Prioritized Norms in Formal Argumentation

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Abstract. To resolve conflicts among norms, various nonmonotonic formalisms can be used to perform prioritized normative reasoning. Meanwhile, formal argumentation provides a way to represent nonmonotonic logics. In this paper, we propose a representation of prioritized normative reasoning by argumentation. Using hierarchical abstract normative systems, we define three kinds of prioritized normative reasoning approaches, called *Greedy*, *Reduction*, and *Optimization*. Then, after formulating an argumentation theory for a hierarchical abstract normative system, we show that for a totally ordered hierarchical abstract normative system, Greedy and Reduction can be represented in argumentation by applying the weakest link and the last link principles respectively, and Optimization can be represented by introducing additional defeats capturing implicit conflicts between arguments.

1 Introduction

Since the work of Alchourrón and Makinson [1] on hierarchies of regulations and their logic, in which a partial ordering on a code of laws or regulations is used to overcome logical imperfections in the code itself, reasoning with prioritized norms has been a central challenge in deontic logic [13,4,12].

The goal of this paper is to study the open issue of reasoning with priorities over norms through the lens of argumentation theory [10]. More precisely, we focus on reasoning with the abstract normative system proposed by Tosatto *et al.* [26], which in turn is based on Makinson and van der Torre's approach to input/output logic [20]. In this system, an abstract norm is represented by an ordered pair (a, x), where the body of the norm a is thought of as an input, representing some kind of condition or situation, and the head of the norm x is thought of as an output, representing what the norm tells us to be obligatory in that situation a. As a consequence, an abstract normative system is a directed graph (L, N) together with a context $C \subseteq L$, where L is a set of nodes, and $N \subseteq L \times L$ is the set of abstract norms. When the edge of an abstract normative system is associated with a number to indicate its priority over the other norms in the system, we obtain a hierarchical abstract normative system (HANS), which will be formally defined and studied in the remainder of this paper.

Let us clarify how a hierarchical abstract normative system is defined by considering the well known Order Puzzle [18] example from the deontic logic literature, which revolves around three norms.

> "Suppose that there is an agent, called Corporal O'Reilly, and that he is subject to the commands of three superior officers: a Captain, a Major, and a Colonel. The Captain, who does not like to be cold, issues a standing order that, during the winter, the heat should be turned on. The Major, who is concerned about energy conservation, issues an order that, during the winter, the window should not be opened. And the Colonel, who does not like to be too warm and does not care about energy conservation, issues an order that, whenever the heat is on, the window should be opened."

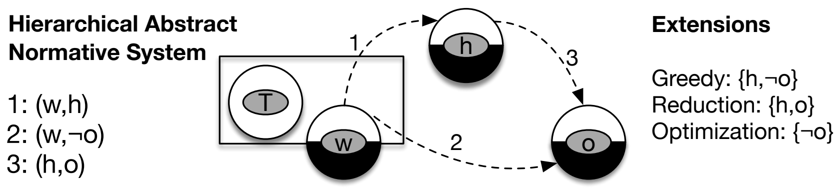

Fig. 1. The Order puzzle example, represented using the graphical notation of Tosatto *et al.* [27] with edges annotated by norm strength.

Let w, h and o respectively denote the propositions that it is winter, the heat is turned on, and the window is open. There are three norms (w, h), $(w, \neg o)$ and (h, o).

These three norms are visualized in Figure 1, extending the graphical notation described in Tosatto *et al.* [27] by associating edges with numbers denoting priorities of norms. These priorities are obtained from the rank of the issuer, since Colonels outrank Majors, and Majors outrank Captains. Within the figure, each circle denotes a proposition; the light part of the circle is the proposition itself, while the dark part denotes a negated proposition. Dashed lines represent the conditional obligations. Within Figure 1, the line from the light part of w to the dark part of o denotes $(w, \neg o)$. The box on the left represents the context, in the example containing $\top$ and w.

The central notion of inference in normative systems is called detachment. For example, in the Order Puzzle, the question is whether we can detach o, or we can detach $\neg o$, or both. In the example, the formulas which can be derived from a normative system are obligations. In general, permissions and institutional facts can also be detached from normative systems, but we do not consider these aspects in this paper. A detachment procedure defines the way deontic facts are derived from a normative system. Different detachment procedures have been defined and studied in deontic logic, as well as in other rule based systems. Moreover, even in hierarchical normative systems, not all conflicts may be resolved. In such a case, the detachment procedure may derive several so-called *extensions*, each representing a set of obligations, permissions and institutional facts.

In formalizing examples, one challenge in applied logic is that the representation may be challenged. For example, it may be argued that the Colonel implies that if the window is closed, then the heating should be turned off. However, in normative systems, such pragmatic considerations are usually not part of the detachment procedure

[17], with only explicitly given norms and commands being considered. Any additional interpretation or other pragmatic concerns are out of the scope of this paper.

Abstract normative systems [26] were introduced as a common core representation for normative systems, which is still expressive enough to define the main detachment procedures. In particular, analogous to the main input/output logics, they have factual detachment built in, and have reasoning by cases, deontic detachment and identity as optional inference patterns [3,17]. Such systems are called 'abstract', because negation is the only logical connective that is defined in the language. Furthermore, Tosatto *et al*. [26] considered elements and anti-elements rather than literals and propositions. It is straightforward to define more connectives within such systems, and it is also possible to define structured normative systems where the abstract elements are instantiated with logical formulas, for example with formulas of a propositional or modal logic. The latter more interesting representation of logical structure is analogous to the use of abstract arguments in formal argumentation. An advantage of abstract normative systems over structured ones is that the central inference of detachment can be visualized by walking paths in the graph. In other words, inference is represented by graph reachability. For example, node o is reachable from the context, and thus it can be detached. Moreover, a conflict is represented by a node where both the light and the dark side are reachable from the context, like node o in Figure 1.

There are several optional inference patterns for abstract normative systems, because, as is well known, most principles of deontic logic have been criticized for some examples and applications. However, the absence of the same inference patterns is criticized as well due to lack of explanations and predictions of the resulting detachment procedures. Therefore, current approaches to represent and reason with normative systems, such as input/output logic as well as abstract normative systems, do not restrict themselves to a single logic, but define a family of logics which can be embedded within them. Deciding which logic to use in a specific context depends on the requirements of the application. Similarly, with regards to permissions, there is an even larger diversity of deontic logics [16] which adopt different representations. For each input/output logic, various notions of permission have been defined, in terms of their relation to obligation. We refer to the handbook of deontic logic and normative systems [12] for further explanation, discussion and motivation.

Now let us consider how variants of detachment procedures might apply norms in hierarchical normative systems in different orders, and result in different outcomes or extensions. We examine three approaches describing well known procedures defined in the literature, such as procedures defined in artificial intelligence [30,5,15]. However, our procedures have one important distinction: the context itself is not necessarily part of the output. It is precisely this feature which distinguishes input/output logics from traditional rule based languages like logic programming or default logic [24,6]. Such traditional rule based languages where the input is part of the output, are called throughput operators in input/output logic research.

Greedy: The context contains propositions that are known to hold. This procedure always applies the norm with the highest priority that does not introduce inconsistency to an extension and the context. Here we say that a norm is applicable when its body is in the context or has been produced by other norms and added to the extension. In this example, we begin with the context $\{w\}$, and $(w, \neg o)$ is first applied. Then (w, h) is applied. Finally, (h, o) cannot be applied as this would result in a conflict, and so, by using Greedy, we obtain the extension $\{h, \neg o\}$.

Reduction: in this approach, a candidate extension is guessed. All norms that are applicable according to this candidate extension are selected and transformed into unconditional or body-free norms. For example, a norm (a, b) selected in this way is transformed to a norm $(\top, b)$. The modified hierarchical abstract normative system, with the transformed norms is evaluated using Greedy. The candidate extension is selected as an extension by Reduction if it is identified as an extension according to this application of Greedy. Applied to our example, we select a candidate extension $\{h, o\}$, obtaining a set of body-free norms $\{(\top, h), (\top, \neg o), (\top, o)\}$. The priorities assigned to these norms are carried through from the original hierarchical abstract normative system, and are therefore respectively 1, 2 and 3. After applying Greedy, we get an extension of Reduction: $\{h, o\}$. However, if we had selected the candidate extension $\{h, \neg o\}$, this new extension would not appear in Greedy as $(\top, \neg o)$ has a lower priority than $(\top, o)$, and the latter is therefore not an extension of Reduction.

Optimization: In terms of Hansen's prioritized conditional imperatives, a set of maximally *obeyable* (i.e., minimally violated) norms is selected by choosing norms in order of priority which are consistent with the context. Once these norms are selected, Greedy is applied to identify the extension. In our example, the maximal set of obeyable norms is $\{(h, o), (w, \neg o)\}$. Optimization therefore detaches the unique extension $\{\neg o\}$.

