Multi-Portfolio Optimization: A Potential Game Approach

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***Abstract*—In modern asset management, portfolio managers address the multi-account investment decision problem by optimizing each account's portfolio separately based on the trading requirements and portfolio constraints of the individual clients. However, trades associated with the individual accounts are usually pooled together for execution, therefore amplifying the level of the so-called market impact on all accounts. If this aggregate market impact is not considered when each account is individually optimized, the actual market impact can be severely under-estimated. Multi-portfolio optimization aims at finding the optimal rebalancing of the multiple accounts by considering their joint effects while adhering to account-specific constraints. In this paper, we first model this phenomenon as a Nash Equilibrium problem (NEP) and thereafter consider a generalized NEP (GNEP) for the case where there are global constraints imposed on all accounts, adopting as a desirable outcome the concept of Nash Equilibrium (NE). For both game problems, we give a complete characterization of the NE, including its existence and uniqueness, and devise various distributed algorithms with provable convergence. Interestingly, the proposed methodology heavily hinges on a number of well-known and important signal processing techniques.**

***Index Terms*—Multi-Portfolio Optimization, Market Impact Cost, Game Theory, Nash Equilibrium, Socially Optimal Solution, Convex Optimization, Distributed Algorithms.**

I. INTRODUCTION

In financial engineering, the field of portfolio optimization studies how to allocate funds among a number of risky assets so that a certain utility function, usually given in terms of a measure of achieved risk-adjusted return, is maximized. In a ground-breaking work laying down the foundations of modern portfolio theory [1], Markowitz introduced the mean-variance framework and justified that the optimal portfolio should be determined based on the trade-off between maximizing the expected return and minimizing the risk.

Let $\mathbf{w}$ be the vector of weights defining the proportion of wealth allocated among a total number of K assets, and assume that the return of the k-th asset over a single-period investment horizon is modeled as a random variable denoted by r_k. Let $\boldsymbol{\mu} = (\mu_k)_{k=1}^K$ be the vector of expected returns where $\mu_k = \mathbb{E}[r_k]$, and $\mathbf{R} = (\mathbf{R}_{kj})_{k,j}$ be the positive definite covariance matrix where $\mathbf{R}_{kj} = \mathbb{E}\left[(r_k - \mu_k)(r_j - \mu_j)\right]$. In Markowitz's mean-variance portfolio optimization framework, the expected return of the portfolio is $\boldsymbol{\mu}^T\mathbf{w}$ while the risk of the portfolio is $\mathbf{w}^T\mathbf{R}\mathbf{w}$. The latter is based on the intuition that the prices of highly correlated assets will likely increase or decrease simultaneously, and it is thus advisable to diversify the investment choices over a variety of assets in order to effectively reduce the risk, which is referred to as diversification principle in finance and investing. Then, considering the trade-off between the expected return and risk, the optimal portfolio is the solution to the following problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \boldsymbol{\mu}^T\mathbf{w} - \frac{1}{2}\rho\mathbf{w}^T\mathbf{R}\mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{W}, \end{array} \tag{1}$$

where ρ is a given positive constant specifying the investor's level of risk aversion, and $\mathcal{W}$ is the set of feasible portfolios specified by various trading constraints (see Section II). This formulation reveals that among the portfolios that have the same risk (expected return, respectively), we should choose the one with largest expected return (smallest risk, respectively). Because of its fundamental role in investment science, (1) is an important source problem in optimization literature [2, 3].

Since the appearance of Markowitz's work in [1], numerous generalizations of the classical mean-variance portfolio optimization framework (1) have been proposed to deal with a variety of practical operating conditions, such as the effect of transaction cost in the investment performance. Basically, transaction cost can be explicit, as defined by taxes, market fees, and brokerage commissions, or implicit, such as bid-ask spread and market impact cost. Specifically, market impact refers to the negative effect on the price of an asset when executing orders that are large relative to the liquidity available in the market [4–6]. The market impact component represents the element most largely contributing to uncertainty in trading cost analysis and portfolio performance measure, and thus should be properly characterized and incorporated into the optimization model.

Let $TC(\bullet)$ be the market impact cost function. The market impact cost associated with rebalancing from the current

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position $\mathbf{w}^0$ to a new position $\mathbf{w}$ is given by $TC(\mathbf{w}-\mathbf{w}^0)$. Then the optimization problem (1) should be revised as

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & \boldsymbol{\mu}^T\mathbf{w}-\frac{1}{2}\rho\mathbf{w}^T\mathbf{R}\mathbf{w}-TC(\mathbf{w}-\mathbf{w}^0)\\ \text{subject to} & \mathbf{w}\in\mathcal{W}.\end{array} \quad (2)$$

In some scenarios, this formulation may still not be satisfactory enough. For example, in a practical framework accommodating multiple accounts, trades of separate accounts are usually pooled and executed together. As a consequence, the market impact cost of any single account depends on the aggregate trades from all accounts [7–10]. Mathematically, suppose there are N accounts and denote as $\mathbf{w}_n$ the portfolio vector of the n-th account, the market impact cost of account n is $TC(\sum_{n=1}^{N}(\mathbf{w}_n-\mathbf{w}_n^0))$ rather than $TC(\mathbf{w}_n-\mathbf{w}_n^0)$. In practice, however, each account is independently optimized while the market impact from other accounts is naively ignored. The actual market impact cost, typically much larger than the one estimated, is subsequently allocated among accounts based on the proportion of each account's amount of trading, known as the *pro rata* scheme [11]. This under-estimation of the market impact cost in the naive approach can result in a reduction in realized returns.

A natural and direct extension of this naive approach is to explicitly consider in each account's single portfolio optimization problem the aggregate trades from other accounts. The resulting multi-portfolio optimization problem is actually a non-cooperative game problem in disguise, see Section II-B. The desirable outcome in this context is the Nash Equilibrium (NE), the most widely used solution in applications of game theory to economics, at which no account has an incentive to unilaterally deviate from it [12].

On the other hand, the NE is not efficient in the sense that it does not necessarily maximize the total welfare over all accounts, an important social welfare function in microeconomics and widely used in practice, cf. [7, 10, 12–14]. Such a solution is referred to as a socially optimal solution and it has been considered first in [7] and further elaborated in [8–10].

In multi-portfolio optimization, a central problem associated with the optimal solution is the fairness issue [11, 14]. It is shown in [9, 10] that to achieve social optimality, some accounts may be forced to sacrifice their own benefits. For example, when one of the accounts is much larger in size than the others, smaller accounts can suffer from a shortage of liquidity. For those small accounts, the socially optimal solution is not fair in the sense that they can achieve a better return profile by acting alone such as when pursing the NE [9, 10]. If the separate accounts belong to individual clients who care about their own utilities only, those "smaller" clients may not be satisfied with the socially optimal solution, and this is where the NE presents itself as a more sensible solution in the sense of fairness: no client can further maximize his payoff by unilaterally deviating from the NE.

Although the NE and the socially optimal solution have been considered in [7, 9], an analytical characterization of these solutions, such as the existence, uniqueness and algorithms, was either left open [7] or only partially addressed using heuristics under a very specific setting [9]. Besides, it is not clear if the connection between the socially optimal solution and the NE in [9] is still valid in a general setting.

Another shortcoming of [7, 9] is that a centralized approach is used to generate optimal trades for all accounts simultaneously. However, the variable dimension in this "one-shot" optimization problem depends on the product of the number of accounts and the number of assets; thus the computational cost is excessively expensive when the number of accounts and/or the number of assets are large. Distributed computation methods are desirable as they can make use of structure of the problem in order to decompose a large problem into a number of smaller problems, which are typically solved either sequentially or simultaneously. The challenging question associated with distributed algorithms is whether they converge to the optimal solution or not, which is however left open in literature. The design of distributed algorithms and convergence analysis becomes more challenging when there are portfolio constraints imposed on all accounts, which often arise due to practical considerations, for instance liquidity limitations indicated by the average daily trading volume (ADV) for a given asset.

In this paper, we fill these gaps in state-of-the-art theory and practice of multi-account optimization by rigorously analyzing the problem in a general setting building on potential game theory. Specifically, our contributions are the following:

- We show that when all accounts are individually constrained, there exists a unique NE. This attractive property in turn provides an additional justification for the NE: the unique NE is an outcome that all accounts can predict and agree on.
- We derive both synchronous and asynchronous distributed algorithms with provable convergence: the multi-portfolio optimization is decomposed into a number of smaller single-account problems which can be solved efficiently by existing infrastructure. The information exchange is maintained at a very low level: each account only needs an aggregate trading vector from the preceding iteration, and it does NOT need the individual trading strategies of other accounts.
- We also consider and analyze the total welfare maximization problem. We show that the socially optimal solution may not be unique, and this could give rise to a fairness issue among accounts as some accounts may prefer one solution while other accounts prefer another. Distributed algorithms are derived to compute the socially optimal solutions efficiently.
- When there are global constraints imposed on all accounts, there exists a unique Variational Equilibrium, a special class of generalized NE (GNE), and distributed algorithms with satisfactory convergence properties are proposed.

Connection to signal processing problems: Interestingly, one can draw a close connection between the multi-portfolio optimization problem and many seemingly different problems in signal processing, communication networks, and power systems; see [15, 16] for Digital Subscriber Line (DSL) systems, [17, 18, 20] for interference channels, [21, 22] for cognitive radio (CR) networks, and [23] for power systems.

In DSL systems, subscriber lines are usually bundled together and they create electromagnetic interference into each other, thus causing crosstalk noise [15, 16]. In this context, crosstalk noise plays a similar role as market impact in multi-portfolio optimization problem. A similar effect also happens in interference channels [17, 18, 20].

In CR networks, secondary users (SU) can coexist with primary users (PU) provided that the interference generated by the SUs is tolerable for the PUs. If SUs naively transmit without considering the interference temperature constraints, they would generate considerable interference to PUs [21, 22]. As a result, neither the SUs nor the PUs can achieve the target transmission rate. This is similar to the ADV (considered in Section V) that all accounts together have to obey in multi-portfolio optimization.

In a smart grid power system, different consumers are interconnected by the price of the electricity [23]. In particular, the price of electricity dynamically depends on the aggregate consumption from all consumers. If one simply ignores the existence of other consumers, the estimate price of electricity is smaller than the actual price, and this may lead to excessive consumption and thus a much higher cost. In this context, the consumption of electricity plays a similar role as market impact in the multi-portfolio optimization problem.

