

# Associative idempotent nondecreasing functions are reducible\*

Gergely Kiss

University of Luxembourg, Faculty of Science, Mathematical  
Research Unit, 2, avenue de l'Université, L-4365 Esch-sur-Alzette,  
kigergo57@gmail.com

Gábor Somlai

Eötvös Loránd University, Faculty of Science, Institute of  
Mathematics, Pázmány Péter sétány. 1/c, Budapest, H-1117  
Hungary, gsomlai@cs.elte.hu

Communicated by Mikhail Volkov

## Abstract

An  $n$ -ary associative function is called reducible if it can be written as a composition of a binary associative function. We summarize known results when the function is defined on a chain and is nondecreasing. Our main result shows that associative idempotent and nondecreasing functions are uniquely reducible.

## 1 Introduction

In this paper we investigate the class of functions  $F: X^n \rightarrow X$  ( $n \geq 2$ ) defined on a chain (i.e., totally ordered set)  $X$  that are nondecreasing, idempotent and associative. For arbitrary set  $X$ , the study of associativity stemmed

---

\*The research was supported by the internal research project R-AGR-0500 of the University of Luxembourg. The first author was partially supported by the Hungarian Scientific Research Fund (OTKA) K104178. The second author was partially supported by the Hungarian Scientific Research Fund (OTKA) K115799.

back to the pioneering work of Dörnte [5] and Post [9]. Dudek and Mukhin [6,7] gave a characterization of reducibility using the terminology of a neutral element (see Theorem 3.5). While their result is essential from a theoretic point of view, it is not easy to apply it for a given situation unless the function originally has a neutral element (for further details see also [8]). Ackerman [1] made a complete characterization of quasitrivial associative functions. In his paper it was shown that every quasitrivial associative function is derived from a binary or a ternary function.

Couceiro and Marichal showed in [2] that continuous symmetric cancellative and associative  $n$ -ary functions defined on a nonempty real interval are reducible (see Remark 4 of [2]). Although they established reducibility under some hypotheses that are not related to those of the present paper, it also shows that reducibility is an important property in the study of associative  $n$ -ary functions. Reducibility and extremality<sup>1</sup> of quasitrivial associative symmetric nondecreasing functions were studied in [4].

The paper is organized as follows. Section 2 contains the basic definitions and notation. In Section 3.1 we collect the preliminary results in the case when  $F: X^n \rightarrow X$  is idempotent, monotone, associative and has a neutral element. This part is based on [7] and [8]. In Section 3.2 we complete the study of reducibility of quasitrivial nondecreasing associative  $n$ -ary functions (without the assumption of symmetry). In Section 4 we present the main results about the reducibility of idempotent, nondecreasing, associative functions. Because of its simplicity we present the symmetric case with useful lemmas (see Lemma 4.1 and 4.2) in Section 4.1. In Section 4.2 we prove the general result. The main technicality is that we have to divide the proof into two subcases. Theorem 4.4 can be used only for  $n = 3$ , and another inductive proof (Theorem 4.8) works for  $n > 3$ . In Section 5 we discuss extremality which holds in many special cases but not for every associative idempotent nondecreasing function. We also and monotonicity as a relaxation of the property of the nondecreasingness.

## 2 Definitions and notation

Let  $X$  be an arbitrary set and  $F: X^n \rightarrow X$  an  $n$ -ary function. We denote by  $S_n$  the symmetric group on the set  $\{1, \dots, n\}$ . Now we give a sequence of definitions:

**Definition 2.1.** The function  $F: X^n \rightarrow X$  is called

---

<sup>1</sup>The definition of extremality stems from [10].

- (i) *idempotent* if  $F(x, \dots, x) = x$  for every  $x \in X$ ,
- (ii) *symmetric* if  $F(x_1, \dots, x_n) = F(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  for all  $x_1, \dots, x_n \in X$  and every permutation  $\sigma \in S_n$ ,
- (iii) *quasitrivial* (or *conservative*) if for all  $x_1, \dots, x_n \in X$

$$F(x_1, \dots, x_n) \in \{x_1, \dots, x_n\},$$

- (iv) *n-associative* if for all  $x_1, \dots, x_{2n-1} \in X$  and  $1 \leq i \leq n-1$  we have

$$\begin{aligned} F(F(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) = \\ F(x_1, \dots, x_i, F(x_{i+1}, \dots, x_{i+n}), x_{i+n+1}, \dots, x_{2n-1}). \end{aligned} \quad (1)$$

We usually say that  $F: X^n \rightarrow X$  is associative and we only write that  $F$  is  $n$ -associative if we want to emphasize the number of variables in  $F$ .

We say that  $e \in X$  is a *neutral element* for  $F: X^n \rightarrow X$  if for every  $x \in X$  and  $1 \leq i \leq n$  we have  $F(e, \dots, e, x, e, \dots, e) = x$ , where  $x$  is in the  $i$ -th coordinate of  $F$ .

