HARNACK ESTIMATES FOR NONLINEAR PARABOLIC EQUATIONS UNDER THE RICCI FLOW

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ABSTRACT. In this paper, we consider first the Li-Yau Harnack estimates for a nonlinear parabolic equation $\partial_t u = \Delta_t u - qu - au(\ln u)^\alpha$ under the Ricci flow, where $\alpha > 0$ is a constant. To extend these estimates to a more general situation, in the second part, we consider the gradient estimates for a positive solution of the nonlinear parabolic equation $\partial_t u = \Delta_t u + hu^p$ on a Riemannian manifold whose metrics evolve under the geometric flow $\partial_t g(t) = -2S_{g(t)}$. To obtain these estimates, we introduce a quantity $\underline{\boldsymbol{S}}$ along the flow which measures whether the tensor S_{ij} satisfies the second contracted Bianchi identity. Under conditions on $\text{Ric}_{g(t)}, S_{g(t)}$, and $\underline{\boldsymbol{S}}$, we obtain the gradient estimates.

1. Introduction

The nonlinear parabolic equation is a classical subject that has been extensively studied, which leads to lots of important results especially in researches of differential geometry. One of the important technique in studying the heat equation is the differential Harnack inequality developed by Li and Yau [8]. This is also applied to Ricci flow by Hamilton [6], and plays an important role in solving the Poincaré conjecture [12].

1.1. **Gradient estimates for (1.2) under the Ricci flow.** Consider first positive solutions of a nonlinear parabolic equation on an n-dimensional complete manifold M, which evolves under the Ricci flow. A series of gradient estimates are obtained for such solutions, including several Li-Yau-type inequalities. Let $(M, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\partial_t g(t) = -2\,\text{Ric}_{g(t)}, \quad t \in [0, T]. \tag{1.1}$$

We assume that its Ricci curvature remains uniformly bounded for all $t \in [0, T]$. Consider a positive function $u = u(x, t)$ defined on $M \times [0, T]$ solving the equation

$$\left(\Delta_{g(t)} - q - \partial_t\right) u = au(\ln u)^\alpha, \quad t \in [0, T], \tag{1.2}$$

which has been first studied in [19] where $g(t) \equiv g$ is a fixed metric. Qian [13] and Wu [18] got a series of similar conclusions. Here $\Delta_{g(t)}$ stands for the Laplacian of $g(x, t)$ defined on $M \times [0, T]$ and $q(x, t)$ is a C^2 function defined on $M \times [0, T]$. Notice that the Laplacian $\Delta_{g(t)}$ depends on the parameter t, and we should study

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the nonlinear parabolic equation (1.2) coupled with the Ricci flow (1.1).

We introduce notions used throughout this paper. Let $B_{\rho,T} = \{(x,t) \in M \times [0,T] : \mathrm{dist}_{g(t)}(x,x_0) < \rho\}$, where $\mathrm{dist}_{g(t)}(x,x_0)$ denotes the distance between x to a fixed point x_0 with respect to $g(t)$. $\nabla_{g(t)}$ and $|\cdot|_{g(t)}$ stand for the Levi-Civita connection and norm with respect to $g(t)$ respectively. For the simplicity, we always omit the subscripts $g(t)$ or t in the concrete computations.

We divide the study of the equation (1.2) into two cases: (1) $\alpha = 1$ and (2) $\alpha \neq 1$. For the first case, we have

Theorem 1.1. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M with $\sup_{B_{\rho,T}} |\mathrm{Ric}_{g(t)}|_{g(t)} \leq K$ for some $K > 0$, and u is a smooth positive function on $M \times [0,T]$ satisfying the nonlinear parabolic equation (1.2) where the function $q(x,t)$ is defined on $M \times [0,T]$ which is C^2 in the x-variable and C^1 in the t-variable. If $u(x,t) \leq A$ for some $A > 0$ on $B_{\rho,T}$, $\alpha = 1, |\nabla_{g(t)} q|_{g(t)} \leq \gamma$, and*

$$b := \frac{1}{8} + \min_{M\times[0,T]} q - \max\{a, 0\} \in \left(0, \frac{1}{2}\right]$$

then there exists a constant C that depends only on n such that

$$\frac{|\nabla_{g(t)} u|_{g(t)}}{u} \leq \frac{C}{b}\left(\frac{1}{\rho} + \frac{1}{\sqrt{t}} + \sqrt{K} + 1 + \sqrt{\gamma} + \sqrt{|a|}\right)\left(1 + \ln\frac{A}{u}\right) \tag{1.3}$$

on $B_{\rho/2,T}$ with $t \neq 0$.

When q is nonnegative and $a \leq 0$, the constant C/b can be a universal constant which means a constant depending only on the dimension n. The number $1/8$ in b is not essential, because in the following proof we shall see that we can replace $1/8$ by $1/2$.

For the general value of α, we have the following version of estimates.

Theorem 1.2. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M with $-K_1 g(t) \leq \mathrm{Ric}_{g(t)} \leq K_2 g(t)$ for some $K_1, K_2 > 0$ on $B_{\rho,T}$. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then there exists a constant C depending only on n such that, on $B_{\rho/2,T}$,*

(1) *for $a \geq 0$, we have*

$$\begin{aligned}|\nabla_{g(t)} f|^2_{g(t)} - \beta f_t - \beta q - \beta a f^\alpha \quad &\leq \quad \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3 C^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} \\ &+ \frac{n\beta[\beta K_1 + a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)} \\ &+ \frac{n\beta^2 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2c(\beta-1)(1-\epsilon)} \\ &+ \sqrt{\frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}\end{aligned}$$

(2) *for $a \le 0$, we have*

$$
\begin{aligned}
|\nabla_{g(t)} f|^2_{g(t)} - \beta f_t - \beta q - \beta a f^\alpha \quad \le \quad & \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2(1-c\epsilon)} + \frac{n^2\beta^3 C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} \\
& + \frac{n\beta[\beta K_1 - \frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)} \\
& + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}.
\end{aligned}
$$

Here $f := \ln u$, $|f|_\infty := \max_M |f|$, $\overline{K} := \max\{K_1, K_2\}$, $\beta > 1$, $0 < \epsilon < 1$, $|\nabla_{g(t)} q|_{g(t)} \le \gamma$, $\Delta_{g(t)} q \le \theta$,

$$
A = C\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \frac{1}{t} + \overline{K}\right)
$$

and $c, d > 0$ with $c + d = 1/\beta$.

1.2. **Gradient estimates for (1.4) under the geometric flow.** In the second part, we consider the gradient estimates of (1.2) under a general geometric flow; these results generalize our previous works [9, 21]. More generally, let $(M, g(t))_{t\in[0,T]}$ be a complete solution to the geomtric flow

$$
\partial_t g(t) = -2S_{g(t)}, \quad t \in [0, T]. \tag{1.4}
$$

on a complete and noncompact n-dimensional manifold M and consider a positive function $u = u(x,t)$ defined on $M \times [0,T]$ solving the equation

$$
\partial_t u = \Delta_{g(t)} u + h u^p, \quad t \in [0, T], \tag{1.5}
$$

where $\Delta_{g(t)}$ stands for the Laplacian of $g(t)$, h is a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, and p is a positive constant. When metrics are fixed, the study on the gradient estimates of (1.5) arose from [7]. If $h = 0$, Sun [17] derived the gradient estimates and the Harnack inequalities for the positive solutions of the linear parabolic equation $\partial_t u = \Delta_{g(t)} u$ under the geometric flow. In this paper, we consider the general case for the nonlinear parabolic equation. Notice that the $\Delta_{g(t)}$ depends on the parameter t, and we should study the equation (1.5) coupled with the geometric flow (1.4).

Introduce a 1-form $\underline{\boldsymbol{S}}_{g(t)}$ on $M \times [0,T]$ by

$$
\underline{\boldsymbol{S}}_{g(t)} := \mathrm{div}_{g(t)} S_{g(t)} - \frac{1}{2}\nabla_{g(t)}\left(\mathrm{tr}_{g(t)} S_{g(t)}\right).
$$

Locally, one has

$$
\underline{\boldsymbol{S}}_i = \nabla^j S_{ij} - \frac{1}{2}\nabla_i\left(\mathrm{tr}_{g(t)} S_{g(t)}\right).
$$

For example, if $S_{ij} = R_{ij}$, that is, (1.4) is the Ricci flow, we have

$$
\underline{\boldsymbol{S}}_i = \nabla^j R_{ij} - \frac{1}{2}\nabla_i R_{g(t)} = \frac{1}{2}\nabla_i R_{g(t)} - \frac{1}{2}\nabla_i R_{g(t)} = 0
$$

by the second contracted Bianchi identity. Thus, the quantity $\underline{\boldsymbol{S}}_{g(t)}$ measures whether S_{ij} satisfies the second contracted Bianchi identity.

Theorem 1.3. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M with $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, and $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ for some $K_1, K_2, K_3, K_4 > 0$ on $B_{2R,T}$, with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, satisfying $\Delta_{g(t)} h \geq -\theta$ and $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$ on $B_{2R,T} \times [0,T]$ for some nonnegative constants θ and γ. If $u(x,t)$ is a positive smooth solution of (1.5) on $M \times [0,T]$, then*

(i) *for $0 < p < 1$, we have*

$$
\begin{aligned}
\frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p} u^{p-1} - \frac{1}{p}\frac{u_t}{u} \quad \leq \quad & \frac{C_1}{p^2 t} + \frac{n(1-p)}{p^2} M_1 M_2 + \frac{n[3K_1 + 2(K_3+K_4)p]}{2p^2(1-p)} \\
& + \frac{C_1}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{n}{p(1-p)}\right) \\
& + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{n/K_1}}{p}\gamma M_2 + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}},
\end{aligned} \tag{1.6}
$$

where C_1 is a positive constant depending only on n and

$$M_1 := \max_{B_{2R,T}} h_-, \quad M_2 := \max_{B_{2R,T}} u^{p-1}, \quad h_- := \max(-h, 0).$$

(ii) *for $p \geq 1$, we have*

$$
\begin{aligned}
\frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p} u^{p-1} - \frac{1}{p}\frac{u_t}{t} \quad \leq \quad & \frac{k^2 C_2}{p^2 t} + \frac{nk^2(p-1)}{p^2} M_4 M_5 + \frac{k^3 n}{k-p} M_3 M_4 \\
& + \frac{k^2 C_2}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{k^2 n}{p(k-p)}\right) \\
& + \frac{2k^3 n}{(k-p)p^2}\left[K_1 + \frac{p}{k}(K_3+K_4)\right] + \frac{k^2\sqrt{n}\gamma}{p} M_4 \\
& + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4} + \frac{k^2 n}{p^2}\left(\overline{K} + \sqrt{\frac{K_4}{2n}}\right),
\end{aligned} \tag{1.7}
$$

where $k > p$, C_2 is a positive constant depending only on n and

$$M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h.$$

As an immediate consequence of the above theorem we have

Theorem 1.4. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2in x and C^1 in t.*

(i) For $0 < p < 1$, assume that $h \geq 0$, $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$, $\Delta_{g(t)} h \geq 0$ along the geometric flow with $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ for some positive constants $\gamma, K_1, K_2, K_3, K_4$ with $\overline{K} := \max\{K_1, K_2\}$, along the geometric flow. If u is a smooth positive function satisfying

the nonlinear parabolic equation (1.5), then

$$\begin{aligned}\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{u} \leq & \frac{C_1}{p^2 t}+\frac{C_1}{p^3(1-p)}+\frac{C_1}{p^2}\overline{K}+\frac{2nK_1}{p^2(1-p)}\\ & +\frac{\sqrt{n/K_1}}{p}\gamma M+\frac{n}{p^2}\sqrt{\frac{K_4}{2n}}+\frac{n(K_3+K_4)}{p(1-p)} \end{aligned} \tag{1.8}$$

for some positive constant C_1 *depending only on* n, *where* $M := \max_{M\times[0,T]} u^{p-1}$.

(ii) For $p=1$, *assume that* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ *for some positive constants* K_1, K_2, K_3, K_4 *with* $\overline{K} := \max\{K_1, K_2\}$, $h \geq 0$, $\Delta_{g(t)} h \geq -\theta$ *(θ is nonnegative), and* $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$ *(γ is nonnegative), along the geometric flow. If* u *is a smooth positive function satisfying the nonlinear parabolic equation (1.5), then*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+h-\frac{u_t}{u} \leq \frac{C_2}{t}+C_2\left(1+K_1+K_2+K_3+K_4+\overline{K}+\gamma+\sqrt{\theta}\right) \tag{1.9}$$

for some positive constant C_2 *depending only on* n.

(iii) For $p>1$, *assume that* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ *for some positive constants* $\gamma, K_1, K_2, K_3, K_4$ *with* $\overline{K} := \max\{K_1, K_2\}$. $\Delta_{g(t)} h \geq -\theta$, $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$, *and* $-k_1 \leq h \leq k_2$, *where* $\theta, \gamma, k_1, k_2 > 0$, *along the geometric flow. If* u *is a bounded smooth positive function satisfying the nonlinear parabolic equation (1.5), then*

$$\begin{aligned}\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{u} \leq & \left(\frac{k}{p}\right)^2\frac{C_3}{t}+\left(\frac{k}{p}\right)^3\frac{k}{k-p}C_3+\left(\frac{k}{p}\right)^2 C_3\Bigg(\overline{K}+\\ & +\frac{k}{k-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}\Bigg)\\ & +\left(\frac{k}{p}\right)^2 n(p-1)k_2 M+\frac{k^3 n}{k-p}k_1 M\\ & +\frac{k^2\sqrt{n}}{p}\gamma M+\left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M}, \end{aligned} \tag{1.10}$$

for some positive constant C_3 *depending only on* n, *where* $M := \max_{M\times[0,T]} u^{p-1}$ *and* $k>p$. *In particular, taking* $k=2p$, *we get*

$$\begin{aligned}\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{u} \leq & \frac{C_4}{t}+C_5\left(1+K_1+K_2+K_3+K_4+\overline{K}\right)\\ & +C_4 p^2\left[(k_1+k_2)M+\gamma M+\sqrt{\theta M}\right], \end{aligned} \tag{1.11}$$

for some positive constant C_4 *depending only on* n.

Another type of Harnack inequality is the following

Theorem 1.5. *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the geometric flow (1.4) on* M, *satisfying* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_{g(t)}\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$, *for some* $K_1, K_2, K_3, K_4 > 0$, *with* $\overline{K} := \max\{K_1, K_2\}$. *Let* $h(x,t)$ *be a nonnegative function defined on* $M\times[0,T]$ *which is* C^2 *in* x *and* C^1 *in* t, $\Delta_{g(t)} h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ *on* $M\times[0,T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p>1$ *and*

$C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.5), then

$$\begin{aligned}\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{u} \quad &\leq \quad \frac{C}{p^2 t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} \\ &+ \frac{4n}{p(2-p)}(K_1 + K_3 + K_4) + \frac{1}{p^2}\sqrt{8nK_4},\end{aligned} \tag{1.12}$$

for some positive constant C depending only on n.

This theorem has three important consequences.

Corollary 1.6. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M, satisfying $0 \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_{g(t)}\boldsymbol{\underline{S}}_{g(t)}|_{g(t)} \leq K_4$, for some positive constants K_2, K_3, K_4. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.5), then*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{t} \leq \frac{C}{p^2 t} + \frac{8n}{p^2}K_2 + \frac{4n}{p(2-p)}(K_3 + K_4) + \frac{1}{p^2}\sqrt{8nK_4} \tag{1.13}$$

for some positive constant C depending only on n.

Corollary 1.7. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M, satisfying $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_{g(t)}\boldsymbol{\underline{S}}_{g(t)}|_{g(t)} \leq K_4$, for some $K_1, K_2, K_3, K_4 > 0$, with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.5), then*

$$\begin{aligned}\frac{u(x_2,t_2)}{u(x_1,t_1)} \quad \geq \quad & \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\Bigg[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot{\gamma}(t)|^2_{g(t)}dt - 2n(t_2 - t_1) \\ & \left(\frac{1}{p}\overline{K} + \frac{2}{p}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{1}{2-p}(K_1 + K_3 + K_4) + \frac{1}{p}\sqrt{2nK_4}\right)\Bigg]\end{aligned} \tag{1.14}$$

for some positive constant C depending only on n, where $(x_1,t_1), (x_2,t_2) \in M \times [0,T]$ with $t_1 < t_2$.

