Enumerating quasitrivial semigroups

MALOTEC

Jimmy Devillet

in collaboration with Miguel Couceiro and Jean-Luc Marichal

University of Luxembourg

Connectedness and Contour Plots

Let X be a nonempty set and let $F \colon X^2 \to X$

Definition

- The points $(x, y), (u, v) \in X^2$ are *connected for F* if

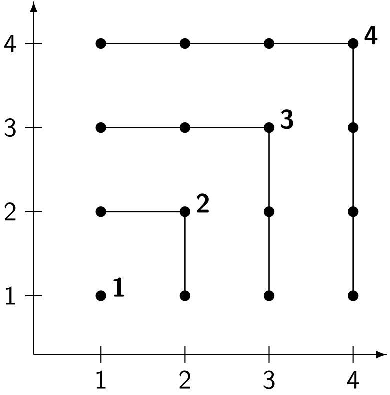
$$F(x, y) \; = \; F(u, v)$$

- The point $(x, y) \in X^2$ is *isolated for F* if it is not connected to another point in X^2

Connectedness and Contour Plots

For any integer $n \geq 1$, let $X_n = \{1, ..., n\}$ endowed with $\leq$

Example. $F(x, y) = \max\{x, y\}$ on $(X_4, \leq)$



Quasitriviality and Idempotency

Definition

$F\colon X^2 \to X$ is said to be

- *quasitrivial* if

$$F(x,y) \in \{x,y\}$$

- *idempotent* if

$$F(x,x) = x$$

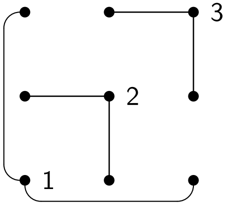
Graphical interpretation of quasitriviality

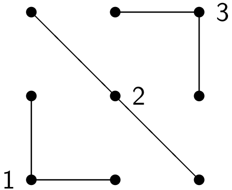
Let $\Delta_X = \{(x,x) \mid x \in X\}$

Proposition

$F\colon X^2 \to X$ is quasitrivial iff

- it is idempotent
- every point $(x,y) \notin \Delta_X$ is connected to either (x,x) or (y,y)





Graphical interpretation of the neutral element

Definition. An element $e \in X$ is said to be a *neutral element* of $F\colon X^2 \to X$ if

$$F(x,e) \;=\; F(e,x) \;=\; x$$

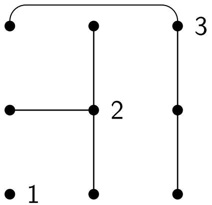
Proposition

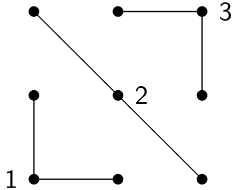
Assume $F\colon X^2 \to X$ is idempotent.
If $(x,y) \in X^2$ is isolated, then it lies on Δ_X, that is, $x = y$

Graphical interpretation of the neutral element

Proposition

Assume $F\colon X^2 \to X$ is quasitrivial and let $e \in X$.
Then e is a neutral element iff (e,e) is isolated





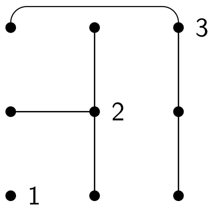
Graphical test for associativity under quasitriviality

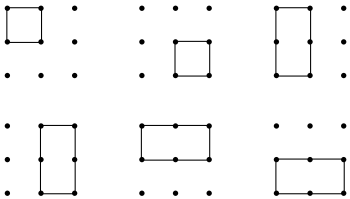
Proposition

Assume $F\colon X^2 \to X$ is quasitrivial. The following assertions are equivalent.

(i) F is associative

(ii) For every rectangle in X^2 that has only one vertex on Δ_X, at least two of the remaining vertices are connected





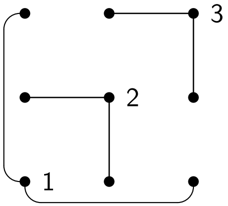
Graphical test for non associativity under quasitriviality

Proposition

Assume $F\colon X^2 \to X$ is quasitrivial. The following assertions are equivalent.

(i) F is not associative

(ii) There exists a rectangle in X^2 with only one vertex on Δ_X and whose three remaining vertices are pairwise disconnected



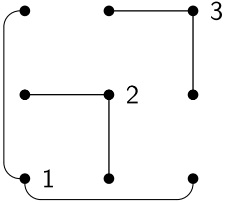
Graphical test for non associativity under quasitriviality

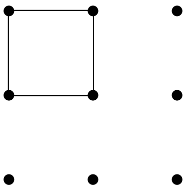
Proposition

Assume $F\colon X^2 \to X$ is quasitrivial. The following assertions are equivalent.