We can also consider the example in terms of formal argumentation. Given a hierarchical abstract normative system, we may construct an argumentation framework as illustrated in Figure 2. An argumentation framework is a directed graph in which nodes denote arguments, and edges denote attacks between arguments. In the setting of a hierarchical abstract normative system, an argument is represented as a path within the directed graph starting from a node in the context. In this example, there are four arguments A_0, A_1, A_2 and A_3, represented as $[w], [w, h], [w, h, o]$ and $[w, \neg o]$, respectively. Since the conclusions of A_2 and A_3 are inconsistent, A_2 attacks A_3 and vice versa. Priorities allow us to transform these attacks into *defeats* according to different principles. While the *last link* principle ranks an argument based on the strength of its last inference, the *weakest link* ranks an argument based on the strength of its weakest inference. In this example, if the last link principle is applied, then $[w, h, o]$ defeats $[w, \neg o]$. Furthermore, if the weakest link principle is used instead, $[w, \neg o]$ defeats $[w, h, o]$. As a result, the former principle allows us to conclude $\{h, o\}$, while the latter concludes $\{h, \neg o\}$. In turn, the first result coincides with that obtained by Reduction, while the second is the same as that obtained by Greedy.

Inspired by the example above, we wish to investigate the links between the three detachment procedures for prioritized normative reasoning and argumentation theory. More specifically, our main research question is as follows.

> *How can the three detachment procedures (Reduction, Greedy, and Optimization) proposed in the context of abstract normative reasoning be represented in formal argumentation?*

To answer this research question, we propose a formal framework to connect hierarchical normative reasoning with argumentation theory. More precisely, our framework

Fig. 2. The argumentation framework obtained from the Order puzzle hierarchical normative system, with the four arguments and the attacks among them visualized as directed arrows.

represents the above-mentioned detachment procedures by lifting priorities from rules to arguments, with the underlying goal of making as few commitments as possible to specific argumentation systems. For this reason, we build on a structured argumentation framework which admits undercuts and rebuts between arguments, and allows for priorities between norms making up arguments. We show that variants of approaches to lifting priorities from rules to arguments allow us to capture both Greedy and Reduction, while the introduction of additional defeats capturing implicit conflicts between arguments allows us to obtain Optimization.

The layout of the paper is as follows. Section 2 formalizes the above-mentioned three detachment procedures of hierarchical normative reasoning (i.e., Greedy, Reduction, and Optimization). In Section 3, we introduce an argumentation theory for a hierarchical abstract normative system. Sections 4, 5 and 6 show how Greedy, Reduction and Optimization can be represented in argumentation. Finally, in Section 7 we discuss open problems and compare the proposed approach with related work, and in Section 8 we point out possible directions for future work.

2 Hierarchical abstract normative systems

In this section, we formally introduce the notion of hierarchical abstract normative system and three detachment procedures to compute what normative conclusions hold. A hierarchical abstract normative system captures the context of a system and the norms in force in such a system. There is an element in the universe called $\top$, contained in every context. In this paper, we consider only a finite universe. A hierarchical abstract normative system also encodes a ranking function over the norms to allow for the resolution of conflicts.

Based on the notion of *abstract normative system* defined by Tosatto *et al.* [26], a hierarchical abstract normative system can be defined as follows.

Definition 1 (Hierarchical abstract normative system). *A hierarchical abstract normative system is a tuple* $\mathcal{H} = \langle L, N, C, r \rangle$*, where*

- $L = \mathrm{E} \cup \{\neg e \mid e \in \mathrm{E}\} \cup \{\top\}$ *is the universe, a set of literals based on some finite set* E *of atomic elements;*
- $C \subseteq L$ *is a subset of the universe, called a context, such that* $\top \in C$ *and for all* e *in* E, $\{e, \neg e\} \not\subseteq C$;
- $N \subseteq L \times L$ *is a finite set of regulative norms;*
- $r : N \to \mathbb{N}$ *is a function from norms to natural numbers.*

Regulative (ordinary) norms are of the kind "if you turn on the heat, then you should open the window". These norms are *conditional norms*, requiring some condition to hold (e.g., turning on the heat) before their conclusion can be drawn.

We write (a, x) for a regulative norm, where $a, x \in L$ are the antecedent and conclusion of the norm, respectively. Given (a, x), we use $r(a, x)$ to denote $r((a, x))$. Let $u, v \in N$ be two norms, we say that v is at least as preferred as u (denoted $u \leq v$) if and only if $r(u)$ is not larger than $r(v)$ (denoted $r(u) \leq r(v)$), where $r(u)$ is also called the rank of u. We write $u < v$ or $v > u$ if and only if $u \leq v$ and $v \not\leq u$. Given a norm $u = (a, x)$ or $\langle a, x\rangle$, we write *ant*(u) for a to represent the antecedent of the norm, and *cons*(u) for x to represent the consequent of the norm. Given a set of norms $S \subseteq N$, we use *cons*(S) to denote $\{cons(u) \mid u \in S\}$. We say that a hierarchical abstract normative system is totally ordered if and only if the ordering $\leq$ over N is antisymmetric, transitive and total. Due to the finiteness of universe, the set of norms is finite. Note that given this assumption, the notion of total ordering here is identical to that of the full prioritization in Brewka and Eiter's [7] and Hansen's [14] work, and of the linearized ordering of Young *et al*. [30]. For $a \in L$, we write $\overline{a} = \neg a$ if and only if $a \in \mathrm{E}$, and $\overline{a} = e$ for $e \in \mathrm{E}$ if and only if $a = \neg e$. For a set $S \subset L$, we say that a set $S \subseteq L$ is consistent if and only if there exist no $e_1, e_2 \in S$ such that $e_1 = \overline{e_2}$.

Example 1 (Order puzzle). In terms of Definition 1, the set of norms and priorities that are visualized in Figure 1 can be formally represented as a hierarchical abstract normative system $\mathcal{H}_1 = \langle L, N, C, r\rangle$, where $L = \{w, h, o, \neg w, \neg h, \neg o, \top\}$, $N = \{(w, h), (h, o), (w, \neg o)\}$, $C = \{w, \top\}$, $r(w, h) = 1$, $r(h, o) = 3$, $r(w, \neg o) = 2$.

In the hierarchical abstract normative system setting, the three detachment procedures for prioritized normative reasoning can be defined as follows.

First, Greedy detachment for a hierarchical abstract normative system always applies the norm with the highest priority among those which can be applied, if this does not bring inconsistency to the extension and the context. So, we have the following definitions.

Definition 2 (Paths). *Let $\mathcal{H} = \langle L, N, C, r\rangle$ be a hierarchical abstract normative system. For all $R \subseteq N$, a path in $\mathcal{H}$ from x_1 to x_n with respect to R is a sequence of norms $(x_1, x_2), \ldots (x_{n-1}, x_n)$ such that $\{(x_1, x_2), \ldots, (x_{n-1}, x_n)\} \subseteq R$, $n \geq 2$, and all norms of the sequence are distinct.*

Definition 3 (Greedy). *Let $\mathcal{H} = \langle L, N, C, r\rangle$ be a totally ordered hierarchical abstract normative system. For all $R \subseteq N$, let $R(C) = \{x \mid$ there is a path in $\mathcal{H}$ from an element in C to x with respect to $R\}$, and $Appl(N, C, R) := \{(a, x) \in N \mid a \in C \cup cons(R), x, \overline{x} \notin C \cup cons(R)\}$. The extension of $\mathcal{H}$ by Greedy, written as Greedy$(\mathcal{H})$, is the set $R(C)$ such that $R = \cup_{i=0}^{\infty} R_i$ is built inductively as:*

$$R_0 = \emptyset$$
$$R_{i+1} = R_i \cup \max(N, C, R_i, r)$$

where $\max(N, C, R_i, r) = \{u \in Appl(N, C, R_i) \mid \forall v \in Appl(N, C, R_i) : r(u) \geq r(v)\}$.

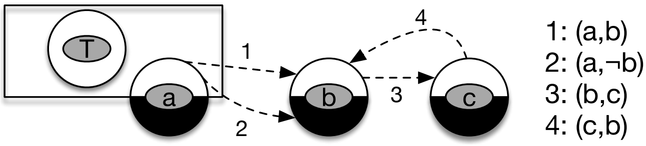

Fig. 3. The hierarchical abstract normative system of Example 3 containing the two Reduction extensions $\{b, c\}$ and $\{\neg b\}$.

Example 2 (Extensions by Greedy). Given $\mathcal{H}_1$ in Example 1, by Greedy, it holds that $R_0 = \emptyset$, $R_1 = \{(w, \neg o)\}$, $R_2 = \{(w, \neg o), (w, h)\}$, $R = \{(w, \neg o), (w, h)\}$. So, $Greedy(\mathcal{H}_1) = \{h, \neg o\}$.

Based on Greedy, Reduction and Optimization are defined as follows.