When $K_1 = 0$, we have the following

Corollary 1.8. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M, satisfying $0 \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_{g(t)}\boldsymbol{\underline{S}}_{g(t)}|_{g(t)} \leq K_4$, for some $K_2, K_3, K_4 > 0$. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$*

if $p \leq 1$*), and* $0 < p \leq \frac{2n}{2n-1}$ $(n \geq 3)$*. If* u *is a positive solution of (1.5), then*

$$\begin{aligned}\frac{u(x_2,t_2)}{u(x_1,t_1)} \quad &\geq \quad \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\Bigg[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot{\gamma}(t)|^2_{g(t)}dt \\ &\quad - 2n(t_2-t_1)\left(\frac{K_2}{p}+\frac{K_3+K_4}{2-p}+\frac{\sqrt{2nK_4}}{p}\right)\Bigg]\end{aligned}$$

for some positive constant C *depending only on* n*, where* $(x_1,t_1),(x_2,t_2) \in M \times [0,T]$ *with* $t_1 < t_2$*.*

2. Gradient estimates for (1.2) under the Ricci flow

Firstly, we introduce a cut-off function (see [3, 8, 10, 11, 19]) on $B_{\rho,T} := \{(\chi,t) \in M\times[0,T] : \mathrm{dist}_{g(t)}(\chi,x_0) < \rho\}$, where $\mathrm{dist}_{g(t)}(\chi,x_0)$ stands for the distance between χ and x_0 with respect to the metric $g(t)$, which satisfies a basic analytical result stated in the following lemma.

Lemma 2.1. *Given* $\tau \in (0,T]$*, there exists a smooth function* $\overline{\Psi} : [0,\infty)\times[0,T] \to R$ *satisfying the following requirements:*

(1) *The support of* $\overline{\Psi}(r,t)$ *is a subset of* $[0,\rho]\times[0,T]$*,* $0 \leq \overline{\Psi}(r,t) \leq 1$ *in* $[0,\rho]\times[0,T]$*, and* $\overline{\Psi}(r,t) = 1$ *holds in* $[0,\frac{\rho}{2}]\times[\tau,T]$*.*
(2) $\overline{\Psi}$ *is decreasing as a radial function in the spatial variables.*
(3) *The estimate* $|\partial_t\overline{\Psi}| \leq \frac{\overline{C}}{\tau}\overline{\Psi}^{1/2}$ *is satisfied on* $[0,\infty)\times[0,T]$ *for some* $\overline{C} > 0$*.*
(4) *The inequalities* $-\frac{C_\alpha}{\rho}\overline{\Psi}^\alpha \leq \partial_r\overline{\Psi} \leq 0$ *and* $|\partial_r^2\overline{\Psi}| \leq \frac{C_\alpha}{\rho^2}\overline{\Psi}^\alpha$ *hold on* $[0,\infty)\times[0,T]$ *for every* $a \in (0,1)$ *with some constant* C_α *dependent on* a*.*

Proof. See [1]. □

These properties are derived from Calabi's argument(see, e.g., [2, 4, 15]). Using this auxiliary function and applying the maximum principle, we are able to establish Li-Yau-type inequality for the system (1.1) – (1.2).

In the following, we always omit the subscripts $g(t)$ or t in concrete computations. For example, we write Δ instead of $\Delta_{g(t)}$.

2.1. Gradient estimates I: $\alpha = 1$.

To prove Theorem 1.1, we need the following crucial lemma.

Lemma 2.2. *Let* $(M,g(t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow (1.1) on an* n*-dimensional manifold* M *with* $\sup_{B_{\rho,T}}|\mathrm{Ric}_{g(t)}|_{g(t)} \leq K$ *for some* $K > 0$*, and* u *is a smooth positive function on* $M\times[0,T]$ *satisfying the nonlinear parabolic equation (1.2) with* $\alpha = 1, a < 0$*. We assume that* $u \leq 1$ *on* $B_{\rho,T}$*. If* $f := \ln u$ *and* $w := |\nabla_{g(t)}\ln(1-f)|^2_{g(t)} = |\nabla_{g(t)}f|^2_{g(t)}/(1-f)^2$*, then the inequality*

$$\begin{aligned}(\Delta-\partial_t)\,w \quad &\geq \quad 2\frac{f}{1-f}\langle\nabla f,\nabla w\rangle + 2(1-f)w^2 \\ &\quad + 2\frac{\langle\nabla f,\nabla q\rangle}{(1-f)^2} + 2a\frac{|\nabla f|^2}{(1-f)^2} + 2\frac{|\nabla f|^2(q+af)}{(1-f)^3}\end{aligned} \tag{2.1}$$

holds on $B_{\rho,T}$*.*

Proof. Since u is a positive solution to the nonlinear parabolic equation (1.2) with $\alpha = 1$ and $a < 0$, direct calculation shows that

$$\Delta f + |\nabla f|^2 - f_t - q - af = 0, \quad f_t := \partial_t f.$$

The partial derivative of w with respect to t is given by

$$\begin{aligned} w_t &= \frac{2\langle \nabla f, \nabla f_t\rangle}{(1-f)^2} + \frac{2|\nabla f|^2 f_t}{(1-f)^3} + \frac{2\mathrm{Ric}(\nabla f, \nabla f)}{(1-f)^2} \\ &= \frac{2\langle \nabla f, \nabla(\Delta f + |\nabla f|^2 - q - af)\rangle}{(1-f)^2} + \frac{2\mathrm{Ric}(\nabla f, \nabla f)}{(1-f)^2} \\ &+ \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2 - q - af)}{(1-f)^3}. \end{aligned}$$

Using Bochner's identity $\langle \nabla f, \nabla \Delta f\rangle = \langle \nabla f, \Delta \nabla f\rangle - \mathrm{Ric}(\nabla f, \nabla f)$ we obtain

$$\begin{aligned} w_t &= \frac{2\langle \nabla f, \Delta \nabla f\rangle + 2\langle \nabla f, \nabla(|\nabla f|^2 - q - af)\rangle}{(1-f)^2} \\ &+ \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2 - q - af)}{(1-f)^3}. \end{aligned}$$

The partial derivative of w with respect to x is given by

$$\Delta w = 2\frac{|\nabla^2 f|^2}{(1-f)^2} + 2\frac{\langle \nabla f, \Delta \nabla f\rangle}{(1-f)^2} + 4\frac{\langle \nabla f, \nabla |\nabla f|^2\rangle}{(1-f)^3} + 6\frac{|\nabla f|^4}{(1-f)^4} + 2\frac{|\nabla f|^2 \Delta f}{(1-f)^3}.$$

Combining those partial derivatives imply

$$\begin{aligned} (\Delta - \partial_t)\, w &= 2\frac{|\nabla^2 f|^2}{(1-f)^2} + 4\frac{\langle \nabla f, \nabla |\nabla f|^2\rangle}{(1-f)^3} + 6\frac{|\nabla f|^4}{(1-f)^4} - 2\frac{\langle \nabla f, \nabla |\nabla f|^2\rangle}{(1-f)^2} \\ &- 2\frac{|\nabla f|^4}{(1-f)^3} + 2a\frac{|\nabla f|^2}{(1-f)^2} + 2\frac{\langle \nabla f, \nabla q\rangle}{(1-f)^2} + 2\frac{q|\nabla f|^2}{(1-f)^3} + 2a\frac{f|\nabla f|^2}{(1-f)^3}. \end{aligned}$$

On the other hand, the gradient term $\langle \nabla f, \nabla w\rangle$ is determined by

$$\langle \nabla f, \nabla w\rangle = \frac{\langle \nabla f, \nabla |\nabla f|^2\rangle}{(1-f)^2} + 2\frac{|\nabla f|^4}{(1-f)^3}.$$

Plugging it into the evolution of $(\Delta - \partial_t)w$ we conclude that

$$\begin{aligned} (\Delta - \partial_t)\, w &= 2\frac{|\nabla^2 f|^2}{(1-f)^2} + 2\frac{\langle \nabla f, \nabla |\nabla f|^2\rangle}{(1-f)^3} + \frac{2}{1-f}\langle \nabla f, \nabla w\rangle \\ &+ 2\frac{|\nabla f|^4}{(1-f)^4} - 2\langle \nabla f, \nabla w\rangle + 2\frac{|\nabla f|^4}{(1-f)^3} + 2a\frac{|\nabla f|^2}{(1-f)^2} \\ &+ 2\frac{\langle \nabla f, \nabla q\rangle}{(1-f)^2} + 2\frac{q|\nabla f|^2}{(1-f)^3} + 2a\frac{f|\nabla f|^2}{(1-f)^3}. \end{aligned}$$

Because of the identity

$$\frac{|\nabla^2 f|^2}{(1-f)^2} + \frac{\langle \nabla f, \nabla |\nabla f|^2\rangle}{(1-f)^3} + \frac{|\nabla f|^4}{(1-f)^4} = \left|\frac{\nabla^2 f}{1-f} + \frac{\nabla f \otimes \nabla f}{(1-f)^2}\right|^2$$

we therefore arrive at

$$\begin{aligned}(\Delta-\partial_t)\,w &= 2\left|\frac{\nabla^2 f}{1-f}+\frac{\nabla f\otimes\nabla f}{(1-f)^2}\right|^2+\frac{2}{1-f}\langle\nabla f,\nabla w\rangle+2\frac{|\nabla f|^4}{(1-f)^3}\\ &\quad -2\langle\nabla f,\nabla w\rangle+\frac{2\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2a\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^2(q+af)}{(1-f)^3}\end{aligned}$$

which immediately implies

$$\begin{aligned}(\Delta-\partial_t)\,w &\geq \frac{2f}{1-f}\langle\nabla f,\nabla w\rangle+\frac{2|\nabla f|^4}{(1-f)^3}+\frac{2\langle\nabla f,\nabla q\rangle}{(1-f)^2}\\ &\quad +2a\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^2(q+af)}{(1-f)^3}.\end{aligned}$$

This complete the proof. □

Theorem 2.3. *Suppose that $(M,g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M with $\sup_{B_{\rho,T}}|\mathrm{Ric}_{g(t)}|_{g(t)}\leq K$ for some $K>0$, and u is a smooth positive function on $M\times[0,T]$ satisfying the nonlinear parabolic equation (1.2) where the function $q(x,t)$ is defined on $M\times[0,T]$ which is C^2 in the x-variable and C^1 in the t-variable. If $u(x,t)\leq A$ for some $A>0$ on $B_{\rho,T}$, $\alpha=1, |\nabla_{g(t)}q|_{g(t)}\leq\gamma$, and*

$$b:=\frac{1}{8}+\min_{M\times[0,T]}q-\max\{a,0\}\in\left(0,\frac{1}{2}\right]$$

then there exists a constant C that depends only on n such that

$$\frac{|\nabla_{g(t)}u|_{g(t)}}{u}\leq\frac{C}{b}\left(\frac{1}{\rho}+\frac{1}{\sqrt{t}}+\sqrt{K}+1+\sqrt{\gamma}+\sqrt{|a|}\right)\left(1+\ln\frac{A}{u}\right)\tag{2.2}$$

on $B_{\rho/2,T}$ with $t\neq 0$.

When q is nonnegative and $a\leq 0$, the constant C/b can be a universal constant which means a constant depending only on the dimension n. The number 1/8 in b is not essential, because in the following proof we shall see that we can replace 1/8 by 1/2.

Proof. Without loss of generality, we may assume that $A=1$; otherwise, we can replace u by u/A. Pick a number $\tau\in(0,T]$ and fix a function $\overline{\Psi}(x,t)$ satisfying the conditions of Lemma 2.1. We will establish (2.2) at (x,τ) for all x such that $\mathrm{dist}_{g(\tau)}(x,x_0)<\rho/2$. Define $\Psi: M\times[0,T]\to\mathbf{R}$ by setting

$$\Psi(x,t):=\overline{\Psi}\left(\mathrm{dist}_{g(t)}(x,x_0),t\right).$$

Then, using the identity (2.1) and a straightforward calculation, one has

$$\begin{aligned}(\Delta-\partial_t)(\Psi w) \quad \geq \quad & \Psi\langle-\Lambda,\nabla w\rangle+\left[2(1-f)w^2+\frac{2\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2a\frac{|\nabla f|^2}{(1-f)^2}\right.\\ & \left.+\,2\frac{|\nabla f|^2(q+af)}{(1-f)^3}\right]\Psi+2\langle\nabla w,\nabla\Psi\rangle+w\Delta\Psi-w\partial_t\Psi\\ \geq \quad & \langle-\Lambda,\nabla(\Psi w)\rangle+2\Psi(1-f)w^2+w\langle\Lambda,\nabla\Psi\rangle\\ & +\,w\Delta\Psi-w\Psi_t+\frac{2}{\Psi}\langle\nabla\Psi,\nabla(\Psi w)\rangle-2\frac{|\nabla\Psi|^2}{\Psi}w\\ & +\,\left[\frac{2\,\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2a\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^2(q+af)}{(1-f)^3}\right]\Psi\end{aligned}$$

where $\Lambda=-\frac{2f}{1-f}\nabla f$. By our assumption that $|\mathrm{Ric}|\leq K$ on $B_{\rho,T}$ and Lemma 2.1 that $-\frac{C_1}{\rho}\overline{\Psi}^{1/2}\leq\overline{\Psi}_r\leq 0$, and the identity

$$-w\Psi_t=-\left[\overline{\Psi}_t+\overline{\Psi}_r\partial_t\mathrm{dist}_{g(t)}(\cdot,x_0)\right]w,$$

we have (because $-\partial_t\mathrm{dist}_t(\cdot,x_0)\leq 4\sqrt{(m-1)K}$, c.f. Lemma 8.33 in [5])

$$-w\Psi_t\geq-\overline{\Psi}_t w-\frac{4C_1\sqrt{(m-1)K}}{\rho}w\overline{\Psi}^{1/2}.$$

Suppose that Ψw achieves its maximum at (x_0,t_0). By [8], without loss of generality, we may assume that x_0 is not in the cut-locus of M. At the point (x_0,t_0), one has $\Delta(\Psi w)\leq 0$, $\nabla(\Psi w)=0$, $(\Psi w)_t\geq 0$. Therefore

$$\begin{aligned}2\Psi(1-f)w^2 \quad \leq \quad & -w\langle\Lambda,\nabla\Psi\rangle+2\frac{|\nabla\Psi|^2}{\Psi}w-w\Delta\Psi+w\Psi_t\\ & -\left[\frac{2\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2a\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^2(q+af)}{(1-f)^3}\right]\Psi\end{aligned}\tag{2.3}$$

at (x_0,t_0). We need to bound each term on the right-hand side of (2.3):

$$|w\langle\Lambda,\nabla\Psi\rangle|\leq 2\frac{w|f|}{1-f}|\nabla f||\nabla\Psi|=2w^{3/2}|f||\nabla\Psi|\leq\Psi(1-f)w^2+\frac{27}{16}\frac{|f|^4|\nabla\Psi|^4}{[\Psi(1-f)]^3}$$

where we used the Young's inequality that $ab\leq\epsilon a^p+b^q/(q(p\epsilon)^{q/p})$ for any $a,b,\epsilon>0$ and $p,q>1$ with $p^{-1}+q^{-1}=1$. This together with Lemma 2.1 implies

$$|w\langle\Lambda,\nabla\Psi\rangle|\leq\Psi(1-f)w^2+C_2\frac{f^4}{\rho^4(1-f)^3}.\tag{2.4}$$

