

LONG TIME EXISTENCE AND BOUNDED SCALAR CURVATURE IN THE RICCI-HARMONIC FLOW

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ABSTRACT. In this paper we study the long time existence of the Ricci-harmonic flow in terms of scalar curvature and Weyl tensor which extends Cao's result [6] in the Ricci flow. In dimension four, we also study the integral bound of the "Riemann curvature" for the Ricci-harmonic flow generalizing a recently result of Simon [38].

1. INTRODUCTION

In this paper we study the long time existence of the Ricci-harmonic flow (RHF)

$$(1.1) \quad \partial_t g(t) = -2\text{Ric}_{g(t)} + 2\alpha(t)\nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t), \quad \partial_t \phi(t) = \Delta_{g(t)}\phi(t).$$

introduced in [25, 26, 30, 32], where $g(t)$ is a family of Riemannian metric, $\phi(t)$ is a family of functions, and $t \in [0, T)$ (with $T \leq +\infty$). Here $\alpha(t)$ is a time-dependent positive constant. In particular, we may choose $\alpha(t) \equiv \alpha$ a positive constant. If all functions $\phi(t) \equiv 0$, we obtain the Ricci flow (RF) introduced by Hamilton in his famous paper [23] and definitely used by Perelman [34, 35, 36] on his work about the Poincaré conjecture. The flow equations (1.1) come from static Einstein vacuum equations arising in the general relativity, and also arise as dimensional reductions of RF in higher dimensions [28].

For the RF, Hamilton [23] showed that the short-time existence and

$$(1.2) \quad T < \infty \implies \limsup_{t \rightarrow T} \left(\max_M |\text{Rm}_{g(t)}|_{g(t)}^2 \right) = \infty.$$

Later, Sesum [37] improved Hamilton's result as

$$(1.3) \quad T < \infty \implies \limsup_{t \rightarrow T} \left(\max_M |\text{Ric}_{g(t)}|_{g(t)}^2 \right) = \infty$$

by blow-up argument. For the integral bounds, Ye [47] and Wang [42] independently proved that

$$(1.4) \quad T < \infty \implies \left(\int_0^T \int_M |\text{Rm}_{g(t)}|_{g(t)}^{\frac{m+2}{2}} dV_{g(t)} dt \right)^{\frac{2}{m+2}} = \infty.$$

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Moreover, Wang [42] proved another version for RH that

$$(1.5) \quad \text{Ric}_{g(t)} \geq -C, T < \infty \implies \left(\int_0^T \int_M |R_{g(t)}|_{g(t)}^{\frac{m+2}{2}} dV_{g(t)} dt \right)^{\frac{2}{m+2}} = \infty.$$

Here C is a uniform constant. For other works on integral bounds, see for example [8, 29, 43, 46].

A well-known conjecture (see [6]) about the extension of RF states that

$$(1.6) \quad \text{Is it true for } \limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) = \infty? \text{ Here } T < \infty.$$

This conjecture was settled for Kähler-Ricci flow by Zhang [48] and for type-I maximal solution of RF by Enders-Müller-Topping [11]. Cao [6] proved the following

$$(1.7) \quad T < \infty \implies \begin{aligned} & \text{either } \limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) = \infty, \text{ or} \\ & \limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) < \infty \text{ but } \limsup_{t \rightarrow T} \frac{|W_{g(t)}|_{g(t)}}{R_{g(t)}} = \infty, \end{aligned}$$

where $W_{g(t)}$ denotes the Weyl tensor of $g(t)$.

For 4D RF, Simon [38] and Bamler-Zhang [5] independently proved

$$(1.8) \quad T < \infty, |R_{g(t)}| \leq C \implies \int_M |\text{Ric}_{g(t)}|_{g(t)}^2 dV_{g(t)}, \int_M |\text{Rm}_{g(t)}|_{g(t)}^2 dV_{g(t)} \leq C'$$

by different methods (for earlier work see [41]).

On the other hand, for the Ricci-harmonic flow (RHF) we have the following results. When $m = \dim M \geq 3$ and $T < +\infty$. Müller [30, 32] showed that (1.2) is also true for RHF. Recently, Cheng and Zhu [9] extended Sesum's result [37] to Ricci-harmonic flow, that is, (1.3) is true for RHF. For more results about the RHF, see [1, 2, 3, 4, 7, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 30, 31, 32, 33, 39, 40, 44, 45, 49].

The first main result is an extension of Cao's result [6] to RHF.

Theorem 1.1. *Let $(M, g(t), \phi(t))_{t \in [0, T]}$ be a solution to RHF (1.1) with $\alpha(t) \equiv \alpha$ a positive constant on a closed manifold M with $m = \dim M \geq 3$ and $T < +\infty$. Either one has*

$$\limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) = \infty$$

or

$$\limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) < \infty \text{ but } \limsup_{t \rightarrow T} \left(\max_M \frac{|W_{g(t)}|_{g(t)} + |\nabla_{g(t)}^2 \phi(t)|_{g(t)}^2}{R_{g(t)}} \right) = \infty.$$

Here $W_{g(t)}$ is the Weyl part of $\text{Rm}_{g(t)}$.

The second main result focuses on the 4D RHF. Write

$$\text{Sic}_{g(t)} := \text{Ric}_{g(t)} - \alpha \nabla_{g(t)} \phi(t) \otimes \nabla_{g(t)} \phi(t), \quad S_{g(t)} := R_{g(t)} - \alpha |\nabla_{g(t)} \phi(t)|_{g(t)}^2.$$

According to Theorem 2.2 below we can find a uniform constant C such that $S_{g(t)} + C > 0$ for all $t \in [0, T)$.

Theorem 1.2. *Let $(M, g(t), \phi(t))_{t \in [0, T]}$ be a solution to RHF (1.1) on a closed manifold M with $m = \dim M = 4$, $T \leq +\infty$, $\alpha(t) \equiv \alpha$ a positive constant. Choose a uniform constant C in Theorem 2.2 such that $S_{g(t)} + C > 0$. Then*

$$(1.9) \quad \int_M |\text{Sic}_{g(s)}|_{g(s)} dV_{g(s)} \leq 2c_0(M, g(0), \phi(0), s) + \frac{C}{2} \text{Vol}(M, g(s)) \\ + 1148e^{36Cs} \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt, \\ \int_0^s \int_M |\text{Sic}_{g(t)}|_{g(t)}^2 dV_{g(t)} dt \leq 8c_0(M, g(0), \phi(0), s) + \frac{C^2}{4} \int_0^s \text{Vol}(M, g(t)) dt \\ (1.10) \quad + 4592e^{36Cs} \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt,$$

for all $s \in [0, T]$. Here, $A_1 = \max_M |\nabla_{g(0)} \phi(0)|_{g(0)}^2$ and

$$(1.11) \quad = \frac{c_0(M, g(0), \phi(0), s)}{36C} \left(e^{36Cs} - 1 \right) + \frac{104\alpha^2 A_1^2 \text{Vol}(M, g(0))}{35C} \left(e^{35Cs} - e^{Cs} \right) \\ + e^{37Cs} \alpha A_1 \text{Vol}(M, g(0)) + e^{36Cs} \int_M \frac{|\text{Sic}_{g(0)}|_{g(0)}^2}{S_{g(0)} + C} dV_{g(0)}.$$

According to Theorem 2.3 below and following [38], we consider the basic assumption **(BA)** for a solution $(M, g(t), \phi(t))_{t \in [0, T]}$ to RHF:

- (a) M is a connected and closed 4-dimensional smooth manifold,
- (b) $(M, g(t), \phi(t))_{t \in [0, T]}$ is a solution to RHF with $\alpha(t) \equiv \alpha$ a positive constant,
- (c) $T < +\infty$,
- (d) $\max_{M \times [0, T]} |S_{g(t)}| \leq 1$.

The upper bound 1 in condition (d) is not essential, since we can rescale the pair $(g(t), \phi(t))$ so that the condition (d) is always satisfied. Furthermore, since

$$|\nabla_{g(t)} \phi(t)|_{g(t)}^2 \leq A_1$$

(by (3.6)) it follows that condition (d) is equivalent to the uniform bound for $R_{g(t)}$.

Theorem 1.3. *If $(M, g(t), \phi(t))_{t \in [0, T]}$ satisfies **BA**, then*

$$(1.12) \quad \int_M |\text{Sic}_{g(s)}|_{g(s)}^2 dV_{g(s)} \leq b(M, g(0), \phi(0), s),$$

$$(1.13) \quad \int_0^s \int_M |\text{Sic}_{g(t)}|_{g(t)}^4 dV_{g(t)} dt \leq b(M, g(0), \phi(0), s),$$

for any $s \in [0, T]$. Here

$$(1.14) \quad b(M, g(0), \phi(0), s) := 9e^{88s} \int_M |\text{Sic}_{g(0)}|_{g(0)}^2 dV_{g(0)} + \frac{1152}{88} \pi^2 \chi(M) \left(e^{88s} - 1 \right) \\ + \frac{468}{86} (\alpha A_1)^2 \text{Vol}(M, g(0)) \left(e^{88s} - e^{2s} \right) \\ + 9(\alpha A_1) \text{Vol}(M, g(0)) e^{90s}.$$

Define

$$(1.15) \quad c(M, g(0), \phi(0), T) := 9e^{90T} \left[\int_M |\text{Sic}_{g(0)}|_{g(0)}^2 dV_{g(0)} + \pi^2 |\chi(M)| + [(\alpha A_1)^2 + (\alpha A_1)] \text{Vol}(M, g(0)) \right].$$

Then $|b(M, g(0), \phi(0), T)| \leq c(M, g(0), \phi(0), T)$. Theorem 1.3 now yields

Theorem 1.4. *If $(M, g(t), \phi(t))_{t \in [0, T]}$ satisfies **BA**, then*

$$(1.16) \quad \sup_{t \in [0, T]} \int_M |\text{Sic}_{g(t)}|_{g(t)}^2 dV_{g(t)} \leq c(M, g(0), \phi(0), T) < +\infty,$$

$$(1.17) \quad \sup_{t \in [0, T]} \int_M |\text{Sm}_{g(t)}|_{g(t)}^2 dV_{g(t)} \leq 32\pi^2 \chi(M) + 8c(M, g(0), \phi(0), T) + 13(\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2T} < +\infty.$$

Remark 1.5. For a time-dependent tensor field, we always omit time variable in its components. Although in Theorem 1.2–Theorem 1.4, we assume that $\alpha(t) \equiv \alpha$ is a positive constant, the same results also hold for $\alpha(t) > 0$ and $\dot{\alpha}(t) < 0$ (see Section 4).