The rest of the paper is structured as follows. In Section II, we introduce the multi-portfolio problem, and model it as a game and a total welfare maximization problem. In Section III, we characterize the NE and the socially optimal solution including existence and uniqueness, and develop various distributed algorithms in Section IV. Section V deals with a generalized NEP (GNEP) where all accounts are subject to global constraints as well. Numerical results are presented in Section VI and conclusions are drawn in Section VII.

Notation: Scalars, vectors, and matrices are denoted by x, $\mathbf{x}$, and $\mathbf{X}$, respectively. $\mathbf{x}_k$ is the k-th block component of $\mathbf{x}$ and x_k is the k-th element of $\mathbf{x}$. The eigenvalues of $\mathbf{X}$ are denoted as $\boldsymbol{\lambda}(\mathbf{X})$, with $\lambda_{\max}(\mathbf{X})$ and $\lambda_{\min}(\mathbf{X})$ representing the largest and smallest eigenvalue, respectively. $\sigma_{\max}(\mathbf{X})$ denotes the largest singular value of $\mathbf{X}$. $\mathbf{e}_n$ is a unit vector where the n-th entry is 1. $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{J}_n$ is a $n \times n$ matrix with all entries 1. $\mathbf{X} \otimes \mathbf{Y}$ denotes the Kronecker product of $\mathbf{X}$ and $\mathbf{Y}$; diag$(\mathbf{X}, \mathbf{Y})$ is a block diagonal matrix with $\mathbf{X}$ and $\mathbf{Y}$ on the diagonal in a descending order; diag$(\boldsymbol{\rho})$ is a diagonal matrix with diagonal vector $\boldsymbol{\rho}$. $[\mathbf{x}]^+ = \max(\mathbf{x}, \mathbf{0})$ and $[\mathbf{x}]^- = \max(-\mathbf{x}, \mathbf{0})$ is the positive and negative decomposition of $\mathbf{x}$, respectively. Note that $[\mathbf{x}]^+ \geq \mathbf{0}$ and $[\mathbf{x}]^- \geq \mathbf{0}$. $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\mathbf{x}^T \mathbf{y}$ are used interchangeably to denote the inner product between $\mathbf{x}$ and $\mathbf{y}$.

II. Problem Model and Formulations

A. Mean-variance utility function and constraints

We analyze the multi-portfolio optimization problem under the mean-variance framework (1)-(2). Specifically, the market impact cost function $TC(\mathbf{w})$ is modeled as

$$TC(\mathbf{w}) = \left\langle [\mathbf{w}]^+, \mathbf{c}^+(\mathbf{w}) \right\rangle + \left\langle [\mathbf{w}]^-, \mathbf{c}^-(\mathbf{w}) \right\rangle \tag{3}$$

where $[\mathbf{w}]^+$ ($[\mathbf{w}]^-$) is the buy (sell) vector, and $\mathbf{c}^+(\mathbf{w})$ ($\mathbf{c}^-(\mathbf{w})$) is the market impact price function for buys (sells) giving the cost per unit traded for each asset.

For the market impact price function $\mathbf{c}^+(\mathbf{w})$, we assume that it is separable among assets [4, 7, 11, 14], i.e., $\mathbf{c}^+(\mathbf{w}) = (c_k^+(w_k))_{k=1}^K$, and $c_k^+(w_k) = \boldsymbol{\Omega}_{kk}^+([w_k]^+)^p$ with $0.5 \leq p \leq 1$ [4, 7], where $\boldsymbol{\Omega}^+$ is a positive diagonal matrix representing market impact coefficients; the modeling is similar for sells. We assume the usual choice $p = 1$: this linear market impact price function is one of the most fundamental models and has been justified in a number of works, see [4–7, 9, 11, 14, 24].

In the presence of multiple accounts, the market impact price function depends on the aggregate trade from all accounts [9], i.e., $\mathbf{c}^+(\mathbf{w}_1, \ldots, \mathbf{w}_N) = \boldsymbol{\Omega}^+(\sum_{n=1}^N [\mathbf{w}_n]^+)$, and the market impact cost for each account is proportional to their individual trade amount (the *pro rata* scheme [11]). Under this consideration, the utility function for account n is[1]

$$\begin{aligned} u_n(\mathbf{w}_n, \mathbf{w}_{-n}) &= \boldsymbol{\mu}^T \mathbf{w}_n - \frac{1}{2} \rho_n \mathbf{w}_n^T \mathbf{R} \mathbf{w}_n \\ &- \frac{1}{2} \Big\langle \left[\mathbf{w}_n - \mathbf{w}_n^0\right]^+, \boldsymbol{\Omega}^+ \sum\nolimits_{m=1}^N \left[\mathbf{w}_m - \mathbf{w}_m^0\right]^+ \Big\rangle \\ &- \frac{1}{2} \Big\langle \left[\mathbf{w}_n - \mathbf{w}_n^0\right]^-, \boldsymbol{\Omega}^- \sum\nolimits_{m=1}^N \left[\mathbf{w}_m - \mathbf{w}_m^0\right]^- \Big\rangle, \end{aligned} \tag{4}$$

where $\mathbf{w}_{-n} = (\mathbf{w}_m)_{m \neq n}$ represents the strategies of account n's competitors, i.e., all accounts except account n. Since the mean-variance framework focuses on a single-period investment, we assume that $\boldsymbol{\mu}$, $\mathbf{R}$, $\boldsymbol{\rho}$, $\boldsymbol{\Omega}^{+(-)}$ are fixed [11, 24].

As in (1)-(2), the feasible trading strategy $\mathbf{w}$ is in a closed and convex constraint set $\mathcal{W}$. In general, these portfolio constraints may consist of two categories: individual constraints and global constraints. They together make sure that the strategies in each account's constraint set are not only feasible for the particular account but also feasible in the market.

Individual constraints:

- Holding constraint: To reduce risk, a portfolio should not exhibit large concentrations in any specific asset. Minimal and maximal holdings can be controlled by constraints of this form: $\boldsymbol{l}_n \leq \mathbf{w}_n \leq \boldsymbol{u}_n$.
- Long-only constraint (no short-selling constraint): In the process of short-selling, we sell an asset that we borrowed from someone else, and repay our loan after buying the asset back at a later date. Short-selling is profitable if the asset price declines. Because of the risky nature, it is prohibited or purposely avoided sometimes. Mathematically, the long-only constraint corresponds to $\mathbf{w}_n \geq \mathbf{0}$ and it is a special case of the holding constraint where $\boldsymbol{l}_n = \mathbf{0}$ and $\boldsymbol{u}_n = \infty$.
- Budget constraint: $\sum_{k=1}^K w_{n,k} \leq b_n$.

Additional constraints, such as tracking error (benchmark exposure) constraints, risk factor constraints, cardinality constraints, and direct transaction costs constraints (including broker commissions and taxes) can be taken into consideration as long as they are convex or they can be approximated using convex techniques [24]; see [14, Ch. 4] for a review of portfolio constraints. It is easy to see that if each account is subject to individual constraints only, one account's strategy set is independent of other accounts' strategies.

[1] In (3)-(4), the market impact costs for buys and sells are separated [9]. There is another model for the market impact price function where buys and sells can be internally crossed, and the corresponding market impact cost for account n is $\mathbf{w}_n^T \boldsymbol{\Omega} \left(\sum_{m=1}^N \mathbf{w}_m\right)$. This model is simpler (as buys and sells cannot always be crossed internally) and can be analyzed by the same methodology to be developed in this paper.

Note that different accounts need not manage the same set of assets, and this can be formulated by proper individual portfolio constraints. For example, setting $l_{n,k} = u_{n,k} = 0$ implies that asset k is not managed by account n. Besides, each account needs not follow the same set of individual constraints either. For example, if account n does not have a budget constraint, one can simply set $b_n = \infty$.

Global constraints: In some circumstances, there may exist regulations on *all* accounts, and these regulations can be modeled as global (coupling) constraints.

• Turnover or transaction size constraints over multiple accounts, which are used to limit the average daily trade volume associated with the k-th asset:

$$\sum\nolimits_{n=1}^{N} \left| w_{n,k} - w_{n,k}^0 \right| \le D_k,\ k = 1, \ldots, K; \tag{5}$$

• Limitations on the amount invested over groups of assets with related characteristics (e.g., industries, sectors, countries, and asset classes, etc.):

$$\sum\nolimits_{n=1}^{N} \sum\nolimits_{k\in\mathcal{J}_l} \left| w_{n,k} - w_{n,k}^0 \right| \le U_l,\ l = 1, \ldots, L; \tag{6}$$

Other convex global constraints such as limit of liquidity can be straightforwardly incorporated as well.

It is easy to see from (5)-(6) that one account's available strategies also depend on other accounts' actions. In other words, the presence of global constraints introduces coupling into each account's strategy set, and complicates the analysis and design dramatically.

B. Problem formulations

In this subsection, we introduce several formulations for the multi-portfolio optimization problem, as detailed next.

Game theoretical formulation under individual constraints: We formulate the optimization as a NEP: each account n competes against the others by choosing a strategy that maximizes his own utility function. Stated in mathematical terms, given the strategies of other accounts $\mathbf{w}_{-n}$, account n solves the following optimization problem:

$$\left.\begin{array}{ll} \underset{\mathbf{w}_n}{\text{maximize}} & u_n(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{W}_n \end{array}\right\} \forall n, \tag{7}$$

where $u_n(\mathbf{w}_n, \mathbf{w}_{-n})$ is defined in (4), and $\mathcal{W}_n$ is a non-empty, closed, and convex set specified by the *individual* portfolio constraints. Since each account's strategy set is independent of the rival accounts, the joint strategy set of all accounts has a Cartesian structure, i.e., $\mathcal{W}_1 \times \ldots \times \mathcal{W}_N$.