(i) F is not associative

(ii) There exists a rectangle in X^2 with only one vertex on Δ_X and whose three remaining vertices are pairwise disconnected





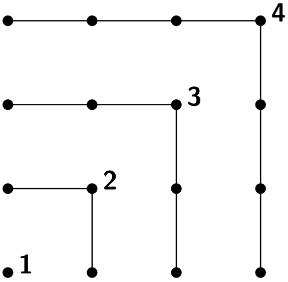
Degree sequence

Recall that $X_n = \{1, \dots, n\}$

Definition. Assume $F \colon X_n^2 \to X_n$ and let $z \in X_n$. The *F-degree of z*, denoted $\deg_F(z)$, is the number of points $(x, y) \in X_n^2 \setminus \{(z, z)\}$ such that $F(x, y) = F(z, z)$

Definition. Assume $F \colon X_n^2 \to X_n$. The *degree sequence of F*, denoted $\deg_F$, is the nondecreasing n-element sequence of the degrees $\deg_F(x)$, $x \in X_n$

Degree sequence



$$\mathbf{1} < \mathbf{2} < \mathbf{3} < \mathbf{4}$$

$$\deg_F = (0, 2, 4, 6)$$

Graphical interpretation of the annihilator

Definition. An element $a \in X$ is said to be an *annihilator* of $F\colon X^2 \to X$ if

$$F(x,a) \;=\; F(a,x) \;=\; a$$

Proposition

Assume $F\colon X_n^2 \to X_n$ is quasitrivial and let $a \in X$.
Then a is an annihilator iff $\deg_F(a) = 2n-2$

A class of associative operations

We are interested in the class of operations $F\colon X^2 \to X$ that are

- associative
- quasitrivial
- nondecreasing w.r.t. some total ordering on X

Note : If we assume further that F is commutative and has a neutral element then it is an idempotent uninorm.

Total orderings and weak orderings

Recall that a binary relation R on X is said to be

- *total* if $\forall x, y$: xRy or yRx
- *transitive* if $\forall x, y, z$: xRy and yRz implies xRz

A *weak ordering on X* is a binary relation $\lesssim$ on X that is total and transitive. We denote the symmetric and asymmetric parts of $\lesssim$ by $\sim$ and $<$, respectively.
Recall that $\sim$ is an equivalence relation on X and that $<$ induces a total ordering on the quotient set $X/\sim$

A result of Maclean and Kimura

Theorem (Mclean, 1954, Kimura, 1958)

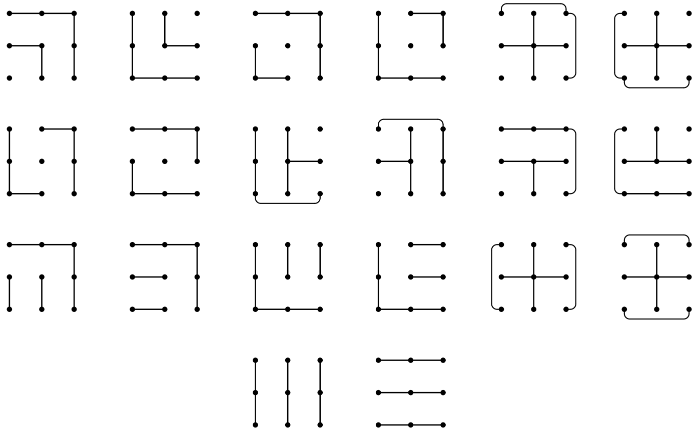
$F\colon X^2 \to X$ is associative and quasitrivial iff there exists a weak ordering $\precsim$ on X such that

$$F|_{A\times B} = \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A, B \in X/\sim$$

Corollary

$F\colon X^2 \to X$ is associative, quasitrivial and commutative iff there exists a total ordering $\preceq$ on X such that $F = \max_{\preceq}$.