The proof of Theorem 1.1 is based on a “curvature pinching estimate” for RHF (see Theorem 2.2). The new ingredient in the proof of Theorem 2.2 is an introduction of “Riemann curvature tensor” $\text{Sm}_{g(t)}$ for RHF, so that we can express the Weyl tensor $W_{g(t)}$ in terms of $\text{Sm}_{g(t)}$.

The proofs of Theorem 1.2–Theorem 1.4 follow from the method of Simon [38]. As in [38] we define

$$Z_{ijk} := \left(\nabla_i S_{jk} \right) (S_{g(t)} + C) - S_{jk} \left(\nabla_i S_{g(t)} \right), \quad Z_{g(t)} := (Z_{ijk}), \quad f := \frac{|\text{Sic}_{g(t)}|_{g(t)}^2}{S_{g(t)} + C}.$$

Analogous to [38], we can show that

$$\begin{aligned} \frac{d}{dt} \int_M f dV_{g(t)} &= \int_M \left[-2 \frac{|Z|^2}{(S+C)^3} - 2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - fS \right. \\ &\quad \left. - 2\alpha \left| \Delta \phi \frac{\text{Sic}}{S+C} - \nabla^2 \phi \right|^2 + 2\alpha |\nabla^2 \phi|^2 \right] dV_{g(t)}. \end{aligned}$$

The main difference is the last term on the right-hand side of the above equation. To control the integral of $|\nabla^2 \phi|^2$ we make use the evolution equation for $|\nabla \phi|^2$ (see (3.6)) so that

$$2 \int_0^t \int_M |\nabla^2 \phi|^2 dV_{g(s)} ds + \int_M |\nabla \phi|^2 dV_{g(t)} \leq e^{Ct} A_1 \text{Vol}(M, g(0)).$$

The above estimate not only controls the space-time integral of $|\nabla^2 \phi|^2$, but also an uniform bound for the integral of $|\nabla \phi|^2$. These two estimates play essential role in the following proof. In dimension four, the famous Gauss-Bonnet-Chern formula (3.10) should transform to (3.13), where the terms involving $|\nabla \phi|^2$ can be bounded by the above discussion. A modification of [38] is now applied to the RHF.

2. CURVATURE PINCHING ESTIMATE FOR RHF

Consider a solution $(M, g(t), \phi(t))_{t \in [0, T]}$ to RHF with coupling time-dependent constant $\alpha(t)$

$$(2.1) \quad \partial_t g(t) = -2\text{Ric}_{g(t)} + 2\alpha(t)\nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t), \quad \partial_t \phi(t) = \Delta_{g(t)}\phi(t).$$

Let

$$\square_{g(t)} := \partial_t - \Delta_{g(t)}.$$

As in [25, 26, 30, 32] we define the following notions

$$(2.2) \quad \text{Sic}_{g(t)} := \text{Ric}_{g(t)} - \alpha(t)\nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t),$$

$$(2.3) \quad S_{g(t)} := \text{tr}_{g(t)}\text{Sic}_{g(t)} = R_{g(t)} - \alpha(t) \left| \nabla_{g(t)}\phi(t) \right|_{g(t)}^2.$$

Here $\alpha(t)$ is a family of time-dependent functions. Motivated by RF, we introduce a ‘‘Riemann curvature’’ type for RHF

$$(2.4) \quad S_{ijkl} := R_{ijkl} - \frac{\alpha}{2} \left(g_{j\ell} \nabla_i \phi \nabla_k \phi + g_{k\ell} \nabla_i \phi \nabla_j \phi \right).$$

Our notation for S_{ijkl} implies that

$$S_{ij} := g^{k\ell} S_{ik\ell j} = R_{ij} - \alpha \nabla_i \phi \nabla_j \phi = g^{k\ell} S_{kij\ell}$$

which coincides with the components of $\text{Sic}_{g(t)}$. The corresponding tensor field for S_{ijkl} is denoted by $\text{Sm}_{g(t)}$.

Lemma 2.1. *For a solution $(M, g(t), \phi(t))_{t \in [0, T]}$ with a time-dependent coupling function $\alpha(t)$, we have*

$$(2.5) \quad \square_{g(t)} S_{g(t)} = 2|\text{Sic}_{g(t)}|_{g(t)}^2 + 2\alpha(t) \left| \Delta_{g(t)}\phi(t) \right|_{g(t)}^2 - \dot{\alpha}(t) \left| \nabla_{g(t)}\phi(t) \right|_{g(t)}^2,$$

$$(2.6) \quad \begin{aligned} \square_{g(t)} \text{Sic}_{g(t)} &= 2\text{Sm}_{g(t)}(\text{Sic}_{g(t)}, \cdot) - 2\text{Sic}_{g(t)}^2 \\ &\quad + 2\alpha(t)\Delta_{g(t)}\phi(t)\nabla_{g(t)}^2\phi(t) - \dot{\alpha}(t)\nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t) \end{aligned}$$

where $\text{Sic}_{g(t)}^2 = (S_{ik}S_{j\ell}g^{k\ell})_{ij}$ and $\text{Sm}_{g(t)}(\text{Sic}_{g(t)}, \cdot) = (S_{kij\ell}S^{k\ell})_{ij}$.

Proof. The first equation can be found in [32], Corollary 4.5. In the same corollary we also have

$$\partial_t S_{ij} = \Delta_{g(t), L} S_{ij} + 2\alpha \Delta_{g(t)} \phi(t) \nabla_i \nabla_j \phi - \dot{\alpha} \nabla_i \phi \nabla_j \phi.$$

Here $\Delta_{g(t), L}$ denotes the Lichnerowicz Laplacian with respect to $g(t)$ defined by

$$\Delta_{g(t), L} S_{ij} = \Delta_{g(t)} S_{ij} + 2R_{kij\ell} S^{k\ell} - R_{ik} S_j^k - R_{jk} S_i^k.$$

Then

$$\square_{g(t)} S_{ij} = 2R_{kij\ell} S^{k\ell} - R_{ik} S_j^k - R_{jk} S_i^k + 2\alpha \Delta_{g(t)} \phi(t) \nabla_i \nabla_j \phi - \dot{\alpha} \nabla_i \phi \nabla_j \phi.$$

Plugging $S_{ij} = R_{ij} - \alpha \nabla_i \phi \nabla_j \phi$ into the above equation yields the second desired equation. \square

As a corollary of Lemma 2.1 we have (see [32], Corollary 5.2)

$$(2.7) \quad \min_M S_{g(t)} \geq \min_M S_{g(0)}, \quad \text{provided } \alpha(t) \geq 0 \text{ and } \dot{\alpha}(t) \leq 0.$$

Theorem 2.2. (Curvature pinching estimate) *Let $(M, g(t), \phi(t))_{t \in [0, T]}$ be a solution to RHF on a closed manifold M with $m = \dim M \geq 3$, $T \leq +\infty$, $\alpha(t) \geq 0$ and $\dot{\alpha}(t) \leq 0$. There exist uniform constants C_1, C_2, C , depending only on $m, g(0), \phi(0), \alpha(0)$ such that*

$$(2.8) \quad S_{g(t)} + C > 0, \quad \frac{|\text{Sin}_{g(t)}|_{g(t)}}{S_{g(t)} + C} \leq C_1 + C_2 \max_{M \times [0, t]} \sqrt{\frac{|W_{g(s)}|_{g(s)} + |\nabla_{g(s)}^2 \phi(s)|_{g(s)}^2}{S_{g(s)} + C}}$$

where $\text{Sin}_{g(t)} = \text{Sic}_{g(t)} - \frac{S_{g(t)}}{m}g(t)$ is the trace-free part of $\text{Sic}_{g(t)}$ and $W_{g(t)}$ is the Weyl tensor field of $g(t)$.

Proof. The first inequality follows from (2.7). Let

$$f := \frac{|\text{Sin}_{g(t)}|_{g(t)}^2}{(S_{g(t)} + C)^\gamma}, \quad \gamma > 0.$$

Since $\text{Sin}_{g(t)} = (\text{Sic}_{g(t)} + \frac{C}{m}g(t)) - (\frac{S_{g(t)} + C}{m})g(t)$, it follows that

$$f = \frac{|\text{Sic}_{g(t)} + \frac{C}{m}g(t)|_{g(t)}^2}{(S_{g(t)} + C)^\gamma} - \frac{1}{m}(S_{g(t)} + C)^{2-\gamma}.$$

In the following we always omit the subscripts t and $g(t)$ and set

$$\text{Sic}'_{g(t)} := \text{Sic}_{g(t)} + \frac{C}{m}g(t), \quad S'_{g(t)} := S_{g(t)} + C = \text{tr}_{g(t)} \text{Sic}'_{g(t)}.$$

In the following we always omit the subscripts $g(t)$ and t . Using the equation (3.21) in [10] we have

$$\begin{aligned} \square \frac{|\text{Sic}'|^2}{(S')^\gamma} &= \frac{1}{(S')^\gamma} \square |\text{Sic}'|^2 - \gamma \frac{|\text{Sic}'|^2}{(S')^{\gamma+1}} \square S - \gamma(\gamma+1) \frac{|\text{Sic}'|^2}{(S')^{\gamma+2}} |\nabla S'|^2 \\ &\quad + \frac{2\gamma}{(S')^{\gamma+1}} \langle \nabla |\text{Sic}'|^2, \nabla S' \rangle. \end{aligned}$$

It is clear from (2.6) that

$$\square |\text{Sic}|^2 = -2 |\nabla \text{Sic}|^2 + 4\text{Sm}(\text{Sic}, \text{Sic}) + 2 \left\langle \text{Sic}, 2\alpha \Delta \phi \nabla^2 \phi - \dot{\alpha} \nabla \phi \otimes \nabla \phi \right\rangle.$$