Using again Lemma 2.1 we have

$$\frac{|\nabla\Psi|^2}{\Psi}w=\Psi^{1/2}w\frac{|\nabla\Psi|^2}{\Psi^{3/2}}\leq\frac{1}{8}\Psi w^2+2\frac{|\nabla\Psi|^4}{\Psi^3}\leq\frac{1}{8}\Psi w^2+\frac{C_3}{\rho^4}.\tag{2.5}$$

Furthermore, by the properties of Ψ and the assumption of the Ricci curvature, one has (c.f., [16, 20])

$$-w\Delta\Psi\leq\frac{1}{8}\Psi w^2+\frac{C_4}{\rho^4}+C_4K^2.\tag{2.6}$$

The estimation for $w\Psi_t$ is given by (c.f. [11])

$$|w\Psi_t|\leq\frac{1}{8}\Psi w^2+\frac{C_5}{\tau^2}+C_5K^2.\tag{2.7}$$

Since $f \le 0$ it follows that

$$\left|\frac{2\langle\nabla f,\nabla q\rangle}{(1-f)^2}\Psi\right| \le \frac{|\nabla f|^2+|\nabla q|^2}{(1-f)^2}\Psi \le w\Psi+\gamma^2\Psi$$

$$\le \frac{1}{8}\Psi w^2+(2+\gamma^2)\Psi, \tag{2.8}$$

$$\left|2a\frac{|\nabla f|^2}{(1-f)^2}\Psi\right| = 2aw\Psi \le \frac{1}{8}\Psi w^2+2a^2\Psi \tag{2.9}$$

and

$$\begin{aligned} -2\frac{|\nabla f|^2(q+af)}{(1-f)^3}\Psi &= 2\Psi w^2\left(\frac{-q}{1-f}+a\frac{-f}{1-f}\right)\\ &\le 2\Psi w^2\left(-\min_{M\times[0,T]}q+a\frac{-f}{1-f}\right)\\ &\le 2\left(\max\{a,0\}-\min_{M\times[0,T]}q\right)\Psi w^2. \end{aligned}\tag{2.10}$$

Substituting (2.5) – (2.10) to the right-hand side of (2.3), we deduce that

$$\begin{aligned}\Psi(1-f)w^2 \le{}& C_6\frac{f^4}{\rho^4(1-f)^3}+\frac{3}{4}\Psi w^2+2\left(\max\{a,0\}-\min_{M\times[0,T]}q\right)\Psi w^2\\ &+\frac{C_6}{\tau^2}+\frac{C_6}{\rho^4}+C_6K^2+\left(2+\gamma^2+2a^2\right)\end{aligned}$$

at (x_0,t_0). Since $f<0$, it follows that $f^4/(1-f)^4\le 1$ and then

$$\begin{aligned}\Psi w^2 \le{}& \frac{C_6}{\rho^4}+\frac{1}{1-f}\left[\frac{3}{4}\Psi w^2+2\left(\max\{a,0\}-\min_{M\times[0,T]}q\right)\Psi w^2\right.\\ &\left.+\frac{C_6}{\tau^2}+\frac{C_6}{\rho^4}+C_6K^2+(2+\gamma^2+2a^2)\right]\\ \le{}& \frac{C_6}{\rho^4}+\frac{3}{4}\Psi w^2+2\left(\max\{a,0\}-\min_{M\times[0,T]}q\right)\Psi w^2\\ &+\frac{C_6}{\tau^2}+\frac{C_6}{\rho^4}+C_6K^2+\left(2+\gamma^2+2a^2\right)\end{aligned}$$

at (x_0,t_0), when

$$\min_{M\times[0,T]}q-\max\{a,0\}\le\frac{3}{8}. \tag{2.11}$$

Therefore, we can conclude that

$$\left[\frac{1}{4}+2\left(\min_{M\times[0,T]}q-\max\{a,0\}\right)\right]\Psi w^2\le\frac{2C_6}{\rho^4}+\frac{C_6}{\tau^2}+C_6K^2+(2+\gamma^2+2a^2)$$

at (x_0,t_0). If we assume in addition that

$$\frac{1}{8}+\min_{M\times[0,T]}q-\max\{a,0\}>0, \tag{2.12}$$

we arrive at

$$\Psi w^2\le\frac{C_7}{\frac{1}{8}+\min_{M\times[0,T]}q-\max\{a,0\}}\left(\frac{1}{\rho^4}+\frac{1}{\tau^2}+K^2+1+\gamma^2+a^2\right)$$

at (x_0, t_0). Because $\Psi(x,\tau)=1$ when $\mathrm{dist}_\tau(x,x_0)<\rho/2$, we finally arrive at

$$w^2(x,\tau) \le \frac{C_7}{\frac{1}{8}+\min_{M\times\times[0,T]} q - \max\{a,0\}}\left(\frac{1}{\rho^4}+\frac{1}{\tau^2}+K^2+1+\gamma^2+a^2\right)$$

on $B_{\rho/2,T}$, which, since $\tau\in(0,T]$ was arbitrary, implies

$$\frac{|\nabla f|}{1-f} \le \frac{C_8}{\frac{1}{8}+\min_{M\times[0,T]} q - \max\{a,0\}}\left(\frac{1}{\rho}+\frac{1}{\sqrt{t}}+\sqrt{K}+1+\sqrt{\gamma}+\sqrt{|a|}\right),$$

provided (2.11) and (2.12), or provided

$$-\frac{1}{8} < \min_{M\times[0,T]} q - \max\{a,0\} \le \frac{3}{8}. \tag{2.13}$$

We have completed the proof of Theorem 1.1 since $f=\ln(u/A)$ with A scaled to 1. □

The number 1/8 in (2.12) is not essential, because in the above argument we can replace 1/8 in (2.5) – (2.9) by a given positive number ϵ, and hence we need only to require that

$$1-6\epsilon+2\min_{M\times[0,T]} q - 2\max\{a,0\}>0, \quad 6\epsilon+2\left(\max\{a,0\}-\min_{M\times[0,T]} q\right)\ge 0$$

or

$$3\epsilon-\frac{1}{2} < \min_{M\times[0,T]} q - \max\{a,0\} \le 3\epsilon$$

instead of (2.13). When

$$\frac{1}{2}+\min_{M\times[0,T]} q - \max\{a,0\} \in \left(0,\frac{1}{2}\right]$$

we can choose ϵ to be any positive number in the interval $[\frac{A}{3},\frac{A}{3}+\frac{1}{6})$, where $A:=\min_{M\times[0,T]} q - \max\{a,0\}\in(-1/2,0]$.

2.2. **Gradient estimates II: general case.** In this section we extend Theorem 1.1 with $\alpha=1$ to the general case.

Lemma 2.4. *Suppose $(M,g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M, with $-K_1 g(t)\le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some $K_1, K_2>0$ on $B_{\rho,T}$. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then, for given $\beta\ge 1$ and any $c,d>0$ with $c+d=1/\beta$, we have*

$$\begin{aligned}\Box F \ \ge\ & -2\langle\nabla f,\nabla F\rangle-\frac{F}{t}+\frac{2c\beta t}{n}\left(|\nabla f|^2-q-f_t-af^\alpha\right)^2\\ & -2(\beta-1)t\langle\nabla f,\nabla q\rangle-2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2\\ & -\beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2-2\beta tK_1|\nabla f|^2-\frac{n\beta t}{2d}\overline{K}^2\\ & -\beta a\alpha t f^{\alpha-1}\left(-|\nabla f|^2+f_t+q+af^\alpha\right)-\beta t\Delta q,\end{aligned} \tag{2.14}$$

where $\overline{K}:=\max\{K_1,K_2\}$, $f:=\ln u$, and $F:=t(|\nabla f|^2-\beta f_t-\beta q-\beta af^\alpha)$.

Proof. The proof of this lemma was original from [11]. Now we will find a convenient bound on ΔF like the way in [21]. Notice that

$$\nabla_i F = t\left(2\nabla^j f \nabla_i \nabla_j f - \beta \nabla_i f_t - \beta \nabla_i q - \beta a\alpha f^{\alpha-1}\nabla_i f\right).$$

Then the Laplace of F equals

$$\begin{aligned} \Delta F &= \nabla^i \nabla_i F \\ &= t\Big[2|\nabla^2 f|^2 + 2\langle \nabla f, \Delta \nabla f\rangle - \beta \Delta f_t - \beta \Delta q \\ &\quad - \beta a\alpha\left((\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right)\Big]. \end{aligned}$$

Using the Bochner's formula $\Delta \nabla f = \nabla \Delta f + \mathrm{Ric}(\nabla f, \cdot)$, we get

$$\begin{aligned} \Delta F &= t\Big[2|\nabla^2 f|^2 + 2\langle \nabla f, \nabla \Delta f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f) - \beta \Delta f_t \\ &\quad - \beta \Delta q - \beta a\alpha\Big((\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\Big)\Big] \\ &\geq t\Big[\frac{2(\Delta f)^2}{n} + 2\langle \nabla f, \nabla \Delta f\rangle - 2K_1|\nabla f|^2 - \beta(\Delta f)_t - \beta \Delta q \\ &\quad - \beta a\alpha\Big((\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\Big) + 2\beta R_{ij}\nabla^i\nabla^j f\Big] \end{aligned}$$

since $|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2$ and $\Delta f_t = (\Delta f)_t - 2R_{ij}\nabla^i\nabla^j f$. Recalling from the result

$$\Delta f = -|\nabla f|^2 + q + f_t + af^\alpha = -\frac{F}{t} - (\beta-1)\left(q + f_t + af^\alpha\right),$$

we arrive at

$$\begin{aligned} \Delta F &\geq \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 + \left(\frac{2d\beta t}{n}(\Delta f)^2 + 2t\beta R_{ij}\nabla^i\nabla^j f\right) \\ &\quad - 2t\left\langle \nabla f, \nabla\left(\frac{F}{t} + (\beta-1)(q + f_t + af^\alpha)\right)\right\rangle \\ &\quad - 2K_1 t|\nabla f|^2 - t\beta\left(-\frac{F}{t} - (\beta-1)(q + f_t + af^\alpha)\right)_t - \beta t\Delta q \\ &\quad - \beta a\alpha t\left[(\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right] \end{aligned} \tag{2.15}$$

in the set $B_{\rho,T}$. Because

$$\begin{aligned} \frac{2d\beta t}{n}(\Delta f)^2 + 2t\beta R_{ij}\nabla^i\nabla^j f &= \frac{2d\beta t}{n}\left[(\Delta f)^2 + \frac{n}{d}R_{ij}\nabla^i\nabla^j f\right] \\ &= \frac{2d\beta t}{n}\left|\nabla^2 f + \frac{n}{2d}\mathrm{Ric}\right|^2 - \frac{n\beta t}{2d}|\mathrm{Ric}|^2 \\ &\geq -\frac{n\beta t}{2d}|\mathrm{Ric}|^2, \end{aligned}$$

the inequality (2.15) can be written as

$$\begin{aligned}\Delta F \quad \geq \quad & \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 - \frac{n\beta t}{2d}|\mathrm{Ric}|^2 \\ & - 2t\left\langle \nabla f, \nabla\left(\frac{F}{t} + (\beta-1)(q + f_t + af^\alpha)\right)\right\rangle \\ & - 2K_1 t|\nabla f|^2 - t\beta\left(-\frac{F}{t} - (\beta-1)(q + f_t + af^\alpha)\right)_t - \beta t\Delta q \\ & - \beta a\alpha t\left[(\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right]\end{aligned} \tag{2.16}$$

To get the time derivative of F, we shall use the identity

$$F_t = \frac{F}{t} + t\bigg(|\nabla f|^2 - \beta f_t - \beta q - a\beta f^\alpha\bigg)_t.$$

Subtracting this from (2.16), we get

$$\begin{aligned}(\Delta - \partial_t)F \quad \geq \quad & -2\langle \nabla f, \nabla F\rangle - \frac{F}{t} + \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 \\ & - \beta t\Delta q - 2(\beta-1)t\langle \nabla f, \nabla q\rangle - \frac{n\beta t}{2d}|\mathrm{Ric}|^2 \\ & - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 - \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 \\ & - \beta a\alpha t f^{\alpha-1}\bigg(- |\nabla f|^2 + f_t + q + af^\alpha\bigg) - 2\beta K_1 t|\nabla f|^2.\end{aligned}$$

Now the inequality (2.14) follows immediately by noting that $|\mathrm{Ric}| \leq \overline{K}$. □

Now we can consider the local space-time gradient estimate with Lemma 2.3. In the following part, n is the dimension of M.

Theorem 2.5. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M with $-K_1 g(t) \leq \mathrm{Ric}_{g(t)} \leq K_2 g(t)$ for some $K_1, K_2 > 0$ on $B_{\rho,T}$. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then there exists a constant C depending only on n such that, on $B_{\rho/2,T}$,*

(1) *for $a \geq 0$, we have*

$$\begin{aligned}|\nabla_{g(t)}f|^2_{g(t)} - \beta f_t - \beta q - \beta af^\alpha \quad \leq \quad & \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3C^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} \\ & + \frac{n\beta[\beta K_1 + a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)} \\ & + \frac{n\beta^2 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2c(\beta-1)(1-\epsilon)} \\ & + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}\end{aligned}$$

(2) *for $a \le 0$, we have*

$$
\begin{aligned}
|\nabla_{g(t)} f|^2_{g(t)} - \beta f_t - \beta q - \beta a f^\alpha \quad \le \quad & \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2(1-c\epsilon)} + \frac{n^2\beta^3 C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} \\
& + \frac{n\beta[\beta K_1 - \frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)} \\
& + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}.
\end{aligned}
$$

Here $f := \ln u$, $|f|_\infty := \max_M |f|$, $\overline{K} := \max\{K_1, K_2\}$, $\beta > 1$, $0 < \epsilon < 1$, $|\nabla_{g(t)} q|_{g(t)} \le \gamma$, $\Delta_{g(t)} q \le \theta$,

$$
A = C\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \frac{1}{t} + \overline{K}\right)
$$

and $c, d > 0$ *with* $c + d = 1/\beta$.