For any integer  $k \geq 0$  and any  $x \in X$ , we set  $k \cdot x = \underbrace{x, \dots, x}_{k \text{ times}}$ . For instance, the idempotency of  $F$  can be written in the form  $F(n \cdot x) = x$ .

From now on,  $X$  will be a totally ordered set. For any  $n \in \mathbb{N}$  the function  $F: X^n \rightarrow X$  is called *nondecreasing* (resp. *nonincreasing*) if

$$F(a_1, \dots, a_n) \geq F(b_1, \dots, b_n) \quad (\text{resp. } F(a_1, \dots, a_n) \leq F(b_1, \dots, b_n)), \quad (2)$$

for every pair of  $n$ -tuples  $(a_1, \dots, a_n), (b_1, \dots, b_n) \in X^n$  with  $a_i \geq b_i$  for  $1 \leq i \leq n$ .

The function  $F$  is called *monotone in the  $i$ -th variable* if for all fixed elements  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$  of  $X$ , the 1-ary function defined as

$$f_i(x) := F(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$$

is nondecreasing or nonincreasing. The function  $F$  is called *monotone* if it is monotone in each of its variables.

We use the lattice notation for the minimum ( $\wedge$ ) and for maximum ( $\vee$ ) of a set. Hence we introduce the notation

$$\wedge_{i=1}^n x_i = \min\{x_1, \dots, x_n\}, \quad \vee_{i=1}^n x_i = \max\{x_1, \dots, x_n\}.$$

### 3 Preliminary results

**Definition 3.1.** We say that  $F: X^n \rightarrow X$  is derived from  $G: X^2 \rightarrow X$  if  $F$  can be written in the form

$$F(x_1, \dots, x_n) = x_1 \circ \dots \circ x_n,$$

where  $x \circ y = G(x, y)$ . We note that this expression is well-defined for  $n \geq 3$  if and only if  $G$  is associative. If such a  $G$  exists, then we say that  $F$  is *reducible*.

We note that if  $n = 2$  then the function  $F$  is derived from itself.

The previous definition only deals with the existence of a binary function from which a given  $n$ -associative function can be derived. The uniqueness of the binary function follows from certain conditions. The following result was proved first in [4, Proposition 3.5].

**Proposition 3.2.** *Assume that the function  $F: X^n \rightarrow X$  is associative and derived from an associative idempotent binary function. Then the binary function is unique.*

In our case, when  $X$  is a totally ordered set and  $F$  is monotone, we can strengthen the previous statement. The result presented here follows from [8, Lemma 3.4] when  $F$  is chosen to be monotone.

**Proposition 3.3.** *Let  $X$  be a totally ordered set and  $F: X^n \rightarrow X$  an associative idempotent monotone function, which is derived from an associative binary function  $G$ . Then  $G$  is idempotent as well.*

Combining the previous statements we get:

**Corollary 3.4.** *Let  $X$  be a totally ordered set. If an associative idempotent monotone function  $F$  is derived from a binary function  $G: X^2 \rightarrow X$ , then  $G$  is uniquely determined by  $F$ .*

#### 3.1 Neutral element

Suppose that  $F: X^n \rightarrow X$  is an associative function having a neutral element  $e \in X$ . Then one can define  $G: X^2 \rightarrow X$  by

$$G(a, b) = F(a, (n-2) \cdot e, b) \tag{3}$$

for every  $a, b \in X$ . The following theorem of Dudek and Mukhin [7] shows a general result for an arbitrary set  $X$ .

**Theorem 3.5.** *Let  $X$  be a nonempty set. Let  $F : X^n \rightarrow X$  be an associative function. Then  $F$  is derived from a binary function  $G$  if and only if  $F$  has a neutral element or one can adjoin<sup>2</sup> a neutral element to  $X$  for  $F$ . In this case such a  $G$  can be defined by (3).*

We note that the previous statement also holds for  $n = 2$ . Indeed, every associative binary function is reducible and if an associative function  $F$  has no neutral element, then we can adjoin one. Let  $e \notin X$  and let  $\bar{F}$  be defined as  $\bar{F}(x, y) = F(x, y)$  for  $x, y \in X$  and  $\bar{F}(z, e) = \bar{F}(e, z) = z$  for every  $z \in X \cup \{e\}$ . It is easy to check that  $\bar{F}$  is associative on  $X \cup \{e\}$ .

The following statement was proved in [8, Proposition 3.13] as an application of the previous structural theorem.

**Proposition 3.6.** *Let  $X$  be a totally ordered set and  $F : X^n \rightarrow X$  an associative monotone idempotent function with a neutral element  $e$ . Let  $G$  be defined by (3). Then  $F$  is derived from the binary function  $G$ , which is also associative, idempotent, monotone and has the same neutral element  $e$ .*

Since every monotone, idempotent associative binary function is non-decreasing by [8, Lemma 3.10], the previous statement immediately has a simple consequence.