Therefore

$$\begin{aligned} \square |\text{Sic}'|^2 &= \square \left[|\text{Sic}|^2 + \frac{C^2}{m} + \frac{2C}{m} S \right] \\ &= \square |\text{Sic}|^2 + \frac{2C}{m} \left[2|\text{Sic}|^2 + 2\alpha |\Delta \phi|^2 - \dot{\alpha} |\nabla \phi|^2 \right] \\ &= -2 |\nabla \text{Sic}'|^2 + 4\text{Sm}(\text{Sic}, \text{Sic}) + \frac{4C}{m} |\text{Sic}|^2 \\ &\quad + 2 \left\langle \text{Sic}', 2\alpha \Delta \phi \nabla^2 \phi - \dot{\alpha} \nabla \phi \otimes \nabla \phi \right\rangle \\ &= -2 |\nabla \text{Sic}'|^2 + 4\text{Sm}(\text{Sic}', \text{Sic}') - \frac{4C}{m} |\text{Sic}'|^2 + \frac{4C^2}{m^2} S' \\ &\quad + 2 \left\langle \text{Sic}', 2\alpha \Delta \phi \nabla^2 \phi - \dot{\alpha} \nabla \phi \otimes \nabla \phi \right\rangle \end{aligned}$$

and

$$\begin{aligned}
\Box \frac{|\text{Sic}'|^2}{(S')^\gamma} &= -\frac{2}{(S')^\gamma} |\nabla \text{Sic}'|^2 - \frac{2\gamma |\text{Sic}'|^4}{(S')^{\gamma+1}} + \frac{4}{(S')^\gamma} \text{Sm}(\text{Sic}', \text{Sic}') \\
&\quad - \gamma(\gamma+1) \frac{|\text{Sic}'|^2 |\nabla S|^2}{(S')^{\gamma+2}} + \frac{2\gamma}{(S')^{\gamma+1}} \langle \nabla |\text{Sic}'|^2, \nabla S' \rangle \\
&\quad + \frac{4C^2}{m^2} \frac{S'}{(S')^\gamma} - \frac{4C}{m} \frac{(1-\gamma)S' + \frac{1}{2}\gamma C}{(S')^{\gamma+1}} |\text{Sic}'|^2 \\
&\quad + \frac{2}{(S')^\gamma} \langle \text{Sic}', 2\alpha \Delta \phi \nabla^2 \phi - \dot{\alpha} \nabla \phi \otimes \nabla \phi \rangle \\
&\quad - \frac{\gamma |\text{Sic}'|^2}{(S')^{\gamma+1}} \left[2\alpha |\Delta \phi|^2 - \dot{\alpha} |\nabla \phi|^2 \right],
\end{aligned}$$

according to

$$\Box S' = 2|\text{Sic}'|^2 - \frac{4C}{m} S' + \frac{2C^2}{m} + 2\alpha |\Delta \phi|^2 - \dot{\alpha} |\nabla \phi|^2.$$

Using the identity

$$\left\langle \nabla \frac{|\text{Sic}'|^2}{(S')^\gamma}, \nabla S' \right\rangle = \frac{1}{(S')^\gamma} \langle \nabla |\text{Sic}'|^2, \nabla S' \rangle - \frac{\gamma}{(S')^{\gamma+1}} |\nabla S|^2 |\text{Sic}'|^2,$$

we arrive at

$$\begin{aligned}
\Box \frac{|\text{Sic}'|^2}{(S')^\gamma} &= \frac{2\gamma}{S'} \left\langle \nabla \frac{|\text{Sic}'|^2}{(S')^\gamma}, \nabla S' \right\rangle - \frac{2}{(S')^{\gamma+2}} |S' \nabla \text{Sic}'|^2 \\
&\quad + \frac{\gamma(\gamma-1)}{(S')^{\gamma+2}} |\text{Sic}'|^2 |\nabla S|^2 + \frac{4}{(S')^\gamma} \text{Sm}(\text{Sic}', \text{Sic}') - \frac{2\gamma}{(S')^{\gamma+1}} |\text{Sic}'|^4 \\
&\quad + \frac{4C^2}{m^2} \frac{S'}{(S')^\gamma} - \frac{4C}{m} \frac{(1-\gamma)S' + \frac{1}{2}\gamma C}{(S')^{\gamma+1}} |\text{Sic}'|^2 \\
&\quad + \frac{2}{(S')^\gamma} \langle \text{Sic}', 2\alpha \Delta \phi \nabla^2 \phi - \dot{\alpha} \nabla \phi \otimes \nabla \phi \rangle \\
&\quad - \frac{\gamma |\text{Sic}'|^2}{(S')^{\gamma+1}} \left[2\alpha |\Delta \phi|^2 - \dot{\alpha} |\nabla \phi|^2 \right].
\end{aligned}$$

On the other hand, the following two identities

$$\begin{aligned}
|S' \nabla \text{Sic}'|^2 &= |S' \nabla_i S'_{jk} - S'_{jk} \nabla_i S' + S'_{jk} \nabla_i S'|^2 \\
&= |S' \nabla_i S'_{jk} - S'_{jk} \nabla_i S'|^2 - |\text{Sic}'|^2 |\nabla S'|^2 + S' \langle \nabla |\text{Sic}'|^2, \nabla S' \rangle, \\
\langle \nabla |\text{Sic}'|^2, \nabla S' \rangle &= \left\langle \nabla \left[(S')^\gamma \frac{|\text{Sic}'|^2}{(S')^\gamma} \right], \nabla S' \right\rangle \\
&= \frac{\gamma}{S'} |\text{Sic}'|^2 |\nabla S'|^2 + (S')^\gamma \left\langle \nabla \frac{|\text{Sic}'|^2}{(S')^\gamma}, \nabla S' \right\rangle
\end{aligned}$$

implies

$$\begin{aligned} \square \frac{|\text{Sic}'|^2}{(S')^\gamma} &= \frac{2(\gamma-1)}{S'} \left\langle \nabla \frac{|\text{Sic}'|^2}{(S')^\gamma}, \nabla S' \right\rangle - \frac{2}{(S')^{\gamma+2}} |S' \nabla_i S'_{jk} - S'_{jk} \nabla_i S'|^2 \\ &\quad - \frac{(2-\gamma)(\gamma-1)}{(S')^{\gamma+2}} |\text{Sic}'|^2 |\nabla S'|^2 + \frac{4}{(S')^\gamma} \text{Sm}(\text{Sic}', \text{Sic}') - \frac{2\gamma |\text{Sic}'|^4}{(S')^{\gamma+1}} \\ &\quad + \frac{4C^2}{m^2} \frac{S'}{(S')^\gamma} - \frac{4C(1-\gamma)S' + \frac{1}{2}\gamma C}{m(S')^{\gamma+1}} |\text{Sic}'|^2 \\ &\quad + \frac{2}{(S')^\gamma} \langle \text{Sic}', 2\alpha \Delta \phi \nabla^2 \phi - \dot{\alpha} \nabla \phi \otimes \nabla \phi \rangle - \frac{\gamma |\text{Sic}'|^2}{(S')^{\gamma+1}} [2\alpha |\Delta \phi|^2 - \dot{\alpha} |\nabla \phi|^2]. \end{aligned}$$

It is clear that $\square(S')^{2-\gamma} = (2-\gamma)(S')^{1-\gamma} \square S' - (2-\gamma)(1-\gamma)(S')^{-\gamma} |\nabla S'|^2$ and $|\text{Sic}|^2 = |\text{Sic}'|^2 + \frac{C^2}{m} - \frac{2CS'}{m}$. These yield

$$\begin{aligned} \square(S')^{2-\gamma} &= \frac{2(\gamma-1)}{S'} \left\langle \nabla(S')^{2-\gamma}, \nabla S' \right\rangle + 2(2-\gamma)(S')^{1-\gamma} |\text{Sic}'|^2 \\ &\quad - \frac{(2-\gamma)(\gamma-1)}{(S')^2} (S')^{2-\gamma} |\nabla S'|^2 + (2-\gamma)(S')^{1-\gamma} \left[\frac{2C^2}{m} - \frac{4C}{m} S' \right] \\ &\quad + (2-\gamma)(S')^{1-\gamma} (2\alpha |\Delta \phi|^2 - \dot{\alpha} |\nabla \phi|^2). \end{aligned}$$

Then

$$\begin{aligned} \square f &= \frac{2(\gamma-1)}{S'} \langle \nabla f, \nabla S' \rangle - \frac{2}{(S')^{\gamma+2}} |S' \nabla_i S'_{jk} - S'_{jk} \nabla_i S'|^2 \\ (2.9) \quad &\quad - \frac{(2-\gamma)(\gamma-1)}{(S')^2} |\nabla S'|^2 f + \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 \end{aligned}$$

where

$$(2.10) \quad \mathcal{Q}_1 := -\frac{2(2-\gamma)}{m} (S')^{1-\gamma} |\text{Sic}'|^2 + \frac{4}{(S')^\gamma} \text{Sm}(\text{Sic}', \text{Sic}') - \frac{2\gamma |\text{Sic}'|^4}{(S')^{1+\gamma}},$$

$$(2.11) \quad \mathcal{Q}_2 := \frac{4C}{m} \left[\frac{CS'}{m(S')^2} - \frac{(1-\gamma)S' + \frac{1}{2}\gamma C}{(S')^{\gamma+1}} |\text{Sic}'|^2 - \frac{2-\gamma}{2m} \frac{C-2S'}{(S')^{\gamma-1}} \right]$$

$$\begin{aligned} \mathcal{Q}_3 &:= \frac{2}{(S')^\gamma} \langle \text{Sic}', 2\alpha \Delta \phi \nabla^2 \phi - \dot{\alpha} \nabla \phi \otimes \nabla \phi \rangle \\ (2.12) \quad &\quad - \frac{2\alpha |\Delta \phi|^2 - \dot{\alpha} |\nabla \phi|^2}{(S')^{\gamma+1}} \left[\gamma |\text{Sic}'|^2 + \frac{2-\gamma}{m} (S')^2 \right]. \end{aligned}$$

Observe that

$$(2.13) \quad \mathcal{Q}_2 = \frac{4C}{m^2} \left[\frac{C}{S'} + \frac{C}{(S')^{\gamma-1}} + \frac{1}{(S')^{\gamma-2}} + mf \left(\gamma - 1 - \frac{\gamma C}{sS'} \right) \right].$$