Definition 4 (Reduction). *Given a totally ordered hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r\rangle$, and a set X, let $\mathcal{H}^X = \langle L, N', C, r'\rangle$, where $N' = max\{r(l_1, l_2) \mid (l_1, l_2) \in N; l_1 \in C \cup X\}$ and $r'(\top, l_2) = r(l_1, l_2)$ for all $(l_1, l_2) \in N$ are priorities over norms. An extension of $\mathcal{H}$ by Reduction is a set U such that U is Greedy$(\mathcal{H}^U)$. The set of extensions of $\mathcal{H}$ by Reduction is denoted as Reduction$(\mathcal{H})$.*

Example 3 (Extensions by Reduction). Consider again $\mathcal{H}_1$ in Example 1. By using Reduction, given $X = \{h, o\}$, we have $\mathcal{H}_1^X = \langle L, N', C, r'\rangle$, where $N' = \{(\top, h), (\top, o), (\top, \neg o)\}$, $r'(\top, h) = 1$, $r'(\top, o) = 3$ and $r'(\top, \neg o) = 2$. Since $X \in$ *Greedy*$(\mathcal{H}_1^X)$, and no other set can be an extension, we have that *Reduction*$(\mathcal{H}_1) = \{\{h, o\}\}$.

Given the hierarchical abstract normative system in Figure 3, assume that we have a context $C = \{a\}$. We then consider $X_1 = \{b, c\}$ and $X_2 = \{\neg b\}$. In the first case, we have $\mathcal{H}^{X_1} = \langle L, N', C, r'\rangle$ where $N' = \{(\top, b), (\top, \neg b), (\top, c)\}$ and $r'(\top, b) = 4$, $r'(\top, c) = 3$, $r'(\top, \neg b) = 2$. Here, *Greedy*$(\mathcal{H}^{X_1}) = \{b, c\}$ i.e., X_1.

In the second case, we obtain $\mathcal{H}^{X_2} = \langle L, N', C, r'\rangle$ where $N' = \{(\top, b), (\top, \neg b)\}$ and $r'(\top, b) = 1$, $r'(\top, \neg b) = 2$. Now, *Greedy*$(\mathcal{H}^{X_2}) = \{\neg b\}$, i.e., X_2. In this case, we therefore obtain two extensions using Reduction.

Definition 5 (Optimization). *Given a totally ordered hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r\rangle$, let $T = \{u_1, u_2, \ldots, u_n\}$ be the linear order on N such that $u_1 > u_2 > \cdots > u_n$. We define a set R as $R = R_n$ such that:*

$$R_0 = \emptyset$$
$$R_{i+1} = \begin{cases} R_i \cup \{u_i\}, & \text{if } cons(C \cup R_i \cup \{u_i\}) \text{ is consistent} \\ R_i, & \text{else} \end{cases}$$

An extension by Optimization is a set $O = R(C)$. The set of extensions of $\mathcal{H}$ by Optimization is denoted as Optimization$(\mathcal{H})$.

Example 4 (Extensions by Optimization). Regarding $\mathcal{H}_1$ in Example 1, by Optimization, let $u_1 = (h, o)$, $u_2 = (w, \neg o)$, and $u_3 = (w, h)$, and $T = \{u_1, u_2, u_3\}$. Then, it holds that $R_0 = \emptyset$, $R_1 = \{u_1\}$, $R_2 = \{u_1, u_2\}$, and $R = R_3 = R_2 = \{u_1, u_2\}$. So, we obtain that *Optimization*$(\mathcal{H}_1) = \{\{\neg o\}\}$.

3 Argumentation theory for a hierarchical abstract normative system

In this section, we introduce an argumentation theory on prioritized norms. Given a hierarchical abstract normative system, we first define arguments and defeats between them, then compute extensions of arguments in terms of Dung's theory [10], and from these, obtain conclusions.

3.1 Arguments

In a hierarchical abstract normative system, an argument is an acyclic path in the graph starting in an element of the context. We assume minimal arguments—no norm can be applied twice in an argument and no redundant norm is included in an argument. We use $concl(\alpha)$ to denote the conclusion of an argument α, and $concl(E) = \{concl(\alpha) \mid \alpha \in E\}$ for the conclusions of a set of arguments E.

Definition 6 (Arguments and sub-arguments). *Let $\mathcal{H} = \langle L, N, C, r\rangle$ be a hierarchical abstract normative system.*

A context argument *in $\mathcal{H}$ is an element $a \in C$, and its conclusion is $concl(a) = a$.*
An ordinary argument *is a path α in $\mathcal{H}$ from u_1 to u_n, $n \geq 1$, such that:*

1. *$ant(u_1) \in C$; and*
2. *$\{ant(u_1), \ldots, ant(u_n), cons(u_n)\}$ is consistent.*

Moreover, we have that $concl(\alpha) = cons(u_n)$.
The sub-arguments *of argument $[u_1, \ldots, u_n]$ are, for $1 \leq i \leq n$, $[u_1, \ldots, u_i]$. Note that context arguments do not have sub-arguments.*

The set of all arguments constructed from $\mathcal{H}$ is denoted as $Arg(\mathcal{H})$. For readability, $[(a_1, a_2), \ldots, (a_{n-1}, a_n)]$ may be written as $(a_1, a_2, \ldots, a_{n-1}, a_n)$. The set of sub-arguments of an argument α is denoted as $sub(\alpha)$.

3.2 Defeat relation between arguments

We follow the tradition in much of preference-based argumentation [2,21], where *attack* captures a relation among arguments which ignores preferences, and *defeat* is a preference-aware relation on which the semantics is based. To define the defeat relation among prioritized arguments, we assume that *only* the priorities of the norms are used to compare arguments. In other words, we assume a lifting of the ordering on norms to a binary relation on sequences of norms (i.e., arguments), written as $\alpha \succeq \beta$, where α and β are two arguments, indicating that α is at least as preferred as β.

There is no common agreement about the best way to lift $\geq$ to $\succeq$. In argumentation, there are at least two ways to introduce weights. As an argumentation framework consists of a set of arguments and an attack relation between them, we can either assign the weights to the arguments, or we can assign the weights to the attacks. Traditionally, weights are assigned to arguments. Two common approaches to give the strength of an argument are the *weakest link* and the *last link* principles, combined with the elitist and democratic ordering [21]. For example, in the weakest link principle the weight of the argument is the weight of the weakest rule used in the argument. However, Young *et al.* [30] showed that elitist weakest link cannot be used to calculate $\succeq$ for Greedy, and proposes a *disjoint elitist order* which ignores shared rules. It is worth noticing that the strength of an argument may depend on the argument it is attacking, as identified by Young *et al*. [30]. Based on this idea, we define the orderings between arguments by assigning a strength to the attacks between the arguments, to reflect the priority of the norms used in the arguments, following the same insights of the weakest link and last link principles (denoted as $\succeq_w$ and $\succeq_l$ respectively). By taking the way of defining the weakest link ordering from [30], we have the following definition:

Definition 7 (Weakest link and last link). *Let $\mathcal{H} = \langle L, N, C, r\rangle$ be a hierarchical abstract normative system, and $\alpha = [u_1, \ldots, u_n]$ and $\beta = [v_1, \ldots, v_m]$ be two arguments in Arg($\mathcal{H}$). Let $\Phi_1 = \{u_1, \ldots, u_n\}$ and $\Phi_2 = \{v_1, \ldots, v_m\}$. By the weakest link principle, $\alpha \succeq_w \beta$ iff $\exists v \in \Phi_2 \setminus \Phi_1$ s.t. $\forall u \in \Phi_1 \setminus \Phi_2, v \leq u$. By the last link principle, $\alpha \succeq_l \beta$ iff $u_n \geq v_m$.*

When the context is clear, we write $\succeq$ for $\succeq_w$, or $\succeq_l$. We write $\alpha \succ \beta$ for $\alpha \succeq \beta$ without $\beta \succeq \alpha$.

Proposition 1 (Transitivity). *It holds that the relations $\succeq_w$ and $\succeq_l$ are transitive.*

Proof. Let $\alpha = [u_1, \ldots, u_n]$, $\beta = [v_1, \ldots, v_m]$ and $\gamma = [w_1, \ldots, w_k]$ be three arguments in *Arg*($\mathcal{H}$). For the case of the weakest link, let $\Phi_1 = \{u_1, \ldots, u_n\}$, $\Phi_2 = \{v_1, \ldots, v_m\}$, and $\Phi_3 = \{w_1, \ldots, w_k\}, n, m, k \geq 1$. Let $x_{12} \in \Phi_1 \cap \Phi_2, x_{13} \in \Phi_1 \cap \Phi_3$, and $x_{23} \in \Phi_2 \cup \Phi_3$, $x_1 \in \Phi_1 \setminus (\Phi_2 \cup \Phi_3)$, $x_2 \in \Phi_2 \setminus (\Phi_1 \cup \Phi_3)$, and $x_3 \in \Phi_3 \setminus (\Phi_1 \cup \Phi_2)$. Assume that $\alpha \succeq_w \beta$ and $\beta \succeq_w \gamma$. There are only the following four possible cases.