Naive solution: We can mathematically recover the naive solution from the proposed formulation (7), in which the aggregate effect from other accounts is simply ignored and the optimization problem for each account is [9]

$$\left.\begin{array}{ll} \underset{\mathbf{w}_n\in\mathcal{W}_n}{\text{maximize}} & u_n\left(\mathbf{w}_n, \mathbf{w}_{-n}^0\right) = \boldsymbol{\mu}^T\mathbf{w}_n - \frac{1}{2}\rho_n \mathbf{w}_n^T\mathbf{R}\mathbf{w}_n \\ & -\frac{1}{2}\left\langle \left[\mathbf{w}_n - \mathbf{w}_n^0\right]^+, \boldsymbol{\Omega}^+\left[\mathbf{w}_n - \mathbf{w}_n^0\right]^+ \right\rangle \\ & -\frac{1}{2}\left\langle \left[\mathbf{w}_n - \mathbf{w}_n^0\right]^-, \boldsymbol{\Omega}^-\left[\mathbf{w}_n - \mathbf{w}_n^0\right]^- \right\rangle \\ \text{subject to} & \mathbf{w}_n \in \mathcal{W}_n \end{array}\right\} \forall n. \tag{8}$$

Now it is clear that the NEP is a natural and direct improvement and extension of the naive solution by explicitly considering the aggregate trades from others accounts.

With the NEP formulation, the desirable outcome is the well-known notion of NE, which is achieved when no account has an incentive to deviate from it unilaterally:

Definition 1. A (pure) strategy profile $\mathbf{w}_{\text{ne}} = (\mathbf{w}_n^\star)_{n=1}^N$ is a NE of the NEP (7) if

$$u_n(\mathbf{w}_n^\star, \mathbf{w}_{-n}^\star) \ge u_n(\mathbf{w}_n, \mathbf{w}_{-n}^\star), \quad \forall\, \mathbf{w}_n \in \mathcal{W}_n, \forall\, n. \tag{9}$$

Game theoretical formulation under global constraints: When there are global constraints imposed on all accounts, there is coupling in both utility functions and constraint sets [25]. This can be modeled as a GNEP, in which account n solves the following problem:

$$\left.\begin{array}{ll} \underset{\mathbf{w}_n}{\text{maximize}} & u_n(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{W}_n \\ & \mathbf{g}(\mathbf{w}_n, \mathbf{w}_{-n}) \le \mathbf{0} \end{array}\right\} \forall n, \tag{10}$$

where $\mathbf{g}(\mathbf{w})$ denotes the global constraints (5)-(6):

$$\mathbf{g}(\mathbf{w}) \triangleq \left[\begin{array}{c} \left(\sum_{n=1}^N \left| w_{n,k} - w_{n,k}^0 \right| - D_k\right)_{k=1}^K \\ \left(\sum_{n=1}^N \sum_{j\in\mathcal{J}_l} \left| w_{n,j} - w_{n,j}^0 \right| - U_l\right)_{l=1}^L \end{array}\right]. \tag{11}$$

The joint strategy set of all accounts of the GNEP (10) is thus

$$\left\{\mathbf{w} : \mathbf{w} \in \mathcal{W}_1 \times \ldots \times \mathcal{W}_N,\ \mathbf{g}(\mathbf{w}) \le 0\right\}. \tag{12}$$

A solution of the GNEP (10), termed GNE, can be defined similarly to a NE of the NEP (7). Specifically, a (pure) strategy profile $\mathbf{w}_{\text{ne}} = (\mathbf{w}_n^\star)_{n=1}^N$ is a GNE of the GNEP (10) if for all $\mathbf{w}_n \in \mathcal{W}_n$ and $\mathbf{g}(\mathbf{w}_n, \mathbf{w}_{-n}^\star) \le \mathbf{0}$ we have

$$u_n(\mathbf{w}_n^\star, \mathbf{w}_{-n}^\star) \ge u_n(\mathbf{w}_n, \mathbf{w}_{-n}^\star), \quad n = 1, \ldots, N. \tag{13}$$

The lack of Cartesian structure in (12) makes the analysis of the GNEP much more difficult than that for the NEPs [21, 25, 26]. We study the GNE in Section V.

Total welfare maximization problem: In general, the NE is not efficient in the sense that it is not necessarily socially optimal. The social problem is defined as:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \sum_{n=1}^N u_n(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{W}_n,\ n = 1, \ldots, N, \end{array} \tag{14}$$

and $\mathbf{w}$ also has to satisfy the global constraints $\mathbf{g}(\mathbf{w}) \le \mathbf{0}$ if they are present.

Remark 2. In addition to the above formulations, there also exist other fairness concepts such as proportional fairness solutions, max-min fairness solutions and equitable efficient solutions [11, 13]. In this paper, however, we do not aim at obtaining the most fair solution. We instead provide a thorough and rigorous theoretical analysis on two existing solution concepts, namely the NE and the socially optimal solution with roots in microeconomics [12], where the market impact cost, allocated by the *pro rata* scheme [11], is explicitly modeled as a part of the utility function.

III. Multi-Portfolio Optimization Problem with Individual Constraints

In this section, we provide a complete characterization of the NE and socially optimal solutions, when each account is only individually constrained (we will discuss the globally constrained case in Section V). The analysis is carried out under the framework of potential game theory, an important class of games that allows us to infer the properties of the NEP by solving standard optimization problems [19, 27–29].

A potential game is formally defined below [27]; see [19] for a general and recent developments of potential games.

Definition 3. The NEP (7) is called a(n exact) potential game if there exists a function $P:\mathcal{W}_1\times\ldots\times\mathcal{W}_N\rightarrow\mathbb{R}$ such that for all accounts n and $(\mathbf{x},\mathbf{w}_{-n})$, $(\mathbf{y},\mathbf{w}_{-n})\in\mathcal{W}_1\times\ldots\times\mathcal{W}_N$:

$$u_n(\mathbf{x},\mathbf{w}_{-n})-u_n(\mathbf{y},\mathbf{w}_{-n})=P(\mathbf{x},\mathbf{w}_{-n})-P(\mathbf{y},\mathbf{w}_{-n}). \quad (15)$$

A key rule in the study of potential games is played by the following standard optimization problem, where the objective function is just the potential function $P(\mathbf{w})$:

$$\begin{array}{ll}\underset{\mathbf{w}}{\text{maximize}} & P(\mathbf{w})\\ \text{subject to} & \mathbf{w}\in\mathcal{W}_1\times\ldots\times\mathcal{W}_N.\end{array} \quad (16)$$

The relationship between the NEP (7) and problem (16) is given in the following lemma [28].

Lemma 4. *Let the NEP (7) be a potential game with a concave potential function $P(\mathbf{w})$. If $\mathbf{w}^\star$ is an optimal solution of (16), then it is a NE of the NEP (7). Conversely, if $P(\mathbf{w})$ is continuously differentiable and $\mathbf{w}_{\text{ne}}$ is a NE of the NEP (7), then $\mathbf{w}_{\text{ne}}$ is an optimal solution of (16).*

A. Reformulation of the single account problem

Before studying the NEP, let us rewrite (7) in a more convenient form. In fact, the projections in the utility functions $[\bullet]^+$ and $[\bullet]^-$ are generally difficult to handle because of the nonconvexity and nondifferentiability they bring about. To cope with these difficulties, we introduce new *nonnegative* variables $\widetilde{\mathbf{w}}_n=[\widetilde{\mathbf{w}}_n^+;\widetilde{\mathbf{w}}_n^-]$ and make the following variable substitutions:

$$\left[\mathbf{w}_n-\mathbf{w}_n^0\right]^{+(-)}=\widetilde{\mathbf{w}}_n^{+(-)},\ \mathbf{w}_n-\mathbf{w}_n^0=\widetilde{\mathbf{w}}_n^+-\widetilde{\mathbf{w}}_n^-,\quad\forall n.$$

Then the utility function (4) in terms of the new variable $\widetilde{\mathbf{w}}$ is (some constants are added)

$$\begin{aligned}&\widetilde{u}_n(\widetilde{\mathbf{w}}_n,\widetilde{\mathbf{w}}_{-n})=\\ &\underbrace{\left[\boldsymbol{\mu}-\rho_n\mathbf{R}\mathbf{w}_n^0;-\boldsymbol{\mu}+\rho_n\mathbf{R}\mathbf{w}_n^0\right]}_{\triangleq\widetilde{\boldsymbol{\mu}}_n}{}^T\underbrace{\left[\widetilde{\mathbf{w}}_n^+;\widetilde{\mathbf{w}}_n^-\right]}_{\triangleq\widetilde{\mathbf{w}}_n}\\ &-\tfrac{1}{2}\rho_n\left[\widetilde{\mathbf{w}}_n^+;\widetilde{\mathbf{w}}_n^-\right]^T\underbrace{\begin{bmatrix}\mathbf{R}&-\mathbf{R}\\-\mathbf{R}&\mathbf{R}\end{bmatrix}}_{\triangleq\widetilde{\mathbf{R}}}\left[\widetilde{\mathbf{w}}_n^+;\widetilde{\mathbf{w}}_n^-\right]\\ &-\tfrac{1}{2}\left[\widetilde{\mathbf{w}}_n^+;\widetilde{\mathbf{w}}_n^-\right]^T\underbrace{\begin{bmatrix}\boldsymbol{\Omega}^+&\\&\boldsymbol{\Omega}^-\end{bmatrix}}_{\triangleq\widetilde{\boldsymbol{\Omega}}}\left(\textstyle\sum_{m=1}^N\left[\widetilde{\mathbf{w}}_m^+;\widetilde{\mathbf{w}}_m^-\right]\right)\\ &=\widetilde{\boldsymbol{\mu}}_n\widetilde{\mathbf{w}}_n-\tfrac{1}{2}\rho_n\widetilde{\mathbf{w}}_n^T\widetilde{\mathbf{R}}\widetilde{\mathbf{w}}_n-\tfrac{1}{2}\widetilde{\mathbf{w}}_n^T\widetilde{\boldsymbol{\Omega}}\left(\textstyle\sum_{m=1}^N\widetilde{\mathbf{w}}_m\right).\end{aligned} \quad (17)$$

With this change of variable, the new constraint set is

$$\widetilde{\mathcal{W}}_n\triangleq\left\{\widetilde{\mathbf{w}}_n:\left[\mathbf{I}\ -\mathbf{I}\right]\widetilde{\mathbf{w}}+\mathbf{w}_n^0\in\mathcal{W},\widetilde{\mathbf{w}}\geq\mathbf{0}\right\},$$

which is convex in $\widetilde{\mathbf{w}}_n$.