The \mathcal{Q}_1 terms can be written as

$$\begin{aligned} \mathcal{Q}_1 &= \frac{2}{(S')^{\gamma+1}} \left[\frac{\gamma-2}{m} |\text{Sic}'|^2 (s')^2 + 2S' \text{Sm}(\text{Sic}', \text{Sic}') - \gamma |\text{Sic}'|^4 \right] \\ (2.14) \quad &= \frac{2}{(S')^{\gamma+1}} \left[(2-\gamma) |\text{Sic}'|^2 |\text{Sin}|^2 - 2 \left(|\text{Sic}'|^4 - S' \text{Sm}(\text{Sic}', \text{Sic}') \right) \right], \end{aligned}$$

where $\text{Sin} = \text{Sic} - \frac{S}{m}g = \text{Sic}' - \frac{C}{m}g - \frac{S}{m}g = \text{Sic}' - \frac{S'}{m}g =: \text{Sin}'$. Recall the decomposition of Rm that

$$R_{ijkl} = \frac{1}{m-2}(R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik}) - \frac{R(g_{il}g_{jk} - g_{ik}g_{jl})}{(m-1)(m-2)} + W_{ijkl}$$

where W_{ijkl} stands for the Weyl tensor field. Then

$$\begin{aligned} S_{ijkl} &= W_{ijkl} + \frac{1}{m-2}(S'_{il}g_{jk} + S'_{jk}g_{il} - S'_{ik}g_{jl} - S'_{jl}g_{ik}) - \frac{S'(g_{il}g_{jk} - g_{ik}g_{jl})}{(m-2)(m-1)} \\ &+ \frac{C}{m(m-1)(m-2)}(g_{il}g_{jk} - g_{ik}g_{jl}) - \frac{C}{m(m-2)}(g_{jk}g_{il} - g_{jl}g_{ik}) \\ (2.15) \quad &+ \frac{\alpha}{m-2}(g_{jk}\nabla_i\phi\nabla_\ell\phi + g_{il}\nabla_j\phi\nabla_k\phi - g_{jl}\nabla_i\phi\nabla_k\phi - g_{ik}\nabla_j\phi\nabla_\ell\phi) \\ &- \frac{\alpha|\nabla\phi|^2}{(m-2)(m-1)}(g_{il}g_{jk} - g_{ik}g_{jl}) - \frac{\alpha}{2}(g_{jl}\nabla_i\phi\nabla_k\phi + g_{kl}\nabla_i\phi\nabla_j\phi). \end{aligned}$$

Together with (2.15), we get

$$\begin{aligned} \text{Sm}(\text{Sic}', \text{Sic}') &= \frac{1}{m-2} \left[\frac{2m-1}{m-1} S' |\text{Sic}'|^2 - 2\text{Sic}'^3 - \frac{S'^3}{m-1} \right] + W(\text{Sic}', \text{Sic}') \\ (2.16) \quad &- \frac{1}{m-1} \left(\frac{C}{m} + \frac{\alpha}{m-2} |\nabla\phi|^2 \right) (|S'|^2 - |\text{Sic}'|^2) \\ &+ \frac{2\alpha}{m-2} \left\langle S'\text{Sic}' - \frac{m}{2}\text{Sic}'^2, \nabla\phi \otimes \nabla\phi \right\rangle \end{aligned}$$

where $\text{Sic}'^3 = S'_{ij}S'^j{}_k S'^{ki}$. Substituting (2.16) into (2.14) we arrive at

$$\begin{aligned} \mathcal{Q}_1 &= \frac{2}{(S')^{\gamma+1}} \left[(2-\gamma)|\text{Sic}'|^2|\text{Sin}'|^2 - 2\mathcal{Q}_4 + 2S'W(\text{Sic}', \text{Sic}') \right. \\ (2.17) \quad &- \frac{2}{m-1} \left(\frac{C}{m} + \frac{\alpha}{m-2} |\nabla\phi|^2 \right) (S'^3 - S'|\text{Sic}'|^2) \\ &\left. + \frac{2\alpha}{m-2} \left\langle S'^2\text{Sic}' - \frac{m}{2}S'\text{Sic}'^2, \nabla\phi \otimes \nabla\phi \right\rangle \right] \end{aligned}$$

where

$$(2.18) \quad \mathcal{Q}_4 = |\text{Sic}'|^4 - \frac{S'}{m-2} \left(\frac{2m-1}{m-1} S' |\text{Sic}'|^2 - 2\text{Sic}'^3 - \frac{S'^3}{m-1} \right).$$

The first term

$$(2-\gamma)|\text{Sic}'|^2|\text{Sin}'|^2 - 2\mathcal{Q}_4 + 2S'W(\text{Sic}', \text{Sic}')$$

on the right-hand side of (2.17) can be written as

$$\begin{aligned} -\gamma|\text{Sin}'|^4 + \left[\frac{2(m^2+2m-2)}{m(m-1)(m-2)} - \frac{\gamma}{m} \right] S'^2|\text{Sin}'|^2 \\ + \frac{4}{m^2(m-2)} S'^4 - \frac{4}{m-2} S'\text{Sic}'^3 + 2S'W(\text{Sic}', \text{Sic}') \end{aligned}$$

by using the fact that $|\text{Sic}'|^2 = |\text{Sin}'|^2 + \frac{S'^2}{m}$, where Sic'^3 is equals to

$$\text{Sic}'^3 = \text{Sin}'^3 + \frac{3S'}{m}|\text{Sin}'|^2 + \frac{S'^3}{m^2},$$

and therefore

$$-\gamma|\text{Sin}|^4 + \left[\frac{2(m-2)}{m(m-1)} - \frac{\gamma}{m} \right] S'^2 |\text{Sin}|^2 - \frac{4}{m-2} S' \text{Sin}^3 + 2S'W(\text{Sin}, \text{Sin}),$$

because of $W(\text{Sic}', \text{Sic}') = W(\text{Sin}, \text{Sin})$. Finally, we obtain

$$(2.19) \quad \begin{aligned} \square f &= 2(\gamma-1)\langle \nabla f, \nabla \ln S' \rangle - \frac{2}{(S')^\gamma} |\nabla_i S'_{jk} - S'_{jk} \nabla_i \ln S'|^2 \\ &\quad - (2-\gamma)(\gamma-1) |\nabla \ln S'|^2 f + \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 \end{aligned}$$

where

$$\begin{aligned} \mathcal{Q}_1 &= \frac{2}{(S')^{\gamma+1}} \left[-\gamma(S')^{2\gamma} f^2 + \left(\frac{2m-4}{n(m-1)} - \frac{\gamma}{m} \right) (S')^{\gamma+2} f - \frac{4S'^4}{m-2} \frac{\text{sin}^3}{S'^3} \right. \\ &\quad \left. + 2(S')^3 W\left(\frac{\text{Sin}}{S'}, \frac{\text{Sin}}{S'}\right) - \frac{2}{m-1} \left(\frac{C}{m} + \frac{\alpha|\nabla\phi|^2}{m-2} \right) \left(\frac{m-1}{m} S'^3 - (S')^{\gamma+1} f \right) \right. \\ &\quad \left. + \frac{2\alpha}{m-2} \left\langle S'^2 \text{Sic}' - \frac{m}{2} S' \text{Sic}'^2, \nabla\phi \otimes \nabla\phi \right\rangle \right], \\ \mathcal{Q}_2 &= \frac{4C}{m^2} \left[\frac{C}{S'} + \frac{C}{(S')^{\gamma+1}} + \frac{1}{(S')^{\gamma-2}} + mf \left(\gamma - 1 - \frac{\gamma C}{2S'} \right) \right], \\ \mathcal{Q}_3 &= \frac{2}{(S')^\gamma} \langle \text{Sic}', 2\alpha\Delta\phi\nabla^2\phi - \dot{\alpha}\nabla\phi \otimes \nabla\phi \rangle \\ &\quad - \frac{2\alpha|\Delta\phi|^2 - \dot{\alpha}|\nabla\phi|^2}{(S')^{\gamma+1}} \left[\gamma(S')^\gamma f + \frac{2S'^2}{m} \right]. \end{aligned}$$

from (2.9)–(2.13) and (2.17)–(2.18).

In particular, when $\gamma = 2$, we have $f = |\text{Sin}|^2/S'^2 = |\text{Sic}'|^2/S'^2$ and

$$(2.20) \quad \square f = 2\langle \nabla f, \nabla \ln S' \rangle - 2 \left| \nabla \left(\frac{\text{Sin}}{S'} \right) \right|^2 + \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3$$

where

$$\begin{aligned} \mathcal{Q}_1 &= 4S' \left[-f^2 - \frac{f}{m(m-1)} - \frac{2}{m-2} \frac{\text{Sin}^3}{S'^3} + \frac{1}{S'} W\left(\frac{\text{Sin}}{S'}, \frac{\text{Sin}}{S'}\right) \right. \\ &\quad \left. - \frac{1}{S'} \left(\frac{C}{m} + \frac{\alpha|\nabla\phi|^2}{m-2} \right) \left(\frac{1}{m} - \frac{f}{m-1} \right) + \frac{1}{S'} \frac{\alpha}{m-2} \left\langle \frac{\text{Sin}^2}{S'^2}, \nabla\phi \otimes \nabla\phi \right\rangle \right. \\ &\quad \left. + \frac{1}{S'} \frac{\alpha|\nabla\phi|^2}{2m(m-2)} \right], \\ \mathcal{Q}_2 &= \frac{4C}{m^2} \left[\frac{C}{S'} + \frac{C}{S'^3} + mf \left(1 - \frac{C}{S'} \right) \right], \\ \mathcal{Q}_3 &= \frac{2}{S'} \left[\langle \text{Sin}, 2\alpha\Delta\phi\nabla^2\phi - \dot{\alpha}\nabla\phi \otimes \nabla\phi \rangle - \left(2\alpha|\Delta\phi|^2 - \dot{\alpha}|\nabla\phi|^2 \right) f \right]. \end{aligned}$$

Since $\alpha(t) \geq 0$ and $\dot{\alpha}(t) \leq 0$, it follows from [32], Proposition 5.5, that $|\nabla\phi|^2$ is bounded from above by a uniform constant. Consequently

$$\begin{aligned}\mathcal{Q}_1 &\leq 4S' \left(-f^2 - \frac{f}{m(m-1)} + \frac{2}{m-2}f^{3/2} + C_m \frac{|W|}{S'}f + C_0f \right), \\ \mathcal{Q}_2 &\leq 4S'(C_0 + C_0f), \\ \mathcal{Q}_3 &\leq 4S' \left(C_0 + C_0 \frac{|\nabla^2\phi|^2}{S'}f^{1/2} \right)\end{aligned}$$

where $C_m = C(m) > 0$ and $C_0 = C(m, g(0), \phi(0), \alpha(0)) > 0$. Without loss of generality, we may assume that $f \geq 1$. In this case, we have

$$(2.21) \quad \begin{aligned}\square f &\leq 2\langle \nabla f, \nabla \ln S' \rangle + 4S'f \left[-f - \frac{1}{m(m-1)} + \frac{2}{m-2}f^{1/2} \right. \\ &\quad \left. + C_0 + C_0 \frac{|W| + |\nabla^2\phi|^2}{S'} \right]\end{aligned}$$

Applying the maximum principle to (2.21) yields

$$f - \frac{2}{m-2}f^{1/2} + \frac{1}{m(m-1)} - C_0 - C_0 \frac{|W| + |\nabla^2\phi|^2}{S'} \leq 0$$

at the point where f achieves its maximum. Thus we obtain (2.8). \square

As an immediate consequence of Theorem 2.2 we obtain the following theorem that is an extension of Cao's result ([6], Corollary 3.1).