Proof. We will use the same notation $f = \ln u$ and $F = t(|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha)$ as in lemma 2.4. For the fixed $\tau \in (0, T]$, chose the cut-off function $\overline{\Psi}$ constructed in Lemma 2.1. Define $\Psi : M \times [0, T] \to \mathbf{R}$ by setting

$$
\Psi(x,t) = \overline{\Psi}\left(\mathrm{dist}_{g(t)}(x, x_0), t\right).
$$

Lemma 2.4 implies that

$$
\begin{aligned}
(\Delta - \partial_t)(\Psi F) \quad \ge \quad & -2\langle \nabla f, \nabla(\Psi F)\rangle + 2F\langle \nabla f, \nabla \Psi\rangle \\
& + \Bigg[\frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - a f^\alpha\right)^2 - \frac{F}{t} - \beta t \Delta q \\
& - 2(\beta-1)t\langle \nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\
& - \beta t a\alpha(\alpha-1) f^{\alpha-2}|\nabla f|^2 - 2\beta t K_1 |\nabla f|^2 - \frac{n\beta t}{2d}\overline{K}^2 \\
& - \beta a\alpha t f^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + a f^\alpha\right)\Bigg]\Psi \\
& + F\Delta\Psi + \frac{2}{\Psi}\langle \nabla\Psi, \nabla(\Psi F)\rangle - 2\frac{|\nabla\Psi|^2}{\Psi}F - F\frac{\partial \Psi}{\partial t}.
\end{aligned}
$$

Let (x_0, t_0) be a maximum point for the function ΨF in the set $\{(x,t) | 0 \le t \le \tau, d_t(x, x_0) \le \rho\}$. Then at the point (x_0, t_0) we have

$$
\begin{aligned}
0 \quad \ge \quad & 2F\langle \nabla f, \nabla\Psi\rangle + F(\Delta - \partial_t)\Psi - 2\frac{|\nabla\Psi|^2}{\Psi}F \\
& + \Bigg[-\frac{F}{t} + \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - a f^\alpha\right)^2 - \beta t\Delta q \\
& - 2(\beta-1)t\langle \nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\
& - \beta t a\alpha(\alpha-1) f^{\alpha-2}|\nabla f|^2 - \beta a\alpha t f^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + a f^\alpha\right) \\
& - 2\beta t K_1 |\nabla f|^2 - \frac{n\beta t}{2d}\overline{K}^2\Bigg]\Psi.
\end{aligned}
$$

By Lemma 2.1 and the Laplacian comparison theorem, we have

$$\begin{aligned}
\frac{|\nabla\Psi|^2}{\Psi} &\leq \frac{C_{1/2}^2}{\rho^2},\\
\Delta\Psi &\geq -\frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho^2}-\frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho}(n-1)\sqrt{K_1}\coth(\sqrt{k_1}\rho)\\
&\geq -\frac{d_1}{\rho^2}-\frac{d_1\Psi^{\frac{1}{2}}}{\rho}\sqrt{K_1},\\
-\partial_t\Psi &\geq -\frac{\overline{C}\Psi^{\frac{1}{2}}}{\tau}-C_{1/2}\overline{K}\Psi^{\frac{1}{2}}
\end{aligned}$$

where $C_{1/2}, \overline{C}$ and d_1 are positive constants depending only on n. Plugging those estimates into above inequality yields that

$$\begin{aligned}
0 \geq\ & d_2\left(-\frac{1}{\rho^2}-\frac{\Psi^{1/2}}{\rho}\sqrt{K_1}-\frac{\Psi^{1/2}}{\tau}-\overline{K}\Psi^{1/2}\right)F-2F|\nabla f||\nabla\Psi|\\
&+\left[\frac{2c\beta t}{n}\left(|\nabla f|^2-q-f_t-af^\alpha\right)^2-\frac{F}{t}-\ \beta t\Delta q-\frac{n\beta t}{2d}\overline{K}^2\right.\\
&-2(\beta-1)t\langle\nabla f,\nabla q\rangle-2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2-2\beta tk_1|\nabla f|^2\\
&\left.-\beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2-\beta a\alpha tf^{\alpha-1}\left(-|\nabla f|^2+f_t+q+af^\alpha\right)\right]\Psi
\end{aligned}\tag{2.17}$$

at (x_0,t_0), where d_2 is equal to $\max\{d_1,\overline{C},C_{1/2},2C_{1/2}^2\}$. Introduce a function

$$A:=d_2\left(\frac{1}{\rho^2}+\frac{\Psi^{1/2}}{\rho}\sqrt{K_1}+\frac{\Psi^{1/2}}{\tau}+\overline{K}\Psi^{1/2}\right).$$

If one multiplies by $t\Psi$ and makes a few elementary manipulations, one will obtain

$$\begin{aligned}
0 \geq\ & -2F|\nabla f||\nabla\Psi|\Psi t-AF\Psi t+\left[\frac{2c\beta t}{n}\left(|\nabla f|^2-q-f_t-af^\alpha\right)^2\right.\\
&-\beta t\Delta q-2(\beta-1)t\langle\nabla f,\nabla q\rangle-2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2\\
&-\beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2-\beta a\alpha tf^{\alpha-1}\left(-|\nabla f|^2+f_t+q+af^\alpha\right)\\
&\left.-2\beta tK_1|\nabla f|^2-\frac{n\beta t}{2d}\overline{K}^2\right]\Psi^2t-F\Psi^2
\end{aligned}\tag{2.18}$$

at (x_0,t_0). As in [3, 19], we set

$$\mu:=\frac{|\nabla f|^2(x_0,t_0)}{F(x_0,t_0)}\geq 0.$$

Because $|\nabla f|=\mu^{1/2}F^{1/2}$ and

$$\begin{aligned}
|\nabla f|^2-f_t-q-af^\alpha &= F\left(\mu-\frac{\mu t-1}{\beta t}\right),\\
\langle\nabla f,\nabla\Psi\rangle \leq |\nabla f||\nabla\Psi| &\leq \frac{C_1}{\rho}\Psi^{1/2}|\nabla f|
\end{aligned}$$

we can simplify (2.18) into the following inequality

$$\begin{aligned} AFt\Psi \quad \geq \quad & -\frac{2C_1 t}{\rho}\Psi^{3/2}\mu^{1/2}F^{3/2} - \Psi^2 F + \frac{2c\Psi^2}{n\beta}[1+(\beta-1)\mu t]^2F^2 - \frac{n\beta}{2d}\overline{K}^2(\Psi t)^2 \\ & - 2(t\Psi)^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu F + at\Psi^2\alpha f^{\alpha-1}[1+(\beta-1)t\mu]F \\ & - \beta(t\Psi)^2\theta - 2(\beta-1)(t\Psi)^2\gamma(\mu F)^{1/2} - \beta(t\Psi)^2 a\alpha(\alpha-1)f^{\alpha-2}\mu F \end{aligned} \tag{2.19}$$

at (x_0, t_0). If we set $G := \Psi F$, then at the point (x_0, t_0) the inequality (2.19) becomes

$$\begin{aligned} AtG \quad \geq \quad & -\frac{2C_1 t}{\rho}\mu^{1/2}G^{3/2} - \Psi G + \frac{2c}{n\beta}[1+(\beta-1)\mu t]^2G^2 - \frac{n\beta}{2d}\overline{K}^2(\Psi t)^2 \\ & - 2\Psi t^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu G + a\Psi t\alpha f^{\alpha-1}[1+(\beta-1)\mu t]G \\ & - \beta(\Psi t)^2\theta - 2(\beta-1)t^2\Psi^{3/2}\gamma\mu^{1/2}G^{1/2} - \beta t^2\Psi a\alpha(\alpha-1)f^{\alpha-2}\mu G \end{aligned} \tag{2.20}$$

at (x_0, t_0). According to the Cauchy inequality, where $0 < \epsilon < 1$,

$$\begin{aligned} \frac{2C_1 t}{R}\mu^{1/2}G^{3/2} \quad & \leq \quad \frac{2\epsilon c}{n\beta}[1+(\beta-1)\mu t]^2G^2 + \frac{n\beta C_1^2 t^2\mu G}{2\epsilon c\rho^2[1+(\beta-1)\mu t]^2}, \\ 2\mu^{1/2}G^{1/2} \quad & \leq \quad 1 + \mu G, \end{aligned}$$

we can simplify (2.20)as

$$\begin{aligned} AtG \quad \geq \quad & \frac{2c(1-\epsilon)}{n\beta}[1+(\beta-1)\mu t]^2G^2 - \Psi G - \frac{n\beta^2 C_1^2 t^2\mu}{2\epsilon c\rho^2[1+(\beta-1)\mu t]^2}G \\ & - 2\Psi t^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu G + a\Psi t\alpha f^{\alpha-1}[1+(\beta-1)\mu t]G \\ & - \beta\Psi^2 t^2\theta - (\beta-1)t^2\Psi^{3/2}\gamma - (\beta-1)t^2\Psi^{3/2}\gamma\mu G \\ & - \beta t^2\Psi a\alpha(\alpha-1)f^{\alpha-2}\mu G - \frac{n\beta}{2d}\overline{K}^2(\Psi t)^2, \end{aligned}$$

at (x_0, t_0), or equivalently,

$$\begin{aligned} \frac{2c(1-\epsilon)[1+(\beta-1)\mu t]^2G^2}{n\beta} \quad \leq \quad & \Bigg[\frac{n\beta^2 C_1^2 t^2\mu}{2\epsilon c\rho^2[1+(\beta-1)\mu t]^2} + \beta t^2\Psi a\alpha(\alpha-1)f^{\alpha-2}\mu \\ & - a\Psi t\alpha f^{\alpha-1}[1+(\beta-1)\mu t] + (\beta-1)t^2\Psi^{3/2}\gamma\mu \\ & + 2\Psi t^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu + At + \Psi\Bigg]G \\ & + \left[\beta\Psi^2\theta + (\beta-1)\Psi^{3/2}\gamma + \frac{n\beta}{2d}\overline{K}^2\Psi^2\right]t^2 \end{aligned}$$

at (x_0,t_0). Note that $0\le\Psi\le 1$ and $1+(\beta-1)\mu t_0\ge 1$. Therefore

$$\begin{aligned}\frac{2c(1-\epsilon)G^2}{n\beta} &\le \Big[At+1+\frac{n\beta^2C_1^2t^2\mu}{2\epsilon c\rho^2[1+(\beta-1)\mu t]}+\frac{2\Psi t^2[\beta K_1+a(\beta-1)\alpha f^{\alpha-1}]\mu}{[1+(\beta-1)\mu t]^2}\\ &\quad -\frac{a\Psi t\alpha f^{\alpha-1}}{1+(\beta-1)\mu t}+\frac{(\beta-1)\gamma t^2\mu}{1+(\beta-1)\mu t}+\frac{\beta t^2\Psi a\alpha|\alpha-1|f^{\alpha-2}\mu}{1+(\beta-1)\mu t}\Big]G\\ &\quad +\Big[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2\Big]t^2\\ &\le \Big[At+1+\frac{n\beta^2C_1^2t}{2\epsilon c\rho^2(\beta-1)}+\frac{2\Psi t^2[\beta K_1+a(\beta-1)\alpha|f^{\alpha-1}|_\infty]\mu}{[1+(\beta-1)\mu t]^2}\\ &\quad +\gamma t-\frac{a\Psi t\alpha f^{\alpha-1}}{1+(\beta-1)\mu t}+\frac{\beta t\Psi a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{\beta-1}\Big]G\\ &\quad +\Big[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2\Big]t^2. \end{aligned}\tag{2.21}$$

Before completing the proof, we recall a fact: if $x^2\le ax+b$ for some $a,b,x\ge 0$, then

$$x\le\frac{a}{2}+\sqrt{b+\Big(\frac{a}{2}\Big)^2}\le\frac{a}{2}+\sqrt{b}+\frac{a}{2}=a+\sqrt{b}.\tag{2.22}$$

If $a\ge 0$ in (2.21), then from (2.21) we deduce that

$$\begin{aligned}G^2 &\le \Big[\frac{(A+\gamma)n\beta t}{2c(1-\epsilon)}+\frac{n\beta}{2c(1-\epsilon)}+\frac{n^2\beta^3C_1^2t}{4\epsilon c^2(1-\epsilon)\rho^2(\beta-1)}\\ &\quad +\frac{n\beta^2a\alpha|\alpha-1||f^{\alpha-2}|_\infty t}{2c(\beta-1)(1-\epsilon)}+\frac{n\beta[\beta K_1+a(\beta-1)\alpha|f^{\alpha-1}|_\infty]t}{c(1-\epsilon)(\beta-1)}\Big]G\\ &\quad +\frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta t^2}{2c(1-\epsilon)}.\end{aligned}\tag{2.23}$$

Applying (2.22) to the inequality (2.23), we get an upper bound for G:

$$\begin{aligned}G &\le \Big[\frac{(A+\gamma)n\beta}{2c(1-\epsilon)}+\frac{n^2\beta^3C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2}+\frac{n\beta[\beta K_1+a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)}\\ &\quad +\frac{n\beta^2a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2c(\beta-1)(1-\epsilon)}\Big]\tau+\sqrt{\frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}\tau+\frac{n\beta}{2c(1-\epsilon)},\end{aligned}$$

since $t_1\le\tau$. By the construction of Ψ, we have $\sup_{B_{\rho/2,T}}F\le\sup_{B_{\rho,T}}(\Psi F)\le G(x_0,t_0)$ for all $t\in[0,\tau]$. Because $\tau\le T$ is arbitrary, it follows that

$$\begin{aligned}|\nabla f|^2-\beta f_t-\beta q-\beta af^\alpha &\le \frac{n\beta}{2c(1-\epsilon)t}+\frac{(A+\gamma)n\beta}{2c(1-\epsilon)}+\frac{n^2\beta^3C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2}\\ &\quad +\frac{n\beta[\beta K_1+a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)}\\ &\quad +\frac{n\beta^2a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2c(\beta-1)(1-\epsilon)}\\ &\quad +\sqrt{\frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}\end{aligned}$$

where $|f|_\infty := \max_M |f|$. Similarly, when $a \le 0$, we have

$$G^2 \le \left[\frac{(A+\gamma)n\beta t}{2c(1-\epsilon)} + \frac{n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3C_1^2 t}{4\epsilon c^2(1-\epsilon)\rho^2(\beta-1)} + \frac{n\beta^2K_1 t}{c(1-\epsilon)(\beta-1)} - \frac{n\beta at\alpha|f^{\alpha-1}|_\infty}{2c(1-\epsilon)}\right]G + \frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta t^2}{2c(1-\epsilon)}. \tag{2.24}$$

From (2.22), (2.24), and above argument, an upper bound for desired quantity in this case is

$$\begin{aligned}|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \le\ & \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} \\ & + \frac{n\beta[\beta K_1 - \frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)} \\ & + \sqrt{\frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}.\end{aligned}$$

Hence, we complete the proof. □

3. Gradient estimates for (1.4) under the geometric flow

Suppose now u is a positive solution of (1.5), and as in [7], we introduce a function

$$W := u^{-q}, \tag{3.1}$$

where q is a positive constant to be determined later. For convenience, we always omit time variable t and write $\mathcal{Q}_t$ for the partial derivative of $\mathcal{Q}$ relative to t. For example, throughout this paper, $\Delta, \nabla, |\cdot|$ mean the correspondence quantities with respect to $g(t)$. Write

$$\Box := \Delta - \partial_t.$$

A simple computation shows that

$$\begin{aligned}\nabla W &= -qu^{-q-1}\nabla u, & |\nabla W|^2 &= q^2u^{-2q-2}|\nabla u|^2, \\ W_t &= -qu^{-q-1}u_t, & \Delta W &= q(q+1)u^{-q-2}|\nabla u|^2 - qu^{-q-1}\Delta u.\end{aligned}$$

The relation (3.1) yields (see [7, 9])

$$|\nabla u|^2 = \frac{|\nabla W|^2}{q^2W^{2+2/q}}, \quad u_t = -\frac{W_t}{qW^{1+1/q}}, \tag{3.2}$$

and hence

$$\Box W = \frac{q+1}{q}\frac{|\nabla W|^2}{W} + qhW^{1+\frac{1-p}{q}}. \tag{3.3}$$

Since $|\nabla W|^2/W^2 = q^2|\nabla u|^2/u^2$ and $hW^{(1-p)/q} = hu^{p-1}$, we consider again the same quantities as in [7, 9],

$$F_0 := \frac{|\nabla W|^2}{W^2} + \alpha hW^{(1-p)/q} = |\nabla \ln W|^2 + \alpha hW^{(1-p)/q}, \tag{3.4}$$

$$F_1 := \frac{W_t}{W} = \partial_t \ln W, \tag{3.5}$$

$$F := F_0 + \beta F_1. \tag{3.6}$$

Here α, β are two positive constants to be fixed later.

To the geometric flow (1.4), introduce a 1-form $\underline{\boldsymbol{S}}_{g(t)}$ on $M \times [0,T]$ by

$$\underline{\boldsymbol{S}}_{g(t)} := \mathrm{div}_{g(t)} S_{g(t)} - \frac{1}{2}\nabla_{g(t)}\left(\mathrm{tr}_{g(t)} S_{g(t)}\right). \tag{3.7}$$

Locally, one has

$$\underline{\boldsymbol{S}}_i = \nabla^j S_{ij} - \frac{1}{2}\nabla_i\left(\mathrm{tr}_{g(t)} S_{g(t)}\right).$$

For example, if $S_{ij} = R_{ij}$, that is, (1.4) is the Ricci flow, we arrive at

$$\underline{\boldsymbol{S}}_i = \nabla^j R_{ij} - \frac{1}{2}\nabla_i R_{g(t)} = \frac{1}{2}\nabla_i R_{g(t)} - \frac{1}{2}\nabla_i R_{g(t)} = 0$$

by the second contracted Bianchi identity. Thus, the quantity $\underline{\boldsymbol{S}}_{g(t)}$ measures whether S_{ij} satisfies the second contracted Bianchi identity.

A analogous quantity like (3.7) also naturally appears in the general relativity, see, for example, Proposition 13.3 in [14].

