Department of Computational Engineering Sciences

Lux

Computed in Luxembourg

Computational Sciences Luxembourg

12th International Conference on Damage Assessment of Structures 2017, Kitakyushu, Japan, 10-12 July 2017
http://www.damas.ugent.be/

UNIVERSITÉ DU LUXEMBOURG

Stéphane P. A. Bordas, 2017 07 10 Tokyo Institute of Technology, Japan
You can download the slides here: http://hdl.handle.net/10993/31487

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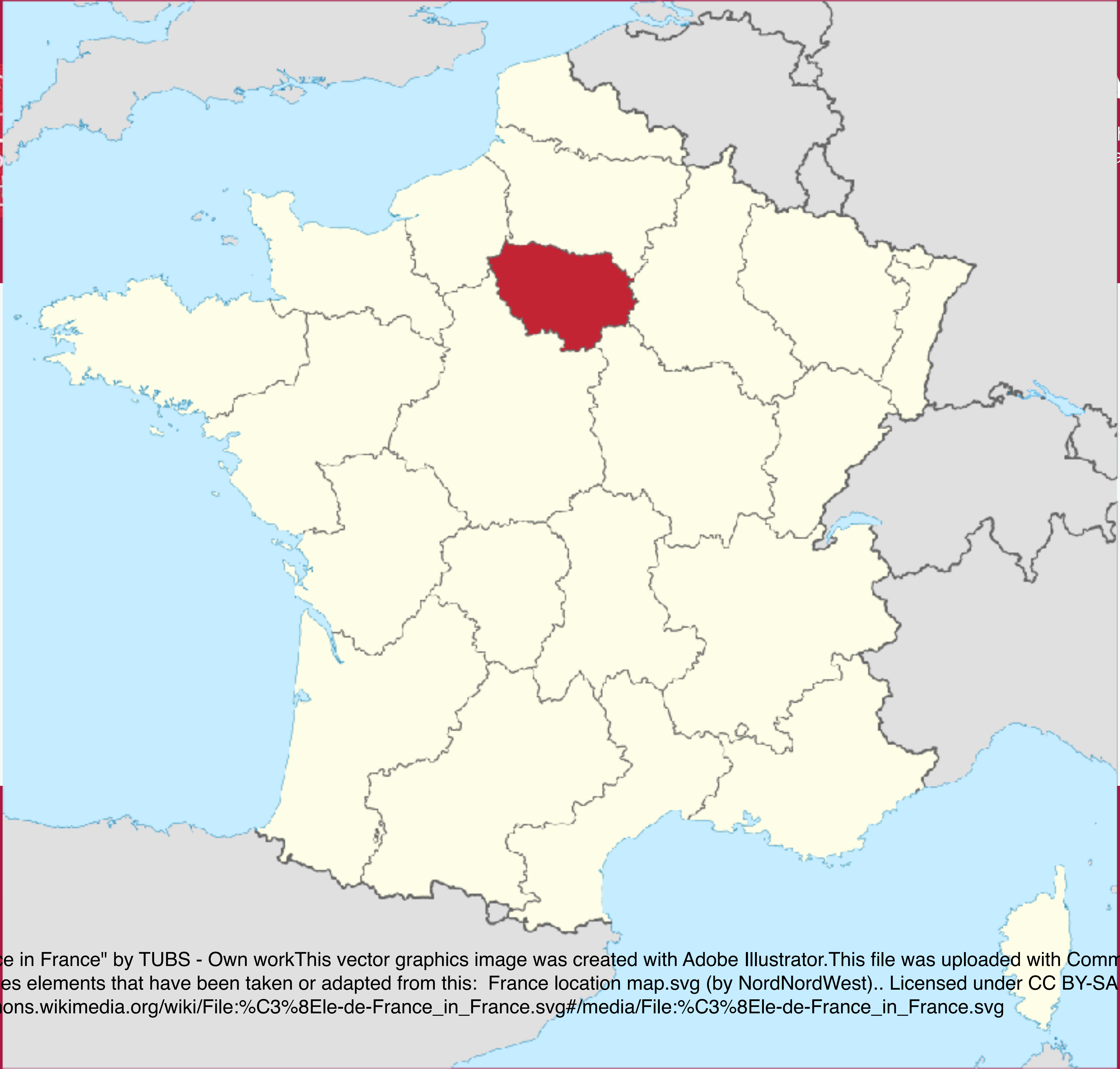