**Corollary 3.7.** *Let  $X$  and  $F$  be as in Proposition 3.6. Then  $F$  is nondecreasing.*

**Observation 3.8.** *Let  $X$  and  $F$  be as in Proposition 3.6. If  $F$  is symmetric, then  $G$  defined by (3) is also symmetric.*

Lemma 3.9 shows a connection between the existence of a neutral element and quasitriviality. The base of the idea appears in Czogała–Drewniak’s theorem [3] where  $X = [0, 1]$ . For the sake of completeness we present a short proof here.

**Lemma 3.9.** *Let  $X$  be a totally ordered set and  $F : X^n \rightarrow X$  an associative, idempotent, monotone function having a neutral element  $e$ . Then  $F$  is quasitrivial.*

*Proof.* By Corollary 3.7, we can automatically assume that  $F$  is nondecreasing. For  $n = 2$  and  $x, y \in X$ , we distinguish two different cases:

---

<sup>2</sup>Adjoining a neutral element to  $X$  for an  $n$ -associative function  $F$  means to define an  $n$ -associative function  $\bar{F}$  on the set  $X \cup \{e\}$  such that  $e \notin X$  is a neutral element for  $\bar{F}$  and  $\bar{F}(x_1, \dots, x_n) = F(x_1, \dots, x_n)$  for all  $x_1, \dots, x_n \in X$ .

1.  $(x \leq e, y \leq e)$  or  $(e \leq x, e \leq y)$ ,
2.  $(x \leq e \leq y)$  or  $(y \leq e \leq x)$ .

We show that in each case  $F(x, y)$  is either the maximum or the minimum, thus it is quasitrivial. In Case 1 if  $x \leq e, y \leq e$ , then by the nondecreasingness of  $F$  we get

$$\begin{aligned} x &= F(x, e) \geq F(x, y) \\ y &= F(e, y) \geq F(x, y). \end{aligned}$$

Thus  $x \wedge y \geq F(x, y)$ .

On the other hand if  $x \leq y$  (the case  $x \geq y$  can be handled similarly), then

$$x = F(x, x) \leq F(x, y) \leq F(y, y) = y,$$

by monotonicity and idempotency. This implies that  $F(x, y) = x \wedge y$ .

Similarly if  $e \leq x, e \leq y$ , it can be obtained that  $F(x, y) = x \vee y$ .

In Case 2 the two subcases can be handled similarly. Now we deal with  $x \leq e \leq y$ . we denote  $F(x, y) = \theta$ . Assume that  $x \leq \theta \leq e \leq y$ , then using associativity, we get

$$F(x, \theta) = F(x, F(x, y)) = F(F(x, x), y) = F(x, y) = \theta \quad (4)$$

On the other hand, since  $x \leq e, \theta \leq e$ , we have already proved that

$$F(x, \theta) = x \wedge \theta = x.$$

This shows that  $\theta = x$ . For  $x \leq e \leq \theta \leq y$  similarly we have

$$\theta = F(\theta, y) = y.$$

Thus we get that the binary function  $F$  is quasitrivial.

If  $n > 2$  and  $F$  is an  $n$ -associative idempotent non-decreasing and have a neutral element, then we can use Proposition 3.6. Thus there exists a binary function  $G$  which is associative, idempotent, non-decreasing and have a neutral element. By the case  $n = 2$  we know that  $G$  is quasitrivial and, since  $F$  is derived from  $G$ ,  $F$  is also quasitrivial.  $\square$

### 3.2 Quasitriviality

In [4, Theorem 3.3 and Corollary 3.4], Devillet, Kiss, and Marichal proved the following characterization for quasitrivial symmetric nondecreasing and associative functions.

**Theorem 3.10.** *Let  $X$  be a totally ordered set and let  $F: X^n \rightarrow X$  be a quasitrivial symmetric nondecreasing associative function. Then  $F$  is reducible. More precisely,  $F$  is derived from  $G: X^2 \rightarrow X$  defined by*

$$G(x, y) = F((n-1) \cdot x, y) = F(x, (n-1) \cdot y). \quad (5)$$

It is easy to see that function  $G$  defined by (5) is quasitrivial, symmetric and nondecreasing. In [4, Theorem 3.3] it was also proved that in this case

$$F(x_1, \dots, x_n) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i). \quad (6)$$

This means that  $F$  is extremal (see Definition 5.1).

One can prove that  $F$  remains reducible if we eliminate the symmetry condition of  $F$ . The result is weaker in the sense that it only shows the existence of such a decomposition (see Theorem 3.12). We note that the analogue of (6) does not hold (for further details see Section 5.1).

The following result is an easy consequence of [1, Theorem 1.4] using the statement therein for  $A_2 = \emptyset$ .

**Theorem 3.11.** *Let  $X$  be an arbitrary set and  $F: X^n \rightarrow X$  a quasitrivial  $n$ -associative function. If  $F$  is not derived from a binary function, then  $n$  is odd and there exist  $b_1, b_2$  ( $b_1 \neq b_2$ ) such that for all  $a_1, \dots, a_n \in \{b_1, b_2\}$ ,*

$$F(a_1, \dots, a_n) = b_i \quad (i = 1, 2), \quad (7)$$

where  $b_i$  occurs amongst  $a_1, \dots, a_n$  an odd number of times.

