The discrete Pompeiu problem on the plane

Gergely Kiss

Joint work with M. Laczkovich and Cs. Vincze

University of Luxembourg

DCG-DISOPT seminar
Lausanne, November 27, 2017

Dimitrie Pompeiu (1873-1954) was a Romanian mathematician. As one of the students of Henri Poincaré he obtained a Ph.D. degree in mathematics in 1905 at the Université de Paris (Sorbonne).

His contributions were mainly connected to the fields of mathematical analysis, especially the theory of complex functions. In one of his article[1] he posed a question of integral geometry.

[1] *Sur certains systémes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables* (Comptes Rendus de l'Académie des Sciences. Série I. Mathématique 188 (1929) pp. 1138-1139)

Dimitrie Pompeiu (1873-1954) was a Romanian mathematician. As one of the students of Henri Poincaré he obtained a Ph.D. degree in mathematics in 1905 at the Université de Paris (Sorbonne).

His contributions were mainly connected to the fields of mathematical analysis, especially the theory of complex functions. In one of his article[1] he posed a question of integral geometry. It is widely known as the Pompeiu problem.

[1] *Sur certains systémes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables* (Comptes Rendus de l'Académie des Sciences. Série I. Mathématique 188 (1929) pp. 1138-1139)

The classical Pompeiu problem

Question

Let f be a continuous function defined on the plane, and let K be a closed set of positive Lebesgue measure. Suppose that

$$\int_{\sigma(K)} f(x,y)d\lambda_x d\lambda_y = 0 \qquad (1)$$

for every rigid motion σ of the plane, where λ denotes the Lebesque measure. Is it true that $f \equiv 0$?

The classical Pompeiu problem

Question

Let f be a continuous function defined on the plane, and let K be a closed set of positive Lebesgue measure. Suppose that

$$\int_{\sigma(K)} f(x,y)d\lambda_x d\lambda_y = 0 \qquad (1)$$

for every rigid motion σ of the plane, where λ denotes the Lebesque measure. Is it true that $f \equiv 0$?

We say that the set K has the **Pompeiu's property** if the answer of this question is affirmative (YES).

The continuous case

1. $K =$ square:
   - Pompeiu showed that the square has the Pompeiu property.

---

[2] L. Brown, B. M. Schreiber and B. A. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* **23** (1973), 125-154.

[3] Chalakov, Sur un problème de D. Pompeiu, *Annaire Univ. Sofia Fac. Phys. Math.*, Livre 1, **40** (1944), 1-44.

[4] *S*. Williams, A partial solutions of Pompeiu problem, *Math. Ann.* **223** (1976), 183-190.

The continuous case

1. K = square:
   - Pompeiu showed that the square has the Pompeiu property.
   - Brown, Schreiber, Taylor[2] generalized this:
     Every convex set K has the Pompeiu property if it has at least one "corner".

[2] L. Brown, B. M. Schreiber and B. A. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* **23** (1973), 125-154.

[3] Chalakov, Sur un problème de D. Pompeiu, *Annaire Univ. Sofia Fac. Phys. Math.*, Livre 1, **40** (1944), 1-44.

[4] *S*. Williams, A partial solutions of Pompeiu problem, *Math. Ann.* **223** (1976), 183-190.

The continuous case

1. $K =$ square:
   - Pompeiu showed that the square has the Pompeiu property.
   - Brown, Schreiber, Taylor[2] generalized this:
     Every convex set K has the Pompeiu property if it has at least one "corner".
2. $K =$ disk:
   - Chakalov[3] showed infinitely many linearly independent solutions of the form $\sin(ax + by)$ for appropriately chosen constants a, b.

---

[2] L. Brown, B. M. Schreiber and B. A. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* **23** (1973), 125-154.

[3] Chalakov, Sur un problème de D. Pompeiu, *Annaire Univ. Sofia Fac. Phys. Math.*, Livre 1, **40** (1944), 1-44.

[4] *S*. Williams, A partial solutions of Pompeiu problem, *Math. Ann.* **223** (1976), 183-190.

The continuous case

1. $K =$ square:
   - Pompeiu showed that the square has the Pompeiu property.
   - Brown, Schreiber, Taylor[2] generalized this:
     Every convex set K has the Pompeiu property if it has at least one "corner".
2. $K =$ disk:
   - Chakalov[3] showed infinitely many linearly independent solutions of the form $\sin(ax + by)$ for appropriately chosen constants a, b.
   - Williams[4] proved that if K is simply-connected and has a sufficiently smooth boundary ∂K then there is a function $f \not\equiv 0$.

---

[2] L. Brown, B. M. Schreiber and B. A. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* **23** (1973), 125-154.

[3] Chalakov, Sur un problème de D. Pompeiu, *Annaire Univ. Sofia Fac. Phys. Math.*, Livre 1, **40** (1944), 1-44.

[4] *S*. Williams, A partial solutions of Pompeiu problem, *Math. Ann.* **223** (1976), 183-190.

Another type of problem - The discrete case

[5] R. Katz, M. Krebs, A. Shaheen, Zero sums on unit square vertex sets and plane colorings, *Amer. Math. Monthly* **121** (2014), no. 7, 610–618.

[6] C. de Groote, M. Duerinckx, Functions with constant mean on similar countable subsets of $\mathbb{R}^2$, *Amer. Math. Monthly* **119** (2012), 603–605.

Another type of problem - The discrete case

- 70th Putman Mathematical Competition problem (2009): Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

[5] R. Katz, M. Krebs, A. Shaheen, Zero sums on unit square vertex sets and plane colorings, *Amer. Math. Monthly* **121** (2014), no. 7, 610–618.

[6] C. de Groote, M. Duerinckx, Functions with constant mean on similar countable subsets of $\mathbb{R}^2$, *Amer. Math. Monthly* **119** (2012), 603–605.

Another type of problem - The discrete case

- 70th Putman Mathematical Competition problem (2009): Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane?
- Similar question is whether $f \equiv 0$ whenever $f(A)+f(B)+f(C)+f(D)=0$ holds for every **unit** square $ABCD$. Katz, Krebs, Shaheen[5] shown a nice elementary proof.

---

[5] R. Katz, M. Krebs, A. Shaheen, Zero sums on unit square vertex sets and plane colorings, *Amer. Math. Monthly* **121** (2014), no. 7, 610–618.

[6] C. de Groote, M. Duerinckx, Functions with constant mean on similar countable subsets of $\mathbb{R}^2$, *Amer. Math. Monthly* **119** (2012), 603–605.