Theorem 2.3. *Let $(M, g(t), \phi(t))_{t \in [0, T]}$ be a solution to RHF with nonincreasing $\alpha(t) \in [\underline{\alpha}, \bar{\alpha}]$, $0 < \underline{\alpha} \leq \bar{\alpha} < +\infty$, on a closed manifold M with $m = \dim M \geq 3$ and $T < +\infty$. Either one has*

$$\limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) = \infty$$

or

$$\limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) < \infty \text{ but } \limsup_{t \rightarrow T} \left(\max_M \frac{|W_{g(t)}|_{g(t)} + |\nabla_{g(t)}^2 \phi(t)|_{g(t)}^2}{R_{g(t)}} \right) = \infty.$$

Proof. Suppose now that

$$\limsup_{t \rightarrow T} \left(\max_M R_{g(t)} \right) < \infty \text{ and } \limsup_{t \rightarrow T} \left(\max_M \frac{|W_{g(t)}|_{g(t)} + |\nabla_{g(t)}^2 \phi(t)|_{g(t)}^2}{R_{g(t)}} \right) < \infty.$$

In this case both $R_{g(t)}$ and $|W_{g(t)}|_{g(t)} + |\nabla_{g(t)}^2 \phi(t)|_{g(t)}^2$ are uniformly bounded. Theorem 2.2 then implies that $|\text{Sin}_{g(t)}|_{g(t)}$ is uniformly bounded. Since $\text{Sin}_{g(t)} = \text{Sic}_{g(t)} - \frac{S_{g(t)}}{m}g(t)$, it follows that $\text{Sic}_{g(t)}$ is uniformly bounded. However, the assumption on $\alpha(t)$ tells us that $|\nabla_{g(t)}\phi(t)|_{g(t)}^2$ is uniformly bounded (e.g., [32], Proposition 5.5). Thus $\text{Ric}_{g(t)}$ is uniformly bounded, contradicting with the fact (1.2). Therefore we prove the theorem. \square

3. 4D RICCI-HARMONIC FLOW WITH BOUNDED $S_{g(t)}$: I

Let the constant C be given in Theorem 2.2 and we assume that α is a positive constant so that $\dot{\alpha}(t) \equiv 0$, and $m = \dim M = 4$. As in [38] we define

$$(3.1) \quad Z_{ijk} := \left(\nabla_i S_{jk} \right) (S_{g(t)} + C) - S_{jk} \left(\nabla_i S_{g(t)} \right), \quad Z_{g(t)} := (Z_{ijk}).$$

In the proof of Theorem 2.2, we actually have proved

$$(3.2) \quad \begin{aligned} \square \frac{|\text{Sic}|^2}{S+C} &= -2 \frac{|Z|^2}{(S+C)^3} - 2 \frac{|\text{Sic}|^4}{(S+C)^2} + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} \\ &\quad - \frac{1}{(S+C)^2} \left[\left(2\alpha |\Delta\phi|^2 - \dot{\alpha} |\nabla\phi|^2 \right) |\text{Sic}|^2 \right. \\ &\quad \left. - 2(S+C) \left\langle \text{Sic}, 2\alpha \Delta\phi \nabla^2\phi - \dot{\alpha} \nabla\phi \otimes \nabla\phi \right\rangle \right]. \end{aligned}$$

The bracket in the right-hand side of (3.2) can be expressed as

$$2\alpha \left[\left| \text{Sic} \Delta\phi - (S+C) \nabla^2\phi \right|^2 - (S+C)^2 |\nabla^2\phi|^2 \right]$$

because of $\dot{\alpha} \equiv 0$. Therefore the identity (3.2) is equal to

$$(3.3) \quad \begin{aligned} \square f &= -2 \frac{|Z|^2}{(S+C)^3} - 2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} \\ &\quad - 2\alpha \left| \Delta\phi \frac{\text{Sic}}{S+C} - \nabla^2\phi \right|^2 + 2\alpha |\nabla^2\phi|^2, \end{aligned}$$

where

$$(3.4) \quad f := \frac{|\text{Sic}|^2}{S+C}$$

which differs from the previous one in the proof of Theorem 2.2. Integrating (3.3) over M yields

$$(3.5) \quad \begin{aligned} \frac{d}{dt} \int_M f dV_{g(t)} &= \int_M \left[-2 \frac{|Z|^2}{(S+C)^3} - 2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - fS \right. \\ &\quad \left. - 2\alpha \left| \Delta\phi \frac{\text{Sic}}{S+C} - \nabla^2\phi \right|^2 + 2\alpha |\nabla^2\phi|^2 \right] dV_{g(t)}. \end{aligned}$$

To control the integral of $|\nabla^2\phi|^2$ we recall the evolution equation for $|\nabla\phi|^2$ (see [32], Proposition 4.3):

$$(3.6) \quad \square |\nabla\phi|^2 = -2\alpha |\nabla\phi \otimes \nabla\phi|^2 - 2|\nabla^2\phi|^2.$$

In particular, we see that

$$(3.7) \quad |\nabla\phi|^2 \leq \max_M \left(|\nabla\phi|^2 \Big|_{t=0} \right) =: A_1.$$

Moreover we have

$$\frac{d}{dt} \int_M |\nabla\phi|^2 dV_{g(t)} \leq \int_M \left[-(S+C) |\nabla\phi|^2 - 2|\nabla^2\phi|^2 + C |\nabla\phi|^2 \right] dV_{g(t)}$$

which shows that

$$(3.8) \quad 2 \int_0^t \int_M |\nabla^2 \phi|^2 dV_{g(s)} ds + \int_M |\nabla \phi|^2 dV_{g(t)} \leq e^{Ct} A_1 \text{Vol}(M, g(0)).$$

Define

$$A_2(t) := \int_M |\nabla^2 \phi|^2 dV_{g(t)}.$$

Plugging (3.8) into (3.5) we arrive at

$$(3.9) \quad \frac{d}{dt} \int_M f dV_{g(t)} \leq \int_M \left[-2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - fS \right] dV_{g(t)} + 2\alpha A_2(t).$$

In the following we restrict ourself in 4D RHF, i.e., $m = \dim M = 4$. In the case, the famous Gauss-Bonnet-Chern formula says that

$$(3.10) \quad 2^5 \pi^2 \chi(M) = \int_M \left[|\text{Rm}|^2 - 4|\text{Ric}|^2 + R^2 \right] dV_{g(t)}$$

for any $t \in [0, T]$. In order to use the formula (3.10) we should translate the integrand in (3.10) into a function in terms of S_{ijkl} .

Lemma 3.1. *For any m -dimensional manifold M , one has*

$$(3.11) \quad \begin{aligned} |\text{Rm}|^2 - 4|\text{Ric}|^2 + R^2 &= |\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \\ &\quad - \frac{m+9}{2} \alpha^2 |\nabla \phi|^4 - 9\alpha \text{Sic}(\nabla \phi, \nabla \phi) + 2\alpha S |\nabla \phi|^2. \end{aligned}$$

Proof. Using $S_{ijkl} = R_{ijkl} - \frac{\alpha}{2}(g_{jl}\nabla_i\phi\nabla_k\phi + g_{kl}\nabla_i\phi\nabla_j\phi)$ we obtain

$$\begin{aligned} |\text{Rm}|^2 &= R_{ijkl}R^{ijkl} \\ &= \left(S_{ijkl} + \frac{\alpha}{2}(g_{jl}\nabla_i\phi\nabla_k\phi + g_{kl}\nabla_i\phi\nabla_j\phi) \right) \\ &\quad \cdot \left(S^{ijkl} + \frac{\alpha}{2}(g^{jl}\nabla^i\phi\nabla^k\phi + g^{kl}\nabla^i\phi\nabla^j\phi) \right) \\ &= |\text{Sm}|^2 + \frac{m+1}{2} \alpha^2 |\nabla \phi|^4 + \alpha \left(S_{ijkl}g^{jl}\nabla^i\phi\nabla^k\phi + S_{ijkl}g^{kl}\nabla^i\phi\nabla^j\phi \right). \end{aligned}$$

Compute

$$\begin{aligned} S_{ijkl}g^{jl} &= -R_{ik} - \frac{m+1}{2} \alpha \nabla_i\phi\nabla_k\phi = -S_{ik} - \frac{m+3}{2} \alpha \nabla_i\phi\nabla_k\phi, \\ S_{ijkl}g^{kl} &= g^{kl}R_{ijkl} - \frac{m+1}{2} \alpha \nabla_i\phi\nabla_j\phi = -\frac{m+1}{2} \alpha \nabla_i\phi\nabla_j\phi \end{aligned}$$

because of the first Bianchi identity $g^{kl}R_{ijkl} = -g^{kl}(R_{jkil} + R_{kijl}) = -(-R_{ji} + R_{ij}) = 0$. Consequently

$$\begin{aligned} |\text{Rm}|^2 &= |\text{Sm}|^2 + \frac{m+1}{2} \alpha^2 |\nabla \phi|^4 \\ &\quad + \alpha \left[-\text{Sic}(\nabla \phi, \nabla \phi) - \frac{m+3}{2} \alpha |\nabla \phi|^4 - \frac{m+1}{2} \alpha |\nabla \phi|^4 \right] \\ &= |\text{Sm}|^2 - \alpha \text{Sic}(\nabla \phi, \nabla \phi) - \frac{m+3}{2} \alpha^2 |\nabla \phi|^4. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} |\text{Ric}|^2 &= |\text{Sic}|^2 + 2\alpha \text{Sic}(\nabla \phi, \nabla \phi) + \alpha^2 |\nabla \phi|^4, \\ R^2 &= S^2 + \alpha^2 |\nabla \phi|^4 + 2\alpha S |\nabla \phi|^2. \end{aligned}$$