As a consequence of this theorem we prove the following:

**Theorem 3.12.** *Let  $X$  be a totally ordered set and let  $F: X^n \rightarrow X$  be an associative quasitrivial nondecreasing function. Then  $F$  is reducible.*

*Proof.* By contradiction we assume that  $F$  is not derived from a binary function. Now we apply the previous theorem since we intend to show that in this case the conditions for  $b_1, b_2$  cannot be satisfied. Thus every associative, quasitrivial, nondecreasing function defined on a totally ordered set  $X$  is reducible.

According to Theorem 3.11, if  $F$  is not reducible, then  $n$  is odd. Hence  $n \geq 3$  and there exist  $b_1, b_2$  satisfying equation (7). Since  $b_1 \neq b_2$ , we may assume that  $b_1 < b_2$  (the case  $b_2 < b_1$  can be handled similarly). By our assumption on  $b_1$  and  $b_2$  we have

$$F(n \cdot b_1) = b_1, \quad F(b_2, (n-1) \cdot b_1) = b_2, \quad F(2 \cdot b_2, (n-2) \cdot b_1) = b_1. \quad (8)$$

Since  $F$  is nondecreasing we have

$$F(n \cdot b_1) \leq F(b_2, (n-1) \cdot b_1) \leq F(2 \cdot b_2, (n-2) \cdot b_1).$$

This implies  $b_1 = b_2$ , a contradiction.  $\square$

## 4 Main results

In this section we prove that every associative idempotent nondecreasing function defined on a totally ordered set  $X$  is derived from a binary function  $G$ . As it was shown in Corollary 3.4,  $G$  is also unique. This result generalizes some of the previous results on reducibility. As a consequence of Theorem 3.5, this means that if an associative idempotent nondecreasing function  $F$  is defined on a totally ordered set  $X$ , then either there is a neutral element for  $F$  or we can adjoin an element to  $X$  which acts as a neutral element for  $F$ . We note that all of our statements also hold for  $n = 2$  but bring no information in this case. Practically, we just deal with the cases when  $n \geq 3$ .

### 4.1 Symmetric case

The symmetric case (as usual) is much simpler than the general one but we present a separate argument here. Our result is based on the following two lemmas.

**Lemma 4.1.** *Let  $X$  be a totally ordered set and  $F: X^n \rightarrow X$  an associative nondecreasing idempotent function. Then for every  $a, c \in X$ ,*

$$F(a, (n-1) \cdot c) = F((n-1) \cdot a, c).$$

*Proof.* If  $a = c$ , then the statement trivially follows from the idempotency of  $F$ . We assume that  $a < c$ . (The case  $a > c$  can be handled similarly.) We denote  $F((n-1) \cdot a, c)$  by  $\theta$ . Since  $F$  is nondecreasing and idempotent, we have  $a \leq \theta \leq c$ . Now we have

$$\begin{aligned} \theta &= F((n-1) \cdot a, c) \leq F(a, (n-1) \cdot c) \leq F(\theta, (n-1) \cdot c) = \\ &F(F((n-1) \cdot a, c), (n-1) \cdot c) = F((n-1) \cdot a, F(n \cdot c)) = \\ &F((n-1) \cdot a, c) = \theta. \end{aligned}$$

Thus, we get  $F(a, (n-1)c) = F((n-1)a, c)$ .  $\square$



*Remark 1.* As a consequence of the previous lemma we obtain that if  $F$  is an associative idempotent nondecreasing function, then  $F(k \cdot a, (n - k) \cdot c)$  is the same for every  $1 \leq k \leq n - 1$ . Indeed, if  $a \leq c$ , then  $F((n - 1) \cdot a, c) \leq F(k \cdot a, (n - k) \cdot c) \leq F(a, (n - 1) \cdot c)$ . If  $a \geq c$ , then  $F((n - 1) \cdot a, c) \geq F(k \cdot a, (n - k) \cdot c) \geq F(a, (n - 1) \cdot c)$ .

**Lemma 4.2.** *Let  $X$  be a totally ordered set and  $F: X^n \rightarrow X$  an associative idempotent and nondecreasing function. Then the function  $G$  defined by*

$$G(a, c) = F(a, (n - 1) \cdot c) = F((n - 1) \cdot a, c). \quad (9)$$

*is associative idempotent and nondecreasing.*

We note that by Lemma 4.1 and Remark 1,  $G$  is well-defined and  $G(a, c) = F(k \cdot a, (n - k) \cdot c)$  for every  $k = 1, \dots, n - 1$ .

*Proof.* It is clear that  $G$  is idempotent and nondecreasing. The following equation shows that  $G$  is associative.

$$\begin{aligned} G(a, G(b, c)) &= F((n - 1) \cdot a, F(b, (n - 1) \cdot c)) = \\ &= F(F((n - 1) \cdot a, b), (n - 1) \cdot c) = G(G(a, b), c). \quad \square \end{aligned}$$

Now we investigate the question of reducibility for the symmetric case.