Another type of problem - The discrete case

- 70th Putman Mathematical Competition problem (2009): Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane?
- Similar question is whether $f\equiv 0$ whenever $f(A)+f(B)+f(C)+f(D)=0$ holds for every **unit** square $ABCD$. Katz, Krebs, Shaheen[5] shown a nice elementary proof.
- Groote, Duerinckx[6] investigated the following version: Given a nonempty finite set $H\subset\mathbb{R}^2$ and a function $f:\mathbb{R}^2\to\mathbb{R}^d$ such that the **arithmetic mean** of f at the elements of any **similar** copy of H is constant. Does it follow that f is constant on $\mathbb{R}^2$?

---

[5] R. Katz, M. Krebs, A. Shaheen, Zero sums on unit square vertex sets and plane colorings, *Amer. Math. Monthly* **121** (2014), no. 7, 610–618.

[6] C. de Groote, M. Duerinckx, Functions with constant mean on similar countable subsets of $\mathbb{R}^2$, *Amer. Math. Monthly* **119** (2012), 603–605.

More abstractly

Let X be a nonempty set and $\mathcal{G}$ be the family of bijections mapping X to itself.

Definition

The finite set $H = \{d_1, \ldots, d_n\} \subset X$ has the *discrete Pompeiu property* w.r.t. $\mathcal{G}$ if for every function $f : X \to \mathbb{C}$ the equation

$$\sum_{i=1}^{n} f(\sigma(d_i)) = 0 \tag{2}$$

holds for all $\sigma \in \mathcal{G}$ implies that $f \equiv 0$.

More abstractly

Let X be a nonempty set and $\mathcal{G}$ be the family of bijections mapping X to itself.

Definition

The finite set $H = \{d_1, \ldots, d_n\} \subset X$ has the *discrete Pompeiu property* w.r.t. $\mathcal{G}$ if for every function $f : X \to \mathbb{C}$ the equation

$$\sum_{i=1}^{n} f(\sigma(d_i)) = 0 \qquad (2)$$

holds for all $\sigma \in \mathcal{G}$ implies that $f \equiv 0$.

In the definition we replace the Lebesque measure by the counting measure.

Weighted version

Definition

The finite set $H = \{d_1, \ldots, d_n\} \subset X$ has the *weighted discrete Pompeiu property* w.r.t. $\mathcal{G}$ if the following condition satisfied: whenever $a_1, \ldots, a_n$ are complex numbers with $\sum_{i=1}^n a_i \neq 0$ and $f : X \to \mathbb{C}$, the equation

$$\sum_{i=1}^{n} a_i f(\sigma(d_i)) = 0 \tag{3}$$

holds for all $\sigma \in \mathcal{G}$, then $f \equiv 0$.

Weighted version

Definition

The finite set $H = \{d_1, \ldots, d_n\} \subset X$ has the *weighted discrete Pompeiu property* w.r.t. $\mathcal{G}$ if the following condition satisfied: whenever $a_1, \ldots, a_n$ are complex numbers with $\sum_{i=1}^{n} a_i \neq 0$ and $f : X \to \mathbb{C}$, the equation

$$\sum_{i=1}^{n} a_i f(\sigma(d_i)) = 0 \tag{3}$$

holds for all $\sigma \in \mathcal{G}$, then $f \equiv 0$.

We focus on the similarity group ($\mathcal{S}$), the translation group ($\mathcal{T}$) and the isometry group ($\mathcal{I}$) of $\mathbb{C}$.

Similarity case

Theorem

Every finite set H has the weighted discrete Pompeiu property w.r.t. $\mathcal{S}$.

[7] G. Kiss and A. Varga, Existence of nontrivial solutions of linear functional equation, *Aequationes Math.* (2014), **88**, no. 1, 151-162.

Similarity case

Theorem

Every finite set H has the weighted discrete Pompeiu property w.r.t. $\mathcal{S}$.

Let $d_1, d_2, \dots, d_n \in \mathbb{C}$ be the points of H. Then the equation (3) can be written in the form

$$a_1 f(x + d_1 y) + a_2 f(x + d_2 y) + \ldots + a_n f(x + d_n y) = 0 \qquad (4)$$

for every $x, y \in \mathbb{C}$, $y \neq 0$ with constant $a_1, \dots, a_n \in \mathbb{C}$ with $\sum_{i=1}^{n} a_i \neq 0$.

[7] G. Kiss and A. Varga, Existence of nontrivial solutions of linear functional equation, *Aequationes Math.* (2014), **88**, no. 1, 151-162.

Similarity case

Theorem

Every finite set H has the weighted discrete Pompeiu property w.r.t. $\mathcal{S}$.

Let $d_1, d_2, \ldots, d_n \in \mathbb{C}$ be the points of H. Then the equation (3) can be written in the form

$$a_1 f(x + d_1 y) + a_2 f(x + d_2 y) + \ldots + a_n f(x + d_n y) = 0 \qquad (4)$$

for every $x, y \in \mathbb{C}$, $y \neq 0$ with constant $a_1, \ldots, a_n \in \mathbb{C}$ with $\sum_{i=1}^{n} a_i \neq 0$.

Theorem (K.- Varga[7], '14)

Assume that (4) holds for every $x, y \in \mathbb{C}$. Then

$$\exists f \not\equiv 0 \text{ solution of } (4) \iff$$

$$\iff \exists \text{ automorphism } \phi \text{ of } \mathbb{C} \text{ which is a solution of } (4) \iff$$

$$\iff \phi \text{ satisfies } \sum_i a_i = 0 \text{ and } \sum_i a_i \phi(b_i) = 0.$$

[7] G. Kiss and A. Varga, Existence of nontrivial solutions of linear functional equation, *Aequationes Math.* (2014), **88**, no. 1, 151-162.

Slight generalization

We say that a family I of subsets of $\mathbb{C}$ is *proper and translation invariant ideal*, if

Slight generalization

We say that a family I of subsets of $\mathbb{C}$ is *proper and translation invariant ideal*, if

- $A, B \in I$ implies that $A \cup B \in I$
- $A \in I$ and $B \subseteq A$, then $B \in I$
- $\mathbb{C} \notin I$
- $A \in I$, then $A + c = \{x + c : x \in A\} \in I$ for every $c \in I$.

Slight generalization

We say that a family I of subsets of $\mathbb{C}$ is *proper and translation invariant ideal*, if

- $A, B \in I$ implies that $A \cup B \in I$
- $A \in I$ and $B \subseteq A$, then $B \in I$
- $\mathbb{C} \notin I$
- $A \in I$, then $A + c = \{x + c : x \in A\} \in I$ for every $c \in I$.

