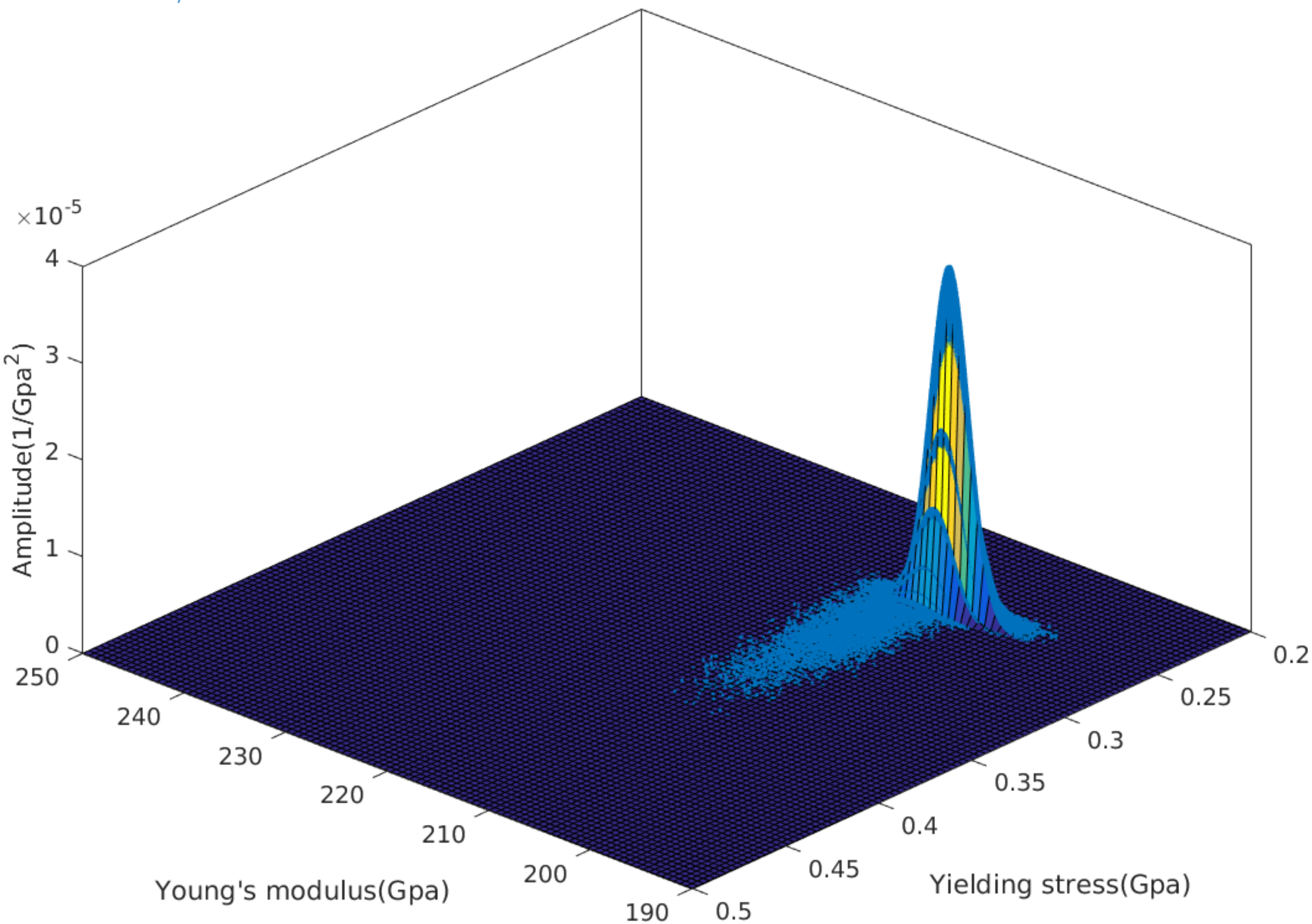
$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Difficult to compute the evidence probability: use MCMC

Markov-Chain Monte Carlo (MCMC) method: parameter space



Perfect plasticity: amplitude plot



$$\mu_{posterior} = [208.6690.2603]\ ; \Gamma_{posterior} = \begin{bmatrix} 4.0918 & 0.0044 \\ 0.0044 & 0.0001 \end{bmatrix}$$

$$N_{obs} = 39$$