**Theorem 4.3.** *Let  $X$  be a totally ordered set and let  $F: X^n \rightarrow X$  be an associative symmetric nondecreasing idempotent function. Then  $F$  is derived from a unique binary function  $G: X^2 \rightarrow X$  which can be obtained as*

$$G(a, c) = F(a, (n - 1) \cdot c). \quad (10)$$

*Moreover*

$$F(x_1, \dots, x_n) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i). \quad (11)$$

*Remark 2.* Equation (11) means that  $F$  is extremal (see Section 5.1).

*Proof.* Applying Lemma 4.1, we can define  $G$  for any  $a, c \in X$  by

$$G(a, c) = F((n - 1) \cdot a, c) = F(a, (n - 1) \cdot c).$$

The uniqueness of the binary function follows from Corollary 3.4 so we only have to verify that  $G$  fulfils our requirements.

Since  $F$  is nondecreasing, we have that

$$G(a, c) = F((n - 1) \cdot a, c) \leq F(a, x_1, \dots, x_{n-2}, c) \leq F(a, (n - 1) \cdot c) = G(a, c) \quad (12)$$

for every  $a \leq x_1, \dots, x_{n-2} \leq c$ . We get that the inequalities in (12) are equalities. Thus by the symmetry of  $F$ , the value of  $F(x_1, \dots, x_n)$  depends only on  $\wedge_{i=1}^n x_i$  and  $\vee_{i=1}^n x_i$ .

Using the symmetry of  $F$  we can reorder the entries of  $F$  and we get

$$F(x_1, \dots, x_n) = F(\wedge_{i=1}^n x_i, \dots, \vee_{i=1}^n x_i) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i).$$

This argument shows that  $F$  is derived from  $G$  (and extremal). □

## 4.2 General case

In this section we do not assume that our functions are symmetric. In Theorem 4.4 and 4.8 we prove the reducibility of associative idempotent nondecreasing  $n$ -ary functions for  $n \geq 3$  which is the main result of this section. It seems from our argument that the cases  $n = 3$  and  $n \geq 4$  should be handled in different ways and separately. First we discuss the case  $n = 3$ .

**Theorem 4.4.** *Let  $X$  be a totally ordered set and let  $F: X^3 \rightarrow X$  be an associative idempotent nondecreasing function. Then  $F$  is derived from a unique binary function denoted by  $G: X^2 \rightarrow X$ . The function  $G$  can be defined by*

$$G(a, c) = F(a, c, c) = F(a, a, c). \quad (13)$$

*Proof.* By Lemma 4.1,  $G$  can be defined by (13). Applying Lemma 4.2 we get that  $G$  is associative nondecreasing and idempotent. We need to show that

$$F(a, b, c) = G(a, G(b, c)) = G(G(a, b), c)$$

for every  $a, b, c \in X$ .

If  $a \leq b \leq c$  (the case  $a \geq b \geq c$  can be handled similarly), then we can directly apply (13) and we obtain

$$G(a, c) = F(a, a, c) \leq F(a, b, c) \leq F(a, c, c) = G(a, c).$$

On the other hand, since  $G$  is nondecreasing and idempotent, we have

$$\begin{aligned} G(a, G(b, c)) &\leq G(a, G(c, c)) = G(a, c), \\ G(G(a, b), c) &\geq G(G(a, a), c) = G(a, c). \end{aligned} \quad (14)$$

By the associativity of  $G$  and equation (14) we get  $G(a, c) \leq G(G(a, b), c) = G(a, G(b, c)) \leq G(a, c)$ . Hence

$$F(a, b, c) = G(a, c) = G(G(a, b), c) = G(a, G(b, c)),$$

as required.

Assume that  $a \leq b$ ,  $c \leq b$  or  $a \geq b$ ,  $c \geq b$  (i.e.,  $b$  is the smallest or the largest among  $a, b, c$ ). We may assume that all of these relations are strict inequalities. Otherwise we are in the previous case. On the other hand the following proof works for non-strict cases, as well.

We introduce the following notation

$$\begin{aligned}\theta_1 &= G(a, b) = F(a, a, b) = F(a, b, b), \\ \theta_2 &= G(b, c) = F(b, b, c) = F(b, c, c).\end{aligned}$$

Then we get

$$\begin{aligned}F(a, b, c) &= F(F(3 \cdot a), F(3 \cdot b), c) = \\ &F(a, F(a, a, b), F(b, b, c)) = F(a, \theta_1, \theta_2).\end{aligned}\tag{15}$$

and

$$\begin{aligned}F(a, b, c) &= F(a, F(3 \cdot b), F(3 \cdot c)) = \\ &F(F(a, b, b), F(b, c, c), c) = F(\theta_1, \theta_2, c).\end{aligned}\tag{16}$$

Suppose that  $b = \max\{a, b, c\}$  ( $b = \min\{a, b, c\}$  can be handled similarly). If  $\theta_1 \leq \theta_2$ , then  $a \leq b$  implies  $G(a, a) = a \leq G(a, b) = \theta_1 \leq \theta_2$ , so  $a, \theta_1, \theta_2$  are in increasing order. Therefore by the previous case

$$F(a, \theta_1, \theta_2) = G(a, \theta_2) = G(a, G(b, c)).$$

Using equation (15) we obtain that  $F(a, b, c) = G(a, G(b, c))$ , which equals to  $G(G(a, b), c)$  since  $G$  is associative.