Combining those identities we obtain (3.11). \square

From (3.10) and (3.11) one has, in dimension $m = 4$,

$$(3.12) \quad \begin{aligned} \int_M \left[|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \right] dV_{g(t)} &= 2^5 \pi^2 \chi(M) + \frac{13}{2} \alpha^2 \int_M |\nabla \phi|^4 dV_{g(t)} \\ &+ 9\alpha \int_M \text{Sic}(\nabla \phi, \nabla \phi) dV_{g(t)} \\ &- 2\alpha \int_M S |\nabla \phi|^2 dV_{g(t)}. \end{aligned}$$

Using the inequality

$$\text{Sic}(\nabla \phi, \nabla \phi) \leq \epsilon |\text{Sic}|^2 + \frac{|\nabla \phi|^4}{4\epsilon}, \quad \epsilon := \frac{9}{26\alpha}$$

we obtain from (3.12) that, in dimension $m = 4$,

$$(3.13) \quad \begin{aligned} \int_M \left[|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \right] dV_{g(t)} &\leq 2^5 \pi^2 \chi(M) + 13\alpha^2 \int_M |\nabla \phi|^4 dV_{g(t)} \\ &+ \frac{81}{26} \int_M |\text{Sic}|^2 dV_{g(t)} \\ &- 2\alpha \int_M S |\nabla \phi|^2 dV_{g(t)}. \end{aligned}$$

For any $\epsilon > 0$ we have

$$4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S + C} \leq \epsilon^2 |\text{Sm}|^2 + \frac{4|\text{Sic}|^2}{\epsilon^2(S + C)^2} = \frac{4}{\epsilon^2} f^2 + \epsilon^2 |\text{Sm}|^2$$

so that

$$\begin{aligned} -2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S + C} - fS &\leq -2f^2 + \frac{4}{\epsilon^2} f^2 + \epsilon^2 |\text{Sm}|^2 - fS \\ &= -2f^2 + \epsilon^2 \left(|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \right) + 4\epsilon^2 |\text{Sic}|^2 - \epsilon^2 S^2 + \frac{4}{\epsilon^2} f^2 - fS \\ &= \epsilon^2 \left(|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \right) + (4\epsilon^2 - 1)fS - \left(2 - \frac{4}{\epsilon^2} \right) f^2 + 4C\epsilon^2 f - \epsilon^2 S^2. \end{aligned}$$

Using the estimate (3.13) we have

$$\begin{aligned} \int_M \left[-2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S + C} - fS \right] dV_{g(t)} &\leq \epsilon^2 \left[32\pi^2 \chi(M) \right. \\ &+ \left. 13\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} + \frac{81}{26} \int_M f(S + C) dV_{g(t)} \right] - \epsilon^2 \int_M f^2 dV_{g(t)} \\ &+ (4\epsilon^2 - 1) \int_M fS dV_{g(t)} - \left(2 - \frac{4}{\epsilon^2} \right) \int_M f^2 dV_{g(t)} + 4C\epsilon^2 \int_M f dV_{g(t)} \\ &= \epsilon^2 \left[32\pi^2 \chi(M) + 13\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} \right] \\ &+ \int_M \left[- \left(2 - \frac{4}{\epsilon^2} \right) f^2 + \left(\frac{55}{26} + 4\epsilon^2 \right) fS + \left(\frac{81}{26} + 4\epsilon^2 \right) Cf - \epsilon^2 S^2 \right] dV_{g(t)}. \end{aligned}$$

For any $\eta > 0$ we have $fS \leq \eta f^2 + \frac{1}{4\eta} S^2$ and then

$$\begin{aligned} \int_M \left[-2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - fS \right] dV_{g(t)} &\leq \epsilon^2 \left[32\pi^2 \chi(M) \right. \\ &+ 13\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} \left. \right] + \int_M \left[- \left(2 - \frac{4}{\epsilon^2} - \left(\frac{55}{26} + 4\epsilon^2 \right) \eta \right) f^2 \right. \\ &+ \left. \left(\frac{81}{26} + 4\epsilon^2 \right) Cf + \left(\frac{55}{26} + 4\epsilon^2 - \epsilon^2 \right) S^2 \right] dV_{g(t)}. \end{aligned}$$

If we choose

$$(3.14) \quad \eta = \frac{\frac{4}{\epsilon^2}}{\frac{55}{26} + 4\epsilon^2}, \quad \epsilon > 2,$$

then

$$\begin{aligned} \int_M \left[-2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - fS \right] dV_{g(t)} &\leq \epsilon^2 \left[32\pi^2 \chi(M) \right. \\ &+ 13\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} \left. \right] + \int_M \left[- \left(2 - \frac{8}{\epsilon^2} \right) f^2 + \left(\frac{81}{26} + 4\epsilon^2 \right) Cf \right. \\ &+ \left. \left(\frac{(\frac{55}{26} + 4\epsilon^2)^2}{16} - 1 \right) \epsilon^2 S^2 \right] dV_{g(t)}. \end{aligned}$$

In particular, when $\epsilon = 2\sqrt{2}$ in (3.14) we arrive at

$$\begin{aligned} \int_M \left[-2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - fS \right] dV_{g(t)} \\ (3.15) \quad &\leq \int_M \left(-f^2 + 36Cf + 574S^2 \right) dV_{g(t)} \\ &+ 8 \left[32\pi^2 \chi(M) + 13\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} \right]. \end{aligned}$$

Plugging (3.15) into (3.9) implies

$$\begin{aligned} \frac{d}{dt} \int_M f dV_{g(t)} &\leq \int_M \left(-f^2 + 36Cf + 574S^2 \right) dV_{g(t)} \\ (3.16) \quad &+ \left[256\pi^2 \chi(M) + 104\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} + 2\alpha A_2(t) \right]. \end{aligned}$$

Theorem 3.2. *Let $(M, g(t), \phi(t))_{t \in [0, T]}$ be a solution to RHF on a closed manifold M with $m = \dim M = 4$, $T \leq +\infty$, $\alpha(t) \equiv \alpha$ a positive constant. Choose a uniform constant C in Theorem 2.2 such that $S_{g(t)} + C > 0$. Then*

$$\begin{aligned} \int_M \frac{|\text{Sic}_{g(s)}|_{g(s)}^2}{S_{g(s)} + C} dV_{g(s)} + \int_0^s \int_M \frac{|\text{Sic}_{g(t)}|_{g(t)}^4}{(S_{g(t)} + C)^2} dV_{g(t)} dt \\ (3.17) \quad &\leq c_0(M, g(0), \phi(0), s) + 574e^{36Cs} \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt, \end{aligned}$$

Furthermore,

$$\begin{aligned}
(3.18) \quad \int_M |\text{Sic}_{g(s)}|_{g(s)} dV_{g(s)} &\leq 2c_0(M, g(0), \phi(0), s) + \frac{C}{2} \text{Vol}(M, g(s)) \\
&\quad + 1148e^{36Cs} \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt, \\
\int_0^s \int_M |\text{Sic}_{g(t)}|_{g(t)}^2 dV_{g(t)} dt &\leq 8c_0(M, g(0), \phi(0), s) + \frac{C^2}{4} \int_0^s \text{Vol}(M, g(t)) dt \\
(3.19) \quad &\quad + 4592e^{36Cs} \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt, \\
\int_0^s \int_M |\text{Sm}_{g(t)}|_{g(t)}^2 dV_{g(t)} dt &\leq \frac{131}{50} C^2 \int_0^s \text{Vol}(M, g(t)) dt \\
&\quad + \frac{13\alpha^2 A_1^2 \text{Vol}(M, g(0))}{C} (e^{Cs} - 1) \\
(3.20) \quad &\quad + \frac{881}{25} c_0(M, g(0), \phi(0), s) + 32\pi^2 \chi(M) s \\
&\quad + \left(\frac{505694}{25} e^{36Cs} + \frac{81}{50} \right) \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt,
\end{aligned}$$

for all $s \in [0, T)$. Here, $A_1 = \max_M |\nabla_{g(0)} \phi(0)|_{g(0)}^2$ and

$$\begin{aligned}
(3.21) \quad &c_0(M, g(0), \phi(0), s) \\
&= \frac{256\pi^2 \chi(M)}{36C} (e^{36Cs} - 1) + \frac{104\alpha^2 A_1^2 \text{Vol}(M, g(0))}{35C} (e^{35Cs} - e^{Cs}) \\
&\quad + e^{37Cs} \alpha A_1 \text{Vol}(M, g(0)) + e^{36Cs} \int_M \frac{|\text{Sic}_{g(0)}|_{g(0)}^2}{S_{g(0)} + C} dV_{g(0)}.
\end{aligned}$$

Note that when $C < 0$, the constant $c_0(M, g(0), \phi(0), s)$ can be chosen to be independent of s .

Proof. By (3.16) one has

$$\begin{aligned}
\frac{d}{dt} \left(e^{-36Ct} \int_M f dV_{g(t)} \right) + e^{-36Ct} \int_M f^2 dV_{g(t)} &\leq 574e^{-36Ct} \int_M S^2 dV_{g(t)} \\
&\quad + e^{-36Ct} A_3(t)
\end{aligned}$$

where $A_3(t) = 256\pi^2 \chi(M) + 104\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} + 2\alpha A_2(t)$. Therefore

$$\begin{aligned}
&e^{-36Cs} \int_M f dV_{g(s)} + e^{-36Cs} \int_0^s \int_M f^2 dV_{g(t)} dt \\
&\leq \int_0^s A_3(t) e^{-36Ct} dt + 574 \int_0^s e^{-36Ct} \int_M S^2 dV_{g(t)} dt + \int_M f dV_{g(0)} \\
&\leq \int_0^s \left[256\pi^2 \chi(M) + 104\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} + 2\alpha \int_M |\nabla^2 \phi|^2 dV_{g(t)} \right] e^{-36Ct} dt \\
&\quad + 574 \int_0^s \int_M S^2 dV_{g(t)} dt + \int_M f dV_{g(0)} \\
&= \frac{256\pi^2 \chi(M)}{36C} (1 - e^{-36Cs}) + \frac{104\alpha^2 A_1^2 \text{Vol}(M, g(0))}{35C} (1 - e^{-35Cs}) \\
&\quad + 2\alpha \int_0^s e^{-36Ct} \int_M |\nabla^2 \phi|^2 dV_{g(t)} dt + 574 \int_0^s \int_M S^2 dV_{g(t)} dt + \int_M f dV_{g(0)}.
\end{aligned}$$

According to the estimate (3.8), we obtain the first inequality.