Note that $u_n(\mathbf{w}_n,\mathbf{w}_{-n})$ is not necessarily equivalent to $\widetilde{u}_n(\widetilde{\mathbf{w}}_n,\widetilde{\mathbf{w}}_{-n})$ because $[\mathbf{w}_n-\mathbf{w}_n^0]^+$ is by definition orthogonal to $[\mathbf{w}_n-\mathbf{w}_n^0]^-$, but such an orthogonality is not imposed between $\widetilde{\mathbf{w}}_n^+$ and $\widetilde{\mathbf{w}}_n^-$; instead, $\widetilde{\mathbf{w}}_n^+$ and $\widetilde{\mathbf{w}}_n^-$ are only assumed to be nonnegative. However, in the following lemma we prove that this orthogonality property is automatically satisfied at the optimal $\widetilde{\mathbf{w}}_n^+$ and $\widetilde{\mathbf{w}}_n^-$.

Lemma 5. *In the optimization problem of account n (7), given any arbitrary but fixed feasible $(\widetilde{\mathbf{w}}_m)_{m\neq n}$, the optimal buy vector $\widetilde{\mathbf{w}}_n^+$ and sell vector $\widetilde{\mathbf{w}}_n^-$ are orthogonal.*

Proof: See Appendix A. ■

Remark 6. The diagonal structure of $\boldsymbol{\Omega}^+$ and $\boldsymbol{\Omega}^-$ is crucial in proving the orthogonal property of optimal $\widetilde{\mathbf{w}}_n^+$ and $\widetilde{\mathbf{w}}_n^-$. However, Lemma 5 still holds without the diagonal structure if buys and sells are not separated.

Lemma 5 states that there is no loss of optimality when we replace $[\mathbf{w}_n-\mathbf{w}_n^0]^{+(-)}$ with $\widetilde{\mathbf{w}}_n^{+(-)}$ without assuming the orthogonality between $\widetilde{\mathbf{w}}_n^+$ and $\widetilde{\mathbf{w}}_n^-$. We can then work with the new utility function $\widetilde{u}_n(\widetilde{\mathbf{w}}_n,\widetilde{\mathbf{w}}_{-n})$, which is a strongly concave and twice differentiable function in $\widetilde{\mathbf{w}}_n$.

B. Characterization of Nash Equilibrium

According to Lemma 5, the NEP (7) is equivalent to

$$\left.\begin{array}{ll}\underset{\widetilde{\mathbf{w}}_n}{\text{maximize}} & \widetilde{u}_n(\widetilde{\mathbf{w}}_n,\widetilde{\mathbf{w}}_{-n})\\ \text{subject to} & \widetilde{\mathbf{w}}_n\in\widetilde{\mathcal{W}}_n.\end{array}\right\}\forall n. \quad (18)$$

The NEP (18) is a potential game, as shown next.

Lemma 7. *The NEP (18) is equivalent to the following optimization problem:*

$$\begin{array}{ll}\underset{\widetilde{\mathbf{w}}}{\text{maximize}} & P_{\text{ne}}(\widetilde{\mathbf{w}})\triangleq\widetilde{\boldsymbol{\mu}}^T\widetilde{\mathbf{w}}-\frac{1}{2}\widetilde{\mathbf{w}}^T\mathbf{M}_{\text{ne}}\widetilde{\mathbf{w}}\\ \text{subject to} & \widetilde{\mathbf{w}}\in\widetilde{\mathcal{W}}_1\times\ldots\times\widetilde{\mathcal{W}}_N,\end{array} \quad (19)$$

where $\widetilde{\boldsymbol{\mu}}\triangleq(\widetilde{\boldsymbol{\mu}}_n)_{n=1}^N$, $\widetilde{\mathbf{w}}\triangleq(\widetilde{\mathbf{w}}_n)_{n=1}^N$,

$$\mathbf{M}_{\text{ne}}\triangleq\text{diag}(\boldsymbol{\rho})\otimes\widetilde{\mathbf{R}}+\frac{1}{2}\left(\mathbf{I}_N+\mathbf{J}_N\right)\otimes\widetilde{\boldsymbol{\Omega}}, \quad (20)$$

and $\mathbf{J}_n$ is an $n\times n$ matrix with all entries equal to 1.

Now we can obtain existence and uniqueness results of the NE by invoking existence and uniqueness results of an optimal solution of (19). They are stated in the following theorem.

Theorem 8. *There exists a unique NE of the NEP (18).*

Proof: The existence and uniqueness results follow from the strong convexity of (19). To prove this, we need to show there exists a positive constant c such that $-\nabla^2_{\widetilde{\mathbf{w}}}P_{\text{ne}}(\widetilde{\mathbf{w}})=\mathbf{M}_{\text{ne}}\succeq c\mathbf{I}$, which is equivalent to showing that $\mathbf{x}^T\mathbf{M}_{\text{ne}}\mathbf{x}\geq$

$c\|\mathbf{x}\|^2$ for any $\mathbf{x} = ([\mathbf{x}_n^+; \mathbf{x}_n^-])_{n=1}^N \in \mathbb{R}^{2NK}$:

$$\begin{aligned}\mathbf{x}^T\mathbf{M}_{\text{ne}}\mathbf{x} = & \sum\nolimits_{n=1}^N \rho_n(\mathbf{x}_n^+ - \mathbf{x}_n^-)^T\mathbf{R}(\mathbf{x}_n^+ - \mathbf{x}_n^-) \\ & + \Big(\sum\nolimits_{n=1}^N \mathbf{x}_n^+\Big)^T \boldsymbol{\Omega}^+ \Big(\sum\nolimits_{n=1}^N \mathbf{x}_n^+\Big) \\ & + \Big(\sum\nolimits_{n=1}^N \mathbf{x}_n^-\Big)^T \boldsymbol{\Omega}^- \Big(\sum\nolimits_{n=1}^N \mathbf{x}_n^-\Big) \\ & + \sum\nolimits_{n=1}^N (\mathbf{x}_n^+)^T\boldsymbol{\Omega}^+\mathbf{x}_n^+ + \sum\nolimits_{n=1}^N (\mathbf{x}_n^-)^T\boldsymbol{\Omega}^-\mathbf{x}_n^- \\ & \geq \min(\lambda_{\min}(\boldsymbol{\Omega}^+), \lambda_{\min}(\boldsymbol{\Omega}^-))\,\|\mathbf{x}\|^2,\end{aligned}$$

where the inequality comes from $\mathbf{R} \succ \mathbf{0}$ and $\boldsymbol{\Omega}^{+(-)} \succ \mathbf{0}$. This completes the proof. ■

The uniqueness property in Theorem 8 is a very general result and it holds whatever the (possibly different) constraints each account is subject to. Thanks to the uniqueness, one can predict the outcome of the game, making the game theoretical formulation a viable approach [12].

C. Total welfare maximization problem

Invoking Lemma 5, we can rewrite the total welfare maximization problem (14) in a more manageable form:

$$\begin{array}{ll}\underset{\widetilde{\mathbf{w}}}{\text{maximize}} & P_{\text{so}}(\widetilde{\mathbf{w}}) \triangleq \widetilde{\boldsymbol{\mu}}^T\widetilde{\mathbf{w}} - \frac{1}{2}\widetilde{\mathbf{w}}^T\mathbf{M}_{\text{so}}\widetilde{\mathbf{w}} \\ \text{subject to} & \widetilde{\mathbf{w}} \in \widetilde{\mathcal{W}}_1 \times \ldots \times \widetilde{\mathcal{W}}_N,\end{array} \tag{21}$$

where

$$\mathbf{M}_{\text{so}} \triangleq \text{diag}(\boldsymbol{\rho}) \otimes \widetilde{\mathbf{R}} + \mathbf{J}_N \otimes \widetilde{\boldsymbol{\Omega}}. \tag{22}$$

Following the proof of Theorem 8, it is not difficult to see that $P_{\text{so}}(\widetilde{\mathbf{w}})$ is a concave (but not strongly concave) function in $\widetilde{\mathbf{w}}$; therefore, (21) is a convex problem, having in general multiple solutions. This gives rise to a fairness issue as different socially optimal solutions may favor different accounts, and incurs additional difficulties in reaching an agreement on the desirable outcome among competing accounts or allocating liquidity among a portfolio manager's constituent accounts.

We see from (19) and (21) that the two functions $P_{\text{ne}}(\widetilde{\mathbf{w}})$ and $P_{\text{so}}(\widetilde{\mathbf{w}})$ are related as

$$\mathbf{M}_{\text{ne}} = \mathbf{M}_{\text{so}} - \frac{1}{2}(\mathbf{J}_N - \mathbf{I}_N) \otimes \widetilde{\boldsymbol{\Omega}}, \tag{23}$$

indicating that we can add a pricing term to each account's utility function $\widetilde{u}_n(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n})$ in (17) so that $\mathbf{M}_{\text{ne}}$ is identical to $\mathbf{M}_{\text{so}}$ and the NE is also socially optimal [9]. This conjecture is consolidated in the following proposition, which can be proved using the definition of potential games.

Proposition 9. *The total welfare maximization problem (21) is equivalent to the following NEP:*

$$\left.\begin{array}{ll}\underset{\widetilde{\mathbf{w}}_n}{\text{maximize}} & \widetilde{\boldsymbol{\mu}}_n^T\widetilde{\mathbf{w}}_n - \frac{1}{2}\rho_n\widetilde{\mathbf{w}}_n^T\widetilde{\mathbf{R}}\widetilde{\mathbf{w}}_n \\ & -\frac{1}{2}\widetilde{\mathbf{w}}_n^T\widetilde{\boldsymbol{\Omega}}\left(\widetilde{\mathbf{w}}_n + 2\sum_{m\neq n}\widetilde{\mathbf{w}}_m\right) \\ \text{subject to} & \widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n.\end{array}\right\}\ \forall n. \tag{24}$$

Note that the equivalence stated in Proposition 9 holds regardless of the types of individual constraints each account is subject to; the observation in [9] is thus greatly generalized. The interesting interpretation of the socially optimal solution as the NE of a modified NEP enables us to regard [7, 9] as special cases of our framework: both the NE and the socially optimal solution can be achieved by same algorithms.

Remark 10. It is beneficial mainly from a conceptual perspective to interpret the socially optimal solution as the NE of the NEP (24): the market impact price in the total welfare maximization problem is higher than the game theoretical formulation (the difference is $\frac{1}{2}\widetilde{\mathbf{w}}_n^T\widetilde{\boldsymbol{\Omega}}(\sum_{m\neq n}\widetilde{\mathbf{w}}_m) \geq 0$), so small accounts tend to trade less in socially optimal solutions.

