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AN INFINITE DESCENDING CHAIN OF BOOLEAN SUBFUNCTIONS CONSISTING OF THRESHOLD FUNCTIONS

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Abstract. For a class $\mathcal{C}$ of Boolean functions, a Boolean function f is a $\mathcal{C}$-subfunction of a Boolean function g, if $f = g(h_1, \dots, h_n)$, where all the inner functions h_i are members of $\mathcal{C}$. Two functions are $\mathcal{C}$-equivalent, if they are $\mathcal{C}$-subfunctions of each other. The $\mathcal{C}$-subfunction relation is a preorder on the set of all functions if and only if $\mathcal{C}$ is a clone. An infinite descending chain of U_∞-subfunctions is constructed from certain threshold functions (U_∞ denotes the clone of clique functions).

1. Introduction

Various notions of subfunctions (or minors) of Boolean functions have been presented in the literature (see, e.g., [8, 13, 14]). We generalize these notions as follows. For a class $\mathcal{C}$ of Boolean functions, we say that a Boolean function f is a *$\mathcal{C}$-subfunction* of a Boolean function g, if $f = g(h_1, \dots, h_n)$, where all the inner functions h_i are members of $\mathcal{C}$. Two functions are *$\mathcal{C}$-equivalent,* if they are $\mathcal{C}$-subfunctions of each other. The $\mathcal{C}$-subfunction relation is a preorder on the set Ω of all Boolean functions if and only if $\mathcal{C}$ is a clone, and it induces a partial order on the quotient set $\Omega/{\equiv_\mathcal{C}}$ of Ω by the $\mathcal{C}$-equivalence relation $\equiv_\mathcal{C}$. We ask whether this partial order satisfies the descending chain condition.

In this paper, we concentrate on the clone U_∞ of all 1-separating (or clique) functions, and we show that there exists an infinite descending chain of U_∞-subfunctions. Based on an explicit construction of certain threshold functions, the current proof is substantially different from the more general proof presented in [5], and we believe it might be of interest in itself.

2. Notation and definitions

Let $\mathbb{B} = \{0, 1\}$. The set $\mathbb{B}^n$ is a Boolean (distributive and complemented) lattice of 2^n elements under the component-wise order of vectors. We write simply $\mathbf{a} \leq \mathbf{b}$ to denote comparison in this lattice. We denote by $\mathbf{0}$ and $\mathbf{1}$ the all-0 and all-1 vectors, respectively. The *Hamming weight* of a vector $\mathbf{a} \in \mathbb{B}^n$, denoted $w(\mathbf{a})$, is the number of 1's in $\mathbf{a}$.

A *Boolean function* is a map $f : \mathbb{B}^n \to \mathbb{B}$, for some positive integer n called the *arity* of f. Because we only discuss Boolean functions, we refer to them simply as *functions.* A *class* of functions is a subset $\mathcal{C} \subseteq \bigcup_{n \geq 1} \mathbb{B}^{\mathbb{B}^n}$.

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For a fixed arity n, the n different *projection maps* $(a_1, \dots, a_n) \mapsto a_i$, $1 \leq i \leq n$, are denoted by $x_1, \dots, x_n$, where the arity is clear from the context.

If f is an n-ary function and $g_1, \dots, g_n$ are all m-ary functions, then the *composition of* f *with* $g_1, \dots, g_n$, denoted $f(g_1, \dots, g_n)$, is an m-ary function, and its value on $(a_1, \dots, a_m) \in \mathbb{B}^m$ is $f(g_1(a_1, \dots, a_m), \dots, g_n(a_1, \dots, a_m))$. The *composition of a class* $\mathcal{I}$ *with a class* $\mathcal{J}$, denoted $\mathcal{I} \circ \mathcal{J}$, is defined as

$$\mathcal{I} \circ \mathcal{J} = \{f(g_1, \dots, g_n) : n, m \geq 1,\ f\ n\text{-ary in } \mathcal{I},\ g_1, \dots, g_n\ m\text{-ary in } \mathcal{J}\}.$$

A *clone* is a class $\mathcal{C}$ that contains all projections and satisfies $\mathcal{C} \circ \mathcal{C} \subseteq \mathcal{C}$ (or equivalently, $\mathcal{C} \circ \mathcal{C} = \mathcal{C}$).