For the second statement, we use the following inequalities

$$|\text{Sic}| \leq \frac{|\text{Sic}|^2}{S+C} + \frac{S+C}{4}, \quad |S| \leq 2|\text{Sic}|$$

so that

$$|\text{Sic}| \leq \frac{|\text{Sic}|^2}{S+C} + \frac{|\text{Sic}|}{2} + \frac{C}{4}$$

and

$$(3.22) \quad |\text{Sic}| \leq 2\frac{|\text{Sic}|^2}{S+C} + \frac{C}{2}.$$

From (3.22) and (3.17) we obtain (3.18).

For (3.19), we observe that

$$\begin{aligned} |\text{Sic}|^2 &\leq 4\frac{|\text{Sic}|^4}{(S+C)^2} + \frac{(S+C)^2}{16} \leq 4\frac{|\text{Sic}|^4}{(S+C)^2} + \frac{S^2+C^2}{8} \\ &\leq 4\frac{|\text{Sic}|^4}{(S+C)^2} + \frac{|\text{Sic}|^2}{2} + \frac{C^2}{8} \end{aligned}$$

so that

$$|\text{Sic}|^2 \leq 8\frac{|\text{Sic}|^4}{(S+C)^2} + \frac{C^2}{4}.$$

For the last one, we use the inequality (3.13) to deduce that

$$\begin{aligned} \int_M |\text{Sm}|^2 dV_{g(t)} &\leq 32\pi^2 \chi(M) + 13\alpha^2 A_1^2 \text{Vol}(M, g(0)) e^{Ct} \\ &\quad + \frac{81}{25} \int_M \left(f^2 + \frac{S^2+C^2}{2} \right) dV_{g(t)} + 4 \int_M |\text{Sic}|^2 dV_{g(t)}. \end{aligned}$$

Now the estimate (3.20) follows from (3.17) and (3.19). \square

Theorem 3.3. *Let $(M, g(t), \phi(t))_{t \in [0, T]}$ be a solution to RHF on a closed manifold M with $m = \dim M = 4$, $T \leq +\infty$, $\alpha(t) \equiv \alpha$ a positive constant. Suppose $\min_M S_{g(0)} > 0$. Then*

$$(3.23) \quad \int_M |\text{Sic}_{g(s)}|_{g(s)} dV_{g(s)} \leq 2a_0(M, g(0), \phi(0), s) + 1148 \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt,$$

$$(3.24) \quad \begin{aligned} \int_0^s \int_M |\text{Sic}_{g(t)}|_{g(t)}^2 dV_{g(t)} dt &\leq 8a_0(M, g(0), \phi(0), s) \\ &\quad + 4592 \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt, \end{aligned}$$

$$(3.25) \quad \begin{aligned} \int_0^s \int_M |\text{Sm}_{g(t)}|_{g(t)}^2 dV_{g(t)} dt &\leq 32\pi^2 \chi(M) s + 13(\alpha A_1)^2 \text{Vol}(M, g(0)) s \\ &\quad + \frac{881}{25} a_0(M, g(0), \phi(0), s) \\ &\quad + \frac{1011469}{50} \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt \end{aligned}$$

for all $s \in [0, T)$. Here

$$(3.26) \quad \begin{aligned} a_0(M, g(0), \phi(0), s) &:= 256\pi^2\chi(M)s + 104(\alpha A_1)^2\text{Vol}(M, g(0))s \\ &+ \alpha A_1\text{Vol}(M, g(0)) + \int_M \frac{|\text{Sic}_{g(0)}|_{g(0)}^2}{S_{g(0)}} dV_{g(0)}. \end{aligned}$$

Proof. We use the fact $|\text{Sic}| \leq 2|\text{Sic}|^2/S$ and Theorem 3.2 (where we take $C = 0$). Then

$$\int_M |\text{Sic}| dV_{g(s)} \leq 2a_0(M, g(0), \phi(0), s) + 1148 \int_0^s \int_M S^2 dV_{g(t)} dt.$$

The inequality (3.24) follows from Theorem 3.2 and $|\text{Sic}|^2 \leq 8|\text{Sic}|^4/S^2$. The last estimate follows from the inequality mentioned at the end of the proof of Theorem 3.2. \square

According to Theorem 2.3 and following [38], we consider the basic assumption **(BA)** for a solution $(M, g(t), \phi(t))_{t \in [0, T)}$ to RHF:

- (a) M is a connected and closed 4-dimensional smooth manifold,
- (b) $(M, g(t), \phi(t))_{t \in [0, T)}$ is a solution to RHF with $\alpha(t) \equiv \alpha$ a positive constant,
- (c) $T < \infty$,
- (d) $\max_{M \times [0, T)} |S_{g(t)}| \leq 1$.

The upper bound 1 in condition (d) is not essential, since we can rescale the pair $(g(t), \phi(t))$ so that the condition (d) is always satisfied. Furthermore, since

$$|\nabla_{g(t)} \phi(t)|_{g(t)}^2 \leq A_1$$

(by (3.6)) it follows that condition (d) is equivalent to the uniform bound for $R_{g(t)}$.

Theorem 3.4. *If $(M, g(t), \phi(t))_{t \in [0, T)}$ satisfies **BA**, then*

$$(3.27) \quad \begin{aligned} \frac{d}{dt} \int_M f dV_{g(t)} &\leq \int_M (-f^2 + 88f) dV_{g(t)} \\ &+ \left[128\pi^2\chi(M) + 52(\alpha A_1)^2\text{Vol}(M, g(0))e^{2t} + 2\alpha A_2(t) \right]. \end{aligned}$$

where $f := |\text{Sic}_{g(t)}|_{g(t)}^2 / (S_{g(t)} + 2)$.

Proof. Recall the estimate

$$\begin{aligned} 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+2} &\leq \frac{4|\text{Sic}|^4}{\epsilon^2(S+2)^2} + \epsilon^2 |\text{Sm}|^2 \\ &\leq \frac{4}{\epsilon^2} f^2 + \epsilon^2 \left(|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \right) + 4\epsilon^2 |\text{Sic}|^2 \\ &\leq \frac{4}{\epsilon^2} f^2 + \epsilon^2 \left(|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \right) + 4\epsilon^2 (S+2)f. \end{aligned}$$

Since $-1 \leq S \leq 1$, it follows that $4\epsilon^2(S+2)f \leq 12\epsilon^2 f$. Hence

$$4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+2} \leq \frac{4}{\epsilon^2} f^2 + \epsilon^2 \left(|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2 \right) + 12\epsilon^2 f.$$

Using the inequality $-fS = -f(s+2) + 2f \leq 2f$ and (3.13), we arrive at

$$\begin{aligned}
& \int_M \left[-2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+2} - fS \right] dV_{g(t)} \\
& \leq \int_M \left[-2f^2 + \frac{4}{\epsilon^2} f^2 + \epsilon^2 (|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2) + 12\epsilon^2 f - fS \right] dV_{g(t)} \\
& = \int_M \left[-\left(2 - \frac{4}{\epsilon^2}\right) f^2 + (12\epsilon^2 + 2)f + \epsilon^2 (|\text{Sm}|^2 - 4|\text{Sic}|^2 + S^2) \right] dV_{g(t)} \\
& \leq \int_M \left[-\left(2 - \frac{4}{\epsilon^2}\right) f^2 + (12\epsilon^2 + 2)f \right] dV_{g(t)} \\
& \quad + \epsilon^2 \left[32\pi^2 \chi(M) + 13\epsilon^2 (\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2t} \right] + \frac{243}{26} \epsilon^2 \int_M f dV_{g(t)} \\
& = \int_M \left[-\left(2 - \frac{4}{\epsilon^2}\right) f^2 + \left(\frac{555}{26} \epsilon^2 + 2\right) f \right] dV_{g(t)} \\
& \quad + \epsilon^2 \left[32\pi^2 \chi(M) + 13\epsilon^2 (\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2t} \right].
\end{aligned}$$

From (3.9) we get (3.27) (compare with (3.16) when $C = 2$). \square

Theorem 3.5. *If $(M, g(t), \phi(t))_{t \in [0, T]}$ satisfies BA, then*

$$\begin{aligned}
(3.28) \quad & \int_M |\text{Sic}_{g(s)}|_{g(s)}^2 dV_{g(s)} \leq b(M, g(0), \phi(0), s), \\
& \int_M |\text{Sm}_{g(s)}|_{g(s)}^2 dV_{g(s)} \leq 32\pi^2 \chi(M) + 13(\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2s} \\
(3.29) \quad & \quad \quad \quad + \frac{185}{26} b(M, g(0), \phi(0), s),
\end{aligned}$$

$$\begin{aligned}
(3.30) \quad & \int_0^s \int_M |\text{Sic}_{g(t)}|_{g(t)}^4 dV_{g(t)} dt \leq b(M, g(0), \phi(0), s), \\
& \int_s^T \int_M |\text{Sic}|_{g(t)}^p dV_{g(t)} dt \leq [b(M, g(0), \phi(0), T)]^{\frac{p}{4}} e^{\frac{T(4-p)}{4}} \\
(3.31) \quad & \quad \quad \quad [\text{Vol}(M, g(0))]^{\frac{4-p}{4}} (T-s)^{\frac{4-p}{4}}
\end{aligned}$$

for any $s \in [0, T]$ and $0 < p < 4$. Here

$$\begin{aligned}
(3.32) \quad & b(M, g(0), \phi(0), s) := 9e^{88s} \int_M |\text{Sic}_{g(0)}|_{g(0)}^2 dV_{g(0)} + \frac{1152}{88} \pi^2 \chi(M) (e^{88s} - 1) \\
& \quad + \frac{468}{86} (\alpha A_1)^2 \text{Vol}(M, g(0)) (e^{88s} - e^{2s}) \\
& \quad + 9(\alpha A_1) \text{Vol}(M, g(0)) e^{90s}.
\end{aligned}$$