Example

I can be the finite subsets of $\mathbb{C}$, all sets of measure 0 or all first category sets of $\mathbb{C}$.

Slight generalization

We say that a family I of subsets of $\mathbb{C}$ is *proper and translation invariant ideal*, if

- $A, B \in I$ implies that $A \cup B \in I$
- $A \in I$ and $B \subseteq A$, then $B \in I$
- $\mathbb{C} \notin I$
- $A \in I$, then $A + c = \{x + c : x \in A\} \in I$ for every $c \in I$.

Example

I can be the finite subsets of $\mathbb{C}$, all sets of measure 0 or all first category sets of $\mathbb{C}$.

Theorem

Let I be a proper and translation invariant ideal. Assume that (4) holds for every $x \in \mathbb{C}$, $y \in \mathbb{C} \setminus A$, where $A \in I$. Then

$$\exists f \not\equiv 0 \text{ solution of } (4) \iff$$

$$\iff \exists \text{ automorphism } \phi \text{ of } \mathbb{C} \text{ which is a solution of } (4) \iff$$

$$\iff \phi \text{ satisfies } \sum_i a_i = 0 \text{ and } \sum_i a_i \phi(b_i) = 0.$$

Translations

Theorem

Let G be a torsion free Abelian group. No finite set $H \subset G, |H| \geq 2$ has the discrete Pompeiu property w.r.t. G.

[8] D. Zeilberger, Pompeiu's problem in discrete space, *Proc. Nat. Acad. Sci. USA* **75** (1978), no. 8, 3555-3556.

Translations

Theorem

Let G be a torsion free Abelian group. No finite set $H \subset G, |H| \geq 2$ has the discrete Pompeiu property w.r.t. G.

Let G_H be the subgroup of G generated by H. Then G_H is a finitely generated torsion free Abelian group, and thus

$$G_H \cong \mathbb{Z}^n \textit{ for some finite } n.$$

[8] D. Zeilberger, Pompeiu's problem in discrete space, *Proc. Nat. Acad. Sci. USA* **75** (1978), no. 8, 3555-3556.

Translations

Theorem

Let G be a torsion free Abelian group. No finite set $H \subset G, |H| \geq 2$ has the discrete Pompeiu property w.r.t. G.

Let G_H be the subgroup of G generated by H. Then G_H is a finitely generated torsion free Abelian group, and thus

$$G_H \cong \mathbb{Z}^n \text{ for some finite } n.$$

Theorem (Zeilberger[8])

For every finite set $H \subset \mathbb{Z}^n$ there is a nonzero function $f : \mathbb{Z}^n \to \mathbb{C}$ such that $\sum_{i=1}^{n} f(\sigma(d_i)) = 0$ for any translate σ of $\mathbb{Z}^n$.

---

[8] D. Zeilberger, Pompeiu's problem in discrete space, *Proc. Nat. Acad. Sci. USA* **75** (1978), no. 8, 3555-3556.

Translations

Theorem

Let G be a torsion free Abelian group. No finite set $H \subset G, |H| \geq 2$ has the discrete Pompeiu property w.r.t. G.

Let G_H be the subgroup of G generated by H. Then G_H is a finitely generated torsion free Abelian group, and thus

$$G_H \cong \mathbb{Z}^n \text{ for some finite } n.$$

Theorem (Zeilberger[8])

For every finite set $H \subset \mathbb{Z}^n$ there is a nonzero function $f : \mathbb{Z}^n \to \mathbb{C}$ such that $\sum_{i=1}^n f(\sigma(d_i)) = 0$ for any translate σ of $\mathbb{Z}^n$.

Easily, we can find such a function on every coset of G/G_H.

---

[8] D. Zeilberger, Pompeiu's problem in discrete space, *Proc. Nat. Acad. Sci. USA* **75** (1978), no. 8, 3555-3556.

Isometries

Proposition

Let E be a finite set in $\mathbb{R}^n$. If there exists an isometry σ such that $|E \cap \sigma(E)| = |E| - 1$, then E has the discrete Pompeiu property w.r.t. isometries.

Isometries

Proposition

Let E be a finite set in $\mathbb{R}^n$. If there exists an isometry σ such that $|E \cap \sigma(E)| = |E| - 1$, then E has the discrete Pompeiu property w.r.t. isometries.

Corollary

For the vertex set of any k - dimensional simplex in $\mathbb{R}^n$ $(n \geq 2)$ has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Isometries

Proposition

Let E be a finite set in $\mathbb{R}^n$. If there exists an isometry σ such that $|E \cap \sigma(E)| = |E| - 1$, then E has the discrete Pompeiu property w.r.t. isometries.

Corollary

For the vertex set of any k - dimensional simplex in $\mathbb{R}^n$ $(n \geq 2)$ has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proposition

Every 2- and 3-element set has the discrete weighted Pompeiu property w.r.t. $\mathcal{I}$.

Main results[9]

Theorem

Let D be the vertex set of **any** *parallelogram. Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.*

[9] G. Kiss, M. Laczkovich and Cs. Vincze, The discrete Pompeiu problem on the plane, accepted at *Monatshefte für Mathematik*, https://arxiv.org/pdf/1612.00284v1.pdf

Main results[9]

Theorem
Let D be the vertex set of **any** *parallelogram. Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.*

Theorem
Let $D \subset \mathbb{R}^2$ be the set of four points with rational coordinates. Then D has the discrete **weighted** *Pompeiu property w.r.t. $\mathcal{I}$.*

[9] G. Kiss, M. Laczkovich and Cs. Vincze, The discrete Pompeiu problem on the plane, accepted at *Monatshefte für Mathematik*, https://arxiv.org/pdf/1612.00284v1.pdf

Main results[9]

Theorem
Let D be the vertex set of **any** *parallelogram. Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.*

Theorem
Let $D \subset \mathbb{R}^2$ be the set of four points with rational coordinates. Then D has the discrete **weighted** *Pompeiu property w.r.t. $\mathcal{I}$.*

Theorem
Let D be an n-tuple of collinear points in the plane with pairwise commensurable distances. Then D has the discrete **weighted** *Pompeiu property w.r.t. $\mathcal{I}$.*

[9] G. Kiss, M. Laczkovich and Cs. Vincze, The discrete Pompeiu problem on the plane, accepted at *Monatshefte für Mathematik*, https://arxiv.org/pdf/1612.00284v1.pdf

Open Questions

A general open question is the following:

Question

Are there any finite set in the plane which has not the discrete Pompeiu property w.r.t. $\mathcal{I}$*?*

Open Questions

A general open question is the following:

Question

Are there any finite set in the plane which has not the discrete Pompeiu property w.r.t. $\mathcal{I}$?