Associative and quasitrivial operations on X_3



Construction of the weak order

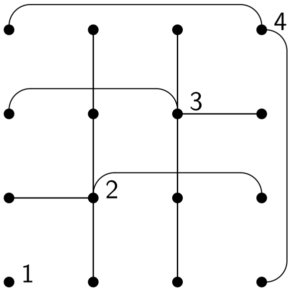
$$F|_{A\times B} \;=\; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim \quad (*)$$

Proposition

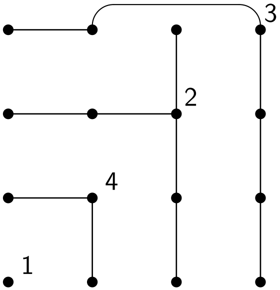
If $F\colon X_n^2 \to X_n$ is of the form $(*)$ for some weak ordering $\precsim$ on X_n, then $\precsim$ is determined by the equivalence

$$x \precsim y \;\Leftrightarrow\; \deg_F(x) \leq \deg_F(y)$$

Construction of the weak order

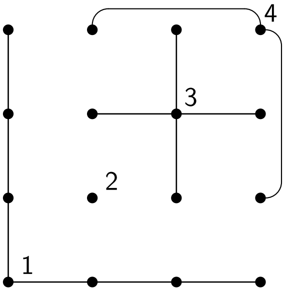


$1 < 2 < 3 < 4$

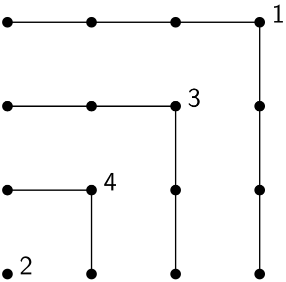


$1 \prec 4 \prec 2 \sim 3$

The commutative case



$1 < 2 < 3 < 4$



$2 \prec 4 \prec 3 \prec 1$

The commutative case

Proposition

Assume $F\colon X_n^2 \to X_n$ is quasitrivial. Then, F is associative and commutative iff $\deg_F = (0, 2, \ldots, 2n-2)$

An alternative characterization

Theorem

Assume $F\colon X^2 \to X$. TFAE

(i) F is associative, quasitrivial, and commutative

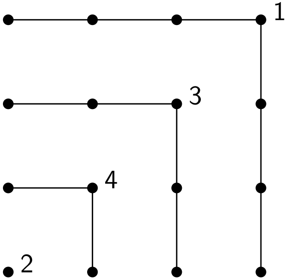
(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X

If $X = X_n$, then any of the assertions (i)–(ii) above is equivalent to the following one

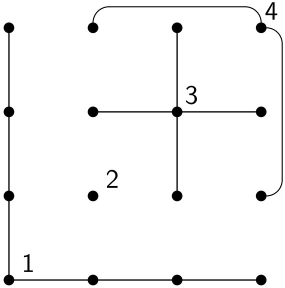
(iii) F is quasitrivial and $\deg_F = (0, 2, 4, \ldots, 2n-2)$

There are exactly $n!$ operations $F\colon X_n^2 \to X_n$ satisfying any of the assertions (i)–(iii). Moreover, the total ordering $\preceq$ considered in assertion (ii) is uniquely determined by the condition: $x \preceq y$ iff $\deg_F(x) \leq \deg_F(y)$. In particular, every of these operations has a unique neutral element $e = \min_{\preceq} X_n$ and a unique annihilator $a = \max_{\preceq} X_n$.

The commutative and nondecreasing case



$2 \prec 4 \prec 3 \prec 1$



$1 < 2 < 3 < 4$

Single-peaked total orderings

Definition. Let $\leq, \preceq$ be total orderings on X. The total ordering $\preceq$ is said to be *single-peaked w.r.t.* $\leq$ if for any $a, b, c \in X$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$

Example. The total ordering $\preceq$ on

$$X_4 = \{1 < 2 < 3 < 4\}$$

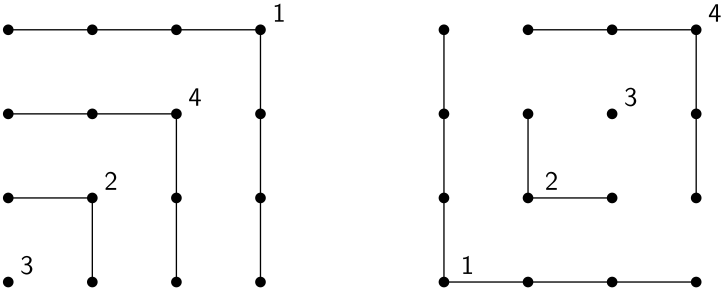
defined by

$$3 \prec 2 \prec 4 \prec 1$$

is single-peaked w.r.t. $\leq$

Note : There are exactly 2^{n-1} single-peaked total orderings on X_n.