Proof. Write $A_3(t) := 128\pi^2 \chi(M) + 52(\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2t} + 2\alpha A_2(t)$. The inequality (3.27) implies

$$\frac{d}{dt} \int_M f dV_{g(t)} \leq A_3(t) + \int_M (-f^2 + 88f) dV_{g(t)}$$

and then

$$\frac{d}{dt} \left(e^{-88t} \int_M f dV_{g(t)} \right) \leq -e^{-88t} \int_M f^2 dV_{g(t)} + e^{-88t} A_3(t).$$

Therefore

$$\begin{aligned}
& e^{-88s} \int_0^s \int_M f^2 dV_{g(t)} dt + e^{-88s} \int_M f dV_{g(s)} \leq \int_M f dV_{g(0)} \\
& + \int_0^s e^{-88t} \left[128\pi^2 \chi(M) + 52(\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2t} + 2\alpha A_2(t) \right] dt \\
& = \int_M f dV_{g(0)} + \frac{128}{88} \pi^2 \chi(M) (1 - e^{-88s}) + \frac{52}{86} (\alpha A_1)^2 \text{Vol}(M, g(0)) (1 - e^{-86s}) \\
& + 2\alpha \int_0^s \int_M |\nabla^2 \phi|^2 dV_{g(t)} dt \\
& \leq \int_M f dV_{g(0)} + \frac{128}{88} \pi^2 \chi(M) (1 - e^{-88s}) + \frac{52}{86} (\alpha A_1)^2 \text{Vol}(M, g(0)) (1 - e^{-86s}) \\
& + (\alpha A_1) \text{Vol}(M, g(0)) e^{2s}
\end{aligned}$$

by (3.8). Because $|S| \leq 1$, we have $\frac{1}{3} |\text{Sic}|^2 \leq f \leq |\text{Sic}|^2$ and hence

$$\begin{aligned}
& \frac{e^{-88s}}{9} \int_0^s \int_M |\text{Sic}|^4 dV_{g(t)} dt + \frac{e^{-88s}}{3} \int_M |\text{Sic}|^2 dV_{g(s)} \\
& \leq \int_M |\text{Sic}|^2 dV_{g(0)} + \frac{128}{88} \pi^2 \chi(M) (1 - e^{-88s}) \\
& + \frac{52}{86} (\alpha A_1)^2 \text{Vol}(M, g(0)) (1 - e^{-86s}) + (\alpha A_1) \text{Vol}(M, g(0)) e^{2s}.
\end{aligned}$$

This estimate yields (3.28) and (3.30).

For (3.29), we use (3.13):

$$\begin{aligned}
\int_M |\text{Sm}|^2 dV_{g(t)} & \leq 32\pi^2 \chi(M) + 13(\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2t} \\
& + \frac{81}{26} \int_M |\text{Sic}|^2 dV_{g(t)} + 4 \int_M |\text{Sic}|^2 dV_{g(t)} \\
& = 32\pi^2 \chi(M) + 13(\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2t} + \frac{185}{26} b(M, g(0), \phi(0), t)
\end{aligned}$$

by (3.28).

For (3.31), we use

$$\frac{d}{dt} \text{Vol}(M, g(t)) = - \int_M S dV_{g(t)} \quad \text{and} \quad -1 \leq S \leq 1$$

to deduce

$$e^{-T} \text{Vol}(M, g(0)) \leq \text{Vol}(M, g(t)) \leq e^T \text{Vol}(M, g(0)).$$

Consequently, for any $0 < s < r < T$,

$$\begin{aligned}
\int_s^r \int_M |\text{Sic}|^p dV_{g(t)} dt & \leq \left(\int_s^r \int_M |\text{Sic}|^4 dV_{g(t)} dt \right)^{\frac{p}{4}} \left(\int_s^r \int_M dV_{g(t)} dt \right)^{\frac{4-p}{4}} \\
& \leq [b(M, g(0), \phi(0), T)]^{\frac{p}{4}} (r-s)^{\frac{4-p}{4}} e^{-\frac{T(4-p)}{4}} [\text{Vol}(M, g(0))]^{\frac{4-p}{4}}.
\end{aligned}$$

Thus we get (3.31). \square

Define

$$(3.33) \quad c(M, g(0), \phi(0), T) := 9e^{90T} \left[\int_M |\text{Sic}_{g(0)}|_{g(0)}^2 dV_{g(0)} + \pi^2 |\chi(M)| + [(\alpha A_1)^2 + (\alpha A_1)] \text{Vol}(M, g(0)) \right].$$

Then $|b(M, g(0), \phi(0), T)| \leq c(M, g(0), \phi(0), T)$. Theorem 3.5 now yields

$$(3.34) \quad \sup_{t \in [0, T]} \int_M |\text{Sic}_{g(t)}|_{g(t)}^2 dV_{g(t)} \leq c(M, g(0), \phi(0), T) < +\infty,$$

$$(3.35) \quad \sup_{t \in [0, T]} \int_M |\text{Sm}_{g(t)}|_{g(t)}^2 dV_{g(t)} \leq 32\pi^2 \chi(M) + 8c(M, g(0), \phi(0), T) + 13(\alpha A_1)^2 \text{Vol}(M, g(0)) e^{2T} < +\infty.$$

4. 4D RICCI-HARMONIC FLOW WITH BOUNDED $S_{g(t)}$: II

We in this section consider the general case of 4D RHF $(M, g(t), \phi(t))_{t \in [0, T]}$ with $T \leq \infty$ and $\alpha(t) \geq 0, \dot{\alpha} \leq 0$. Using the same notions as (3.1) and (3.4) we have

$$(4.1) \quad \begin{aligned} \square f &= -2 \frac{|Z|^2}{(S+C)^3} - 2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - 2\alpha \left| \Delta \phi \frac{\text{Sic}}{S+C} - \nabla^2 \phi \right|^2 \\ &+ 2\alpha |\nabla^2 \phi|^2 + \frac{\dot{\alpha}}{(S+C)^2} |\nabla \phi|^2 |\text{Sic}|^2 + 2\dot{\alpha}(S+C) \langle \text{Sic}, \nabla \phi \otimes \nabla \phi \rangle. \end{aligned}$$

Using $f = |\text{Sic}|^2 / (S+C)$ we get

$$\begin{aligned} &\frac{\dot{\alpha}}{(S+C)^2} |\nabla \phi|^2 |\text{Sic}|^2 + 2\dot{\alpha}(S+C) \langle \text{Sic}, \nabla \phi \otimes \nabla \phi \rangle \\ &= \frac{\dot{\alpha}}{S+C} |\nabla \phi|^2 f - 2\dot{\alpha}(S+C) \left[\epsilon f(S+C) + \frac{1}{4\epsilon} |\nabla \phi|^4 \right] \\ &= -\dot{\alpha} f \left[2\epsilon(S+C)^2 - \frac{|\nabla \phi|^2}{S+C} \right] - \frac{\dot{\alpha}}{2\epsilon} (S+C) |\nabla \phi|^4 \\ &\leq -\dot{\alpha}(S+C)^4 |\nabla \phi|^2. \end{aligned}$$

where $\epsilon := |\nabla \phi|^2 / 2(S+C)^3$. Hence, together with (4.1) yields

$$(4.2) \quad \begin{aligned} \square f &\leq -2 \frac{|Z|^2}{(S+C)^3} - 2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} \\ &+ 2\alpha |\nabla^2 \phi|^2 - \dot{\alpha}(S+C)^4 |\nabla \phi|^2. \end{aligned}$$

As in the proof of (3.9) we arrive at

$$(4.3) \quad \begin{aligned} \frac{d}{dt} \int_M f dV_{g(t)} &\leq \int_M \left[-2f^2 + 4 \frac{\text{Sm}(\text{Sic}, \text{Sic})}{S+C} - fS \right] dV_{g(t)} \\ &+ 2\alpha A_2(t) - \dot{\alpha} A_1 \int_M (S+C)^4 dV_{g(t)}. \end{aligned}$$

Using Lemma 3.1 yields

$$(4.4) \quad \begin{aligned} \frac{d}{dt} \int_M f dV_{g(t)} &\leq \int_M \left(-f^2 + 36Cf + 574S^2 \right) dV_{g(t)} - \dot{\alpha}(t) A_1 \int_M (S+C)^4 dV_{g(t)} \\ &+ \left[256\pi^2 \chi(M) + 104\alpha^2(t) A_1^2 \text{Vol}(M, g(0)) e^{Ct} + 2\alpha(t) A_2(t) \right]. \end{aligned}$$

The above estimate can help use to extend Theorem 3.2 and Theorem 3.3 to the general RHF. For example, we have

$$(4.5) \quad \begin{aligned} &\int_M \frac{|\text{Sic}_{g(s)}|_{g(s)}^2}{S_{g(s)} + C} dV_{g(s)} + \int_0^s \int_M \frac{|\text{Sic}_{g(t)}|_{g(t)}^4}{(S_{g(t)} + C)^2} dV_{g(t)} dt \\ &\leq c_0(M, g(0), \phi(0), s) + 574e^{36Cs} \int_0^s \int_M S_{g(t)}^2 dV_{g(t)} dt \\ &+ e^{36Cs} \int_0^s -\dot{\alpha}(t) \int_M (S_{g(t)} + C)^4 dV_{g(t)} dt. \end{aligned}$$

Similarly, we can extend Theorem 3.5 to the general RHF by the same argument if an analogous result of Theorem 3.4 holds for the general RHF. However, this is obvious, because alone the outline of the proof of Theorem 3.4 we have

$$(4.6) \quad \begin{aligned} \frac{d}{dt} \int_M f dV_{g(t)} &\leq \int_M \left(-f^2 + 88f \right) dV_{g(t)} - \dot{\alpha}(t) A_1 \int_M (S+C)^4 dV_{g(t)} \\ &+ \left[128\pi^2 \chi(M) + 52(\alpha(t) A_1)^2 \text{Vol}(M, g(0)) e^{2t} + 2\alpha(t) A_2(t) \right] \end{aligned}$$

if $(M, g(t), \phi(t))_{t \in [0, T]}$ satisfies **BA**.

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