Question

Do the following sets have the discrete Pompeiu property

1. *Symmetric trapezoid,*
2. *4 points in a line,*
3. *Pentagon, regular n-gon?*

Spectral analysis on discrete Abelian groups

In the continuous case the proofs based on harmonic analysis.

Spectral analysis on discrete Abelian groups

In the continuous case the proofs based on harmonic analysis. Analogously, let G be an Abelian group and we introduce:

Spectral analysis on discrete Abelian groups

In the continuous case the proofs based on harmonic analysis. Analogously, let G be an Abelian group and we introduce:

1. $\mathbb{C}^G = \{f : G \to \mathbb{C}\}$ the linear space of all complex valued functions defined on G equipped with the product topology.

Spectral analysis on discrete Abelian groups

In the continuous case the proofs based on harmonic analysis. Analogously, let G be an Abelian group and we introduce:

1. $\mathbb{C}^G = \{f : G \to \mathbb{C}\}$ the linear space of all complex valued functions defined on G equipped with the product topology.
2. *Exponential function* $g\colon G \to \mathbb{C}$:
   $g \neq 0$ and $g(x+y) = g(x) \cdot g(y)$ for every $x, y \in G$.

Spectral analysis on discrete Abelian groups

In the continuous case the proofs based on harmonic analysis. Analogously, let G be an Abelian group and we introduce:

1. $\mathbb{C}^G = \{f : G \to \mathbb{C}\}$ the linear space of all complex valued functions defined on G equipped with the product topology.
2. *Exponential function* $g\colon G \to \mathbb{C}$:
   $g \neq 0$ and $g(x + y) = g(x) \cdot g(y)$ for every $x, y \in G$.
3. *Variety* V *on* G:
   3.1 translation invariant
   3.2 closed
   3.3 linear

   subspace of $\mathbb{C}^G$.

Spectral analysis on discrete Abelian groups

In the continuous case the proofs based on harmonic analysis. Analogously, let G be an Abelian group and we introduce:

1. $\mathbb{C}^G = \{f : G \to \mathbb{C}\}$ the linear space of all complex valued functions defined on G equipped with the product topology.
2. *Exponential function* $g\colon G \to \mathbb{C}$:
   $g \neq 0$ and $g(x+y) = g(x) \cdot g(y)$ for every $x, y \in G$.
3. *Variety* V *on* G:
   3.1 translation invariant
   3.2 closed
   3.3 linear

   subspace of $\mathbb{C}^G$.
4. *Spectral analysis holds in* G: every $V \neq \{0\}$ on G contains an exponential function.

The *torsion free rank* $r_0(G)$ of G is the cardinality of a maximal independent system of elements of infinite order.

[10] M. Laczkovich and G. Székelyhidi, Harmonic analysis on discrete Abelian groups, *Proc. Am. Math. Soc.* **133** (2004), no. 6, 1581-1586.

The *torsion free rank* $r_0(G)$ of G is the cardinality of a maximal independent system of elements of infinite order.

Theorem (Laczkovich, Székelyhidi[10])

Spectral analysis holds on every Abelian group G iff

$$r_0(G) < \mathfrak{c}.$$

[10] M. Laczkovich and G. Székelyhidi, Harmonic analysis on discrete Abelian groups, *Proc. Am. Math. Soc.* **133** (2004), no. 6, 1581-1586.

The *torsion free rank* $r_0(G)$ of G is the cardinality of a maximal independent system of elements of infinite order.

Theorem (Laczkovich, Székelyhidi[10])

Spectral analysis holds on every Abelian group G iff

$$r_0(G) < \mathfrak{c}.$$

Every function f satisfying

$$\sum_{i=1}^{n} a_i f(x + d_i y) = 0,$$

where $x, y \in \mathbb{C}$, $|y| = 1$, $\sum_{i=1}^{n} a_i \neq 0$ and $H = \{d_1, d_2, \ldots, d_n\}$, constructs a variety V on the additive group of $\mathbb{C}$.

[10] M. Laczkovich and G. Székelyhidi, Harmonic analysis on discrete Abelian groups, *Proc. Am. Math. Soc.* **133** (2004), no. 6, 1581-1586.

Method of the proof

Theorem

Let D be the vertex set of **any** *parallelogram.*
Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proof.

Method of the proof

Theorem

Let D be the vertex set of **any** *parallelogram.*
Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proof.
Let G be a finitely generated additive subgroup of $\mathbb{C}$.

Method of the proof

Theorem

Let D be the vertex set of **any** *parallelogram.*
Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proof.
Let G be a finitely generated additive subgroup of $\mathbb{C}$. The statement can be written in the following form:

$$f(x) + f(x + s_1 y) + f(x + s_2 y) + f(x + (s_1 + s_2)y) = 0 \qquad (5)$$

holds for every $x, y \in G$ and $|y| = 1$.

Method of the proof

Theorem

Let D be the vertex set of **any** *parallelogram.*
Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proof.
Let G be a finitely generated additive subgroup of $\mathbb{C}$. The statement can be written in the following form:

$$f(x) + f(x + s_1 y) + f(x + s_2 y) + f(x + (s_1 + s_2)y) = 0 \qquad (5)$$

holds for every $x, y \in G$ and $|y| = 1$.Assume that $f(0) \neq 0$.