Case 1: There exists $x_{23} \in \Phi_2$, for all $x_{13}, x_1 \in \Phi_1$, $x_{23} \leq x_{13}$ and $x_{23} \leq x_1$; and since $x_{13} \in \Phi_3$, assume that for all $x_{12}, x_2 \in \Phi_2$, $x_{13} \leq x_{12}$, $x_{13} \leq x_2$. It follows that $x_{23} \leq x_{12}$ and $x_{23} \leq x_1$. Since $x_{23} \in \Phi_3$, it means that there exists $x_{23} \in \Phi_3$ such that $x_{23} \leq x_{12}$ and $x_{23} \leq x_1$ where $x_{12}, x_1 \in \Phi_1$. Hence, $\alpha \succeq_w \gamma$.

Case 2: There exist $x_{23} \in \Phi_2$ and $x_3 \in \Phi_3$, for all $x_{13}, x_1 \in \Phi_1$, and $x_{12}, x_2 \in \Phi_2$: $x_{23} \leq x_{13}$, $x_{23} \leq x_1$, $x_3 \leq x_{12}$, $x_3 \leq x_2$. In this case, there are in turn only the following two possible sub-cases: either $x_{23} \leq x_3$ or $x_{23} > x_3$. If $x_{23} \leq x_3$, since $x_3 \leq x_{12}$, it holds that $x_{23} \leq x_{12}$. Since $x_{23} \leq x_1$ and $x_{23} \leq x_{12}$, it holds that $\alpha \succeq_w \gamma$. Second, if $x_{23} > x_3$, since $x_{23} \leq x_1$, $x_3 \leq x_1$. Since $x_3 \leq x_{12}$ and $x_3 \leq x_1$, $\alpha \succeq_w \gamma$.

Case 3: There exists $x_2 \in \Phi_2$, for all $x_{13}, x_1 \in \Phi_1$, $x_2 \leq x_{13}$ and $x_2 \leq x_1$; and since $x_{13} \in \Phi_3$, assume that for all $x_{12}, x_2 \in \Phi_2$, $x_{13} \leq x_{12}$, $x_{13} \leq x_2$. In this case, there are in turn only the following two possible sub-cases: $x_{23} \leq x_{13}$ or $x_3 \leq x_{13}$, or $x_{23} > x_{13}$ and $x_3 > x_{13}$. If $x_{23} \leq x_{13}$ and $x_3 \leq x_{13}$, it holds that $\alpha \succeq_w \gamma$. If $x_{23} > x_{13}$ and $x_3 > x_{13}$, it holds that $x_{13} < x_{23}$ and $x_{13} \leq x_2$, and therefore $\beta \succeq_w \alpha$. Contradiction.

Case 4: There exist $x_2 \in \Phi_2$ and $x_3 \in \Phi_3$, for all $x_{13}, x_1 \in \Phi_1$, and $x_{12}, x_2 \in \Phi_2$: $x_2 \leq x_{13}$, $x_2 \leq x_1$, $x_3 \leq x_{12}$, $x_3 \leq x_2$. In this case, since $x_3 \leq x_{12}$ and $x_3 \leq x_1$, it holds that $\alpha \succeq_w \gamma$.

For last link, it is obvious that $\succeq_l$ is transitive.

Given a way to lift the ordering on norms to an ordering on arguments, the notion of defeat can be defined as follows.

Definition 8 (Defeat among arguments). *Let $\mathcal{H} = \langle L, N, C, r\rangle$ be a hierarchical abstract normative system. For all $\alpha, \beta \in Arg(\mathcal{H})$,*

α **attacks** β *iff β has a sub-argument β' such that*
1. $concl(\alpha) = \overline{concl(\beta')}$

α **defeats** β *iff β has a sub-argument β' such that*
1. $concl(\alpha) = \overline{concl(\beta')}$ *and*
2. *α is a context argument; or α is an ordinary argument and $\beta' \not\succ \alpha$.*

The set of defeats between the arguments in $Arg(\mathcal{H})$ based on a preference ordering $\succeq$ is denoted as $Def(\mathcal{H}, \succeq)$.

In what follows, an argument $\alpha = [u_1, \ldots, u_n]$ with ranking on norms is denoted as $u_1 \ldots u_n : r(\alpha)$, where $r(\alpha) = (r(u_1), \ldots, r(u_n))$.

Example 5 (Order puzzle continued). Consider the hierarchical abstract normative system in Example 1. We have the following arguments (visually presented in Figure 2):

$A_0 : w$	(context argument)
$A_1 : (w, h) : (1)$	(ordinary argument)
$A_2 : (w, h)(h, o) : (1, 3)$	(ordinary argument)
$A_3 : (w, \neg o) : (2)$	(ordinary argument)

We have that A_2 attacks A_3 and vice versa, and there are no other attacks among the arguments. Moreover, A_2 defeats A_3 if $(2) \not\succ (1, 3)$ (last link), and A_3 defeats A_2 if $(1, 3) \not\succ (2)$ (weakest link).

3.3 Argument extensions and conclusion extensions

Given a set of arguments $\mathcal{A} = Arg(\mathcal{H})$ and a set of defeats $\mathcal{R} = Def(\mathcal{H}, \succeq)$, we get an argumentation framework (AF) $\mathcal{F} = (\mathcal{A}, \mathcal{R})$.

Following the seminal work of abstract argumentation by Dung [10], we say that a set $B \subseteq \mathcal{A}$ is *admissible*, if and only if it is *conflict-free* and it can defend each argument within the set. A set $B \subseteq \mathcal{A}$ is *conflict-free* if and only if there exist no arguments α and β in B such that $(\alpha, \beta) \in \mathcal{R}$. Argument $\alpha \in \mathcal{A}$ is *defended* by a set $B \subseteq \mathcal{A}$ (in such a situation α can also be said to be *acceptable* with respect to B) if and only if for all $\beta \in \mathcal{A}$, if $(\beta, \alpha) \in \mathcal{R}$, then there exists $\gamma \in B$ such that $(\gamma, \beta) \in \mathcal{R}$. Based on the notion of admissible sets, some other extensions can be defined. Formally, we have the following.

Definition 9 (Conflict-freeness, defense and extensions). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation framework, and $B \subseteq \mathcal{A}$ a set of arguments.*

- *B is* conflict-free *if and only if $\nexists \alpha, \beta \in B$, s.t. $(\alpha, \beta) \in \mathcal{R}$.*
- *An argument $\alpha \in \mathcal{A}$ is defended by B (equivalently α is* acceptable *w.r.t. B), if and only if $\forall (\beta, \alpha) \in \mathcal{R}$, $\exists \gamma \in B$, s.t. $(\gamma, \beta) \in \mathcal{R}$.*

- *B is* admissible *if and only if B is conflict-free, and each argument in B is defended by B.*
- *B is a* complete *extension if and only if B is admissible and each argument in $\mathcal{A}$ that is defended by B is in B.*
- *B is a* preferred *extension if and only if B is a maximal (w.r.t. set-inclusion) complete extension.*
- *B is a* grounded *extension if and only if B is the minimal (w.r.t. set-inclusion) complete extension.*
- *B is a* stable *extension if and only if B is conflict-free, and $\forall \alpha \in \mathcal{A} \setminus B$, $\exists \beta \in B$ s.t. $(\beta, \alpha) \in \mathcal{R}$.*

A semantics describes the set of extensions one is wishes to obtain. We use $sem \in \{cmp, prf, grd, stb\}$ to denote the complete, preferred, grounded, and stable semantics, respectively. A set of argument extensions of $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ is denoted as $sem(\mathcal{F})$. We write *Outfamily* for the set of conclusions from the extensions of the argumentation theory, as in [28].

Definition 10 (Conclusion extensions). *Given a hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r \rangle$, let $\mathcal{F} = (Arg(\mathcal{H}), Def(\mathcal{H}, \succeq))$ be the AF constructed from $\mathcal{H}$. The conclusion extensions, written as Outfamily$(\mathcal{F}, sem)$, are the conclusions of the ordinary arguments within argument extensions.*

$$Outfamily(\mathcal{F}, sem) = \{\{concl(\alpha) \mid \alpha \in S, \alpha \text{ is an ordinary argument}\} \mid S \in sem(\mathcal{F})\}$$

Multi-extension semantics can yield different conclusions when norms may yield multiple most preferred results.

Example 6 (Order puzzle in argumentation). According to Example 5, let $\mathcal{A} = \{A_0, \ldots, A_3\}$. We have $\mathcal{F}_1 = (\mathcal{A}, \{(A_2, A_3)\})$ where $A_2 \succeq_l A_3$, and $\mathcal{F}_2 = (\mathcal{A}, \{(A_3, A_2)\})$ where $A_3 \succeq_w A_2$. For all $sem \in \{cmp, prf, grd, stb\}$, $Outfamily(\mathcal{F}_1, sem) = \{\{h, o\}\}$, and $Outfamily(\mathcal{F}_2, sem) = \{\{h, \neg o\}\}$.