IV. Synchronous and Asynchronous Iterative Distributed Algorithms

In view of Lemma 7, a NE of the NEP (18) is also an optimal solution of (19) and vice versa. This equivalence is exploited in [7, 9]: to calculate the NE, general-purpose centralized algorithms are applied to solve (19) directly. However, the complexity of (19) depends on the product of the number of accounts and the number of assets; thus they may not be efficient when the number of accounts and/or the number of assets are large. In what follows, we derive distributed algorithms to solve (19). They are desirable as they can make use of problem structure and decompose a large problem into a number of smaller problems, which are typically much easier to solve. Note that the convergence analysis of the following distributed algorithms is similar for the social formulation (21), and it is omitted here due to space constraints.

A. Synchronous algorithms

We consider iterative algorithms based on sequential or simultaneous updates of each account's strategy profile based on single-account best response, given by Algorithm 1. Both sequential and simultaneous algorithms have some desirable properties that make them appealing in practice, namely: low complexity, distributed nature, and fast convergence behavior.

Algorithm 1: Sequential/Simultaneous Best-Response Algorithm

Data: $\widetilde{\mathbf{w}}_n^0 \in \widetilde{\mathcal{W}}_n$ for $n = 1, \ldots, N$. Set $q = 0$.
(S.1): If $\widetilde{\mathbf{w}}^q$ satisfies a termination criterion: STOP.
(S.2): Sequentially or simultaneously update $\widetilde{\mathbf{w}}_n^{q+1}$:

Sequential Update: $\widetilde{\mathbf{w}}_n^{q+1} =$
$$\arg\max_{\widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n} \widetilde{u}_n\left(\widetilde{\mathbf{w}}_{1,\ldots,n-1}^{q+1}, \widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{n+1,\ldots,N}^q\right);$$
Simultaneous Update: $\widetilde{\mathbf{w}}_n^{q+1} =$
$$(1-\tfrac{1}{N})\widetilde{\mathbf{w}}_n^q + \tfrac{1}{N}\arg\max_{\widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n} \widetilde{u}_n\left(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}^q\right).$$
(S.3): Set $q \leftarrow q + 1$; go to (S.1).

The convergence properties of Algorithm 1 are given in the following proposition, whose proof follows standard arguments; see [30, Prop. 2.7.1] and [30, Ex. 1.8.2].

Proposition 11. *Any sequence $\{\widetilde{\mathbf{w}}^q\}_{q=0}^\infty$ generated by the sequential/simultaneous best-response updates in Algorithm 1 converges to the NE of the NEP (18).*

In Algorithm 1, the multi-portfolio optimization problem (19) is decomposed into N small problems, and we refer to each small problem as "single-account problem". These single-account problems are solved either sequentially or simultaneously by the portfolio manager. Each single-account problem is strongly convex, and existing single-account portfolio optimization infrastructure can readily be applied.

Information exchange: Since Algorithm 1 is an iterative algorithm, some information exchange is required among different single-account problems in each iteration, but this is maintained at a very low level. For example, in the context of simultaneous update (the analysis for sequential update is similar), the n-th single-account problem in Step 2 is

$$\begin{aligned} \underset{\widetilde{\mathbf{w}}_n}{\text{maximize}} \quad & \widetilde{\boldsymbol{\mu}}_n^T \widetilde{\mathbf{w}}_n - \tfrac{1}{2}\rho_n \widetilde{\mathbf{w}}_n^T \widetilde{\mathbf{R}} \widetilde{\mathbf{w}}_n \\ & -\tfrac{1}{2}\widetilde{\mathbf{w}}_n^T \widetilde{\boldsymbol{\Omega}} \left(\widetilde{\mathbf{w}}_n + \textstyle\sum_{m\neq n} \widetilde{\mathbf{w}}_m^q\right) \\ \text{subject to} \quad & \widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n. \end{aligned}$$

Therefore, the only information required in iteration $q+1$ is the aggregate trading vector $\sum_{n=1}^N \widetilde{\mathbf{w}}_n^q$ in iteration q. Note that the exact trading strategies $\widetilde{\mathbf{w}}_1^q, \ldots, \widetilde{\mathbf{w}}_N^q$ are not required, so the privacy of individual accounts is preserved.

Complexity analysis: Recall that the variable dimension of (19) is $2NK$ and the number of constraints is MK (suppose for simplicity M is the equal number of individual constraints for each account), while in the single-account problem, the dimension of the single portfolio vector $\widetilde{\mathbf{w}}_n$ is $2K$ and the number of constraints is M; thus the reduction in complexity is notable especially when N and K are large. Although (19) can be solved in "one shot" by general-purpose centralized algorithms, the distributed algorithms often converge reasonably fast, with the advantage that the privacy of each individual account is preserved as each single account problem does not know the specific strategies of other accounts.

B. Asynchronous algorithms

Both sequential and simultaneous algorithms place synchronization requirements among different single-account problems. This requirement may be restrictive when, for example, some single-account problems need more time to solve (this happens when, e.g., some accounts have many trading constraints) and others have to wait for them to finish. This delay could also result in another difficulty, namely, the latest aggregate trading vector may not be available for some single-account problems. To deal with these issues, we introduce in this subsection an asynchronous algorithm (in the sense specified in [31]) in which some portfolio vectors can be updated more frequently than others, and the update can even be based on outdated information.

To provide a formal description of the asynchronous algorithm, we need to introduce some preliminary definitions. Let $\mathcal{T}_n \subseteq \mathcal{T} \subseteq \{0,1,2,\ldots\}$ be the set of times at which $\widetilde{\mathbf{w}}_n$ is updated (thus implying that $\widetilde{\mathbf{w}}_n^q$ is left unchanged if $q \notin \mathcal{T}_n$). Let $\tau_m^n(q)$ denote the most recent time at which the strategy profile of account m is perceived at the n-th single-account problem in the q-th iteration (observe that $0 \leq \tau_m^n(q) \leq q$). Hence, if $\widetilde{\mathbf{w}}_n$ is to be updated at the q-th iteration, then $\widetilde{u}_n(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n})$ is maximized using the outdated strategy profile of the other accounts denoted by $\widetilde{\mathbf{w}}_{-n}^{\tau^n(q)} \triangleq (\widetilde{\mathbf{w}}_m^{\tau_m^n(q)})_{m\neq n}$.

We assume that there exists a positive constant B, called asynchronous measure, such that 1) the strategy variable of each account is updated at least once during any time interval of length B, and 2) the information used by any single-account problem is outdated by at most B time units. The asynchronous algorithm is formally described in Algorithm 2.

Algorithm 2: Asynchronous Best-Response Algorithm

Data: $\widetilde{\mathbf{w}}_n^0 \in \widetilde{\mathcal{W}}_n$ for $n = 1, \ldots, N$; stepsize γ. Set $q = 0$.
(`S.1`): If $\widetilde{\mathbf{w}}^q$ satisfies a termination criterion: `STOP`.
(`S.2`): For $n = 1, \ldots, N$, if $q \in \mathcal{T}_n$, update $\widetilde{\mathbf{w}}_n^{q+1}$ as

$$\widetilde{\mathbf{w}}_n^{q+1} = (1-\gamma)\widetilde{\mathbf{w}}_n^q + \gamma \cdot \underset{\widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n}{\arg\max}\, \widetilde{u}_n\left(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}^{\tau^n(q)}\right).$$

Otherwise it is left unchanged.
(`S.3`): $q \leftarrow q+1$; go to (`S.1`).

Convergence of Algorithm 2 is stated in the following proposition, whose proof follows the same ideas as in [31, Sec. 7.5, Prop. 5.3] and thus is omitted.

Proposition 12. *Any sequence $\{\widetilde{\mathbf{w}}^q\}_{q=0}^\infty$ generated by Algorithm 2 converges to the NE of the NEP (18) if*

$$0 < \gamma < \frac{\min_n \lambda_{\min}\left(\rho_n \widetilde{\mathbf{R}} + \widetilde{\boldsymbol{\Omega}}\right)}{(1+B+NB)\,\lambda_{\max}\left(\mathbf{M}_{\text{ne}}\right)}.$$

It is easy to see that since the stepsize γ is inversely proportional to the asynchronous measure B, there is a tradeoff between asynchronous measure and convergence speed.

Thanks to the asynchronous algorithm, the coordination among different problems is maintained at a minimum level, making the distributed algorithms very practical.

V. Multi-Portfolio Optimization Problem with Global Constraints

A. Nash Equilibrium problems with global constraints

In all previous developments we have considered individual constraints and coupling among accounts is only in the utility functions. In this section, we consider the more general scenario in which there is also coupling in each account's strategy set. For example, one account's trading volume on a particular asset can be limited by other accounts because of the ADV of the assets in the common investment universe.

The coupling in each account's strategy set can be modeled as global constraints over all accounts [19, 21, 25, 26, 29]. This results in a NEP with global coupling constraints, termed as GNEP, defined in (10). It is easy to verify that the conclusions of Lemma 5 still hold when there are global constraints, so we can rewrite (10) in terms of the new variables $\widetilde{\mathbf{w}}$:

$$\left.\begin{aligned} \underset{\mathbf{w}_n}{\text{maximize}} \quad & \widetilde{u}_n(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}) \\ \text{subject to} \quad & \widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n \\ & \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}) \leq 0 \end{aligned}\right\} \forall n, \tag{25}$$

where the utility function is defined in (17), and $\widetilde{\mathbf{g}}(\widetilde{\mathbf{w}})$ is the global constraint (11) written in terms of the new variables $\widetilde{\mathbf{w}}$:

$$\widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}) = \sum_{n=1}^N \widetilde{\mathbf{g}}_n(\widetilde{\mathbf{w}}_n) - \begin{bmatrix} (D_k)_{k=1}^K \\ (U_l)_{l=1}^L \end{bmatrix}, \tag{26a}$$

with

$$\widetilde{\mathbf{g}}_n(\widetilde{\mathbf{w}}_n) = \begin{bmatrix} \left(\widetilde{w}_{n,k}^{+} + \widetilde{w}_{n,k}^{-}\right)_{k=1}^{K} \\ \left(\sum_{j\in\mathcal{J}_l}\left(\widetilde{w}_{n,j}^{+} + \widetilde{w}_{n,j}^{-}\right) - U_l\right)_{l=1}^{L} \end{bmatrix}. \quad (26b)$$

B. Characterization of generalized Nash Equilibrium and socially optimal solutions

It is easy to see from Definition 3 that the definition of potential functions can readily be extended to the GNEPs (so $P_{\text{ne}}(\widetilde{\mathbf{w}})$ in (19) is a potential function of the GNEP (25)) and if $\mathbf{w}^\star$ maximizes the potential function, it is also a GNE of the GNEP. But differently from the NEP (18), the GNE $\mathbf{w}_{\text{ne}}$ of the GNEP (25) does not necessarily maximize the potential function over the joint strategy set (12) [28]:

$$\begin{array}{ll} \underset{\widetilde{\mathbf{w}}}{\text{maximize}} & P_{\text{ne}}(\widetilde{\mathbf{w}}) = \widetilde{\boldsymbol{\mu}}^T\widetilde{\mathbf{w}} - \frac{1}{2}\widetilde{\mathbf{w}}^T\mathbf{M}_{\text{ne}}\widetilde{\mathbf{w}} \\ \text{subject to} & \widetilde{\mathbf{w}} \in \widetilde{\mathcal{W}}_1 \times \ldots \times \widetilde{\mathcal{W}}_N, \\ & \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}) \leq 0. \end{array} \quad (27)$$

This is because the Cartesian structure in the joint strategy set of all accounts is destroyed by the global constraints [cf. (12)].