Method of the proof

Theorem

Let D be the vertex set of **any** *parallelogram.*
Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proof.
Let G be a finitely generated additive subgroup of $\mathbb{C}$. The statement can be written in the following form:

$$f(x) + f(x + s_1 y) + f(x + s_2 y) + f(x + (s_1 + s_2)y) = 0 \qquad (5)$$

holds for every $x, y \in G$ and $|y| = 1$.Assume that $f(0) \neq 0$. There exists an exponential function $g \not\equiv 0$ in V which satisfies

$$g(x + y) = g(x)g(y). \qquad (6)$$

Method of the proof

Theorem

Let D be the vertex set of **any** *parallelogram.*
Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proof.
Let G be a finitely generated additive subgroup of $\mathbb{C}$. The statement can be written in the following form:

$$f(x) + f(x + s_1 y) + f(x + s_2 y) + f(x + (s_1 + s_2)y) = 0 \qquad (5)$$

holds for every $x, y \in G$ and $|y| = 1$.Assume that $f(0) \neq 0$.
There exists an exponential function $g \not\equiv 0$ in V which satisfies

$$g(x + y) = g(x)g(y). \qquad (6)$$

Thus, we get

$$(1 + g(s_1 y))(1 + g(s_2 y)) = 0. \qquad (7)$$

Method of the proof

Theorem

Let D be the vertex set of **any** *parallelogram.*
Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Proof.
Let G be a finitely generated additive subgroup of $\mathbb{C}$. The statement can be written in the following form:

$$f(x) + f(x + s_1 y) + f(x + s_2 y) + f(x + (s_1 + s_2)y) = 0 \qquad (5)$$

holds for every $x, y \in G$ and $|y| = 1$.Assume that $f(0) \neq 0$.
There exists an exponential function $g \not\equiv 0$ in V which satisfies

$$g(x + y) = g(x)g(y). \qquad (6)$$

Thus, we get

$$(1 + g(s_1 y))(1 + g(s_2 y)) = 0. \qquad (7)$$

i.e $g(s_1 y) = -1$ or $g(s_2 y) = -1$ $(\forall y \in G, |y| = 1)$.

$$a = x, \quad b = x + s_1 y,$$
$$d = x + s_2 y, \quad c = x + (s_1 + s_2) y,$$

$$a = x, \quad b = x + s_1 y,$$
$$d = x + s_2 y, \quad c = x + (s_1 + s_2) y,$$

Then g at the points a, b, c, d are

$$\text{either } g(a), -g(a), -g(d), g(d)$$
$$\text{or } g(a), g(b), -g(b), -g(a),$$

respectively.

$$a = x, \quad b = x + s_1 y,$$
$$d = x + s_2 y, \quad c = x + (s_1 + s_2) y,$$

Then g at the points a, b, c, d are

$$\text{either } g(a), -g(a), -g(d), g(d)$$
$$\text{or } g(a), g(b), -g(b), -g(a),$$

respectively.
Let A and B be such that $\mathbb{C} \setminus \{0\} = \mathbb{C}^* = A \cup^* B$ and $A = -B$.

$$a = x, \quad b = x + s_1 y,$$
$$d = x + s_2 y, \quad c = x + (s_1 + s_2) y,$$

Then g at the points a, b, c, d are

$$\text{either } g(a), -g(a), -g(d), g(d)$$
$$\text{or } g(a), g(b), -g(b), -g(a),$$

respectively.
Let A and B be such that $\mathbb{C} \setminus \{0\} = \mathbb{C}^* = A \cup^* B$ and $A = -B$.
We define $h \colon \mathbb{C} \to \{-1, 1\}$ as follows:

$$h(x) = \begin{cases} 1, & \text{if } g(x) \in A \\ -1, & \text{if } g(x) \in B. \end{cases}$$

Euclidean Ramsey theory

Color the points of the plane $\mathbb{R}^2$ with 2 colors.

[11] L. E. Shader, All right triangles are Ramsey in E^2!, *Journ. Comb. Theory (A)* **20** (1976), 385-389.

[12] W. H. Gottschalk, Choice functions and Tychonoff's theorem, *Proc. Amer. Math. Soc.* **2** (1951), 172.

Euclidean Ramsey theory

Color the points of the plane $\mathbb{R}^2$ with 2 colors.
Then we have a 2-coloring h of any finitely generated subgroup $(G, +)$ of the plane such that two-two sides of every parallelogram have the same color.

[11] L. E. Shader, All right triangles are Ramsey in E^2!, *Journ. Comb. Theory (A)* **20** (1976), 385-389.

[12] W. H. Gottschalk, Choice functions and Tychonoff's theorem, *Proc. Amer. Math. Soc.* **2** (1951), 172.

Euclidean Ramsey theory

Color the points of the plane $\mathbb{R}^2$ with 2 colors.
Then we have a 2-coloring h of any finitely generated subgroup $(G, +)$ of the plane such that two-two sides of every parallelogram have the same color.

Theorem (Shader[11])

For every 2-coloring of the plane and every parallelogram H, there is a congruent copy $\sigma(H)$ such that at least three of its vertices has the same color.

---

[11] L. E. Shader, All right triangles are Ramsey in E^2!, *Journ. Comb. Theory (A)* **20** (1976), 385-389.

[12] W. H. Gottschalk, Choice functions and Tychonoff's theorem, *Proc. Amer. Math. Soc.* **2** (1951), 172.

Euclidean Ramsey theory

Color the points of the plane $\mathbb{R}^2$ with 2 colors.
Then we have a 2-coloring h of any finitely generated subgroup $(G, +)$ of the plane such that two-two sides of every parallelogram have the same color.

Theorem (Shader[11])

For every 2-coloring of the plane and every parallelogram H, there is a congruent copy $\sigma(H)$ such that at least three of its vertices has the same color.

For any given parallelogram H, there is a finite witness[12] set R.
Thus, if the generator set of G contains R, we get a contradiction. □

---

[11] L. E. Shader, All right triangles are Ramsey in E^2!, *Journ. Comb. Theory (A)* **20** (1976), 385-389.

[12] W. H. Gottschalk, Choice functions and Tychonoff's theorem, *Proc. Amer. Math. Soc.* **2** (1951), 172.

Easy application on the coloring of the plane

Question (Hadwiger-Nelson's problem[13])

What is the chromatic number χ *of the plane?*

[13] https://en.wikipedia.org/wiki/Hadwiger-Nelson_problem

Easy application on the coloring of the plane

Question (Hadwiger-Nelson's problem[13])

What is the chromatic number χ of the plane?

Theorem

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$

We denote the set of forbidden distances by $\mathcal{FD}$. Previously, $\mathcal{FD} = \{1\}$.

[13] https://en.wikipedia.org/wiki/Hadwiger-Nelson_problem

Easy application on the coloring of the plane

Question (Hadwiger-Nelson's problem[13])

What is the chromatic number χ of the plane?

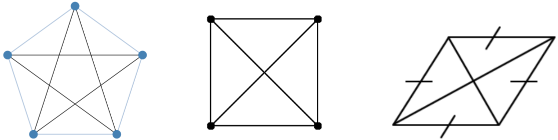
Theorem

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$

We denote the set of forbidden distances by $\mathcal{FD}$. Previously, $\mathcal{FD} = \{1\}$.