We now turn our attention to the properties of the argumentation theory for a hierarchical abstract normative system.

First, according to Definition 8, we have the following proposition.

Proposition 2 (Sub-argument attack and defeat). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AF constructed from a hierarchical abstract normative system. For all $\alpha, \beta \in \mathcal{A}$, if α attacks β, then α attacks arguments that have β as a sub-argument; if α defeats β, α defeats arguments that have β as a sub-argument.*

Proof. When α attacks β, according to Definition 8, β has a sub-argument β' such that $concl(\alpha) = \overline{concl(\beta')}$. Let γ be an argument that has β as a sub-argument. It follows that β' is a sub-argument of γ. Hence, α attacks γ.

When α defeats β, according to Definition 8, β has a sub-argument β' such that $concl(\alpha) = \overline{concl(\beta')}$ and α is a context argument; or α is an ordinary argument and $\beta' \not\succ \alpha$, Since β' is a sub-argument of γ, α defeats γ.

Second, corresponding to properties of sub-argument closure and direct consistency in *ASPIC*$^+$ [21], we have the following two properties.

Proposition 3 (Closure under sub-arguments). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AF constructed from a hierarchical abstract normative system. For all sem $\in \{cmp, prf, grd, stb\}$, $\forall E \in sem(\mathcal{F})$, if an argument $\alpha \in E$, then $sub(\alpha) \subseteq E$.*

Proof. For every $\beta \in sub(\alpha)$, since α is acceptable with respect to E, it holds that β is acceptable with respect to E. This is because for each $\gamma \in \mathcal{R}$, if γ defeats β, then according to Proposition 2, γ defeats α; since $\alpha \in E$, there exists an $\eta \in E$ such that η defeats γ. Given that β is acceptable with respect to E and E is a complete extension, it holds that $\beta \in E$.

Since all norms in a hierarchical abstract normative system are defeasible, we only need to discuss direct consistency.

Proposition 4 (Direct consistency). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AF constructed from a hierarchical abstract normative system. For all sem $\in \{cmp, prf, grd, stb\}$, $\forall E \in sem(\mathcal{F})$, $\{concl(\alpha) \mid \alpha \in E, \alpha$ is an ordinary argument$\}$ is consistent.*

Proof. Assume that there exist $\alpha, \beta \in E$ such that $concl(\alpha) = \overline{concl(\beta)}$. Since both α and β are ordinary arguments, α attacks β, and β attacks α. If $\alpha \succeq \beta$ then α defeats β. Otherwise, β defeats α. In both cases, E is not conflict-free, contradicting the fact that E is a complete extension.

In the next sections, we present representation results for the Greedy, Reduction and Optimization approaches introduced in Section 2, identifying equivalences between these approaches, and an argument semantics based description of a hierarchical abstract normative system.

4 Representation results for Greedy

For a totally ordered hierarchical abstract normative system without permissive norms, we have the following proposition.

Proposition 5 (Greedy is weakest link). *Given a totally ordered hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r \rangle$ and the corresponding argumentation framework $\mathcal{F} = (Arg(\mathcal{H}), Def(\mathcal{H}, \succeq_w))$, it holds that $\mathcal{F}$ is acyclic, and $\{Greedy(\mathcal{H})\} = Outfamily(\mathcal{F}, grd)$.*

Proof. *First, since $\mathcal{H}$ is totally ordered, under $\succeq_w$, the relation $\succeq_w$ among arguments is acyclic. Assume the contrary. Then, there exist three distinct $\alpha, \beta, \gamma \in Arg(\mathcal{H})$ such that $\alpha \succeq_w \beta$, $\beta \succeq_w \gamma$ and $\gamma \succeq_w \alpha$. According to Definition 7, when $\mathcal{H}$ is totally ordered, it holds that $\alpha \succ_w \beta$, $\beta \succ_w \gamma$ and $\gamma \succ_w \alpha$. According to Proposition 1, $\alpha \succ_w \gamma$, contradicting $\gamma \succ_w \alpha$. Hence, $\mathcal{F}$ is acyclic, and therefore has a unique extension under all argumentation semantics mentioned above.*

Second, let $G = Greedy(\mathcal{H})$ be the unique extension of $\mathcal{H}$, and $E = \{(a_1, \ldots, a_n) \in Arg(\mathcal{H}) \mid \{a_1, \ldots, a_n\} \subseteq G \cup C\}$. According to Definition 3, it holds that $G = \{concl(\alpha) \mid \alpha \in E\}$. Now, we verify that E is a stable extension of $\mathcal{F}$:

(1) Since all premises and the conclusion of each argument of E are contained in $G \cup C$ which is conflict-free, it holds that E is conflict-free.

(2) $\forall \beta = (b_1, \ldots, b_m) \in Arg(\mathcal{H}) \setminus E$, $b_m \notin G$ (otherwise, if $b_m \in G$, then $\{b_1, \ldots, b_{m-1}\} \subseteq G \cup C$, and thus $\beta \in E$, contradicting $\beta \notin E$). Then there exist $\alpha = (a_1, \ldots, a_n) \in E$ and $j \in \{2, 3, \ldots, m\}$, such that $\{a_1, \ldots, a_n, b_1, \ldots, b_j\}$ is a minimal inconsistent set, and $a_n = \overline{b_j}$. Then, we have the following two possible cases:

1. *(a_{n-1}, a_n) and (b_{j-1}, b_j) are applicable at the same time: in this case, since $a_n \in G$, $r(a_{n-1}, a_n) \geq r(b_{j-1}, b_j)$. Let $\Phi_1 = \{(a_1, a_2), \ldots, (a_{n-1}, a_n)\}$ and $\Phi_2 = \{(b_1, b_2), \ldots, (b_{j-1}, b_j)\}$. It follows that $(a_1, \ldots, a_n) \succeq_w (b_1, \ldots, b_j)$. This is because it is not the case that $\exists u \in \Phi_1 \setminus \Phi_2$ such that $\forall v \in \Phi_2 \setminus \Phi_1$, $v > u$. Otherwise, u is not applicable until all norms in Φ_2 have been applied. As a result, (a_{n-1}, a_n) is not applicable before (b_{j-1}, b_j) has been applied, contradicting the assumption that (a_{n-1}, a_n) and (b_{j-1}, b_j) are applicable at the same time. Since $(a_1, \ldots, a_n) \succeq_w (b_1, \ldots, b_j)$, it holds that $(a_1, \ldots, a_n) \succeq_w (b_1, \ldots, b_j, b_{j+1}, b_m)$. So, β is defeated by α.*
2. *(a_{n-1}, a_n) is applicable and (b_{j-1}, b_j) is not applicable. In this case, there are two possibilities:*
   - *$(a_1, \ldots, a_n) \succeq_w (b_1, \ldots, b_j)$: β is defeated by α.*
   - *$(b_1, \ldots, b_j) \succ_w (a_1, \ldots, a_n)$: in this case, $\exists \gamma = (c_1, \ldots, c_k) \in E$ s.t.: $c_k = \overline{b_i}$, $(c_1, \ldots, c_k) \succeq_w (b_1, \ldots, b_i)$, $2 \leq i < j$. Then, β is defeated by γ.*

Since E is conflict-free and for all $\beta \in Arg(\mathcal{H}) \setminus E$, β is defeated by an argument in E, E is a stable extension. Finally, since $\mathcal{F}$ is acyclic, the stable and grounded extensions are equivalent, and therefore $\{Greedy(\mathcal{H})\} = Outfamily(\mathcal{F}, grd)$.

Note that Proposition 5 corresponds to Theorem 5.3 of Young *et al*. 2016 [30]. This correspondence arises as follows. First, in the argumentation theory for a hierarchical abstract normative system, we use disjoint elitist order to compare sets of norms, while in the argumentation theory for prioritized default logic, Young *et al.* use a new order called a structure-preference order, which takes into account the structure of how arguments are constructed. Since in the setting of hierarchical abstract normative systems, arguments are acyclic paths, it is not necessary to use the structure-preference order to compare arguments. Second, due to the different ways of constructing argumentation frameworks, the proof of Proposition 5 differs from that of Theorem 5.3 of Young *et al*. 2016 [30]. The former considers the order of the applicability of norms in the proof, while the latter uses the mechanism defined in the structure-preference order.

5 Representation result for Reduction

According to Brewka and Eiter [7], Reduction is based on the following two points.

1) *The application of a rule with nonmonotonic assumptions means jumping to a conclusion. This conclusion is yet another assumption which has to be used globally in the program for the issue of deciding whether a rule is applicable or not.*

2) *The rules must be applied in an order compatible with the priority information.*

This *global* view of deciding whether a rule is applicable coincides with the last-link principle of lifting a preference relation between rules to a priority relation between resulting arguments. According to Definition 4 and the argumentation theory for a hierarchical abstract normative system, we have the following representation result.