Inspired by [26], we use a well-known result in convex analysis to derive the relationship between a GNE of the GNEP (25) and an optimal solution of (27): for a convex optimization problem with strong duality, the pair of primal optimal solution and dual optimal solution is a saddle point of the Lagrangian [32, Th. 28.3]. Specifically, we assume that some constraint qualifications such as Slater's condition are satisfied for (25) and (27). Then let $\widetilde{\mathbf{w}}_{\text{ne}} = (\widetilde{\mathbf{w}}_n^\star)_{n=1}^N$ be a GNE of the GNEP (25), there exists $(\boldsymbol{\lambda}_n^\star)_{n=1}^N \geq \mathbf{0}$ such that

$$\begin{array}{l} \widetilde{\mathbf{w}}_n^\star = \arg\max_{\widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n} \widetilde{u}_n(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}^\star) - \left\langle \boldsymbol{\lambda}_n^\star, \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}^\star)\right\rangle, \\ \mathbf{0} \leq \boldsymbol{\lambda}_n^\star \perp \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}_n^\star, \widetilde{\mathbf{w}}_{-n}^\star) \leq \mathbf{0}, \forall n, \end{array} \quad (28)$$

where $\mathbf{a} \perp \mathbf{b}$ means $\mathbf{a}^T\mathbf{b} = 0$. Similarly, let $\widetilde{\mathbf{w}}^\star$ be an optimal solution of (27), there exists $\boldsymbol{\xi}^\star \geq \mathbf{0}$ such that

$$\begin{array}{l} \widetilde{\mathbf{w}}^\star = \arg\max_{\widetilde{\mathbf{w}} \in \widetilde{\mathcal{W}}_1 \times \ldots \times \widetilde{\mathcal{W}}_N} P_{\text{ne}}(\widetilde{\mathbf{w}}) - \left\langle \boldsymbol{\xi}^\star, \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}})\right\rangle, \\ \mathbf{0} \leq \boldsymbol{\xi}^\star \perp \widetilde{\mathbf{g}}(\widetilde{\mathbf{w}}^\star) \leq \mathbf{0}. \end{array} \quad (29)$$

A comparison of (28) and (29) enables us to give a precise connection between the GNE of a GNEP (25) and the optimal solution of its potential game formulation (27), as summarized in the following proposition.

Proposition 13. *Suppose that $\widetilde{\mathbf{w}}^\star$ is an optimal solution of (27) and $(\widetilde{\mathbf{w}}^\star, \boldsymbol{\xi}^\star)$ satisfies (29). Then $\widetilde{\mathbf{w}}^\star$ is a GNE of the GNEP (25), and (28) holds with $\boldsymbol{\lambda}_1^\star = \boldsymbol{\lambda}_2^\star = \ldots = \boldsymbol{\lambda}_N^\star = \boldsymbol{\xi}^\star$. Conversely, suppose that $\widetilde{\mathbf{w}}_{\text{ne}}$ is a GNE of the GNEP (25), and (28) holds with $\boldsymbol{\lambda}_1^\star = \boldsymbol{\lambda}_2^\star = \ldots = \boldsymbol{\lambda}_N^\star \triangleq \boldsymbol{\xi}^\star$, then $\widetilde{\mathbf{w}}_{\text{ne}}$ is an optimal solution of (27) and $(\widetilde{\mathbf{w}}_{\text{ne}}, \boldsymbol{\xi}^\star)$ satisfies (29).*

To summarize, a GNE of the GNEP (25) is generally not an optimal solution of (27), unless at the GNE, the dual variables associated with the global constraints for all accounts are identical. The GNE of the GNEP (25) that is also the optimal solution of (27) are termed *Variational Equilibrium* (VE), denoted as $\widetilde{\mathbf{w}}_{\text{ve}}$. From now on, we mainly focus on the VE of the GNEP (25), whose (existence and) uniqueness comes readily from the strong convexity of (27).

Corollary 14. *The GNEP (25) has a unique VE.*

Similarly to the NEP case in Section III, the socially optimal solution of GNEP (25) can also be interpreted as a GNE of a modified GNEP, which is the NEP (24) with the additional global constraint $\widetilde{\mathbf{g}}(\widetilde{\mathbf{w}})$.

C. Distributed algorithms

The potential game formulation of the GNEP (25), i.e., (27), not only serves as a direct way to characterize the VE, but also provides us with some intuition to devise distributed algorithms. We develop next a distributed algorithm converging to the VE of the GNEP (25). We introduce the algorithm in a general setting, so that it can be applied to a broader class of GNEPs, including the GNEP (25) as a special case.

Towards this end, consider a generic GNEP where account n solves the following convex optimization problem

$$\left.\begin{array}{ll} \underset{\mathbf{w}_n}{\text{maximize}} & u_n(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{W}_n \\ & \mathbf{g}(\mathbf{w}) \leq \mathbf{0}. \end{array}\right\} \forall n, \quad (30)$$

where $u_n(\bullet, \mathbf{w}_{-n})$ is concave on $\mathcal{W}_n$, $\mathbf{g}(\bullet)$ is convex on $\mathcal{W}_1 \times \ldots \times \mathcal{W}_N$, and $\mathcal{W}_n$ is closed and convex. Suppose the NEP (30) has a differentiable concave potential function $P(\mathbf{w})$ while some constraint qualifications such as Slater's condition are satisfied. We also introduce a new NEP

$$\left.\begin{array}{ll} \underset{\mathbf{w}_n}{\text{maximize}} & u_n(\mathbf{w}_n, \mathbf{w}_{-n}) - \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{W}_n \end{array}\right\} \forall n \quad (31)$$

and denote its NE for a given $\boldsymbol{\lambda}$ as $\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda})$.

The relationship between the GNEP (30) and the NEP (31) is given in the following theorem.

Theorem 15. In the setting above, $\mathbf{w}_{\text{ve}}$ *is a VE of the GNEP (30) if and only if* $\mathbf{w}_{\text{ve}} = \mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\star)$*, where* $(\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\star), \boldsymbol{\lambda}^\star)$ *satisfies*

$$\mathbf{0} \leq \boldsymbol{\lambda}^\star \perp \mathbf{g}(\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\star)) \leq \mathbf{0}. \quad (32)$$

Proof: See Appendix B. ■

Note that in Theorem 15, we have only assumed that (30) is convex and the potential function $P(\mathbf{w})$ is concave on $\mathcal{K}$. It therefore applies to a broad category of potential games including the GNEP (25).

From the perspective of duality theory [32], $\boldsymbol{\lambda}$ is the Lagrange multiplier associated with the constraint $\mathbf{g}(\mathbf{w})$ and $\boldsymbol{\lambda}^\star$ is the dual optimal solution. We can also interpret $\boldsymbol{\lambda}^\star$ as prices paid by the accounts for the common "resources" represented by the global constraints. The complementary slackness condition in (32) says that we only have to pay when the resources become scarce; the price is 0 when there are enough resources.

Thanks to Theorem 15, we have transformed the computation of the VE of the GNEP (30) into that of a NE of the NEP (31). By doing that, we have decoupled the constrains on the accounts by incorporating the global constraints as part of the utility function. This transformation is beneficial because we can achieve the NE of the NEP (31) –the VE of the GNEP (30)– in a distributed manner using Algorithms 1 or 2. Of course, $\boldsymbol{\lambda}^\star$ in (32) is unknown a priori, but it can be found by, for example, subgradient method. Specifically, we can design a double-loop algorithm: in the inner loop, given the price $\boldsymbol{\lambda}$, one computes the unique NE $\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda})$ of the NEP (31); in the outer loop, the price $\boldsymbol{\lambda}$ is updated according to a subgradient-based projection method [note that a subgradient at $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\nu$ is $\mathbf{g}\left(\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\nu)\right)$]. We summarize this double-loop algorithm in Algorithm 3.

Algorithm 3: Subgradient Projection Algorithm

Data: $\boldsymbol{\lambda}^0 \geq \mathbf{0}$; stepsize sequence $\{\lambda^\nu\}$; set $\nu = 0$.
(S.1) : If $\boldsymbol{\lambda}^\nu$ satisfies a termination criterion: STOP.
(S.2) : Compute the unique NE $\mathbf{w}_{\text{ne}}\left(\boldsymbol{\lambda}^\nu\right)$ of NEP (31).
(S.3) : Update $\boldsymbol{\lambda}^\nu$ according to

$$\boldsymbol{\lambda}^{\nu+1} = \left[\boldsymbol{\lambda}^\nu - \gamma^\nu \mathbf{g}\left(\mathbf{w}_{\text{ne}}\left(\boldsymbol{\lambda}^\nu\right)\right)\right]^+ .$$

(S.4) : $\nu \leftarrow \nu + 1$; go to (S.1).