If  $\theta_1 \geq \theta_2$ , then by  $c \leq b$  we get that  $c = G(c, c) \leq G(b, c) = \theta_2 \leq \theta_1$ . Now the sequence  $\theta_1, \theta_2, c$  is in decreasing order, hence

$$F(\theta_1, \theta_2, c) = G(\theta_1, c) = G(G(a, b), c).$$

Using equation (16) we get that  $F(a, b, c) = G(G(a, b), c)$ . Finally, the associativity of  $G$  gives the result, finishing the proof of Theorem 4.4.  $\square$

Now we prove the analogous result for  $n \geq 4$ . The main problem is that in case  $n = 3$  we heavily use the fact that every ordered triple  $(a, b, c)$  is either monotone (i.e.,  $a \leq b \leq c$  or  $a \geq b \geq c$ ) or one of its extrema is in the middle (i.e.,  $a, c \leq b$  or  $b \leq a, c$ ). Generally, for  $n > 3$  there are plenty other cases. Therefore we follow another way to generalize the previous result. We start with two lemmas.

**Lemma 4.5.** *Let  $X$  be a totally ordered set and  $F: X^n \rightarrow X$  an associative idempotent nondecreasing function. Then*

$$F(x_1, \dots, x_{i-1}, 2 \cdot x_i, x_{i+1}, \dots, x_{n-1}) = F(x_1, \dots, x_i, 2 \cdot x_{i+1}, x_{i+2}, \dots, x_{n-1}) \quad (17)$$

holds for every  $i \in \{1, \dots, n-2\}$  and all  $x_1, \dots, x_{n-1} \in X$ .

*Proof.* Lemma 4.1 gives  $F((n-1) \cdot a, c) = F(a, (n-1) \cdot c)$ . Since  $F$  is nondecreasing we obtain

$$F((n-1) \cdot a, c) = F(k \cdot a, (n-k) \cdot c) \quad (18)$$

for every  $1 \leq k \leq n-1$  (as in Remark 1). The following direct calculation proves the statement. We use the idempotency of  $F$  in the first and last equalities, the associativity of  $F$  and in the second and fourth equalities and we use equation (18) for  $x_i$  and  $x_{i+1}$  in the third equality

$$\begin{aligned} F(x_1, \dots, 2 \cdot x_i, x_{i+1}, \dots, x_{n-1}) &= F(x_1, \dots, x_i, F(n \cdot x_i), x_{i+1}, \dots, x_{n-1}) = \\ &= F(x_1, \dots, 2 \cdot x_i, F((n-1) \cdot x_i, x_{i+1}), \dots, x_{n-1}) = \\ &= F(x_1, \dots, 2 \cdot x_i, F((n-2) \cdot x_i, 2 \cdot x_{i+1}), \dots, x_{n-1}) = \\ &= F(x_1, \dots, F(n \cdot x_i), 2 \cdot x_{i+1}, \dots, x_{n-1}) = F(x_1, \dots, x_i, 2 \cdot x_{i+1}, \dots, x_{n-1}). \quad \square \end{aligned}$$

**Corollary 4.6.** *Let  $X$  and  $F$  be as above. One can define  $H: X^{n-1} \rightarrow X$  by the following formula*

$$H(x_1, \dots, x_{n-1}) = F(2 \cdot x_1, x_2, \dots, x_{n-1}) = \dots = F(x_1, \dots, x_{n-2}, 2 \cdot x_{n-1}) \quad (19)$$

*Remark 3.* We note that  $H$  defined by (19) is idempotent and nondecreasing if so is  $F$ .

**Lemma 4.7.** *Let  $X$  be a totally ordered set and  $F: X^n \rightarrow X$  an associative idempotent nondecreasing function. Then the function  $H: X^{n-1} \rightarrow X$  defined in Corollary 4.6 is associative.*

*Proof.* The following equations hold for every  $k \in \{3, \dots, n-1\}$

$$\begin{aligned} H(x_1, \dots, x_{k-1}, H(y_1, \dots, y_{n-1}), x_{k+1}, \dots, x_{n-1}) &= \\ &= F(2 \cdot x_1, \dots, x_{k-1}, F(2 \cdot y_1, \dots, y_{n-1}), x_{k+1}, \dots, x_{n-1}) = \\ &= F(2 \cdot x_1, \dots, x_{k-2}, F(x_{k-1}, 2 \cdot y_1, \dots, y_{n-2}), y_{n-1}, x_{k+1}, \dots, x_{n-1}) = \\ &= H(x_1, \dots, x_{k-2}, H(x_{k-1}, y_1, \dots, y_{n-2}), y_{n-1}, x_{k+1}, \dots, x_{n-1}). \end{aligned}$$