Proposition 6 (Reduction is last link). *Given a totally ordered hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r\rangle$ and the corresponding argumentation framework $\mathcal{F} = (Arg(\mathcal{H}), Def(\mathcal{H}, \succeq_l))$, it holds that $Reduction(\mathcal{H}) = Outfamily(\mathcal{F}, stb)$.*

Proof. ($\Rightarrow$:) *Given every $H \in Reduction(\mathcal{H})$, let $E = \{(a_1, \ldots, a_n) \in Arg(\mathcal{H}) \mid \{a_1, \ldots, a_n\} \subseteq H \cup C\}$. According to Reduction, $H = \{concl(\alpha) \mid \alpha \in E, \alpha$ is an ordinary argument$\}$, because $\forall a \in H$, there exists at least one argument $(a_1, \ldots, a_n)$ s.t. $a_n = a$ and $\{a_1, \ldots, a_{n-1}\} \subseteq H \cup C$, which is in turn because if $a_n \in H$, then (a_{n-1}, a_n) is applicable w.r.t. $H \cup C$, and hence $a_{n-1} \in H \cup C$; recursively, we have $a_i \in H \cup C$ for all $i \in \{1, \ldots, n-1\}$.*

Let $(Args_0, Defeats_0)$ be an argumentation framework, in which $Args_0 = \{\alpha \in Arg(\mathcal{H}) \mid sub(\alpha) \setminus \{\alpha\} \subseteq E\}$, $Defeats_0 \subseteq Args_0 \times Args_0$ that is constructed in terms of the last link principle. It holds that $Defeats_0 \subseteq Def(\mathcal{H}, \succeq_l)$. For all $\alpha \in Args_0 \setminus E$, $concl(\alpha) \notin H$. Then, $\exists \beta \in E$ s.t. $concl(\alpha) = \overline{concl(\beta)}$ and β defeats α by using the last link principle. It follows that E is a stable extension of $(Args_0, Defeats_0)$. Now, let us prove that E is a stable extension of $\mathcal{F}$.

We need only to verify that for all $\alpha \in Arg(\mathcal{H}) \setminus Args_0$, α is defeated by E. It follows that α has at least one sub-argument (otherwise, it should be included in E, contradicting $\alpha \notin Args_0$). Let β be a sub-argument of α and let β have no proper sub-argument, i.e., $sub(\beta) = \{\beta\}$. It follows that β is in $Args_0$. Then we have the following two possible cases:

- *β is defeated by E: In this case, α is defeated by E.*
- *β is not defeated by E: In this case, β is in E (since E is a stable extension). Then, according to the definition of $Args_0$, the direct super argument of β (say β') is in $Args_0$. In turn, we have two possible cases similar to the cases w.r.t. β. Recursively, we may conclude that α is defeated by E or, α is in E (this case does not exist).*

($\Leftarrow$:) *Given every $E \in stb(\mathcal{F})$, let $\mathcal{H}' = \langle L, N', C, r'\rangle$ where $N' = \{(\top, b) \mid (a, b) \in N$ and $a \in concl(E)\}$, and $r'(\top, b) = r(a, b)$ for all $(a, b) \in N$ and $a \in concl(E)$.*

Let $E' = \{(\top, a_n) \mid (a_1, \ldots, a_n) \in E\}$.

In order to prove that $concl(E)$ is an extension of $\mathcal{H}$ in term of Reduction, according to Proposition 5, we only need to verify that E' is a stable extension of $(Arg(\mathcal{H}')$, $Def(\mathcal{H}', \succeq'_w))$ which is an argumentation framework of $\mathcal{H}'$ by using the weakest link principle. This is true, because:

- *Since E is conflict-free, E' is conflict-free.*
- *For all $\beta' \in Arg(\mathcal{H}') \setminus E'$, let β be a corresponding argument in $Arg(\mathcal{H}) \setminus E$ s.t. $\beta = (b_1, \ldots, b_n)$, $\beta' = (\top, b_n)$, and all sub-arguments of β are in E. Since β is not in E, it is defeated by E. Since all sub-arguments of β are not defeated by E, there*

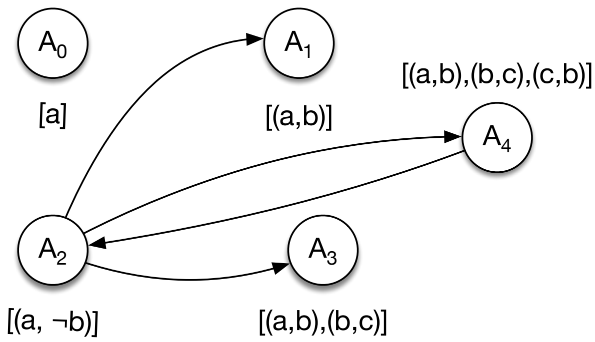

Fig. 4. The argumentation framework obtained for the hierarchical abstract normative system of Figure 3.

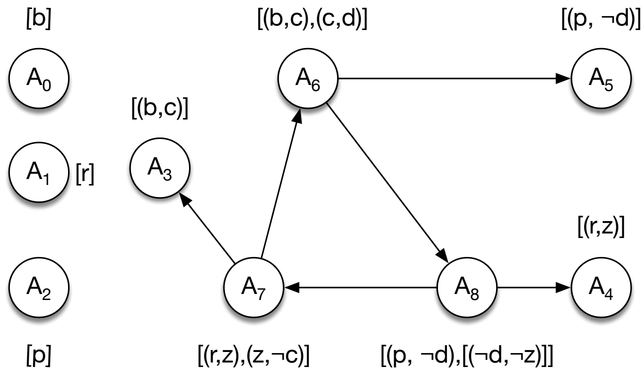

Fig. 5. An argumentation framework with no stable extension.

exists an argument in E whose conclusion is in conflict with $concl(\beta) = concl(\beta')$. So, β' is defeated by E'.

Example 7 (Order puzzle, Reduction). Consider Example 6 when the last link principle is applied, A_2 defeats A_3. Then, we have *Outfamily*$(\mathcal{F}_1,\ stb) = \{\{h, o\}\}$, which is equal to *Reduction*$(\mathcal{H}_1)$.

Example 8 (Multiple extensions). Consider the hierarchical abstract normative system of Figure 3 when the last link principle is applied. We obtain the argumentation framework shown in Figure 4, written as $\mathcal{F}_3$, yielding *Outfamily*$(\mathcal{F}_3, stb) = \{\{b, c\}, \{\neg b\}\}$. Note that here, we have two distinct stable extensions.

Since stable extensions do not necessarily exist for all argumentation frameworks, the Reduction of a hierarchical abstract normative system might not exist.

Example 9 (Empty Reduction). Consider the hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r\rangle$ where $N = \{(c, d), (p, \neg d), (z, \neg c), (\neg d, \neg z), (r, z), (b, c)\}$, $C =$

$\{b, r, p\}$ and $r(c, d) = 5$, $r(p, \neg d) = 4$, $r(z, \neg c) = 6$, $r(\neg d, \neg z) = 2$, $r(r, z) = 1$ and $r(b, c) = 0$. When the last link principle is applied, this hierarchical abstract normative system yields the argumentation framework shown in Figure 5, which has no stable extension.

6 Representation result for Optimization

Intuitively, Optimization involves adding the highest priority norms which are consistent with the context, until no more norms can be added, obtaining a maximal set of obeyable norms, following which conclusions can be computed. For the norms that do not belong to the maximal obeyable set, in the context of argumentation, the arguments represented by these norms should be defeated. For instance, as illustrated in Figure 1, the maximal obeyable set of norms is $\{(h, o), (w, \neg o)\}$. The norm (w, h) does no belong to this set. In order to prevent the conclusion h from being drawn, we need some way to defeat the corresponding argument $[(w, h)]$.

Note that the argument $[(w, h)]$ has an implicit conflict with the argument $[(w, \neg o)]$, since if $[(w, h)]$ is accepted, then its superargument $[(w, h), (h, o)]$ is in conflict with $[(w, \neg o)]$. Since $[(w, h), (h, o)]$ is defeated by $[(w, \neg o)]$ by applying the weakest link principle, corresponding to the detachment procedure of Optimization, $[(w, h)]$ has to be defeated by $[(w, \neg o)]$. Inspired by this idea, we introduce an approach that involves adding auxiliary defeats to the corresponding argumentation framework of an abstract hierarchical normative system. Thus, while the argument $A_1 = [(w, h)]$ is not directly defeated by $A_3 = [(w, \neg o)]$, since the former contains the weakest link of the argument $A_2 = [(w, h), (h, o)]$, we introduce an auxiliary defeat from A_3 to A_1, in addition to the already present defeat from A_3 to A_2. Given an argument α, each subargument of α containing the weakest link is called a *weakest subargument* of α.