There are some well-known stepsize rules to guarantee the convergence to $\boldsymbol{\lambda}^\star$. For example, one can use [33]:

$$\gamma^\nu \to 0, \quad \sum\nolimits_{\nu=0}^{\infty} \gamma^\nu = \infty, \quad \sum\nolimits_{\nu=0}^{\infty} (\gamma^\nu)^2 < \infty.$$

If the potential function of (30) is strongly concave, which is indeed the case for $P_{\text{ne}}(\widetilde{\mathbf{w}})$ in (27), Algorithm 3 would also converge under a constant (but sufficiently small) stepsize [34]:

$$\gamma \leq \frac{\sqrt{2}\lambda_{\min}(\mathbf{M}_{\text{ne}})}{\sqrt{KN + \sum_{l=1}^{L} \sigma_{\max}(\mathbf{1}^T(\mathbf{I}_N \otimes \mathbf{s}_l^T))^2}},$$

where $\mathbf{s}_l = (s_{l,k})_{k=1}^K$ with $s_{l,k} = 1$ if $k \in \mathcal{J}_l$ and 0 otherwise, and $\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\star)$ is always feasible, i.e., $\mathbf{g}(\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\star)) \leq \mathbf{0}$. If the potential function is concave but not strongly concave [the case for $P_{\text{so}}(\widetilde{\mathbf{w}})$ in (21)], $\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\star)$ is not necessarily feasible, but one can deal with this issue by averaging all intermediate variables $\{\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}^\nu)\}_\nu$; see [35] for more details.

In the inner loop (**Step 2**) of Algorithm 3, NEP (31) can be solved by Algorithms 1 or 2, leading to a distributed design. An instance of simultaneous update among accounts for Step 2 of Algorithm 3 is summarized in Algorithm 4.

Algorithm 4 : Distributed Implementations for Step 2 of Algorithm 3

Data: $\boldsymbol{\lambda}^\nu$, $\mathbf{w}_n^0(\boldsymbol{\lambda}^\nu) \in \mathcal{W}_n$ for $n = 1, \ldots, N$. Set $q = 0$.
(S.2a): If $\mathbf{w}^q(\boldsymbol{\lambda}^\nu)$ satisfies a termination criterion: STOP.
(S.2b): Simultaneously update $\mathbf{w}_n^{q+1}(\boldsymbol{\lambda}^\nu)$ as

$$\mathbf{w}_n^{q+1}(\boldsymbol{\lambda}^\nu) = (1 - \frac{1}{N})\mathbf{w}_n^q(\boldsymbol{\lambda}^\nu) + \frac{1}{N}\widehat{\mathbf{w}}_n^q(\boldsymbol{\lambda}^\nu), \quad n = 1, \ldots, N,$$

where $\widehat{\mathbf{w}}_n^q(\boldsymbol{\lambda}^\nu)$ is

$$\underset{\mathbf{w}_n \in \mathcal{W}_n}{\arg\max} \quad u_n(\mathbf{w}_n, \mathbf{w}_{-n}^q(\boldsymbol{\lambda}^\nu)) - \left\langle \boldsymbol{\lambda}^\nu, \mathbf{g}(\mathbf{w}_n, \mathbf{w}_{-n}^q(\boldsymbol{\lambda}^\nu)) \right\rangle .$$

(S.2c): Set $q \leftarrow q + 1$ and go back to (S.2a).

Specializing the general formulation (30) to the multi-portfolio optimization problem in (25), and invoking the separable structure of $\widetilde{\mathbf{g}}(\widetilde{\mathbf{w}})$ in (26), the NEP formulation in (31) can be further simplified as

$$\left.\begin{array}{ll} \underset{\widetilde{\mathbf{w}}_n}{\text{maximize}} & \widetilde{u}_n(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}) - \boldsymbol{\lambda}^T \widetilde{\mathbf{g}}_n(\widetilde{\mathbf{w}}_n) \\ \text{subject to} & \widetilde{\mathbf{w}}_n \in \widetilde{\mathcal{W}}_n \end{array}\right\} \forall n. \quad (33)$$

Therefore, to solve the NEP (33) in each iteration of Algorithms 1 or 2, the only information required by the n-th single-account problem is the aggregate trading vector from the preceding iteration, and the remarks on implementation issues of Algorithms 1-2 in Section IV readily apply here.

In the outer loop (**Step 3**) of Algorithm 3, to update the price vector $\boldsymbol{\lambda}$, the portfolio manager needs to collect the aggregate trading vector of some particular (groups of) assets: $\sum_{n=1}^N \widetilde{\mathbf{g}}_n\left(\widetilde{\mathbf{w}}_n^\star(\boldsymbol{\lambda})\right)$ (recall that $\mathbf{w}_n^\star(\boldsymbol{\lambda})$'s are constituent vectors of $\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda})$: $\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}) = (\mathbf{w}_n^\star(\boldsymbol{\lambda}))_{n=1}^N$). The price vector $\boldsymbol{\lambda}$ is then adjusted according to the (inexpensive) subgradient projection.

VI. Numerical Results and Discussions

In this section, we provide some numerical results to illustrate the efficacy of our multi-portfolio optimization framework, along with the convergence behavior of the proposed distributed iterative algorithms. In our simulations, we consider synthetic data such that $\boldsymbol{\mu}$, $\mathbf{R}$, and $\boldsymbol{\Omega}$ model annual expected values from -5% to 5% and volatility values given in annualized terms in the range of 20% to 30%. We assume that the number of assets is $K = 5$.

Utility improvement: We first compare each account's utility improvement achieved by NE and socially optimal solutions over the naive approach measured by:

$$\left.\frac{u_n(\mathbf{w}) - u_n(\mathbf{w}_{\text{naive}})}{u_n(\mathbf{w}_{\text{naive}})}\right|_{\mathbf{w}=\mathbf{w}_{\text{ne}} \text{ or } \mathbf{w}_{\text{so}}},$$

where $\mathbf{w}_{\text{naive}}$ is the optimal solution of (8) with $\mathbf{w}_n^0 = \mathbf{0}$, and $u_n(\mathbf{w})$ is defined in (4). We assume that there are $N = 5$ accounts and they are subject to the *long-only* constraint and *budget* constraint. The result is plotted in Figure 1. We can see from the red bar that the performance of the NE outperforms the naive design, because the market impact cost incurred from transactions of other accounts are properly counted.

We also compare the NE (red bar on the left) and the socially optimal solution (black bar on the right). We can see that the social optimality is at the price of accounts 1, 3 and 4. This consolidates again what has been observed in [9]: some accounts can probably get better payoff by acting alone than staying in the socially optimal solution. The unilateral optimality and the uniqueness makes the NE a meaningful outcome that can be predicted by all accounts.

To compare the NE and the socially optimal solution from the perspective of total welfare, we also plot in dashed lines the following metric:

$$\left.\frac{\sum_{n=1}^N u_n(\mathbf{w}) - \sum_{n=1}^N u_n(\mathbf{w}_{\text{naive}})}{\sum_{n=1}^N u_n(\mathbf{w}_{\text{naive}})}\right|_{\mathbf{w}=\mathbf{w}_{\text{ne}} \text{ or } \mathbf{w}_{\text{so}}}.$$

As expected, socially optimal solutions can achieve a higher total welfare than NE.

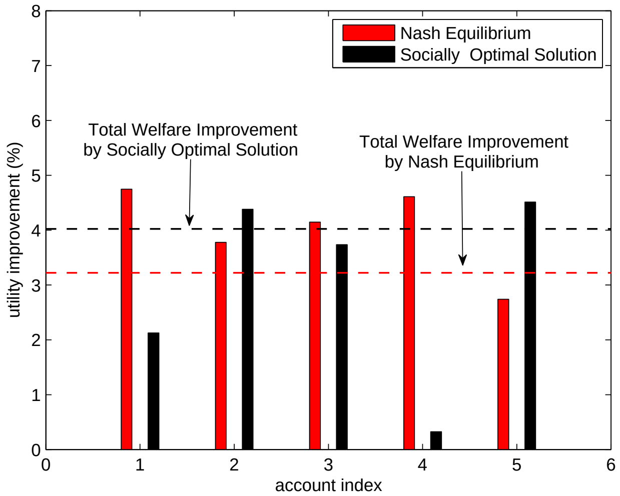

Figure 1. Utility improvement of the NE and socially optimal solution against the naive approach.

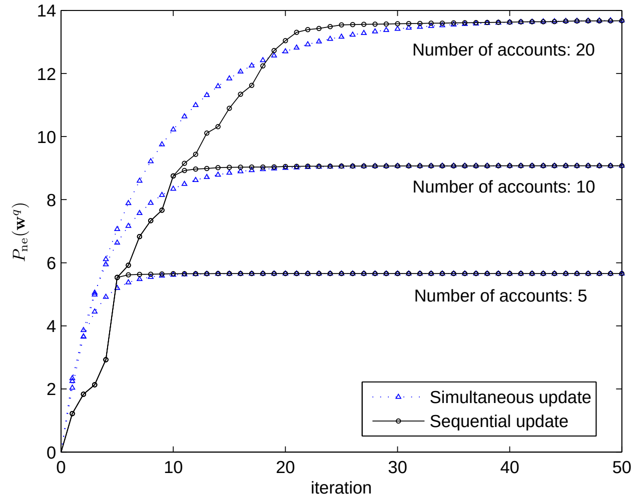

Figure 2. Convergence of Algorithm 1: potential function versus iteration.

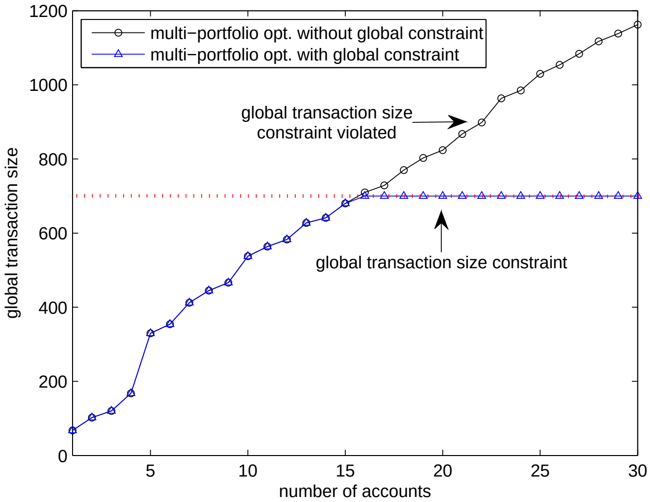

Figure 3. Global transaction-size versus number of accounts.