For  $k = 2$  the previous calculation does not hold. In that case we get the following equation using (19).

$$\begin{aligned}
H(x_1, H(y_1, \dots, y_{n-1}), x_3, \dots, x_{n-1}) &= \\
F(x_1, F(2 \cdot y_1, \dots, y_{n-1}), x_3, \dots, 2 \cdot x_{n-1}) &= \\
F(F(x_1, 2 \cdot y_1, \dots, y_{n-2}), y_{n-1}, x_3, \dots, 2 \cdot x_{n-1}) &= \\
H(H(x_1, y_1, \dots, y_{n-2}), y_{n-1}, x_3, \dots, x_{n-1}). &\quad \square
\end{aligned}$$

Since  $H: X^{n-1} \rightarrow X$  is associative idempotent and nondecreasing, we can use induction for  $n \geq 3$ .

**Theorem 4.8.** *Let  $X$  be a totally ordered set and let  $F: X^n \rightarrow X$  ( $n \geq 2$ ) be an associative idempotent nondecreasing function. Then there exists a unique associative idempotent nondecreasing binary function  $G: X^2 \rightarrow X$  from which  $F$  is derived. Moreover,  $G$  can be defined by*

$$G(a, c) = F(a, (n-1) \cdot c) = F((n-1) \cdot a, c). \quad (20)$$

*Proof.* For  $n = 2$  the statement is automatically true. The statement is proved by induction for  $n \geq 3$ . Theorem 4.4 gives the result for  $n = 3$ .

Assume that  $n > 3$ . By Lemmas 4.1 and 4.2,  $G: X^2 \rightarrow X$  is a well-defined associative idempotent nondecreasing function. Let  $H: X^{n-1} \rightarrow X$  be defined by (19) as in Corollary 4.6. The function  $H$  is associative nondecreasing and idempotent according to Lemma 4.7.

Now we recall the notation  $G(a, b) = a \circ b$  which is well-defined since  $G$  is associative by Lemma 4.2.

By induction,  $H$  is derived from a binary function. Since

$$a \circ b = G(a, b) = F((n-1) \cdot a, b) = H((n-2) \cdot a, b) \quad (21)$$

we have that  $H$  is derived from  $G$ , i.e.:

$$H(x_1, x_2, \dots, x_{n-1}) = x_1 \circ x_2 \circ \dots \circ x_{n-1}. \quad (22)$$

Now we show that  $F$  is also derived from the same binary function  $G$ .

$$\begin{aligned}
F(x_1, x_2, \dots, x_n) &= F(F(n \cdot x_1), x_2, \dots, x_n) = \\
F((n-2) \cdot x_1, F(2 \cdot x_1, x_2, \dots, x_{n-1}), x_n) &= \\
H((n-3) \cdot x_1, H(x_1, x_2, \dots, x_{n-1}), x_n) &= \\
x_1 \circ \dots \circ x_1 \circ (x_1 \circ x_2 \circ \dots \circ x_{n-1}) \circ x_n &= \\
x_1 \circ x_2 \circ \dots \circ x_{n-1} \circ x_n. &
\end{aligned} \quad (23)$$

In the second equation we use the associativity of  $F$ , in the third we substitute  $H$  using that  $n - 2 \geq 2$ , in the fourth equation we apply (22), in the last equation we use the idempotency and associativity of  $G$ . Equation (23) shows that  $F$  is also derived from  $G$ . By (21),  $G$  is of the form (20). The uniqueness of  $G$  comes from Corollary 3.4.  $\square$

**Corollary 4.9.** *Let  $X$  be a totally ordered set and  $n \geq 2$  an integer. An associative idempotent monotone function  $F: X^n \rightarrow X$  is reducible if and only if  $F$  is nondecreasing.*

*Proof.* ( $\Leftarrow$ ): This immediately follows from [8, Corollary 3.12] which states that if  $F: X^n \rightarrow X$  ( $n \geq 2$ ) is associative idempotent monotone (at least in the first and the last variables) and reducible, then  $F$  is nondecreasing (in each of its variables).

( $\Rightarrow$ ): By Theorem 4.8, every associative idempotent nondecreasing  $n$ -ary function ( $n \geq 2$ ) is reducible.  $\square$

**Example 4.10.** Let  $(X, +)$  be a totally ordered Abelian group and let  $G: X \rightarrow X$  be a monotone bijective function on  $X$ . Then the function

$$F(x, y, z) = g^{-1}(g(x) - g(y) + g(z))$$

is idempotent associative monotone but nondecreasing. Thus  $F$  is not reducible.