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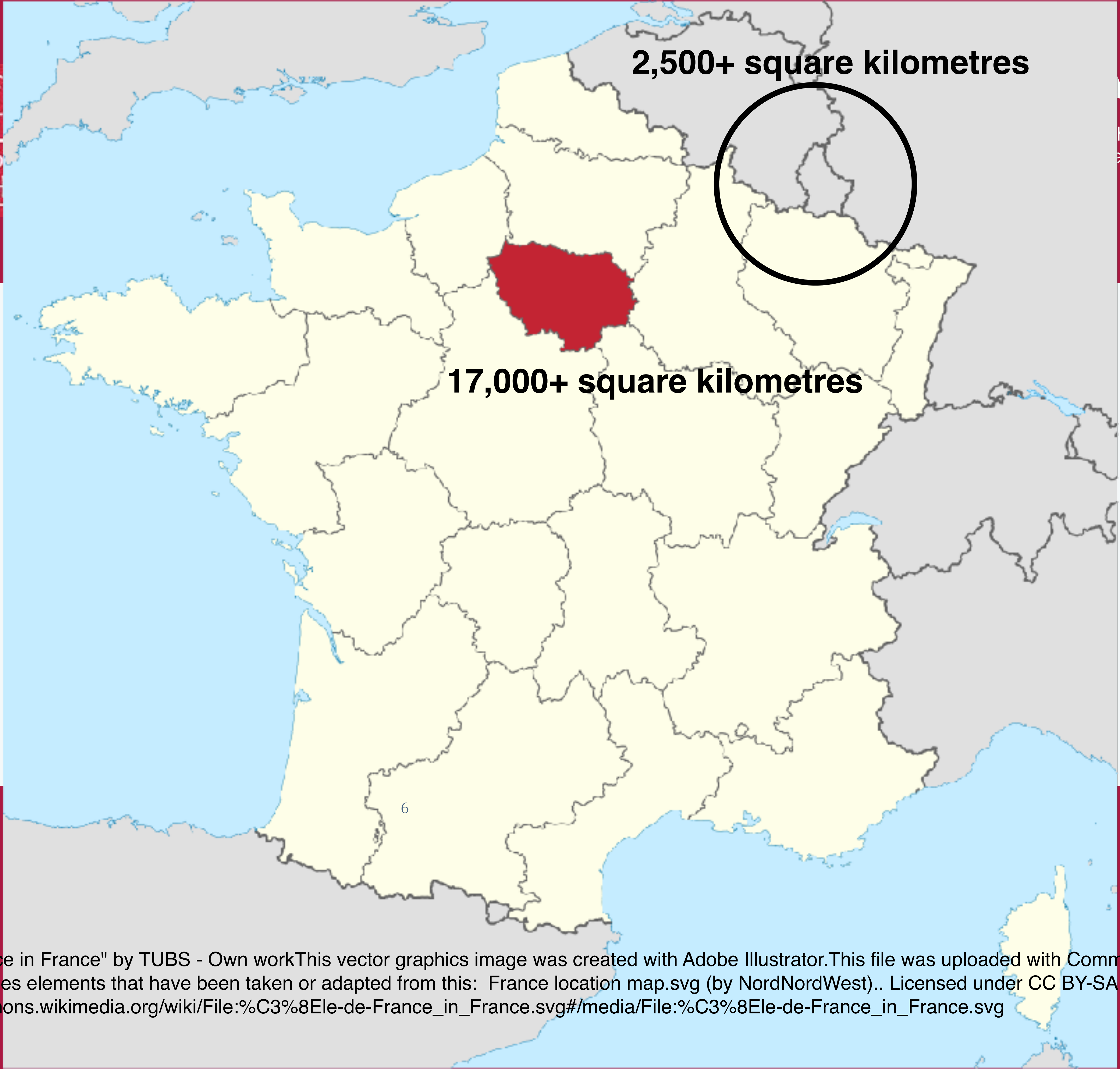