Lemma 3.1. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the geometric flow (1.4) on M. If u is a positive solution of (1.5), then*

$$\begin{aligned} \Box F_1 &= \frac{2}{q}\langle \nabla F_1, \nabla \ln W\rangle + (1-p)hW^{\frac{1-p}{q}}\frac{W_t}{W} + qh_t W^{(1-p)/q} \\ &+ 2\left(1+\frac{1}{q}\right) S(\nabla \ln W, \nabla \ln W) - 2\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle - \frac{2\left\langle S, \nabla^2 W\right\rangle}{W}. \end{aligned} \tag{3.8}$$

Here div *and* tr *are respectively divergence operator and trace operator of $g(t)$.*

Proof. As in [9], we have

$$\Box F_1 = \frac{\Delta W_t - W_{tt}}{W} - \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{W_t(\Delta W - W_t)}{W^2} + \frac{2|\nabla W|^2 W_t}{W^3}. \tag{3.9}$$

Since $g(t)$ evolves under the geometric flow (1.1), it follows that

$$\begin{aligned} (\Delta W)_t &= \partial_t\left(g^{ij}\nabla_i\nabla_j W\right) = \left(\partial_t g^{ij}\right)\nabla_i\nabla_j W + g^{ij}\partial_t\left(\partial_i\partial_j W - \Gamma^k_{ij}\partial_k W\right) \\ &= 2S_{ij}\nabla^i\nabla^j W + \Delta(W_t) - g^{ij}\partial_k W \partial_t \Gamma^k_{ij} \\ &= \Delta(W_t) + 2\left\langle S, \nabla^2 W\right\rangle + 2\left\langle \underline{\boldsymbol{S}}, \nabla W\right\rangle \end{aligned}$$

using the fact that $g^{ij}\partial_t\Gamma^k_{ij} = -2\nabla^j S_j{}^k + \nabla^k(\mathrm{tr}(S)) = -2\underline{\boldsymbol{S}}^k$. The term $\Delta W_t - W_{tt} = (\Delta W - W_t)_t - 2\left\langle S, \nabla^2 W\right\rangle - 2\langle \underline{\boldsymbol{S}}, \nabla W\rangle$ can be simplified as [9] into

$$\begin{aligned} \Delta W_t - W_{tt} &= 2\left(1+\frac{1}{q}\right)\frac{\langle \nabla W, \nabla W_t\rangle}{W} - \left(1+\frac{1}{q}\right)\frac{|\nabla W|^2 W_t}{W^2} + qh_t W^{1+\frac{1-p}{q}} \\ &+ \left(1+\frac{1}{q}\right)\frac{2S(\nabla W, \nabla W)}{W} - 2\left\langle \underline{\boldsymbol{S}}, \nabla W\right\rangle \\ &+ h(q+1-p)W^{\frac{1-p}{q}}W_t - 2\left\langle S, \nabla^2 W\right\rangle. \end{aligned}$$

Plugging it into (3.9) yields

$$\begin{aligned} \Box F_1 &= \frac{2}{q}\frac{\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{2}{q}\frac{|\nabla W|^2 W_t}{W^3} + (1-p)hW^{\frac{1-p}{q}-1}W_t + qh_t W^{\frac{1-p}{q}} \\ &+ \left(1+\frac{1}{q}\right)\frac{2S(\nabla W, \nabla W)}{W} - \frac{2\left\langle S, \nabla^2 W\right\rangle}{W} - \frac{2\langle \underline{\boldsymbol{S}}, \nabla W\rangle}{W}. \end{aligned}$$

The desired equation (3.8) immediately follows. □

Similarly, we can find the evolution equation of (3.5).

Lemma 3.2. *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a complete solution to the geometric flow (1.4) on* M*. If* u *is a positive solution of (1.5), then*

$$\begin{aligned}\Box F_0 \ \geq\ & 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F_0, \nabla \ln W\rangle \\ & - \frac{2\alpha p}{q} W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle + \alpha W^{\frac{1-p}{q}}(\Delta h - h_t) + \frac{2(\mathrm{Ric}-S)(\nabla W, \nabla W)}{W^2} \\ & + (1-p)\left(2-\frac{\alpha p}{q^2}\right) h W^{\frac{1-p}{q}}|\nabla \ln W|^2 + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}}\end{aligned} \tag{3.10}$$

where $\epsilon \in (0,1]$ *is any given constant.*

Proof. Recall from [9] that ΔF_0 satisfies

$$\begin{aligned}\Delta F_0 \ =\ & \frac{2|\nabla^2 W|^2}{W^2} + \frac{2\langle \nabla W, \Delta \nabla W\rangle}{W^2} - 8\frac{\langle \nabla^2 W, \nabla W \otimes \nabla W\rangle}{W^3} - \frac{2|\nabla W|^2 \Delta W}{W^3} \\ & + \frac{6|\nabla W|^4}{W^4} + \alpha W^{\frac{1-p}{q}}\Delta h + 2\alpha\left(\frac{1-p}{q}\right) W^{\frac{1-p}{q}-1}\langle \nabla W, \nabla h\rangle \\ & + \alpha\left(\frac{1-p}{q}\right)\left(\frac{1-p}{q}-1\right) h W^{\frac{1-p}{q}}\frac{|\nabla W|^2}{W^2} + \alpha\left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1}\Delta W.\end{aligned} \tag{3.11}$$

On the other hand, the time derivative of F_0 equals

$$\begin{aligned}\partial_t F_0 \ =\ & \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{2|\nabla W|^2 W_t}{W^3} + \alpha h_t W^{\frac{1-p}{q}} \\ & + \alpha\left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1} W_t + \frac{2S(\nabla W, \nabla W)}{W^2}.\end{aligned} \tag{3.12}$$

From (3.11), (3.12) and the Ricci identity $\Delta \nabla_i W = \nabla_i \Delta W + R_{ij}\nabla^j W$, we have

$$\begin{aligned}\Box F_0 \ =\ & \frac{2\langle \nabla W, \nabla(\Delta W - W_t)\rangle}{W^2} - \frac{2|\nabla W|^2(\Delta W - W_t)\rangle}{W^3} \\ & + \left(\frac{2|\nabla^2 W|^2}{W^2} - \frac{8\langle \nabla^2 W, \nabla W \otimes \nabla W\rangle}{W^3} + \frac{6|\nabla W|^4}{W^4}\right) \\ & + \alpha W^{\frac{1-p}{q}}(\Delta h - h_t) + \alpha\left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1}(\Delta W - W_t) \\ & + 2\alpha\left(\frac{1-p}{q}\right) W^{\frac{1-p}{q}-1}\langle \nabla W, \nabla h\rangle + \frac{2(\mathrm{Ric}-S)(\nabla W, \nabla W)}{W^2} \\ & + \alpha\left(\frac{1-p}{q}\right)\left(\frac{1-p}{q}-1\right) h W^{\frac{1-p}{q}}\frac{|\nabla W|^2}{W^2}.\end{aligned} \tag{3.13}$$

The following argument is the same as Lemma 2.2 in [9]. □

Combing Lemma 3.1 with Lemma 3.2, we get

Proposition 3.3. *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a complete solution to the geometric flow (1.4) on* M*. If* u *is a positive solution of (1.5), Define*

$$W = u^{-q}, \quad F = \frac{|\nabla W|^2}{W^2} + \alpha h W^{\frac{1-p}{q}} + \beta\frac{W_t}{W}.$$

Then for all $\epsilon \in (0,1]$ *we have*

$$\begin{aligned}
\Box F &\geq 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\
&\quad +2\beta\left(1+\frac{1}{q}\right)S(\nabla \ln W, \nabla \ln W) - \frac{2\alpha p}{q}W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
&\quad + (1-p)\left(2-\frac{\alpha p}{q^2}\right)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 + W^{\frac{1-p}{q}}[\alpha\Delta h + h_t(q\beta-\alpha)] \\
&\quad + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} + \beta(1-p)hW^{\frac{1-p}{q}}\frac{W_t}{W} - 2\beta\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \\
&\quad +2\left(\mathrm{Ric}-S\right)(\nabla \ln W, \nabla \ln W) - 2\beta\langle \underline{\boldsymbol{S}}, \nabla \ln W\rangle
\end{aligned} \tag{3.14}$$

3.1. **Two special cases.** As in [9], we consider two special cases. The first special case of (3.14) is to choose

$$\beta := \frac{\alpha}{q}, \quad \alpha = \frac{kq^2}{p}. \tag{3.15}$$

Then $q\beta - \alpha = 0$ so that (3.14) becomes

$$\begin{aligned}
\Box F &\geq 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2\alpha(1+q)}{q^2}S(\nabla \ln W, \nabla \ln W) \\
&\quad + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle - \frac{2\alpha p}{q}W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
&\quad + (1-p)\left(2-\frac{\alpha p}{q^2}\right)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 + \alpha W^{\frac{1-p}{q}}\Delta h \\
&\quad + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} + \frac{\alpha(1-p)}{q}hW^{\frac{1-p}{q}}\frac{W_t}{W} - \frac{2\alpha}{q}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \\
&\quad + 2(\mathrm{Ric}-S)(\nabla \ln W, \nabla \ln W) - \frac{2\alpha}{q}\langle \underline{\boldsymbol{S}}, \nabla \ln W\rangle.
\end{aligned} \tag{3.16}$$

Recall the inequality (3.4) in [9]

$$2\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{2\alpha}{q}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \geq 2\left[\frac{a\alpha}{q}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{\alpha}{4bq}|S|^2\right], \tag{3.17}$$

for any positive real numbers a, b satisfying $a + b = \frac{q}{\alpha}$, with the equality if $S = 2b\nabla^2 W/W$. Using the inequality $|\nabla^2 W|^2 \geq (\Delta W)^2/n$, we conclude from (3.16) and (3.17) that

$$\begin{aligned}
\Box F &\geq \frac{2}{n}\left(\frac{a\alpha}{q}-\epsilon\right)\left|\frac{\Delta W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\
&\quad + \frac{2\alpha(1+q)}{q^2}S(\nabla \ln W, \nabla \ln W) - \frac{2\alpha p}{q}W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
&\quad + \alpha W^{\frac{1-p}{q}}\Delta h + (1-p)\left(2-\frac{\alpha p}{q^2}\right)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\
&\quad + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} + \frac{\alpha(1-p)}{q}hW^{\frac{1-p}{q}}\frac{W_t}{W} - \frac{\alpha}{2bq}|S|^2 \\
&\quad + 2(\mathrm{Ric}-S)(\nabla \ln W, \nabla \ln W) - \frac{2\alpha}{q}|\underline{\boldsymbol{S}}||\nabla \ln W|.
\end{aligned} \tag{3.18}$$

By (3.3), we get

$$\frac{\Delta W}{W} = \frac{q+1}{q}\frac{|\nabla W|^2}{W^2} + \frac{W_t}{W} + qhW^{\frac{1-p}{q}} = \frac{q}{\alpha}F + \left(\frac{1+q}{q} - \frac{q}{\alpha}\right)|\nabla \ln W|^2.$$

Because of the assumption $\alpha = kq^2/p$, we arrive at

$$\frac{\Delta W}{W} = \frac{p}{kq}F + \left(\frac{1+q-p/k}{q}\right)|\nabla \ln W|^2 \tag{3.19}$$

Substituting (3.19) into (3.18), we obtain

Lemma 3.4. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the geometric flow (1.4) on M. If u is a positive solution of (1.5), then*

$$\begin{aligned}
\Box F \quad \geq \quad & \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle + \frac{2}{n}\left(\frac{akq}{p} - \epsilon\right)\frac{p^2}{k^2q^2}F^2 + (1-p)hW^{\frac{1-p}{q}}F \\
& + \frac{4p}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)F|\nabla \ln W|^2 - \frac{kq}{2bp}|S|^2 \\
& + 2\left[\frac{1}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 \\
& + \frac{2k(1+q)}{p}S(\nabla \ln W, \nabla \ln W) - 2qkW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
& + \frac{kq^2}{p}W^{\frac{1-p}{q}}\Delta h + (1-p)(1-k)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\
& + 2(\mathrm{Ric} - S)(\nabla \ln W, \nabla \ln W) - \frac{2kq}{p}|\underline{\boldsymbol{S}}||\nabla \ln W|,
\end{aligned}$$

where ϵ is a positive real number satisfying $\epsilon \in (0,1]$, p, q, k, a, b are positive real numbers such that $a + b = p/kq$, and

$$W = u^{-q}, \quad F = \frac{|\nabla W|^2}{W^2} + \frac{kq^2}{p}hW^{\frac{1-p}{q}} + \frac{kq}{p}\frac{W_t}{W}.$$

The second special case is to choose

$$\beta := \frac{2\alpha}{q}, \quad \alpha = \frac{q^2}{p} \tag{3.20}$$

in (3.14). Then the inequality (3.14) becomes

$$\begin{aligned}
\Box F \quad \geq \quad & 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1 - \frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\
& + \frac{4(1+q)}{p}S(\nabla \ln W, \nabla \ln W) + (1-p)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\
& + \frac{q^2}{p}W^{\frac{1-p}{q}}(\Delta h + h_t) + q^2\left(\frac{1}{p} - 1\right)h^2W^{\frac{2(1-p)}{q}} - 2qW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
& + 2q\left(\frac{1}{p} - 1\right)hW^{\frac{1-p}{q}}\frac{W_t}{W} - \frac{4q}{p}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \\
& + 2\left(\mathrm{Ric} - S\right)(\nabla \ln W, \nabla \ln W) - \frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle.
\end{aligned} \tag{3.21}$$

For any positive real numbers a, b with $a+b = q/2\alpha = p/2q$, we have (cf. [9])

$$2\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{4q}{p}\left\langle S, \frac{\nabla^2 W}{W}\right\rangle \geq \frac{4aq}{p}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{q}{bp}|S|^2. \tag{3.22}$$

Together (3.21), (3.22) with $|\nabla^2 W|^2 \geq (\Delta W)^2/n$ implies

$$\begin{aligned}\Box F \geq\ & \frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left|\frac{\Delta W}{W}\right|^2 + 2\left(1 - \frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\ & + \frac{4(1+q)}{p} S(\nabla \ln W, \nabla \ln W) + (1-p)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 - \frac{q}{bp}|S|^2 \\ & + \frac{q^2}{p} W^{\frac{1-p}{q}}(\Delta h + h_t) + q^2\left(\frac{1}{p} - 1\right)h^2 W^{\frac{2(1-p)}{q}} + 2q\left(\frac{1}{p} - 1\right)hW^{\frac{1-p}{q}}\frac{W_t}{W} \\ & - 2qW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle + 2(\mathrm{Ric} - S)(\nabla \ln W, \nabla \ln W) - \frac{4q}{p}\langle \underline{\boldsymbol{S}}, \nabla \ln W\rangle.\end{aligned} \tag{3.23}$$

Substituting the identity (according to (3.3))

$$\frac{\Delta W}{W} = \frac{p}{2q}F + \frac{q}{2}hW^{\frac{1-p}{q}} + \left(\frac{1+q-p/2}{q}\right)|\nabla \ln W|^2.$$

into (3.23) yields

$$\begin{aligned}\Box F \geq\ & \frac{1}{2n}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2}F^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)F|\nabla \ln W|^2 \\ & + \left[\frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2 + 2\left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{q}{bp}|S|^2 \\ & + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle + \frac{4(1+q)}{p}S(\nabla \ln W, \nabla \ln W) + \frac{q^2}{p}W^{\frac{1-p}{q}}(\Delta h + h_t) \\ & + \frac{q^2}{2n}\left(\frac{2aq}{p} - \epsilon\right)h^2 W^{\frac{2(1-p)}{q}} + \left[\frac{p}{n}\left(\frac{2aq}{p} - \epsilon\right) + (1-p)\right]hW^{\frac{1-p}{q}}F \\ & + \frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 - 2qW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\ & - \frac{4q}{p}\langle \underline{\boldsymbol{S}}, \nabla \ln W\rangle + 2(\mathrm{Ric} - S)(\nabla \ln W, \nabla \ln W).\end{aligned}$$