Proposition

Let $\mathcal{FD}_1 = \{1, \frac{\sqrt{5}+1}{2}\}$, $\mathcal{FD}_2 = \{1, \sqrt{2}\}$ and $\mathcal{FD}_3 = \{1, \sqrt{3}\}$. Then $\chi_{\mathcal{FD}_i}(\mathbb{R}^2) > 4 \quad (i = 1, 2, 3)$.



[13] https://en.wikipedia.org/wiki/Hadwiger-Nelson_problem

Commensurable points in a line

After rescaling the line we can assume that the points are integers: $z_0 = 0 < z_1 < \ldots < z_k = n \in \mathbb{Z}$.

Commensurable points in a line

After rescaling the line we can assume that the points are integers: $z_0 = 0 < z_1 < \ldots < z_k = n \in \mathbb{Z}$.

Theorem

The set $D = \{z_0 = 0, z_1, \ldots, z_k = n\} \subset \mathbb{C}$ has the weighted Pompeiu property.

Commensurable points in a line

After rescaling the line we can assume that the points are integers: $z_0 = 0 < z_1 < \ldots < z_k = n \in \mathbb{Z}$.

Theorem

The set $D = \{z_0 = 0, z_1, \ldots, z_k = n\} \subset \mathbb{C}$ has the weighted Pompeiu property.

Proof. Let M be a sufficiently large positive integer, $I = \{1, \ldots, n^2 + 1\}$ and consider the following *system of triangles*: the triangle $\Delta_{i,j}$ ($(i,j) \in I \times I$) is defined such that

1. the sides of $\Delta_{i,j}$ are of positive integer length(s) $M, a_{i,j}, b_{i,j}$.
2. for any fixed i the number of $a_{i,1}, \ldots, a_{i,n^2+1}$ are different.
3. $a_{i,j} + b_{i,j} = M + i$ for every $(i,j) \in I \times I$.
4. the triangles are lying on the segment $[0, M]$ in the complex plane and the third vertex is contained in the upper half-plane.

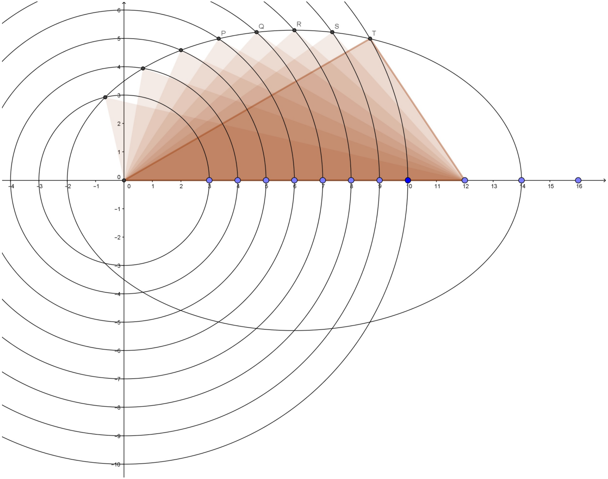


Figure: System of triangles; M=12, i=4

Remark

The length of the major axis is $M + i$ for any fixed $i = 1 \ldots n^2 + 1$.

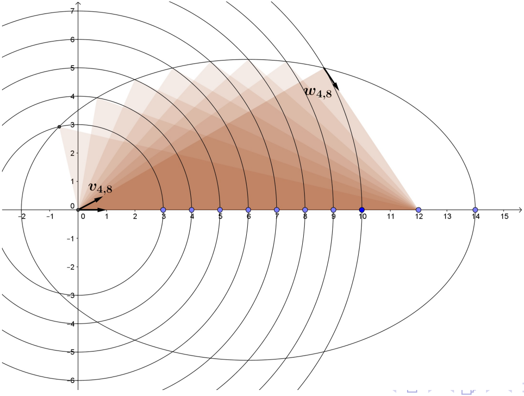
Let us denote the unit vectors parallel to the sides $a_{i,j}$ and $b_{i,j}$ in $\mathbb{C}$ by $v_{i,j}$ and $w_{i,j}$, respectively: $a_{i,j} \cdot v_{i,j} + b_{i,j} \cdot w_{i,j} = M \cdot 1$.

Let us denote the unit vectors parallel to the sides $a_{i,j}$ and $b_{i,j}$ in $\mathbb{C}$ by $v_{i,j}$ and $w_{i,j}$, respectively: $a_{i,j} \cdot v_{i,j} + b_{i,j} \cdot w_{i,j} = M \cdot 1$. An element y is chosen such that $f(y) \neq 0$, where f is a non-zero solution of the Pompeiu equation on the congruent copies of D.

Let us denote the unit vectors parallel to the sides $a_{i,j}$ and $b_{i,j}$ in $\mathbb{C}$ by $v_{i,j}$ and $w_{i,j}$, respectively: $a_{i,j} \cdot v_{i,j} + b_{i,j} \cdot w_{i,j} = M \cdot 1$. An element y is chosen such that $f(y) \neq 0$, where f is a non-zero solution of the Pompeiu equation on the congruent copies of D. Consider the integer grid G_Δ generated by the elements

$$1, \; v_{i,j}, \; w_{i.j}, \; y.$$

G_Δ is a group such that all of the elements are linear combinations with integer coefficients of the generating vectors.



Since G_Δ is finitely generated, spectral analysis holds in the varieties of the vector space of complex valued functions defined on G_Δ.

Since G_Δ is finitely generated, spectral analysis holds in the varieties of the vector space of complex valued functions defined on G_Δ. This means that the variety formed by the solutions of the Pompeiu equation

$$\sum_{i=0}^{k} a_i f(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)) \tag{8}$$

on G_Δ contains a non-zero exponential element $g : G_\Delta \to \mathbb{C}$, i.e.

$$\sum_{i=0}^{k} a_i g(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)).$$

Since G_Δ is finitely generated, spectral analysis holds in the varieties of the vector space of complex valued functions defined on G_Δ. This means that the variety formed by the solutions of the Pompeiu equation

$$\sum_{i=0}^{k} a_i f(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)) \tag{8}$$

on G_Δ contains a non-zero exponential element $g : G_\Delta \to \mathbb{C}$, i.e.

$$\sum_{i=0}^{k} a_i g(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)).$$

If $\sigma = 1$ then the exponential property shows that

$$\sum_{i=0}^{k} a_i (g(1))^{z_i} = 0. \tag{9}$$

Since G_Δ is finitely generated, spectral analysis holds in the varieties of the vector space of complex valued functions defined on G_Δ. This means that the variety formed by the solutions of the Pompeiu equation

$$\sum_{i=0}^{k} a_i f(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)) \tag{8}$$

on G_Δ contains a non-zero exponential element $g : G_\Delta \to \mathbb{C}$, i.e.

$$\sum_{i=0}^{k} a_i g(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)).$$

If $\sigma = 1$ then the exponential property shows that

$$\sum_{i=0}^{k} a_i (g(1))^{z_i} = 0. \tag{9}$$

i.e. $g(1)$ is the root of the polynomial
$P(x) = a_0 \cdot 1 + a_1 \cdot x^{z_1} + \ldots + a_k \cdot x^{z_k}$.