Definition 11 (Weakest subargument). *Given a totally ordered hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r\rangle$ and the corresponding argumentation framework $\mathcal{F} = (Arg(\mathcal{H}), Def(\mathcal{H}, \succeq_w))$, for all $\alpha = (a_1, \ldots, a_n) \in Arg(\mathcal{H})$, if (a_i, a_{i+1}) is the weakest link, then every subargument of α containing (a_i, a_{i+1}) is a weakest subargument.*

When a weakest subargument of α is defeated by an argument β, all superarguments of the weakest subargument are also defeated by β. A weakest subargument of α or a superargument of this weakest subargument is called a *weakest argument* of α. The set of weakest arguments of α is denoted as *wks*(α).

Building on this concept, an expanded argumentation framework of $\mathcal{F}$ with auxiliary defeats on weakest arguments is defined as follows.

Definition 12 (Expanded argumentation framework with additional defeats on weakest arguments). *Given an argumentation framework $\mathcal{F} = (Arg(\mathcal{H}), Def(\mathcal{H}, \succeq_w))$ that is constructed from a totally ordered hierarchical abstract normative system $\mathcal{H} = \langle L, N, C, r\rangle$, the expanded argumentation framework of $\mathcal{F}$ with auxiliary defeats on weakest arguments is $\mathcal{F}' = (Arg(\mathcal{H}), Def(\mathcal{H}, \succeq_w)')$ where $Def(\mathcal{H}, \succeq_w)' = Def(\mathcal{H}, \succeq_w) \cup \{(\alpha, \gamma) \mid \gamma \in wks(\beta), (\alpha, \beta) \in Def(\mathcal{H}, \succeq_w)\}$.*

Example 10 (Prioritized triangle, continued). Consider the argumentation framework in Figure 2, when applying the weakest link principle, A_3 defeats A_2. Since the set of weakest subarguments of A_2 is $\{A_1\}$, A_3 defeats A_1.

The following proposition shows that Optimization can be represented in formal argumentation by the weakest link together with adding auxiliary defeats on weakest arguments to an argumentation framework.

Proposition 7 (Optimization is weakest link plus auxiliary defeats). *Let $\mathcal{H} = \langle L, N, C, r\rangle$ be a totally ordered hierarchical abstract normative system, and $\mathcal{F}' = (Arg(\mathcal{H}), Def(\mathcal{H}, \succeq_w)')$ be an argumentation framework of $\mathcal{H}$ with additional defeats on weakest subarguments. It holds that Optimization$(\mathcal{H})$ = Outfamily$(\mathcal{F}', prf)$.*

Proof. *First, it holds that $Def(\mathcal{H}, \succeq_w)'$ is acyclic for different arguments in the sense that for all $\alpha, \beta \in Arg(\mathcal{H})$, if $\alpha \neq \beta$, then if α defeats β, then there is no path from β to α with respect to $Def(\mathcal{H}, \succeq_w)'$. This is because, $Def(\mathcal{H}, \succeq_w)$ is acyclic, and each auxiliary defeat is from an attacker who has higher priority to a weakest subargument of the attackee who has lower priority. Hence, $\mathcal{F}'$ has a unique extension.*

Second, let O = Optimization$(\mathcal{H})$ be the unique extension of $\mathcal{H}$, and $E = \{(a_1, \ldots, a_n) \in Arg(\mathcal{O}) \mid \{a_1, \ldots, a_n\} \subseteq O \cup C\}$. According to Definition 5, it holds that $O = \{concl(\alpha) \mid \alpha \in E\}$. Now, we verify that E is a preferred extension of $\mathcal{F}'$:

(1) Since all premises and the conclusion of each argument of E are contained in $O \cup C$ which is conflict-free, it holds that E is conflict-free.

(2) For each $\alpha = (a_1, \ldots, a_n) \in E$ and for each $\beta = (b_1, \ldots, b_m) \in Arg(\mathcal{H}) \setminus E$, if β defeats α, then there exists $1 \leq i \leq n$ such that $b_m = \overline{a_i}$. In this case, there exists $\gamma = (c_1, \ldots, c_k) \in E$, $k \geq 1$, and $1 \leq j \leq m$, such that $c_k = \overline{b_j}$ and γ defeats β.

(3) Assume that there exists $E' \supset E$ such that E' is a preferred extension of $\mathcal{F}'$. Then, for each $\alpha = (a_1, \ldots, a_n) \in E'$, $a_n \in \mathcal{H}$, contradicting $\alpha \notin E$.

Given that E is a maximal admissible set, it turns out that E is a preferred extension.

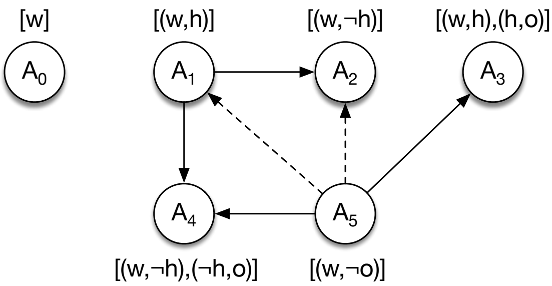

Fig. 6. Argument framework extended with auxiliary defeats (denoted using dashed lines).

Example 11 (Order puzzle, Optimization). Let $\mathcal{H}_1' = \langle L, N, C, r\rangle$ be a hierarchical abstract normative system, where $L = \{w, h, o, \neg w, \neg h, \neg o, \top\}$, $N = \{(w, h), (w, \neg h),$

$(h,o), (\neg h,o), (w,\neg o)\}$, $C = \{w,\top\}$, $r(w,h) = 1$, $r(w,\neg h) = 0$, $r(h,o) = 3$, $r(\neg h,o) = 4$, $r(w,\neg o) = 2$. *Here,* $R_3 = \{(\neg h,o), (h,o), (w,\neg o)\}$ *and so* $Optimization(\mathcal{H}'_1) = \{\neg o\}$. *Figure 6 illustrates the argument framework obtained from this hierarchical abstract normative system. Again, auxiliary defeats are shown as dashed lines. Here,* $Outfamily(\mathcal{H}'_1, prf) = \{\{\neg o\}\}$.

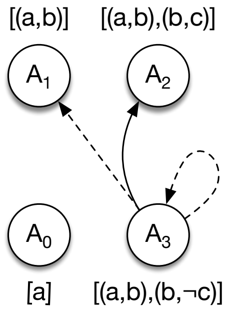

Fig. 7. Argument framework extended with auxiliary defeats (denoted using dashed lines).

Example 12 (Empty Optimization). Let $\mathcal{H} = \langle L, N, C, r\rangle$ *be a hierarchical abstract normative system, where* $L = \{a,b,c,\neg a,\neg b,\neg c,\top\}$, $N = \{(a,b),(b,c),(b,\neg c)\}$, $C = \{a,\top\}$, $r(a,b) = 1$, $r(b,c) = 2$, $r(b,\neg c) = 3$. *Evaluating this system using Optimization,* $R_2 = \{(b,c),(b,\neg c)\}$, *and* $Optimization(\mathcal{H}) = \emptyset$. *Figure 7 illustrates the resulting argument framework. Arguments containing* (a,b) *are weakest arguments, and dashed lines represent the auxiliary defeats generated against these arguments. Since this framework has one preferred extension which is an empty set,* $Outfamily(\mathcal{H}, prf) = \emptyset$.

7 Discussions and Related work

The role of examples in the study of logic has a long and rich history. Traditionally, a logic was proposed to model some example problem, following which examples were introduced to highlight paradoxes or inconsistencies in the logic, whereupon a new logic — addressing these problems — was proposed, and the cycle repeated. While this approach has significantly enriched the field, it is not without its problems. For example, there is still a debate regarding deontic detachment[5] (deriving an obligation from another obligation) within the community, as in some cases, deontic detachment intuitively holds, and in other cases it does not [23]. Given this, we do not seek to claim that the logic we present in this paper is in any sense the “right” logic. Instead, our goal is to answer the following questions.

[5] We note in passing that deontic detachment occurs in the logics we consider, but can be disabled through the introduction of higher priority norms, or relevant contextual information.

1. What are the general properties of systems considered relevant to some problem?
2. Given an application, what choices should be made in order to obain a solution?

In this work, the systems we considered are three different logics, encoded in the general framework of hierarchical abstract normative systems. The property we considered is then the conclusions that one can draw from each of the logics (in the context of prioritized norms), which we describe in the context of an argumentation system.

Our results then characterize the outputs of Greedy, Reduction and Optimization in terms of argumentation, allowing one to decide which approach is relevant to their needs by understanding the effects of each approach through the argumentation literature. The semantics associated with each approach also shed light on the complexity of computing conclusions in the normative context.