The term $2qW^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle$ is bounded from above by (where we assume that h is nonnegative)

$$\eta hW^{\frac{1-p}{q}}|\nabla \ln W|^2 + \frac{q^2}{\eta}W^{\frac{1-p}{q}}\frac{|\nabla h|^2}{h}$$

for any given $\eta > 0$. Therefore

$$\begin{aligned}\Box F \quad \geq \quad & \frac{1}{2n}\left(\frac{2aq}{p}-\epsilon\right)\frac{p^2}{q^2}F^2+\frac{2p}{nq^2}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)F|\nabla \ln W|^2 \\ & + \left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2+2\left(1-\frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4-\frac{q}{bp}|S|^2 \\ & + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle + \frac{4(1+q)}{p}S(\nabla \ln W, \nabla \ln W) \\ & + \frac{q^2}{p}W^{\frac{1-p}{q}}\left(\Delta h + h_t - \frac{p}{\eta}\frac{|\nabla h|^2}{h}\right) \\ & + \frac{q^2}{2n}\left(\frac{2aq}{p}-\epsilon\right)h^2W^{\frac{2(1-p)}{q}}+\left[\frac{p}{n}\left(\frac{2aq}{p}-\epsilon\right)+(1-p)\right]hW^{\frac{1-p}{q}}F \\ & + \left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)-\eta\right]hW^{\frac{1-p}{q}}|\nabla \ln W|^2 \\ & +2\left(\mathrm{Ric}-S\right)\left(\nabla \ln W, \nabla \ln W\right)-\frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle.\end{aligned} \tag{3.24}$$

By choosing the same conditions on positive real numbers p, q, a, b, ϵ as in [9], Lemma 3.2, we obtain

Lemma 3.5. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the geometric flow (1.4) on an n-dimensional manifold M. Let $h(x,t)$ be a nonnegative function defined on $M\times[0,T]$ which is C^2 in x and C^1 in t, and $\Delta_{g(t)}h+h_t-\frac{p}{\eta}\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h}\geq 0$ on $M\times[0,T]$ for some $p,\eta>0$. Let p,q,a,b,ϵ be positive real numbers satisfying*

- (i) *q is a priori given positive real number;*
- (ii) *$0<\epsilon\leq 1$;*
- (iii) *$a+b=p/2q$;*
- (iv) *either $0<\epsilon\leq\frac{2aq-n(p-1)}{p}$ and $1<p<1+\frac{2aq}{n}$ (then we choose $0<\eta\leq\frac{p-1}{2p}$), or $0<p\leq 1$ and $\frac{2aq}{p}-\epsilon>0$ (then we choose $0<\eta\leq\frac{1}{n}(\frac{2aq}{p}-\epsilon)$).*

If u is a positive solution of (1.5), $F(x_0,t_0)>0$ for some point $(x_0,t_0)\in M\times[0,T]$, where

$$F=\frac{|\nabla W|^2}{W^2}+\frac{q^2}{p}hW^{\frac{1-p}{q}}+\frac{2q}{p}\frac{W_t}{W},$$

then at the point (x_0,t_0) we have

$$\begin{aligned}\Box F \quad \geq \quad & \frac{1}{2n}\left(\frac{2aq}{p}-\epsilon\right)\frac{p^2}{q^2}F^2+\frac{2p}{nq^2}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)F|\nabla \ln W|^2 \\ & + \left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2+2\left(1-\frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4-\frac{q}{bp}|S|^2 \\ & + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle + \frac{4(1+q)}{p}S(\nabla \ln W, \nabla \ln W) \\ & + 2\left(\mathrm{Ric}-S\right)\left(\nabla \ln W, \nabla \ln W\right)-\frac{4q}{p}\left\langle \underline{\boldsymbol{S}}, \nabla \ln W\right\rangle.\end{aligned} \tag{3.25}$$

3.2. **Gradient estimates and some relative results.** In this section, we will use previous lemmas to get the gradient estimates for the positive solution of the equation (1.5) under the geometric flow.

Theorem 3.6. *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the geometric flow (1.4) on* M *with* $-K_1 g(t) \le S_{g(t)} \le K_2 g(t)$, $-K_3 g(t) \le \mathrm{Ric}_{g(t)} - S_{g(t)} \le K_3 g(t)$, *and* $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \le K_4$ *for some* $K_1, K_2, K_3, K_4 > 0$ *on* $B_{2R,T}$, *with* $\overline{K} := \max\{K_1, K_2\}$. *Let* $h(x,t)$ *be a function defined on* $M \times [0,T]$ *which is* C^2 *in* x *and* C^1 *in* t, *satisfying* $\Delta_{g(t)} h \ge -\theta$ *and* $|\nabla_{g(t)} h|_{g(t)} \le \gamma$ *on* $B_{2R,T} \times [0,T]$ *for some nonnegative constants* θ *and* γ. *If* $u(x,t)$ *is a positive smooth solution of (1.5) on* $M \times [0,T]$, *then*

(i) *for* $0 < p < 1$, *we have*

$$\begin{aligned}\frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \quad \le \quad & \frac{C_1}{p^2 t} + \frac{n(1-p)}{p^2} M_1 M_2 + \frac{n[3K_1 + 2(K_3+K_4)p]}{2p^2(1-p)} \\ & + \frac{C_1}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{n}{p(1-p)}\right) \\ & + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{n/K_1}}{p}\gamma M_2 + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}},\end{aligned} \tag{3.26}$$

where C_1 *is a positive constant depending only on* n *and*

$$M_1 := \max_{B_{2R,T}} h_-, \quad M_2 := \max_{B_{2R,T}} u^{p-1}, \quad h_- := \max(-h, 0).$$

(ii) *for* $p \ge 1$, *we have*

$$\begin{aligned}\frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{t} \quad \le \quad & \frac{k^2 C_2}{p^2 t} + \frac{nk^2(p-1)}{p^2} M_4 M_5 + \frac{k^3 n}{k-p} M_3 M_4 \\ & + \frac{k^2 C_2}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{k^2 n}{p(k-p)}\right) \\ & + \frac{2k^3 n}{(k-p)p^2}\left[K_1 + \frac{p}{k}(K_3+K_4)\right] + \frac{k^2\sqrt{n}\gamma}{p} M_4 \\ & + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4} + \frac{k^2 n}{p^2}\left(\overline{K} + \sqrt{\frac{K_4}{2n}}\right),\end{aligned} \tag{3.27}$$

where $k > p$, C_2 *is a positive constant depending only on* n *and*

$$M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h.$$

Proof. The proof is along the outline in [1, 7, 8] and is identically as it in [9]. For completeness, we give a proof here. Firstly, we introduce a cut-off function (see [3, 1, 8, 9, 10, 19]) on $B_{\rho,T} = \{(x,t) \in M \times [0,T] : \mathrm{dist}_{g(t)}(x, x_0) < \rho\}$, where $\mathrm{dist}_{g(t)}(x, x_0)$ stands for the distance between x and x_0 with respect to the metric $g(t)$, which satisfies a basic analytical result stated in the following lemma.

For the fixed $\tau \in (0,T]$, choose the above cut-off function $\overline{\Psi}$. Define $\Psi : M \times [0,T] \to \mathbf{R}$ by setting

$$\Psi(x,t) := \overline{\Psi}\left(\mathrm{dist}_{g(t)}(x, x_0), t\right)$$

with $\rho := 2R$ in Lemma 2.1. Consider the function $\varphi(x,t) = tF(x,t)$. Using the argument of Calabi [2], we may assume that the function $G(x,t) := \varphi(x,t)\Psi(x,t)$

with support in $B_{2R,T}$ is smooth. Let (x_0,t_0) be the point where G achieves its maximum in the set $\{(x,t): 0\le t\le\tau,\ d_{g(t)}(x,x_0)\le\rho\}$. Without loss of generality, assuming $G(x_0,t_0)>0$, we have

$$\nabla G=0,\quad \partial_t G\ge 0,\quad \Delta G\le 0$$

at (x_0,t_0). Now apply Lemma 2.1 and the Laplacian comparison theorem (observe that the hypothesis implies that $-(K_1+K_3)g(t)\le \mathrm{Ric}_{g(t)}\le (K_2+K_3)g(t)$), we have

$$\begin{aligned}
\frac{|\nabla\Psi|^2}{\Psi} &\le \frac{C_{1/2}^2}{\rho^2},\quad -\partial_t\Psi \ \ge\ -\frac{\overline{C}\Psi^{1/2}}{\tau}-C_{1/2}\overline{K}\Psi^{1/2},\\
\Delta\Psi &\ge -\frac{C_{1/2}\Psi^{1/2}}{\rho^2}-\frac{C_{1/2}\Psi^{1/2}}{\rho}(n-1)\sqrt{K_1+K_3}\coth(\sqrt{K_1+K_3}\rho)\\
&\ge -\frac{d_1}{\rho^2}-\frac{d_1\Psi^{1/2}}{\rho}\sqrt{K_1+K_3},
\end{aligned}$$

where $C_{1/2},\overline{C}$ and d_1 are positive constants depending only on n. It is easy to show that

$$0\ge \Box G=\varphi\Box\Psi+2\langle\nabla\varphi,\nabla\Psi\rangle+\Psi\Box\varphi \tag{3.28}$$

at (x_0,t_0). Setting $p\in(0,1)$ and $k=1$ in Lemma 3.4, we obtain from $\Box\varphi=t\Box F-\varphi/t$ that

$$\begin{aligned}
\Box\varphi \ \ge\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla\ln W|^2-\frac{q^2\theta t}{p}W^{\frac{1-p}{q}}\\
&+2t\left[\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q}\right)^2+\left(1-\frac{1}{\epsilon}\right)\right]|\nabla\ln W|^4-\frac{nq\overline{K}^2}{2bp}t\\
&-\frac{2(1+q)K_1t+2pK_3t}{p}|\nabla\ln W|^2+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle+(1-p)hW^{\frac{1-p}{q}}\varphi\\
&-2q\gamma tW^{\frac{1-p}{q}}|\nabla\ln W|-\frac{2qt}{p}K_4|\nabla\ln W|-\frac{\varphi}{t}.
\end{aligned}\tag{3.29}$$

According to Hölder's inequality,

$$\begin{aligned}
2q\gamma tW^{\frac{1-p}{q}}|\nabla\ln W| &\le \frac{(1+q)K_1t}{p}|\nabla\ln W|^2+\frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}}\\
\frac{2qt}{p}K_4|\nabla\ln W| &\le 2K_4t|\nabla\ln W|^2+\frac{q^2K_4}{2p^2}t
\end{aligned}$$

we have

$$\begin{aligned}
\Box\varphi \ \ge\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla\ln W|^2-\frac{q^2\theta t}{p}W^{\frac{1-p}{q}}\\
&+2t\left[\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q}\right)^2+\left(1-\frac{1}{\epsilon}\right)\right]|\nabla\ln W|^4-\frac{nq\overline{K}^2}{2bp}t\\
&-\frac{3(1+q)K_1t+2p(K_3+K_4)t}{p}|\nabla\ln W|^2+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle-\frac{\varphi}{t}\\
&+(1-p)hW^{\frac{1-p}{q}}\varphi-\frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}}-\frac{q^2K_4}{2p^2}t.
\end{aligned}$$

Using Hölder's inequality again we have

$$\frac{3(1+q)K_1t+2p(K_3+K_4)t}{p}|\nabla \ln W|^2 \leq \frac{n[3(1+q)K_1+2p(K_3+K_4)]^2t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 + \frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1-p}{q}\right)^2 2t|\nabla \ln W|^4. \tag{3.30}$$

Substituting (3.30) into (3.29) yields

$$\begin{aligned}\Box\varphi \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2 \\ & + 2t\left[\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{2-2p+q}{q}\right)+\left(1-\frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{nq\overline{K}^2}{2bp}t \\ & - \frac{n[3(1+q)K_1+2p(K_3+K_4)p]^2t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle-\frac{\varphi}{t} \\ & + (1-p)hW^{\frac{1-p}{q}}\varphi - \frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}} - \frac{q^2\theta t}{p}W^{\frac{1-p}{q}} - \frac{q^2K_4}{2p^2}t.\end{aligned}$$

Take $\epsilon \in (0,1/4)$ and choose q so that $1/q \geq n(1-\epsilon)/2\epsilon^2(1-p)$. For such a pair (p,q), we may choose a positive real number a such that $aq/p \geq 2\epsilon$ and then the condition $a+b=p/q$ holds for some $b>0$ (because in this case $0<aq/p<1$). Under the above assumption, we have (as in [9])

$$\frac{1}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{2-2p+q}{q}\right)+\left(1-\frac{1}{\epsilon}\right)\geq 0.$$

and hence

$$\begin{aligned}\Box\varphi \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2 \\ & -\frac{nq\overline{K}^2}{2bp}t-\frac{n[3(1+q)K_1+2p(K_3+K_4)p]^2t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2-\frac{\varphi}{t}-\frac{q^2K_4}{2p^2}t \\ & - \frac{pq^2\gamma^2}{(1+q)K_1}tW^{2\frac{1-p}{q}}-\frac{q^2\theta t}{p}W^{\frac{1-p}{q}}+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle+(1-p)hW^{\frac{1-p}{q}}\varphi.\end{aligned} \tag{3.31}$$

Plugging (3.31) into (3.28) and using the estimate for $\Box\Psi$, we arrive at, where $\rho := 2R$,

$$\begin{aligned}0 \geq\ & \varphi\Box\Psi-\frac{2\varphi}{\Psi}|\nabla\Psi|^2+\Psi\Box\varphi \\ \geq\ & \varphi d_1\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2d_1}{\rho^2}\varphi+\Psi\Box\varphi \\ \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\Psi\varphi^2+\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi\Psi|\nabla\ln W|^2 \\ & - \frac{n[3(1+q)K_1+2(K_3+K_4)p]^2t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2-\left(\frac{\theta M_2}{p}+\frac{pM_2^2\gamma^2}{(1+q)K_1}\right)q^2t\Psi \\ & - (1-p)M_1M_2\varphi\Psi-\frac{nq\overline{K}^2}{2bp}t\Psi-\frac{\varphi\Psi}{t}+\varphi d_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right) \\ & - \frac{K_4q^2}{2p^2}t\Psi-\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle\varphi,\end{aligned}$$

where d_1, d_2 are positive constants depending only on n, and

$$M_1 := \sup_{B_{2R,T}} h_-, \quad M_2 := \sup_{B_{2R,T}} u^{p-1}.$$

Multiplying the above inequality by Ψ on both sides, we get, where $G = \varphi\Psi$

$$\begin{aligned} 0 \ \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)G^2 + \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)G\Psi|\nabla \ln W|^2 \\ & - \frac{n[3(1+q)K_1+2(K_3+K_4)p]^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 - \left(\frac{\theta M_2}{p}+\frac{pM_2^2\gamma^2}{(1+q)K_1}\right)tq^2 \\ & - \frac{nq\overline{K}^2}{2bp}t - \frac{G}{t} + Gd_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right) - \frac{2}{q}\langle\nabla\Psi, \nabla\ln W\rangle G \\ & -(1-p)M_1M_2G - \frac{K_4q^2}{2p^2}t. \end{aligned} \tag{3.32}$$