It follows from the definition of class composition that $(\mathcal{I} \circ \mathcal{J}) \circ \mathcal{K} \subseteq \mathcal{I} \circ (\mathcal{J} \circ \mathcal{K})$ for any classes $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$. The following is a corollary to the Associativity Lemma of [1]: if $\mathcal{J}$ is a clone, then $(\mathcal{I} \circ \mathcal{J}) \circ \mathcal{K} = \mathcal{I} \circ (\mathcal{J} \circ \mathcal{K})$.

The clones of Boolean functions, originally described by E. Post [9] (see also [10, 12, 14] for recent shorter proofs), form an algebraic lattice, where the lattice operations are the following: meet is the intersection, join is the smallest clone that contains the union. The greatest element is the clone Ω of all Boolean functions; the least element is the clone I_c of projections. These clones and the lattice are often called the *Post classes* and the *Post lattice,* respectively. We adopt the nomenclature used in [3, 4] for the Post classes. For an exposition on compositions of Post classes, see [2].

We denote by T_c the clone of all constant-preserving functions, i.e., $f \in T_c$ if and only if $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$. We denote by M the clone of monotone functions, i.e., $f \in M$ if and only if $f(\mathbf{a}) \leq f(\mathbf{b})$ whenever $\mathbf{a} \leq \mathbf{b}$. We denote by M_c the clone of monotone constant-preserving functions, i.e., $M_c = M \cap T_c$.

Let $a \in \mathbb{B}$. A set $A \subseteq \mathbb{B}^n$ is said to be *a-separating* if there is i, $1 \leq i \leq n$, such that for every $(a_1, \dots, a_n) \in A$ we have $a_i = a$. A function f is said to be *a-separating* if $f^{-1}(a)$ is a-separating. The function f is said to be *a-separating of rank* $k \geq 2$ if every subset $A \subseteq f^{-1}(a)$ of size at most k is a-separating. 1-separating functions are sometimes also called *clique functions.*

For $k \geq 2$, we denote by U_k the clone of all 1-separating functions of rank k; and we denote by U_∞ the clone of all 1-separating functions, i.e., $U_\infty = \bigcap_{k \geq 2} U_k$. For any $k \geq 2$, $U_\infty \subset U_{k+1} \subset U_k$. For $k = 2, \dots, \infty$, we denote $T_c U_k = T_c \cap U_k$, $MU_k = M \cap U_k$, $M_c U_k = M_c \cap U_k$.

Let $\mathcal{C}$ be a class of functions. We say that a function f is a *$\mathcal{C}$-subfunction* of a function g, denoted $f \preceq_{\mathcal{C}} g$, if $f \in \{g\} \circ \mathcal{C}$. Functions f and g are *$\mathcal{C}$-equivalent,* denoted $f \equiv_{\mathcal{C}} g$, if they are $\mathcal{C}$-subfunctions of each other. If $f \preceq_{\mathcal{C}} g$ but $g \not\preceq_{\mathcal{C}} f$, we say that f is a *proper $\mathcal{C}$-subfunction* of g and denote $f \prec_{\mathcal{C}} g$. If both $f \not\preceq_{\mathcal{C}} g$ and $g \not\preceq_{\mathcal{C}} f$, we say that f and g are *$\mathcal{C}$-incomparable* and denote $f \parallel_{\mathcal{C}} g$.

The $\mathcal{C}$-subfunction relation $\preceq_{\mathcal{C}}$ is a preorder on Ω if and only if $\mathcal{C}$ is a clone. If $\mathcal{C}$ is a clone, then $\equiv_{\mathcal{C}}$ is an equivalence relation on Ω, and the *$\mathcal{C}$-equivalence class* of f is denoted by $[f]_{\mathcal{C}}$. As for preorders, the $\mathcal{C}$-subfunction relation induces a partial order $\preccurlyeq_{\mathcal{C}}$ on $\Omega/{\equiv_{\mathcal{C}}}$: $[f]_{\mathcal{C}} \preccurlyeq_{\mathcal{C}} [g]_{\mathcal{C}}$ if and only if $f \preceq_{\mathcal{C}} g$.