Single-peaked total orderings



$3 \prec 2 \prec 4 \prec 1$ $\quad\quad$ $1 < 2 < 3 < 4$

Single-peaked total orderings

Proposition

Assume $\leq, \preceq$ are total orderings on X and let $F\colon X^2 \to X$ such that $F = \max_{\preceq}$. Then F is nondecreasing w.r.t. $\leq$ iff $\preceq$ is single-peaked w.r.t. $\leq$

A characterization

Theorem

Let $\leq$ be a total order on X and assume $F\colon X^2 \to X$. TFAE

(i) F is associative, quasitrivial, commutative, and nondecreasing (associativity can be ignored)

(ii) $F = \max_{\preceq}$ for some total ordering $\preceq$ on X that is single-peaked w.r.t. $\leq$

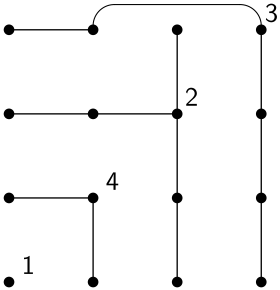
If $(X, \leq) = (X_n, \leq)$, then any of the assertions (i)–(ii) above is equivalent to the following one

(iii) F is quasitrivial, nondecreasing, and $\deg_F = (0, 2, 4, \ldots, 2n-2)$

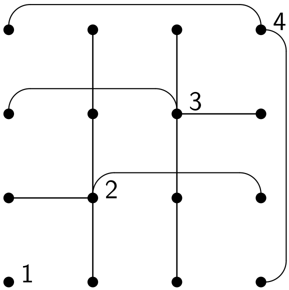
(iv) F is associative, idempotent, commutative, nondecreasing, and has a neutral element.

There are exactly 2^{n-1} operations $F\colon X_n^2 \to X_n$ satisfying any of the assertions (i)–(iv).

The nondecreasing case



$1 \prec 4 \prec 2 \sim 3$



$1 < 2 < 3 < 4$

Weakly single-peaked weak orderings

Definition. Let $\leq$ be a total ordering on X and let $\precsim$ be a weak ordering on X. The weak ordering $\precsim$ is said to be *weakly single-peaked w.r.t.* $\leq$ if for any $a, b, c \in X$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$ or $a \sim b \sim c$

Example. The weak ordering $\precsim$ on

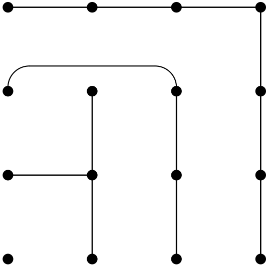
$$X_4 = \{1 < 2 < 3 < 4\}$$

defined by

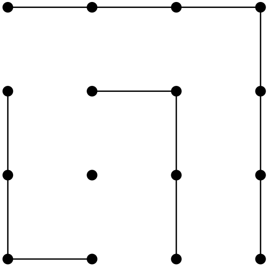
$$2 \prec 1 \sim 3 \prec 4$$

is weakly single-peaked w.r.t. $\leq$

Weakly single-peaked weak orderings



$2 \prec 1 \sim 3 \prec 4$



$1 < 2 < 3 < 4$

Weakly single-peaked weak orderings

$$F|_{A\times B} \;=\; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \qquad \forall A, B \in X/\sim \qquad (*)$$

Proposition

Let $\leq$ be a total ordering on X and let $\precsim$ be a weak ordering on X. Assume $F\colon X^2 \to X$ is of the form $(*)$. Then F is nondecreasing w.r.t. $\leq$ iff $\precsim$ is weakly single-peaked w.r.t. $\leq$

A characterization

$$F|_{A\times B} \;=\; \begin{cases} \max_{\precsim}|_{A\times B}, & \text{if } A \neq B, \\ \pi_1|_{A\times B} \text{ or } \pi_2|_{A\times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X/\sim \qquad (*)$$

Theorem

Let $\leq$ be a total ordering on X. $F\colon X^2 \to X$ is associative, quasitrivial, and nondecreasing w.r.t. $\leq$ iff F is of the form $(*)$ for some weak ordering $\precsim$ on X that is weakly single-peaked w.r.t. $\leq$

Enumeration of associative and quasitrivial operations

Recall that if the generating function (GF) or the exponential generating function (EGF) of a given sequence $(s_n)_{n\geq 0}$ exist, then they are respectively defined as the power series

$$S(z) \;=\; \sum_{n\geq 0} s_n\, z^n \quad \text{and} \quad \hat{S}(z) \;=\; \sum_{n\geq 0} s_n\, \frac{z^n}{n!}\,.$$