Using Hölder's inequality implies

$$\begin{aligned} \frac{2}{q}\langle\nabla\Psi, \nabla\ln W\rangle G \ \leq\ & \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)G\Psi|\nabla\ln W|^2 \\ & + \frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G, \end{aligned}$$

the inequality (3.32) gives us the estimate (because $t \leq \tau$)

$$\begin{aligned} 0 \ \geq\ & \frac{2p^2}{nq^2}\left(\frac{aq}{p}-\epsilon\right)G^2 - (1-p)M_1M_2Gt - d_3G \\ & - t\left[\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\right]d_3G \\ & - t^2\left[\frac{n[3(1+q)K_1+2(K_3+K_4)p]^2}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2+\frac{q^2}{p}M_2\theta \right. \\ & \left. +\frac{q^2p}{(1+q)K_1}(M_2\gamma)^2+\frac{nq}{2bp}\overline{K}^2+\frac{K_4q^2}{2p^2}\right]. \end{aligned} \tag{3.33}$$

for some positive constant d_3 depending only on n. The following inequality

$$aG^2 - bG - c \leq 0 \quad (a,b,c>0) \Longrightarrow G \leq \frac{b}{a}+\sqrt{\frac{c}{a}},$$

implies

$$\begin{aligned} G \ \leq\ & \frac{d_3+(1-p)M_1M_2t+td_3\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\right)}{\frac{2p^2}{nq^2}\left(\frac{aq}{p}-\epsilon\right)} \\ & + t\sqrt{\frac{\frac{n[3(1+q)K_1+2(K_3+K_4)p]^2}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2+\frac{q^2M_2\theta}{p}+\frac{q^2p(M_2\gamma)^2}{(1+q)K_1}+\frac{nq\overline{K}^2}{2bp}+\frac{K_4q^2}{2p^2}}{\frac{2p^2}{nq^2}\left(\frac{aq}{p}-\epsilon\right)}}. \end{aligned}$$

Recall the conditions on p, q, ϵ, a, b that

$$0 < p < 1, \quad 0 < \epsilon < \frac{1}{4}, \quad \frac{1}{q} \geq \frac{n(1-\epsilon)}{2\epsilon^2(1-p)}, \quad a + b = \frac{p}{q}, \quad a \geq 2\epsilon \frac{p}{q}.$$

Choose p, ϵ, q as above and

$$a = \left(\frac{1}{2} + 2\epsilon\right)\frac{p}{q}, \quad b = \left(\frac{1}{2} - 2\epsilon\right)\frac{p}{q}. \tag{3.34}$$

The additional condition (3.34), plugging into the inequality for G, yields

$$\begin{aligned} G \quad \leq \quad & \frac{tnq^2\left[\frac{d_3}{t} + (1-p)M_1M_2 + d_3\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1+K_3}}{\rho} + \overline{K} + \frac{n}{2(1+2\epsilon)p(1-p)}\right)\right]}{p^2(1+2\epsilon)} \\ & + t\sqrt{\frac{nq^4}{p^2(1+2\epsilon)}\begin{pmatrix} \frac{n[3(1+q)K_1+2(K_3+K_4)p]^2}{4(1+2\epsilon)p^2(1-p)^2} + \frac{M_2\theta}{p} \\ +\frac{p(M_2\gamma)^2}{(1+q)K_1} + \frac{n\overline{K}^2}{p^2(1-4\epsilon)} + \frac{K_4}{2p^2}\end{pmatrix}} \end{aligned}$$

at (x_0, t_0). Since

$$G = tq^2\left(\frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u}\right)\Psi$$

and $q \leq 2\epsilon^2(1-p)/n(1-\epsilon)$, it follows that, by letting $\epsilon \to 0$,

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \quad \leq \quad & \frac{d_4}{p^2 t} + \frac{n(1-p)}{p^2}M_1M_2 \\ & + \frac{d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{n}{2p(1-p)}\right) \\ & + \frac{n}{p^2}\sqrt{\begin{matrix} \frac{[3(1+q)K_1+2(K_3+K_4)p]^2}{4(1-p)^2} + \frac{p\theta}{n}M_2 \\ +\frac{(p\gamma)^2}{nK_1}M_2^2 + \overline{K}^2 + \frac{K_4}{2n}\end{matrix}} \end{aligned}$$

on $B_{R,\tau}$, for some positive constant d_4 depending only on n. Because $\tau \in (0, T]$ was arbitrary, we arrive at

$$\begin{aligned} \frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \quad \leq \quad & \frac{d_4}{p^2 t} + \frac{n(1-p)}{p^2}M_1M_2 + \frac{n[3K_1 + 2(K_3+K_4)p]}{2p^2(1-p)} \\ & + \frac{d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1+K_3}}{R} + \overline{K} + \frac{n}{2p(1-p)}\right) \\ & + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{\frac{n}{K_1}}}{p}\gamma M_2 + \frac{n}{p^2}\overline{K} + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}}, \end{aligned}$$

on $B_{R,T}$. Arranging terms yields (3.26).

When $p \geq 1$, applying Lemma 3.4, we have

$$\begin{aligned}
\Box\varphi \geq\ & \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi|\nabla\ln W|^2-\frac{\varphi}{t}\\
&-\frac{kq^2\theta t}{p}W^{\frac{1-p}{q}}+2t\left[\frac{1}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2+\left(1-\frac{1}{\epsilon}\right)\right]|\nabla\ln W|^4\\
&-\frac{knq\overline{K}^2}{2bp}t+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle+(1-p)hW^{\frac{1-p}{q}}\varphi\\
&-2qk\gamma tW^{\frac{1-p}{q}}|\nabla\ln W|+(1-p)(1-k)thW^{\frac{1-p}{q}}|\nabla\ln W|^2\\
&-\frac{2k(1+q)K_1t+2p(K_3+K_4)t}{p}|\nabla\ln W|^2-\frac{k^2q^2K_4t}{2p^2},
\end{aligned}$$

where $\epsilon\in(0,1]$ and p,q,k,a,b are positive real numbers such that $a+b=p/kq$ and $k\geq 1$. Define

$$M_3:=\max_{B_{2R,T}}h_-,\quad M_4:=\max_{B_{2R,T}}u^{p-1},\quad M_5:=\max_{B_{2R,T}}h,$$

and

$$\begin{aligned}
M_6:=\ & \min_{q\geq 0}\min_{y\geq 0}\frac{1}{q^2}\Bigg\{2\left[\frac{1}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2+\left(1-\frac{1}{\epsilon}\right)\right]y^2\\
&-(p-1)(k-1)M_3M_4y-\frac{2k(1+q)K_1+2p(K_3+K_4)}{p}y-2qkM_4\gamma y^{\frac{1}{2}}\Bigg\}.
\end{aligned}$$

Observe that $M_6\leq 0$. Therefore, we arrive at the following inequality

$$\begin{aligned}
\Box\varphi \geq\ & \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi|\nabla\ln W|^2-\frac{\varphi}{t}\\
&-\frac{kq^2\theta t}{p}M_4+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle-(p-1)M_4M_5\varphi+M_6q^2t\\
&-\frac{k^2q^2K_4}{2p^2}t-\frac{knq\overline{K}^2}{2bp}t.
\end{aligned}$$

As before, using $0=\nabla G=\Psi\nabla\varphi+\varphi\nabla\Psi$ at (x_0,t_0), we arrive at, where $\rho:=2R$,

$$\begin{aligned}
0 \geq\ & \varphi\Box\Psi-2\varphi\frac{|\nabla\Psi|^2}{\Psi}+\Psi\Box\varphi\\
\geq\ & \varphi d_1\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2d_1}{\rho^1}\varphi+\Psi\Box\varphi\\
\geq\ & \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)\Psi\varphi^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi\Psi|\nabla\ln W|^2\\
&+M_6q^2\Psi t-\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle\varphi-\frac{kq^2\theta M_4}{p}\Psi t-(p-1)M_4M_5\Psi\varphi\\
&-\frac{\Psi\varphi}{t}+\varphi d_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{t}-\overline{K}\right)-\left(\frac{k^2q^2K_4}{2p^2}+\frac{knq\overline{K}^2}{2bp}\right)\Psi t
\end{aligned}$$

for some positive constants d_1, d_2. Multiplying the above inequality by Ψ on both sides, we get, where $G = \varphi\Psi$,

$$\begin{aligned} 0 \geq\ & \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)G^2 + \frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)G\Psi|\nabla \ln W|^2 \\ & + M_6q^2t - \frac{kq^2\theta t}{p}M_4 - (p-1)M_4M_5G - \left(\frac{k^2q^2K_4}{2p^2} + \frac{knq\overline{K}^2}{2bp}\right)t - \frac{G}{t} \\ & + Gd_2\left(-\frac{1}{\rho^2} - \frac{\sqrt{K_1+K_3}}{\rho} - \frac{1}{\tau} - \overline{K}\right) - \frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle G. \end{aligned} \tag{3.35}$$

Using Hölder's inequality, where we choose $akq > \epsilon p$ and $k + kq > p$,

$$\begin{aligned} \frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle G \leq\ & \frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)G\Psi|\nabla \ln W|^2 \\ & + \frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G, \end{aligned}$$

the inequality (3.35) gives the following estimate

$$\begin{aligned} 0 \geq\ & \frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)G^2 - (p-1)M_4M_5Gt - d_3G \\ & - t\left[\frac{1}{\rho^2} + \frac{\sqrt{K_1+K_3}}{\rho} + \overline{K} + \frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\right]d_3G \\ & - t^2\left(\frac{knq\overline{K}^2}{2bp} + \frac{k^2q^2K_4}{2p^2} + \frac{kq^2}{p}M_4\theta - M_6q^2\right) \end{aligned} \tag{3.36}$$

that is similar to (3.33) at (x_0, t_0), where d_3 is a positive constant. Hence

$$\begin{aligned} G \leq\ & \frac{d_3 + (p-1)M_4M_5t + td_3\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1+K_3}}{\rho} + \overline{K} + \frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\right)}{\frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)} \\ & + t\sqrt{\frac{\frac{knq\overline{K}^2}{2bp} + \frac{k^2q^2K_4}{2p^2} + \frac{kq^2}{p}M_4\theta - M_6q^2}{\frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)}} \end{aligned}$$

at (x_0, t_0). Finally, we obtain

$$\begin{aligned} G \leq\ & \frac{tnk^2q^2}{p^2}\left[\frac{d_3}{t} + (p-1)M_4M_5 + d_3\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1+K_3}}{\rho} + \overline{K} + \frac{k^2n}{2p(k+kq-p)}\right)\right] \\ & + tq^2\left[\frac{nk^2}{p^2}\left(\frac{k^2n\overline{K}^2}{p^2(1-2\epsilon)} + \frac{k^2K_4}{2p^2} + \frac{k}{p}M_4\theta - M_6\right)\right]^{1/2} \end{aligned}$$

by taking $a = (\epsilon + \frac{1}{2})\frac{p}{kq}$, $b = (\frac{1}{2} - \epsilon)\frac{p}{kq}$ with $\epsilon \in (0, 1/2)$ and $k \geq p$. As before, we conclude that

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{t} \quad &\leq \quad \frac{k^2 d_4}{p^2 t} + \frac{nk^2(p-1)}{p^2}M_4 M_5 \\ &+ \frac{k^2 d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1 + K_3}}{R} + \overline{K} + \frac{k^2 n}{2p(k-p)}\right) \\ &+ \frac{k^2 n}{p^2}\sqrt{-M_6\frac{p^2}{k^2 n} + \frac{p\theta}{kn}M_4 + \overline{K}^2 + \frac{K_4}{2n}}\end{aligned}$$

on $B_{R,\tau}$, for some positive constant d_4 depending only on n. Because $\tau \in (0, T]$ was arbitrary, we arrive at

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{t} \quad &\leq \quad \frac{k^2 d_4}{p^2 t} + \frac{nk^2(p-1)}{p^2}M_4 M_5 + \frac{k^2 n}{p^2}\left(\overline{K} + \sqrt{\frac{K_4}{2n}}\right) \\ &+ \frac{k^2 d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1 + K_3}}{R} + \overline{K} + \frac{k^2 n}{2p(k-p)}\right) \\ &+ \frac{k\sqrt{n}}{p}\sqrt{-M_6} + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4}\end{aligned}$$

on $B_{R,\tau}$. In the following we shall show that $-M_6 > 0$ is bounded from above by some constant. For any $q, y \geq 0$ we have

$$q^2 M_6 \geq \left[\frac{1}{n}\left(1 + \frac{k-p}{kq}\right)^2 + 2\left(1 - \frac{1}{\epsilon}\right)\right]y^2 - Ay - By^{1/2}$$

where

$$A := (p-1)(k-1)M_3 M_4 + \frac{2k(1+q)K_1 + 2p(K_3 + K_4)}{p}, \quad B := 2qkM_4\gamma.$$

Since $Ay \leq \eta_1 y^2 + A^2/4\eta_1$ and $By^{1/2} \leq \eta_2 y + B^2/4\eta_2$ for any $\eta_1, \eta_2 > 0$, it follows that (as in [9]), where we choose $\eta_1 = [(k-p)/kq]^2/2n$,

$$\begin{aligned}-M_6 \quad &\leq \quad \frac{nk^2}{2(k-p)^2}\eta_2^2 + \frac{nk^2}{2(k-p)^2}\Bigg[(p-1)(k-1)M_3 M_4 \\ &+ \frac{2k(1+q)K_1 + 2p(K_3 + K_4)}{p}\Bigg]^2 + \frac{k^2 M_4^2\gamma^2}{\eta_2}\end{aligned} \tag{3.37}$$

holds for any $q > 0$. Because the right-hand side of (3.37) as a function of q is increasing, letting $q \to 0$ yields

$$\begin{aligned}-M_6 \quad &\leq \quad \frac{nk^2}{2(k-p)^2}\eta^2 + \frac{k^2\gamma^2 M_4^2}{\eta} \\ &+ \frac{nk^2}{2(k-p)^2}\left[(p-1)(k-1)M_3 M_4 + \frac{2kK_1}{p} + 2(K_3 + K_4)\right]^2\end{aligned} \tag{3.38}$$

where $\eta > 0$. Using (3.38), we prove (3.27). □

As an immediate consequence of the above theorem we have

Theorem 3.7. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t.*

(i) For $0 < p < 1$, *assume that* $h \geq 0$, $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$, $\Delta_{g(t)} h \geq 0$ *along the geometric flow with* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ *for some positive constants* $\gamma, K_1, K_2, K_3, K_4$ *with* $\overline{K} := \max\{K_1, K_2\}$, *along the geometric flow. If* u *is a smooth positive function satisfying the nonlinear parabolic equation (1.5), then*

$$\begin{aligned} \frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p} u^{p-1} - \frac{1}{p}\frac{u_t}{u} \quad &\leq \quad \frac{C_1}{p^2 t} + \frac{C_1}{p^3(1-p)} + \frac{C_1}{p^2}\overline{K} + \frac{2nK_1}{p^2(1-p)} \\ &\quad + \frac{\sqrt{n/K_1}}{p}\gamma M + \frac{n}{p^2}\sqrt{\frac{K_4}{2n}} + \frac{n(K_3+K_4)}{p(1-p)} \end{aligned} \tag{3.39}$$

for some positive constant C_1 *depending only on* n, *where* $M := \max_{M\times[0,T]} u^{p-1}$.

(ii) For $p = 1$, *assume that* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ *for some positive constants* K_1, K_2, K_3, K_4 *with* $\overline{K} := \max\{K_1, K_2\}$, $h \geq 0$, $\Delta_{g(t)} h \geq -\theta$ *(*θ *is nonnegative), and* $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$ *(*γ *is nonnegative), along the geometric flow. If* u *is a smooth positive function satisfying the nonlinear parabolic equation (1.5), then*

$$\frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + h - \frac{u_t}{u} \leq \frac{C_2}{t} + C_2\left(1 + K_1 + K_2 + K_3 + K_4 + \overline{K} + \gamma + \sqrt{\theta}\right) \tag{3.40}$$

for some positive constant C_2 *depending only on* n.