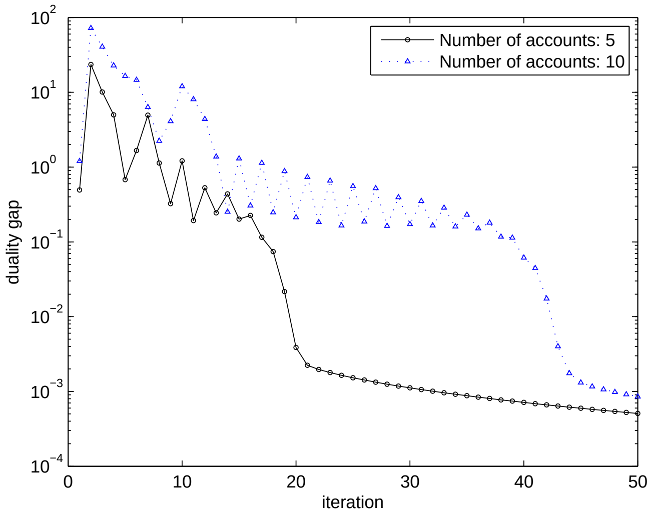

Figure 4. Convergence of Algorithm 3: duality gap versus iteration.

Convergence of Algorithm 1: We assume that the number of accounts is 5, 10 and 20, respectively, and each account is subject to the *long-only* constraint. The results are illustrated in Figure 2, where we update the portfolio in each iteration and the resulting value of the potential function $P_{\text{ne}}(\mathbf{w})$ is plotted. We can see that the algorithm converges reasonably fast for both sequential and simultaneous update, with the convergence speed depending as expected upon the number of accounts.

Global constraint: We assume that each account is subject to the *long-only* constraint. In Figure 3, we can see that, as the number of accounts increases, the global transaction-size constraint may be violated if it is not properly considered. Motivated by liquidity problems for a specific asset in practice, the issue is specially aggravated due to the aggregate effect over accounts.

Convergence of outer loop of Algorithm 3: We assume that the number of accounts is 5 and 10, respectively. Each account is subject to the *long-only* constraint, and the accounts are also subject to the *global transaction-size* constraint as (5). The convergence behavior of the outer loop of Algorithm 3 is illustrated in Figure 4, where in each iteration we generate the NE for a fixed $\boldsymbol{\lambda}$ and the corresponding duality gap (defined as $P_{\text{ne}}(\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}))-\boldsymbol{\lambda}^T\mathbf{g}(\mathbf{w}_{\text{ne}}(\boldsymbol{\lambda}))-P_{\text{ne}}(\mathbf{w}_{\text{ve}})$ where $\mathbf{w}_{\text{ve}}$ is obtained a priori from solving (27) by CVX [36]) is plotted. We see that the asymptotic convergence speed of $\boldsymbol{\lambda}$ is fast and independent of the number of accounts, since the GNEP (25) is solved in its dual domain and the dimension of the dual variable is equal to the number of global constraints.

VII. Concluding Remarks

In this paper, we have studied the multi-portfolio optimization problem where multiple accounts are coupled through the market impact cost, which is modeled as an affine function of the aggregate trades from all accounts. The analysis is from the perspective of non-cooperative game theory, and we have shown that there always exists a unique NE, and moreover devised (synchronous and asynchronous) distributed algorithms with satisfactory convergence properties. Then we have analyzed the NEP with global constraints imposed on all accounts, resulting in a GNEP. We have shown as well that there always exists a unique VE which can be computed in a distributed manner. Finally, we have considered the maximization of the total welfare along with distributed schemes.

APPENDIX A
PROOF OF LEMMA 5

Proof: In (17), the utility function of account n is

$$\begin{aligned}\widetilde{u}_n(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}) = &\left(\boldsymbol{\mu} - \rho_n \mathbf{R}\mathbf{w}_n^0\right)^T \left(\widetilde{\mathbf{w}}_n^+ - \mathbf{w}_n^-\right) \\ &- \frac{1}{2}\rho_n \left(\widetilde{\mathbf{w}}_n^+ - \widetilde{\mathbf{w}}_n^-\right)^T \mathbf{R} \left(\widetilde{\mathbf{w}}_n^+ - \widetilde{\mathbf{w}}_n^-\right) \\ &- \frac{1}{2}\left(\widetilde{\mathbf{w}}_n^+\right)^T \boldsymbol{\Omega}^+ \Big(\sum\nolimits_{m=1}^N \widetilde{\mathbf{w}}_m^+\Big) \\ &- \frac{1}{2}\left(\widetilde{\mathbf{w}}_n^-\right)^T \boldsymbol{\Omega}^- \Big(\sum\nolimits_{m=1}^N \widetilde{\mathbf{w}}_m^-\Big),\end{aligned} \tag{34}$$

and the constraint is $\widetilde{\mathbf{w}}_n^+ - \widetilde{\mathbf{w}}_n^- \in \mathcal{W}_n$. The former two terms of (34) depend only on the difference between $\widetilde{\mathbf{w}}_n^+$ and $\widetilde{\mathbf{w}}_n^-$. We use contradiction to show that at the optimal solution, $\widetilde{\mathbf{w}}_n^{+\star}$ and $\widetilde{\mathbf{w}}_n^{-\star}$ are orthogonal.

First assume that there exists k such that $c_{n,k} \triangleq \min(\widetilde{w}_{n,k}^{+\star}, \widetilde{w}_{n,k}^{-\star}) > 0$. It is easy to see that the variable $\left[\overline{\mathbf{w}}_n^+; \overline{\mathbf{w}}_n^-\right] \triangleq [\widetilde{\mathbf{w}}_n^{+\star} - c_{n,k}\mathbf{e}_k; \widetilde{\mathbf{w}}_n^{-\star} - c_{n,k}\mathbf{e}_k]$ is feasible since $\overline{\mathbf{w}}_n^+ - \overline{\mathbf{w}}_n^- = \widetilde{\mathbf{w}}_n^+ - \widetilde{\mathbf{w}}_n^- \in \mathcal{W}_n$. Consider a new function $f(x)$ with $x \geq 0$ defined as

$$\begin{aligned}f(x) \triangleq &\frac{1}{2}\left(\overline{\mathbf{w}}_n^+ + x\mathbf{e}_k\right)^T \boldsymbol{\Omega}^+ \Big(\overline{\mathbf{w}}_n^+ + x\mathbf{e}_k + \sum\nolimits_{m\neq n} \widetilde{\mathbf{w}}_m^+\Big) \\ &+ \frac{1}{2}\left(\overline{\mathbf{w}}_n^- + x\mathbf{e}_k\right)^T \boldsymbol{\Omega}^- \Big(\overline{\mathbf{w}}_n^- + x\mathbf{e}_k + \sum\nolimits_{m\neq n} \widetilde{\mathbf{w}}_m^-\Big),\end{aligned}$$

which is convex in x. The convexity of $f(x)$ infers that $x^\star = 0$ minimizes $f(x)$ over $x \geq 0$ iff $\nabla f(0) \geq 0$:

$$\begin{aligned}\nabla f(0) = &\boldsymbol{\Omega}_{kk}^+ \Big(\overline{w}_{n,k}^{+\star} + \frac{1}{2}\sum\nolimits_{m\neq n}^N \widetilde{w}_{m,k}^+\Big) \\ &+ \boldsymbol{\Omega}_{kk}^- \Big(\overline{w}_{n,k}^{-\star} + \frac{1}{2}\sum\nolimits_{m\neq n}^N \widetilde{w}_{m,k}^-\Big) \geq 0,\end{aligned}$$

where we have made use of the fact that $\boldsymbol{\Omega}^{+(-)}$ are positive diagonal matrices and $\widetilde{\mathbf{w}}_{-n}^{+(-)} \geq \mathbf{0}$. This establishes that $x^\star = 0$ minimizes $f(x)$ over $x \geq 0$, and $\left[\overline{\mathbf{w}}_n^+; \overline{\mathbf{w}}_n^-\right]$ is the maximizing variable of $\widetilde{u}_n\left(\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_{-n}\right)$ in (34), contradicting the optimality of $[\widetilde{\mathbf{w}}_n^{+\star}; \widetilde{\mathbf{w}}_n^{-\star}]$. This completes the proof. ■

APPENDIX B
PROOF OF THEOREM 15

Proof: A variable $\mathbf{w}^\star$ is a VE of the GNEP (30) if and only if it solves the following optimization problem:

$$\begin{aligned}\underset{\mathbf{w}}{\text{maximize}} \quad & P(\mathbf{w}) \\ \text{subject to} \quad & \mathbf{w} \in \mathcal{W}_1 \times \ldots \times \mathcal{W}_N \\ & \mathbf{g}(\mathbf{w}) \leq \mathbf{0}.\end{aligned} \tag{35}$$

Since (35) is a convex optimization problem, the optimal solution of (35) can be equally achieved from its dual problem, provided Slater's condition is satisfied [2]:

$$\underset{\boldsymbol{\lambda}\geq\mathbf{0}}{\text{minimize}} \quad Q(\boldsymbol{\lambda}) \tag{36}$$

where $Q(\boldsymbol{\lambda}) \triangleq \max_{\mathbf{w}\in\mathcal{W}_1\times\ldots\times\mathcal{W}_N} P(\mathbf{w}) - \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{w})$ and $\boldsymbol{\lambda}$ is the Lagrange multiplier associated with $\mathbf{g}(\mathbf{w}) \leq 0$.

For a fixed $\boldsymbol{\lambda}$, the inner maximization problem in (36) is a potential game equivalent to the following NEP:

$$\left.\begin{aligned}\underset{\mathbf{w}_n}{\text{maximize}} \quad & u_n(\mathbf{w}_n, \mathbf{w}_{-n}) - \boldsymbol{\lambda}^T\mathbf{g}(\mathbf{w}) \\ \text{subject to} \quad & \mathbf{w}_n \in \mathcal{W}_n.\end{aligned}\right\} \forall n. \tag{37}$$

Since $(\mathbf{w}^\star, \boldsymbol{\lambda}^\star)$ is a saddle point of the minimax problem (36) [32], $\mathbf{w}^\star$ can be obtained by solving (37) with $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\star$ while $(\mathbf{w}^\star, \boldsymbol{\lambda}^\star)$ are primal feasible, dual feasible and satisfy the complementary slackness condition. ■

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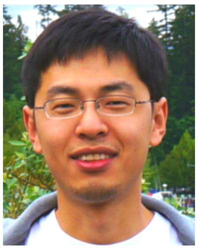

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