We have investigated the $\mathcal{C}$-subfunction relations of Boolean functions in [5]; in particular, we have determined, for each Post class $\mathcal{C}$, whether there exists an infinite descending chain of $\mathcal{C}$-subfunctions and what is the size of the largest antichain of $\mathcal{C}$-incomparable functions. In the remainder of this paper, we will show that there exists an infinite descending chain of U_∞-subfunctions. The current proof differs substantially from the proof presented in [5], which is based on homomorphisms between hypergraphs associated with functions.

3. Infinite descending chain of U_∞-subfunctions

An n-ary function f is a *threshold function* (or a *linearly separable function*), if there are *weights* $w_1, \ldots, w_n \in \mathbb{R}$ and a *threshold* $w_0 \in \mathbb{R}$ such that $f(\mathbf{a}) = 1$ if and only if $\sum_{i=1}^n w_i a_i \geq w_0$. Threshold functions have been studied extensively in the 1960's; see, e.g., [6, 7, 11]. We call the special case where $w_1 = \cdots = w_n = 1$ and $w_0 = k$ for an integer k with $0 \leq k \leq n$ the *n-ary k-threshold function* and denote it by θ_k^n. The following are equivalent definitions of θ_k^n:

$$\theta_k^n(\mathbf{a}) = 1 \iff w(\mathbf{a}) \geq k, \qquad \theta_k^n = \bigvee_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \bigwedge_{i \in S} x_i.$$

An n-ary function f is a *near-unanimity* function, if $f(a_1, \ldots, a_n) = a$ whenever at least $n-1$ of the a_i's equal a. For $n \geq 3$, the n-ary $(n-1)$-threshold function θ_{n-1}^n is a near-unanimity function. It is straightforward to verify that for any $n \geq 3$, $\theta_{n-1}^n \in U_{n-1} \setminus U_n$.

Theorem 1. *For $\mathcal{C} = M_c U_\infty$, MU_∞, $T_c U_\infty$, U_∞, there is an infinite descending chain of $\mathcal{C}$-subfunctions.*

Proof. Let $n > m$, and define for $i = 1, \ldots, m$ the n-ary function $\phi_i = x_i \theta_{n-1}^n$. We observe that $\theta_{m-1}^m(\phi_1, \ldots, \phi_m) = \theta_{n-1}^n$. For, let $\mathbf{a} \in \mathbb{B}^n$. If $w(\mathbf{a}) < n-1$, then $\theta_{n-1}^n(\mathbf{a}) = 0$, so $\phi_i(\mathbf{a}) = 0$ for all i, and $\theta_{m-1}^m(\phi_1(\mathbf{a}), \ldots, \phi_m(\mathbf{a})) = \theta_{m-1}^m(0, \ldots, 0) = 0$. If $w(\mathbf{a}) \geq n-1$, then $\theta_{n-1}^n(\mathbf{a}) = 1$, and at most one of $a_1, \ldots, a_n$ is 0, so $\theta_{m-1}^m(\phi_1(\mathbf{a}), \ldots, \phi_m(\mathbf{a})) = \theta_{m-1}^m(a_1, \ldots, a_m) = 1$. Thus $\theta_{n-1}^n \preceq_{U_\infty} \theta_{m-1}^m$. Since $U_\infty \subseteq U_n \subset U_m$, we have that $\theta_{n-1}^n \circ U_\infty \subseteq U_n$, and consequently θ_{m-1}^m cannot be a U_∞-subfunction of θ_{n-1}^n. We conclude that

$$\theta_2^3 \succ_{U_\infty} \theta_3^4 \succ_{U_\infty} \theta_4^5 \succ_{U_\infty} \cdots$$

is an infinite descending chain of U_∞-subfunctions.

The same argument holds for $M_c U_\infty$-, MU_∞-, $T_c U_\infty$-subfunctions. □

We emphasize that the above proof relies essentially on the fact that there is an infinite descending chain of clones $U_2 \supset U_3 \supset \cdots$ above U_∞ in the Post lattice. Even though the above proof implies that, for any k, $\theta_{n-1}^n \preceq_{U_k} \theta_{m-1}^m$ whenever $n > m$, these functions certainly do *not* constitute an infinite descending chain of U_k-subfunctions for any finite k, because all functions θ_{m-1}^m with $m > k$ are members of U_k and are thus U_k-equivalent.

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