Since G_Δ is finitely generated, spectral analysis holds in the varieties of the vector space of complex valued functions defined on G_Δ. This means that the variety formed by the solutions of the Pompeiu equation

$$\sum_{i=0}^{k} a_i f(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)) \tag{8}$$

on G_Δ contains a non-zero exponential element $g : G_\Delta \to \mathbb{C}$, i.e.

$$\sum_{i=0}^{k} a_i g(\sigma(z_i)) = 0 \quad (\sigma \in \mathrm{Iso}(G_\Delta)).$$

If $\sigma = 1$ then the exponential property shows that

$$\sum_{i=0}^{k} a_i (g(1))^{z_i} = 0. \tag{9}$$

i.e. $g(1)$ is the root of the polynomial
$P(x) = a_0 \cdot 1 + a_1 \cdot x^{z_1} + \ldots + a_k \cdot x^{z_k}$.
Similarly, the same is true for $g(v_{i,j})$ and $g(w_{i,j})$.

Lemma

The root $g(1)$ must be a root of unity.

Proof. Consider the pairs $(g(v_{i,j}), g(w_{i,j}))$.

Lemma

The root $g(1)$ must be a root of unity.

Proof. Consider the pairs $(g(v_{i,j}), g(w_{i,j}))$.
Since the maximal number of the pairs is n^2 but $i, j \in \{1, \ldots, n^2 + 1\}$, using the pigeon-hole principle we get one of the pairs of the roots is taken at least two times in each "row". Repeating the argument for the "column" we have that there exists a pair of roots belonging to at least four triangles.

Lemma

The root $g(1)$ must be a root of unity.

Proof. Consider the pairs $(g(v_{i,j}), g(w_{i,j}))$.
Since the maximal number of the pairs is n^2 but $i, j \in \{1, \ldots, n^2+1\}$, using the pigeon-hole principle we get one of the pairs of the roots is taken at least two times in each "row". Repeating the argument for the "column" we have that there exists a pair of roots belonging to at least four triangles. For the sake of simplicity let $\lambda_1, \ldots, \lambda_n$ be the roots of

$$P(x) = a_0 \cdot 1 + a_1 \cdot x^{z_1} + \ldots + a_k \cdot x^{z_k},$$

$g(1) = \lambda_1$ and suppose that (λ_2, λ_3) is the pair of the roots along the (non-horizontal) directions chosen above.

This means that

$$g(v_{i,j_1}) = g(v_{i,j_2}) = g(v_{i',j'_1}) = g(v_{i',j'_2}) = \lambda_2,$$

$$g(w_{i,j_1}) = g(w_{i,j_2}) = g(w_{i',j'_1}) = g(w_{i',j'_2}) = \lambda_3.$$

This means that

$$g(v_{i,j_1}) = g(v_{i,j_2}) = g(v_{i',j'_1}) = g(v_{i',j'_2}) = \lambda_2,$$

$$g(w_{i,j_1}) = g(w_{i,j_2}) = g(w_{i',j'_1}) = g(w_{i',j'_2}) = \lambda_3.$$

Since g is exponential and $a_{i,j}v_{i,j} + b_{i,j}w_{i,j} = M$ we have

$$\lambda_1^M = \lambda_2^{a_{i,j_1}}\lambda_3^{b_{i,j_1}} = \lambda_2^{a_{i,j_2}}\lambda_3^{b_{i,j_2}},$$

$$\lambda_1^M = \lambda_2^{a_{i',j'_1}}\lambda_3^{b_{i',j'_1}} = \lambda_2^{a_{i',j'_2}}\lambda_3^{b_{i',j'_2}}.$$

This means that

$$g(v_{i,j_1}) = g(v_{i,j_2}) = g(v_{i',j_1'}) = g(v_{i',j_2'}) = \lambda_2,$$

$$g(w_{i,j_1}) = g(w_{i,j_2}) = g(w_{i',j_1'}) = g(w_{i',j_2'}) = \lambda_3.$$

Since g is exponential and $a_{i,j}v_{i,j} + b_{i,j}w_{i,j} = M$ we have

$$\lambda_1^M = \lambda_2^{a_{i,j_1}}\lambda_3^{b_{i,j_1}} = \lambda_2^{a_{i,j_2}}\lambda_3^{b_{i,j_2}},$$

$$\lambda_1^M = \lambda_2^{a_{i',j_1'}}\lambda_3^{b_{i',j_1'}} = \lambda_2^{a_{i',j_2'}}\lambda_3^{b_{i',j_2'}}.$$

Using that $a_{i,j_1} + b_{i,j_1} = a_{i,j_2} + b_{i,j_2} = M + i$, these equations imply that λ_1 is a root of unity.

This means that

$$g(v_{i,j_1}) = g(v_{i,j_2}) = g(v_{i',j_1'}) = g(v_{i',j_2'}) = \lambda_2,$$

$$g(w_{i,j_1}) = g(w_{i,j_2}) = g(w_{i',j_1'}) = g(w_{i',j_2'}) = \lambda_3.$$

Since g is exponential and $a_{i,j}v_{i,j} + b_{i,j}w_{i,j} = M$ we have

$$\lambda_1^M = \lambda_2^{a_{i,j_1}}\lambda_3^{b_{i,j_1}} = \lambda_2^{a_{i,j_2}}\lambda_3^{b_{i,j_2}},$$

$$\lambda_1^M = \lambda_2^{a_{i',j_1'}}\lambda_3^{b_{i',j_1'}} = \lambda_2^{a_{i',j_2'}}\lambda_3^{b_{i',j_2'}}.$$

Using that $a_{i,j_1} + b_{i,j_1} = a_{i,j_2} + b_{i,j_2} = M + i$, these equations imply that λ_1 is a root of unity.