Regarding related work, Young *et al.* [30] endowed Brewka's prioritized default logic (PDL) with argumentation semantics using the ASPIC$^+$ framework for structured argumentation [22]. More precisely, their goal is to define a preference ordering over arguments $\succsim$, based on the strict total order over defeasible rules defined to instantiate $ASPIC^+$ to PDL, so as to ensure that an extension within PDL corresponds to the justified conclusions of its $ASPIC^+$ instantiation. Several options are investigated, and they demonstrate that the standard $ASPIC^+$ *elitist* ordering cannot be used to calculate $\succsim$ as there is no correspondence between the argumentation-defined inferences and PDL, and the same holds for a disjoint elitist preference ordering. The authors come up with a new argument preference ordering definition which captures both preferences over arguments and also *when* defeasible rules become applicable in the arguments' construction, leading to the definition of a strict total order on defeasible rules and corresponding non-strict arguments. Their representation theorem shows that a correspondence always exists between the inferences made in PDL and the conclusions of justified arguments in the $ASPIC^+$ instantiation under stable semantics.

Brewka and Eiter [7] consider programs supplied with priority information, which is given by a supplementary strict partial ordering of the rules. This additional information is used to solve potential conflicts. Moreover, their idea is that conclusions should be only those literals that are contained in at least one answer set. They propose to use preferences on rules for selecting a subset of the answer sets, called the *preferred answer sets*. In their approach, a rule is applied unless it is defeated via its assumptions by rules of higher priorities. Our definition (Def. 4) and the original formalism of Brewka and Eiter [7] are different, in the sense that in our definition we do not make use of default negation to represent the exceptions, i.e., the defeasibility, of a (strict) rule. Rather, we use defeasible rules and the notion of the applicability of such rules. This means that the correct translation of the Order Puzzle of Example 1 ends up with the following logic program[6]:

$\mathtt{r}_0 : \mathtt{w}.$
$\mathtt{r}_1 : \mathtt{h} \ \mathtt{:-}\ \mathtt{not}\ \neg\mathtt{h}, \mathtt{w}.$
$\mathtt{r}_2 : \neg\mathtt{o} \ \mathtt{:-}\ \mathtt{not}\ \mathtt{o}, \mathtt{w}.$
$\mathtt{r}_3 : \mathtt{o} \ \mathtt{:-}\ \mathtt{not}\ \neg\mathtt{o}, \mathtt{h}.$

[6] Note that in [7] $\mathtt{r}_0 < \mathtt{r}_3$ means that $\mathtt{r}_0$ has higher priority than $\mathtt{r}_3$.

$\mathtt{r}_0 < \mathtt{r}_3 < \mathtt{r}_2 < \mathtt{r}_1$

If preferences are disregarded, then this logic program has two answer sets: $\{w, h, \neg o\}$ and $\{w, h, o\}$. Thus, considering preferences, the latter is the unique preferred answer set. After dropping the context w from the answer set, we get an extension $\{h, o\}$, which is identical to the result obtained in Example 3.

Dung [11] presents an approach to deal with contradictory conclusions in defeasible reasoning with priorities. More precisely, he starts from the observation that often, the proposed approaches to defeasible reasoning with priorities (e.g., [5,25,21]) sanction contradictory conclusions, as exemplified by $ASPIC^+$ using the weakest link principle together with the elitist ordering which returns contradictory conclusions with respect to its other three attack relations, and the conclusions reached with the well known approach of Brewka and Eiter [7]. Dung shows then that the semantics for any complex interpretation of default preferences can be characterized by a subset of the set of stable extensions with respect to the normal attack relation assignments, i.e., a normal form for ordinary attack relation assignments. Dung's *normal attack relation* satisfies some desirable properties (Credulous cumulativity and Attack monotonicity) that cannot be satisfied by the *ASPIC*$^+$ semantics [11], i.e., the semantics of structured argumentation with respect to a given ordering of structured arguments (elitist or democratic pre-order) in *ASPIC*$^+$. In the setting of this paper, this notion could be defined as follows. Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ be arguments constructed from a hierarchical abstract normative system. Since we have no Pollock style undercutting argument (as in *ASPIC*$^+$) and each norm is assumed to be defeasible, α is said to normally attack argument β if and only if β has a sub-argument β' such that $concl(\alpha) = \overline{concl(\beta')}$, and $r((a_{n-1}, a_n)) \geq r((b_{m-1}, b_m))$. According to Definitions 7 and 8, the normal defeat relation is equivalent to the defeat relation using the last link principle in this paper.

Kakas *et al.* [19] present a logic of arguments called *argumentation logic*, where the foundations of classical logical reasoning are represented from an argumentation perspective. More precisely, their goal is to integrate into the single argumentative representation framework both classical reasoning, as in propositional logic, and defeasible reasoning.

You *et al.* [29] define a prioritized argumentative characterization of non-monotonic reasoning, by casting default reasoning as a form of prioritized argumentation. They illustrate how the parameterized formulation of priority may be used to allow various extensions and modifications to default reasoning.

We, and all these approaches, share the idea that an argumentative characterization of NMR formalisms, like prioritized default logic in Young's case and hierarchical abstract normative systems in our approach, contributes to make the inference process more transparent to humans. However, the targeted NMR formalism is different, leading to different challenges in the representation results. To the best of our knowledge, no other approach addressed the challenge of an argumentative characterization of prioritized normative reasoning.

The reason we study prioritized normative reasoning *in the setting of formal argumentation* is twofold. First, formal argumentation has been recognized as a popular research area in AI, thanks to its ability to make the inference process more intuitive and provide natural explanations for the reasoning process [8]; its flexibility in dealing with the dynamics of the system; and its appeal in sometimes being more computationally efficient than competing approaches. Second, while some progress has been made on the use of priorities within argumentation (e.g., [2,21]), how to represent different approaches for prioritized normative reasoning in argumentation is still a challenging issue.

8 Conclusions

In this paper we embedded three approaches to prioritized normative reasoning—namely the Greedy [30], Reduction [5] and Optimization [14] approaches—within the framework of a hierarchical abstract normative system. Within such a system, conditional norms are represented by a binary relation over literals, and priorities are represented by natural numbers. Hierarchical abstract normative systems provide an elegant visualisation of a normative system, with conflicts shown as two paths to a proposition and its negation. Since both conflicts and exceptions can be encoded, such systems are inherently non-monotonic. In his seminal paper, Dung [10] pointed out that "many of the major approaches to nonmonotonic reasoning in AI and logic programming are different forms of argumentation", and inspired by this, we described how arguments can be instantiated as paths through a hierarchical abstract normative system; demonstrated that this instantiation satisfies certain desirable properties; and described how attacks and defeats between these arguments can be identified. Defeats in particular are dependent on the priorities associated with the arguments, and several different techniques have been proposed to lift priorities from argument components — made up of norms in the context of our work — to the arguments themselves [21]. We demonstrated that for a total ordering of priorities, lifting priorities to arguments based on the weakest link principle, evaluated using the stable semantics, is equivalent to Greedy; that lifting priorities to arguments based on last link and using the stable semantics is equivalent to Reduction; and that the Optimization approach can be encoded by an argumentation system which uses weakest link together with the preferred semantics, and which introduces additional defeats capturing implicit conflicts between arguments.

This last result—which requires a relatively complex argumentative representation—opens up an interesting avenue for future work, namely in determining which nonmonotonic logics can be easily captured through standard formal argumentation techniques, and which require additional rules or axioms in order to be represented. We note that on the argumentation side, work on bipolar argumentation (e.g., [9]) has considered introducing additional defeats between arguments based on some notion of support, and we intend to investigate how the additional defeats we introduced can be categorized in such frameworks.

Apart from our representation results, the use of argumentation allows us to make some useful observations, such as that Reduction will sometimes not reach any conclusions. Furthermore, the use of argumentation can aid in explanation [8]. When implemented, a system building on our approach can help users understand what norms they should comply with, and why. For large normative systems, the use of stable semantics to compute Reduction and Optimization results in a high computational overhead, while

Greedy is computationally efficient. Ultimately, selecting the correct reasoning procedure thus requires giving consideration to both reasoning complexity, and the domain in which the system will be used.

In closing, our main observations can be summarized as follows. First, from a normative systems perspective, we know there are many many logics of prioritized rules/norms, and we consider only three here. The choice we make (Greedy, Reduction and Optimization) may seem arbitrary. However, many other examples (in particular detachment procedures not satisfying defeasible deontic detachment) are much easer to characterize, while the three throughput variants of Greedy, Reduction and Optimization can be derived from the existing results. Furthermore, these three alternatives display quite diverse behavior, and are illustrative of the various kind of approaches around.

Second, the results we present are interesting and promising, but the work on representing prioritized rules/norms using argumentation has only begun, and there are many open issues. In particular, the restriction to totally ordered systems must be relaxed in future work.

Third, given the large number of possibilities and the vast existing literature on normative rules/norms, a different methodology is needed for dealing with prioritized rules/norms in formal argumentation. Following the work of Dung 2016 [11], a more axiomatic approach—as we pursue—seems most promising.

Finally, one may wonder why our results have not been shown before, given the long standing discussion on weakest vs last link at least since the work of Pollock, and the central role of prioritized rules in many structured argumentation theories like ASPIC$^+$. The reason, we believe, is that it is easier to study these issues on a small fragment, like hierarchical abstract normative systems, than on a very general theory like ASPIC$^+$.

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