Recall also that for any integers $0 \leq k \leq n$ the *Stirling number of the second kind* $\left\{ n \atop k \right\}$ is defined as

$$\left\{ n \atop k \right\} \;=\; \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\, i^n.$$

Enumeration of associative and quasitrivial operations

For any integer $n \geq 1$, let $q(n)$ denote the number of associative and quasitrivial operations $F\colon X_n^2 \to X_n$ (OEIS : A292932)

Theorem

For any integer $n \geq 0$, we have the closed-form expression

$$q(n) \;=\; \sum_{i=0}^{n} 2^i \sum_{k=0}^{n-i} (-1)^k \binom{n}{k} \left\{ {n-k \atop i} \right\} (i+k)!\,, \qquad n \geq 0.$$

Moreover, the sequence $(q(n))_{n\geq 0}$ satisfies the recurrence equation

$$q(n+1) \;=\; (n+1)\,q(n) + 2\sum_{k=0}^{n-1} \binom{n+1}{k} q(k)\,, \qquad n \geq 0,$$

with $q(0) = 1$. Finally, its EGF is given by $\hat{Q}(z) = 1/(z+3-2e^z)$.

Enumeration of associative and quasitrivial operations

In arXiv:1709.09162 we found also explicit formulas for

- $q_e(n)$: number of associative and quasitrivial operations $F\colon X_n^2 \to X_n$ that have a neutral element (OEIS : A292933)
- $q_a(n)$: number of associative and quasitrivial operations $F\colon X_n^2 \to X_n$ that have an annihilator (OEIS : A292933)
- $q_{ea}(n)$: number of associative and quasitrivial operations $F\colon X_n^2 \to X_n$ that have a neutral element and an annihilator (OEIS : A292934)

Enumeration of associative quasitrivial and nondecreasing operations

For any integer $n \geq 0$ we denote by $v(n)$ the number of associative, quasitrivial, and nondecreasing operations $F\colon X_n^2 \to X_n$ (OEIS : A293005)

Theorem

The sequence $(v(n))_{n\geq 0}$ satisfies the second order linear recurrence equation

$$v(n+2) - 2\,v(n+1) - 2\,v(n) \;=\; 2\,, \qquad n \geq 0,$$

with $v(0) = 0$ and $v(1) = 1$, and we have

$$3\,v(n) + 2 \;=\; \textstyle\sum_{k\geq 0} 3^k \big(2\binom{n}{2k} + 3\binom{n}{2k+1}\big)\,, \quad n \geq 0.$$

Moreover, its GF is given by $V(z) = z(z+1)/(2z^3 - 3z + 1)$.

Enumeration of associative and quasitrivial operations

In arXiv:1709.09162 we found also explicit formulas for

- $v_e(n)$: number of associative, quasitrivial and nondecreasing operations $F\colon X_n^2 \to X_n$ that have a neutral element (OEIS : A002605)
- $v_a(n)$: number of associative, quasitrivial and nondecreasing operations $F\colon X_n^2 \to X_n$ that have an annihilator (OEIS : A293006)
- $v_{ea}(n)$: number of associative, quasitrivial and nondecreasing operations $F\colon X_n^2 \to X_n$ that have a neutral element and an annihilator (OEIS : A293007)

Selected references

- N. L. Ackerman. A characterization of quasitrivial n-semigroups. To appear in *Algebra Universalis*.
- D. Black. On the rationale of group decision-making. *J Polit Economy*, 56(1):23–34, 1948
- M. Couceiro, J. Devillet, and J.-L. Marichal. Characterizations of idempotent discrete uninorms. *Fuzzy Sets and Systems*. In press. https://doi.org/10.1016/j.fss.2017.06.013
- M. Couceiro, J. Devillet, and J.-L. Marichal. Quasitrivial semigroups: characterizations and enumerations. Submitted for publication. arXiv:1709.09162.
- B. De Baets, J. Fodor, D. Ruiz-Aguilera, and J. Torrens. Idempotent uninorms on finite ordinal scales. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17(1):1–14, 2009.
- J. Devillet, G. Kiss, and J.-L. Marichal. Characterizations of quasitrivial symmetric nondecreasing associative operations. Submitted for publication. arXiv:1705.00719.
- N. Kimura. The structure od idempotent semigroups. I. *Pacific J. Math.*, 8:257–275, 1958.
- D. McLean. Idempotent semigroups. *Amer. Math. Monthly*, 61:110–113, 1954.