(iii) For $p > 1$, *assume that* $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$ *for some positive constants* $\gamma, K_1, K_2, K_3, K_4$ *with* $\overline{K} := \max\{K_1, K_2\}$. $\Delta_{g(t)} h \geq -\theta$, $|\nabla_{g(t)} h|_{g(t)} \leq \gamma$, *and* $-k_1 \leq h \leq k_2$, *where* $\theta, \gamma, k_1, k_2 > 0$, *along the geometric flow. If* u *is a bounded smooth positive function satisfying the nonlinear parabolic equation (1.5), then*

$$\begin{aligned} \frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p} u^{p-1} - \frac{1}{p}\frac{u_t}{u} \quad &\leq \quad \left(\frac{k}{p}\right)^2 \frac{C_3}{t} + \left(\frac{k}{p}\right)^3 \frac{k}{k-p} C_3 + \left(\frac{k}{p}\right)^2 C_3 \Bigg(\overline{K} + \\ &\quad + \frac{k}{k-p}(K_1 + K_3 + K_4) + \sqrt{\frac{K_4}{2n}}\Bigg) \\ &\quad + \left(\frac{k}{p}\right)^2 n(p-1)k_2 M + \frac{k^3 n}{k-p} k_1 M \\ &\quad + \frac{k^2\sqrt{n}}{p}\gamma M + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M}, \end{aligned} \tag{3.41}$$

for some positive constant C_3 *depending only on* n, *where* $M := \max_{M\times[0,T]} u^{p-1}$ *and* $k > p$. *In particular, taking* $k = 2p$, *we get*

$$\begin{aligned} \frac{|\nabla_{g(t)} u|^2_{g(t)}}{u^2} + \frac{h}{p} u^{p-1} - \frac{1}{p}\frac{u_t}{u} \quad &\leq \quad \frac{C_4}{t} + C_5\left(1 + K_1 + K_2 + K_3 + K_4 + \overline{K}\right) \\ &\quad + C_4 p^2\left[(k_1 + k_2)M + \gamma M + \sqrt{\theta M}\right], \end{aligned} \tag{3.42}$$

for some positive constant C_4 *depending only on* n.

In Lemma 3.5, we required that

$$\Delta_{g(t)} h + h_t - \frac{p}{\eta}\frac{|\nabla_{g(t)} h|^2_{g(t)}}{h} \geq 0$$

for some positive constant p, η. In the following proof, we shall see that when $0 < p \leq \frac{2n}{2n-1}$, we need only to assume that

$$\Delta_{g(t)} + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$$

where

$$C_{n,p} = \begin{cases} n, & p \leq 1, \\ \frac{p}{p-1}, & p > 1. \end{cases}$$

Theorem 3.8. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M, satisfying $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_t \underline{\boldsymbol{S}}_{g(t)}|_t \leq K_4$, for some $K_1, K_2, K_3, K_4 > 0$, with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.5), then*

$$\begin{aligned} \frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{u} \quad &\leq \quad \frac{C}{p^2 t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} \\ &\quad + \frac{4n}{p(2-p)}(K_1 + K_3 + K_4) + \frac{1}{p^2}\sqrt{8nK_4}, \end{aligned} \tag{3.43}$$

for some positive constant C depending only on n.

Proof. As in the proof of Theorem 3.6, we have

$$\begin{aligned} \Box\varphi \quad \geq \quad & \frac{1}{2nt}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2}\varphi^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)\varphi|\nabla \ln W|^2 - \frac{\varphi}{t} \\ & + 2t\left[\frac{1}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1 + q - p/2}{q}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{qnt}{bp}\overline{K}^2 \\ & + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{4(1+q)K_1 + 2(K_3 + K_4)p}{p}t|\nabla \ln W|^2 - \frac{2q^2 K_4 t}{p^2}, \end{aligned}$$

where $\varphi = tF$, from Lemma 3.5. Using Hölder's inequality

$$\begin{aligned} \frac{4(1+q)K_2 + 2K_1}{p}t|\nabla \ln W|^2 \quad \leq \quad & \frac{1}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1 - p/2}{q}\right)^2 2t|\nabla \ln W|^4 \\ & + \frac{2nt[(1+q)K_1 + \frac{1}{2}(K_3 + K_4)p]^2}{p(2aq - p\epsilon)}\left(\frac{q}{1 - p/2}\right)^2, \end{aligned}$$

we see that

$$\begin{aligned} \Box\varphi \quad \geq \quad & \frac{1}{2nt}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2}\varphi^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)\varphi|\nabla \ln W|^2 \\ & - \frac{nq\overline{K}^2 t}{bp} - \frac{2nt[(1+q)K_1 + \frac{1}{2}(K_3 + K_4)p]^2}{p(2aq - p\epsilon)}\left(\frac{q}{1 - p/2}\right)^2 \\ & + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{2q^2 K_4 t}{p^2} - \frac{\varphi}{t}. \end{aligned}$$

Writing $G := \varphi\Psi$ and using $\Box G = \varphi\Box\Psi - 2\varphi|\nabla\Psi|^2/\Psi + \Psi\Box\varphi$, as before, we arrive at

$$\begin{aligned} 0 \geq\ & \frac{p^2}{2ntq^2}\left(\frac{2aq}{p}-\epsilon\right)G^2 + \frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)G\Psi|\nabla\ln W|^2 \\ & - \frac{2nt[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2 - \frac{nq\overline{K}^2}{bp}t - \frac{G}{t} \\ & - \frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle G + Gd_1\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1+K_3}}{\rho}-\frac{1}{\tau}-\overline{K}\right) - \frac{2q^2K_4}{p^2}t, \end{aligned} \tag{3.44}$$

for some positive constant d_1 depending only on n. Plugging the inequality

$$\begin{aligned} \frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle G \leq\ & \frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)G\Psi|\nabla\ln W|^2 \\ & + \frac{\frac{1}{q^2}}{\frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G \end{aligned}$$

into (3.44) yields

$$\begin{aligned} 0 \geq\ & \frac{p^2}{2nq^2}\left(\frac{2aq}{p}-\epsilon\right)G^2 - d_2G \\ & - t\left[\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{2(2aq-p\epsilon)(1+q-p/2)}\right]d_2G \\ & - t^2\left[\frac{2n[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2+\frac{nq}{bp}\overline{K}^2+\frac{2q^2K_4}{p^2}\right] \end{aligned}$$

for some positive constant d_2 depending only on n. Hence

$$\begin{aligned} G \leq\ & \frac{d_2+t\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{2(2aq-p\epsilon)(1+q-p/2)}\right)}{\frac{p^2}{2nq^2}\left(\frac{2aq}{p}-\epsilon\right)} \\ & + t\sqrt{\frac{\frac{2n[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2+\frac{nq}{bp}\overline{K}^2+\frac{2q^2K_4}{p^2}}{\frac{p^2}{2nq^2}\left(\frac{2aq}{p}-\epsilon\right)}}. \end{aligned}$$

The above calculation is based on the assumption that

$$\Delta_{g(t)}h + h_t - \frac{p}{\eta}\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$$

for some positive constant $\eta, p > 0$. We now choose appropriate constants, together with the our assumption that

$$\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \geq 0$$

to verify this assumption in Lemma 3.5. Recall the conditions on p, q, ϵ, a, b. First we consider the case,

$$q > 0, \quad 0 < \epsilon \leq 1, \quad a + b = \frac{p}{2q}, \quad 0 < p \leq 1, \quad 0 < \epsilon < \frac{2aq}{p}. \tag{3.45}$$

Choose

$$a=\left(\epsilon+\frac{1}{2}\right)\frac{p}{2q},\quad b=\left(\frac{1}{2}-\epsilon\right)\frac{p}{2q},\quad 0<\epsilon<\frac{1}{2}. \tag{3.46}$$

Then we can choose $\eta=\frac{1}{n}(\frac{2aq}{p}-\epsilon)=\frac{1}{2n}$ so that $p/\eta=2np>2p$, and furthermore

$$\begin{aligned} G &\le \frac{4nq^2t}{p^2}\left[\frac{d_2}{t}+\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}\right)\right] \\ &\quad+\frac{4nq^2t}{p^2}\sqrt{\frac{1}{1-2\epsilon}\overline{K}^2+\frac{[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{\left(1-\frac{p}{2}\right)^2}+\frac{K_4}{2n}}. \end{aligned}$$

Letting $\epsilon\to 0$ and $R\to\infty$ implies

$$\begin{aligned} \frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) &\le \frac{d_2}{t}+\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)} \\ &\quad+\sqrt{\overline{K}^2+\frac{[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{\left(1-\frac{p}{2}\right)^2}+\frac{K_4}{2n}}. \end{aligned}$$

Now we minimize the above inequality for any $q>0$ by the following observation

$$\begin{aligned} \frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) &\le \frac{d_2}{t}+2\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}+\frac{1+q-\frac{p}{2}}{1-\frac{p}{2}}K_1 \\ &\quad+\frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) &\le \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{n}{p\left(1-\frac{p}{2}\right)}K_1} \\ &\quad+\frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}} \\ &= \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{2n}{p(2-p)}K_1} \\ &\quad+\frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}. \end{aligned}$$

Next we consider the second case; that is,

$$q>0,\quad 0<\epsilon\le 1,\quad a+b=\frac{p}{2q},\quad 1<p<1+\frac{2aq}{n},\quad 0<\epsilon\le\frac{2aq-n(p-1)}{p}. \tag{3.47}$$

We have proved that $1<p<\frac{n}{n-1}\le 2$ and $1+q-\frac{p}{2}>0$ in this case. Choose

$$a=\left(\epsilon+\frac{1}{2}\right)\frac{p}{2q},\quad b=\left(\frac{1}{2}-\epsilon\right)\frac{p}{2q},\quad 0<\epsilon<\frac{1}{2},\quad 1<p\le\frac{2n}{2n-1} \tag{3.48}$$

and $\eta=(p-1)/2p\in(0,1/4n]$ so that $p/\eta=2p\frac{p}{p-1}>2p$. This choice of positive constants a,b,p,q,ϵ satisfies the mentioned condition (3.47). Then we obtain the same inequality

$$\begin{aligned} G &\leq \frac{4nq^2t}{p^2}\left[\frac{d_2}{t}+\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1+K_3}}{\rho}+\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}\right)\right] \\ &+\frac{4nq^2t}{p^2}\sqrt{\frac{1}{1-2\epsilon}\overline{K}^2+\frac{[(1+q)K_1+\frac{1}{2}(K_3+K_4)p]^2}{\left(1-\frac{p}{2}\right)^2}+\frac{K_4}{2n}}. \end{aligned}$$

Letting $\epsilon\to 0$ and $R\to\infty$, and minimizing over all $q>0$, we obtain

$$\begin{aligned} \frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) &\leq \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{2n}{p(2-p)}K_1} \\ &+\frac{p}{2-p}(K_1+K_3+K_4)+\sqrt{\frac{K_4}{2n}}. \end{aligned}$$

In both cases, we proved Theorem 3.8. □

Corollary 3.9. *Suppose that $(M,g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M, satisfying $0\leq S_{g(t)}\leq K_2 g(t)$, $-K_3\leq \mathrm{Ric}_{g(t)}-S_{g(t)}\leq K_3 g(t)$, $|\nabla_t \underline{\boldsymbol{S}}_{g(t)}|_{g(t)}\leq K_4$, for some positive constants K_2,K_3,K_4. Let $h(x,t)$ be a non-negative function defined on $M\times[0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h+h_t-2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h}\geq 0$ on $M\times[0,T]$ (where $C_{n,p}=\frac{p}{p-1}$ if $p>1$ and $C_{n,p}=n$ if $p\leq 1$), and $0<p\leq\frac{2n}{2n-1}$ ($n\geq 3$). If u is a positive solution of (1.5), then*

$$\frac{|\nabla_{g(t)}u|^2_{g(t)}}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\leq\frac{C}{p^2t}+\frac{8n}{p^2}K_2+\frac{4n}{p(2-p)}(K_3+K_4)+\frac{1}{p^2}\sqrt{8nK_4} \tag{3.49}$$

for some positive constant C depending only on n.

Under the hypotheses of Theorem 3.8, we let $f:=\ln u$. Then

$$\begin{aligned} |\nabla_{g(t)}f|^2_{g(t)}-\frac{2}{p}f_t &\leq \frac{C}{p^2t}+\frac{8n}{p^2}\overline{K}+\frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} \\ &+\frac{4n}{p(2-p)}(K_1+K_3+K_4)+\frac{1}{p^2}\sqrt{8nK_4} \end{aligned} \tag{3.50}$$

on $M\times[0,T]$. For any two points $(x_1,t_1),(x_2,t_2)\in M\times[0,T]$ with $t_1<t_2$, as [1], we let $\Theta(x_1,t_1,x_2,t_2)$ the set of all the smooth paths $\gamma:[t_1,t_2]\to M$ that connect x_1 to x_2. Using the same argument in the proof of Lemma 2.10 in [1] and the inequality (3.50), for any $\gamma\in\Theta(x_1,t_1,x_2,t_2)$ we have

$$\begin{aligned} \frac{d}{dt}f(\gamma(t),t) &= \nabla_{g(t)}f(\gamma(t),t)\dot\gamma(t)+\left.\frac{\partial}{\partial s}f(\gamma(t),s)\right|_{s=t} \\ &\geq -|\nabla_{g(t)}f(\gamma(t),t)|_{g(t)}|\dot\gamma(t)|_{g(t)}+\frac{p}{2}\left(|\nabla_{g(t)}f(\gamma(t),t)|^2_{g(t)}-\frac{C}{p^2t}-A\right) \\ &\geq -\frac{1}{2p}|\dot\gamma(t)|^2_{g(t)}-\frac{p}{2}\left(\frac{C}{p^2t}+A\right), \end{aligned}$$

where

$$A:=\frac{8n}{p^2}\overline{K}+\frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1}+\frac{4n}{p(2-p)}(K_1+K_3+K_4)+\frac{1}{p^2}\sqrt{8nK_4}.$$

Therefore, we arrive at

$$\begin{aligned} f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2} \frac{d}{dt} f(\gamma(t),t)\,dt \\ &\geq -\frac{1}{2p}\int_{t_1}^{t_2} |\dot\gamma(t)|^2_{g(t)}dt - \frac{pA}{2}(t_2-t_1) - \frac{C}{2p}\ln\frac{t_2}{t_1}. \end{aligned}$$

Corollary 3.10. *Suppose that $(M,g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.4) on M, satisfying $-K_1 g(t) \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_t \underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$, for some $K_1, K_2, K_3, K_4 > 0$, with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a nonnegative function defined on $M\times[0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ on $M\times[0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p>1$ and $C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.5), then*

$$\begin{aligned} \frac{u(x_2,t_2)}{u(x_1,t_1)} \geq & \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\left[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot\gamma(t)|^2_{g(t)}dt - 2n(t_2-t_1)\right. \\ & \left.\left(\frac{1}{p}\overline{K} + \frac{2}{p}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{1}{2-p}(K_1+K_3+K_4) + \frac{1}{p}\sqrt{2nK_4}\right)\right] \end{aligned} \tag{3.51}$$

for some positive constant C depending only on n, where $(x_1,t_1),(x_2,t_2) \in M\times[0,T]$ with $t_1 < t_2$.

When $K_1 = 0$, we have the following

Corollary 3.11. *Suppose that $(M,g(t))_{t\in[0,T]}$ is a solution to the geometric flow (1.12) on M, satisfying $0 \leq S_{g(t)} \leq K_2 g(t)$, $-K_3 g(t) \leq \mathrm{Ric}_{g(t)} - S_{g(t)} \leq K_3 g(t)$, $|\nabla_t \underline{\boldsymbol{S}}_{g(t)}|_{g(t)} \leq K_4$, for some $K_2, K_3, K_4 > 0$. Let $h(x,t)$ be a nonnegative function defined on $M\times[0,T]$ which is C^2 in x and C^1 in t, $\Delta_{g(t)}h + h_t - 2C_{n,p}p\frac{|\nabla_{g(t)}h|^2_{g(t)}}{h} \geq 0$ on $M\times[0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p>1$ and $C_{n,p} = n$ if $p\leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.5), then*

$$\begin{aligned} \frac{u(x_2,t_2)}{u(x_1,t_1)} \geq & \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\left[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot\gamma(t)|^2_{g(t)}dt\right. \\ & \left. - 2n(t_2-t_1)\left(\frac{K_2}{p} + \frac{K_3+K_4}{2-p} + \frac{\sqrt{2nK_4}}{p}\right)\right] \end{aligned}$$

for some positive constant C depending only on n, where $(x_1,t_1),(x_2,t_2) \in M\times[0,T]$ with $t_1 < t_2$.

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