## 5 Further remarks

### 5.1 Extremality

**Definition 5.1.** We say that  $F: X^n \rightarrow X$  is *extremal*<sup>3</sup> if there exists a  $G: X^2 \rightarrow X$  such that for every  $x_1, \dots, x_n \in X$  we have that  $F(x_1, \dots, x_n)$  equals to either  $G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i)$  or  $G(\vee_{i=1}^n x_i, \wedge_{i=1}^n x_i)$ . In particular, if  $F: X^n \rightarrow X$  is symmetric and extremal, then there exists a symmetric  $G: X^2 \rightarrow X$  such that  $F(x_1, \dots, x_n) = G(\wedge_{i=1}^n x_i, \vee_{i=1}^n x_i)$ .

In [4] it was shown (as we have already stated in equation (6)) that if  $F: X^n \rightarrow X$  is associative, quasitrivial, symmetric and nondecreasing defined on the chain  $X$  then  $F$  is extremal. As a possible generalization it was shown in Theorem 4.3 that instead of quasitriviality it is enough to assume idempotency (see also Remark 2). Namely:

<sup>3</sup>In [10] a mean  $\mu: (\bigcup_{n \in \mathbb{N}} \mathbb{R}^n) \rightarrow \mathbb{R}$  was called *extremal* if for all elements  $a_1, a_2, \dots, a_n \in \mathbb{R}$  with  $a_1 \leq a_2 \leq \dots \leq a_n$ , we have  $\mu(a_1, a_2, \dots, a_n) = \mu(a_1, a_n)$ .

**Proposition 5.2.** *Let  $X$  be a totally ordered set. Then every associative symmetric nondecreasing idempotent function  $F: X^n \rightarrow X$  is extremal.*

In [8, Theorem 2.6.], it was shown that every associative nondecreasing idempotent function having a neutral element is extremal.

If  $F: X^n \rightarrow X$  is associative quasitrivial and nondecreasing, then  $F$  is not necessarily extremal. It can be shown easily that the projection to the  $i$ -th coordinate is not extremal for all  $i = 1, \dots, n$ . If  $i = 1$  or  $i = n - 1$ , then this gives an example of associative idempotent nondecreasing function, which is not extremal.

## 5.2 Monotonicity

Although in the binary case it cannot happen, Example 4.10 shows that there exists an associative idempotent monotone function, which is not nondecreasing (so it is not reducible by Corollary 4.9). The characterization of these functions are not known yet. We conjecture the following (in the spirit of Aczélian  $n$ -ary semigroups [2]):

**Conjecture 5.3.** Let  $(X, +)$  be a totally ordered Abelian group. An associative idempotent strictly<sup>4</sup> monotone function  $F: X^n \rightarrow X$  is not reducible if and only if  $n$  is odd and there exists a monotone bijection  $G: X \rightarrow X$  such that

$$F(x_1, x_2, \dots, x_n) = g^{-1}\left(\sum_{i=1}^n (-1)^i g(x_i)\right). \quad (24)$$

The ‘if’ part of the statement is clear. We note that if Conjecture 5.3 holds for  $X = \mathbb{R}$ , then such an  $F$  must be automatically continuous, since every monotone bijection on an interval is continuous.

**Acknowledgement.** The authors would like to thank the referee for the valuable comments and suggestions which improved the quality of this paper.

## References

- [1] N. L. Ackerman, A characterization of quasitrivial  $n$ -semigroups, to appear.
- [2] M. Couceiro, J.-L. Marichal, Aczélian  $n$ -ary semigroups, *Semigroup Forum*, **85**: 81–90, 2012.

---

<sup>4</sup>A monotone function is *strictly monotone* if every inequality in the definition of monotonicity (see equation (2)) is strict.

- [3] E. Czogała, J. Drewniak, Associative monotonic operations in fuzzy set theory. *Fuzzy Sets and Systems*, **12**(3): 249–269, 1984.
- [4] J. Devillet, G. Kiss, J.-L. Marichal, Characterizations of quasitrivial symmetric nondecreasing associative operations, arXiv:1705.00719.
- [5] W. Dörnte, Untersuchungen über einen verallgemeinerten Gruppenbegriff. *Math. Z.*, **29**: 1–19, 1928.
- [6] W. A. Dudek, V. V. Mukhin, On topological  $n$ -ary semigroups. *Quasigroups and Related Systems*, **3**: 373–388, 1996.
- [7] W. A. Dudek, V. V. Mukhin, On  $n$ -ary semigroups with adjoint neutral element. *Quasigroups and Related Systems*, **14**: 163–168, 2006.
- [8] G. Kiss, G. Somlai, A characterization of  $n$ -associative, monotone, idempotent functions on an interval that have neutral elements, *Semigroup Forum*, **3**: 438–451, 2018.
- [9] E. L. Post, Polyadic groups, *Trans. Amer. Math. Soc.*, **48**: 208–350, 1940.
- [10] C. D. Bennett, W. C. Holland, G. J. Székely, Integer valued means, *Aequat. Math.*, **88** 137–149, 2014.