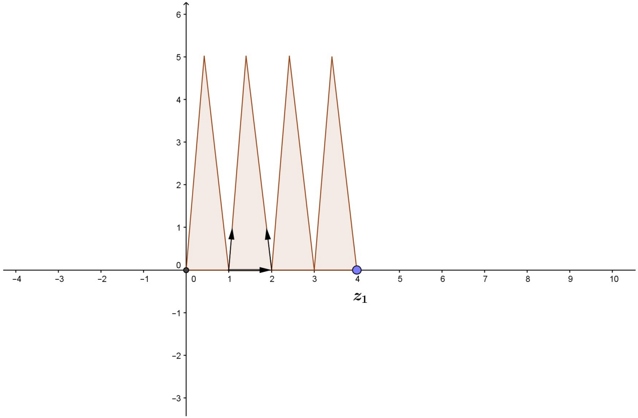
Corollary

The polynomial $P(x) = a_0 \cdot 1 + a_1 \cdot x^{z_1} + \ldots + a_k \cdot x^{z_k}$ *has a root of unity.*

There is an $N \in \mathbb{Z}^+$ such that $(\lambda_k)^N = 1$ for each root λ_k of $P(x)$ which is a root of unity.

End of the proof

Consider the original sample $z_0 = 0, z_1, \ldots, z_k = n$. Choose M such that $N|M$, i.e. $\lambda_k^M = 1$.



Draw an isosceles triangles T_i with side lengths $M, M, 1$ indicated by the vectors $v_1, v_2, \underline{1}$.

Extending G_Δ, similarly as above we get that $g(v_i)$ is a root of the polynomial

$$P(x) = a_0 \cdot 1 + a_1 \cdot x^{z_1} + \ldots + a_k \cdot x^{z_k}$$

and also a root of unity. By the choice of M and since g is exponential

$$g(M \cdot v_j) = [g(v_j)]^M = 1 \quad (j = 1, 2). \tag{10}$$

Extending G_Δ, similarly as above we get that $g(v_i)$ is a root of the polynomial

$$P(x) = a_0 \cdot 1 + a_1 \cdot x^{z_1} + \ldots + a_k \cdot x^{z_k}$$

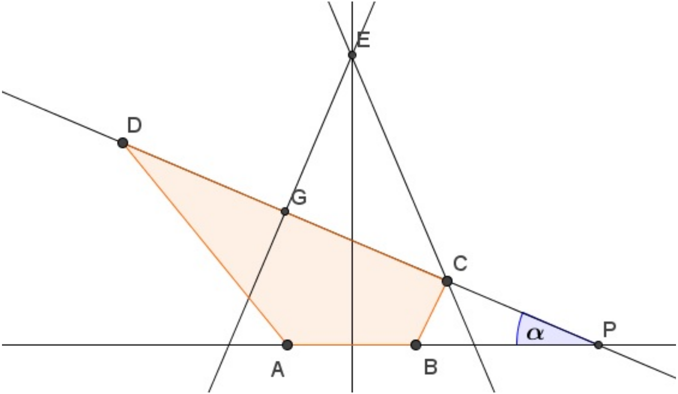
and also a root of unity. By the choice of M and since g is exponential

$$g(M \cdot v_j) = [g(v_j)]^M = 1 \quad (j = 1, 2). \tag{10}$$

This implies that $1 = g(0) = g(z_0) = \cdots = g(z_k)$ which contradicts to $\sum_{i=0}^{k} a_i g(z_i) = 0$, where $\sum_{i=0}^{k} a_i \neq 0$.

A one-parameter family of quadrangles with the discrete Pompeiu property

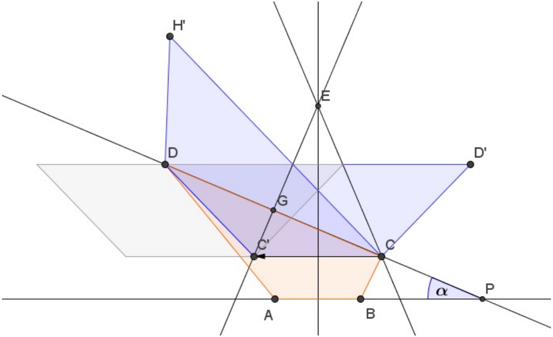
Let $ABC\triangle$ be a non-isosceles triangle and C and B be on the same side of the perpendicular bisector of AB. Let $0 < \alpha < 45^\circ$ be a given angle and choose a point P on the line AB as in the figure below.



Lemma

For any given angle $0 < \alpha < 45^\circ$ *the set* $H = \{A, B, C, D\}$ *has the Pompeiu property.*

Proof. For any point X let X' be the image of X under the reflection on the perpendicular bisector of AB. Then $A' = B$, $B' = A$ and the points C, C', D and D' form a symmetric trapezium such that $D'C = CC' = C'D$.



Taking the difference of equations

$$f(A)+f(B)+f(C)+f(D) = 0 \text{ and } f(A')+f(B')+f(C')+f(D') = 0.$$

Taking the difference of equations

$$f(A)+f(B)+f(C)+f(D) = 0 \text{ and } f(A')+f(B')+f(C')+f(D') = 0.$$

It follows that

$$f(C) - f(C') + f(D) - f(D') = 0. \tag{11}$$

Since equation (11) holds on any congruent copy of the trapezium $CC'DD'$ we have

$$f(C)-f(C')+f(D)-f(D') = 0 \text{ and } f(C)-f(C')+f(D)-f(H') = 0. \tag{12}$$

Taking the difference of equations

$$f(A)+f(B)+f(C)+f(D)=0 \text{ and } f(A')+f(B')+f(C')+f(D')=0.$$

It follows that

$$f(C)-f(C')+f(D)-f(D')=0. \tag{11}$$

Since equation (11) holds on any congruent copy of the trapezium $CC'DD'$ we have

$$f(C)-f(C')+f(D)-f(D')=0 \text{ and } f(C)-f(C')+f(D)-f(H')=0. \tag{12}$$

This implies that $f(D')=f(H')$.

Group acting on $\mathbb{R}^2$	Set	P	WP
Translations	finite sets with at least 2 elements	No	No
Rigid motions	sets with at most 3 elements	Yes	?
	parallelograms	Yes	?
	finite sets of collinear points with commensurable distances	Yes	Yes
Isometries	non-collinear sets with at most 3 elements	Yes	Yes
	rational 4-point sets	Yes	Yes
	Pompeiu quadrangles	Yes	?
Direct similarities	all finite sets	Yes	Yes

Thank you for your kind attention.