2,500+ square kilometres

17,000+ square kilometres

"Île-de-France in France" by TUBS - Own workThis vector graphics image was created with Adobe Illustrator.This file was uploaded with Commonist.This vector image includes elements that have been taken or adapted from this: France location map.svg (by NordNordWest).. Licensed under CC BY-SA 3.0 via Commons https://commons.wikimedia.org/wiki/File:%C3%8Ele-de-France_in_France.svg#/media/File:%C3%8Ele-de-France_in_France.svg

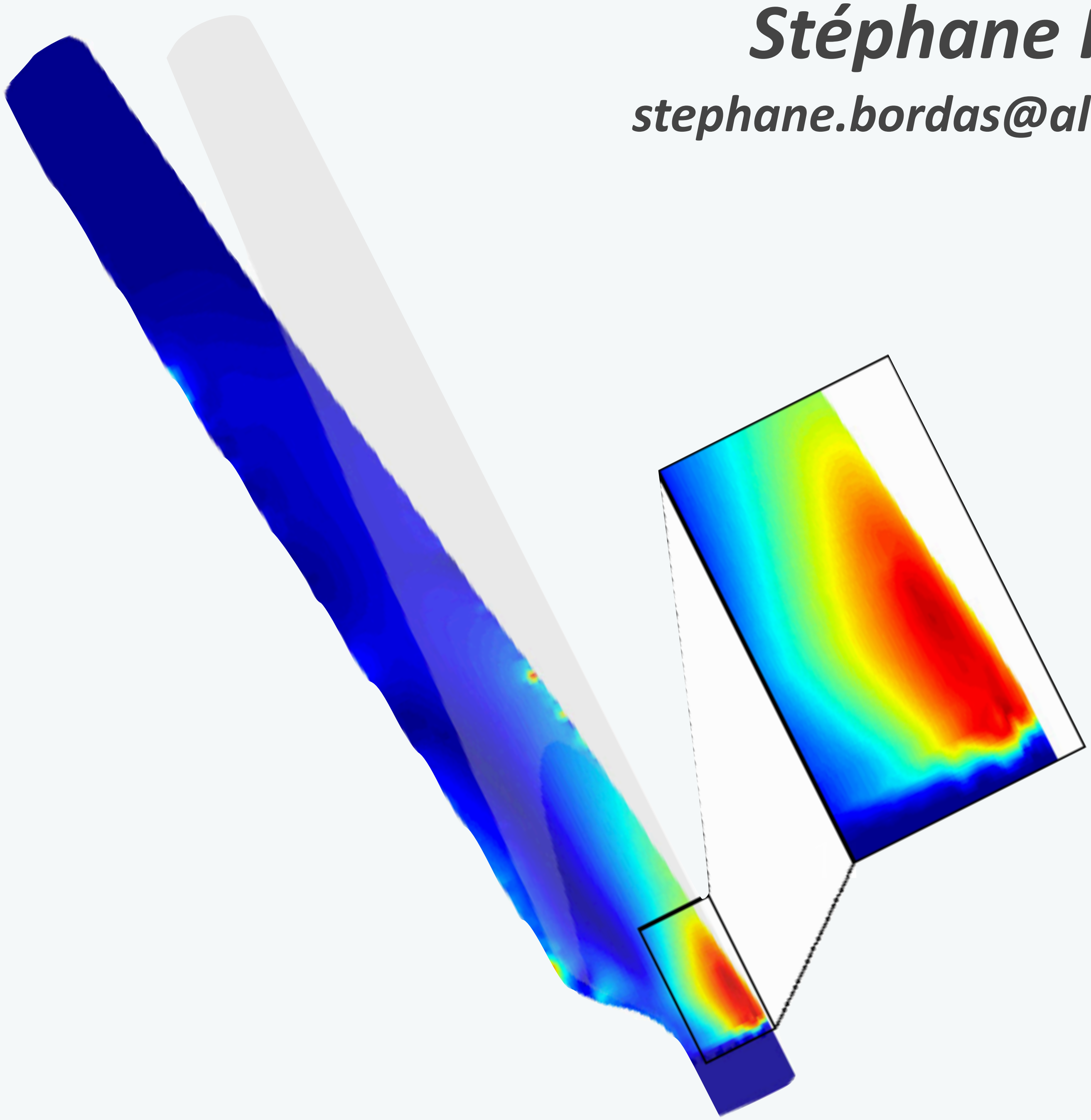
Tokyo 13,572 km^2 13.5 million people…
Luxembourg 2,500 km^2 500,000 people…

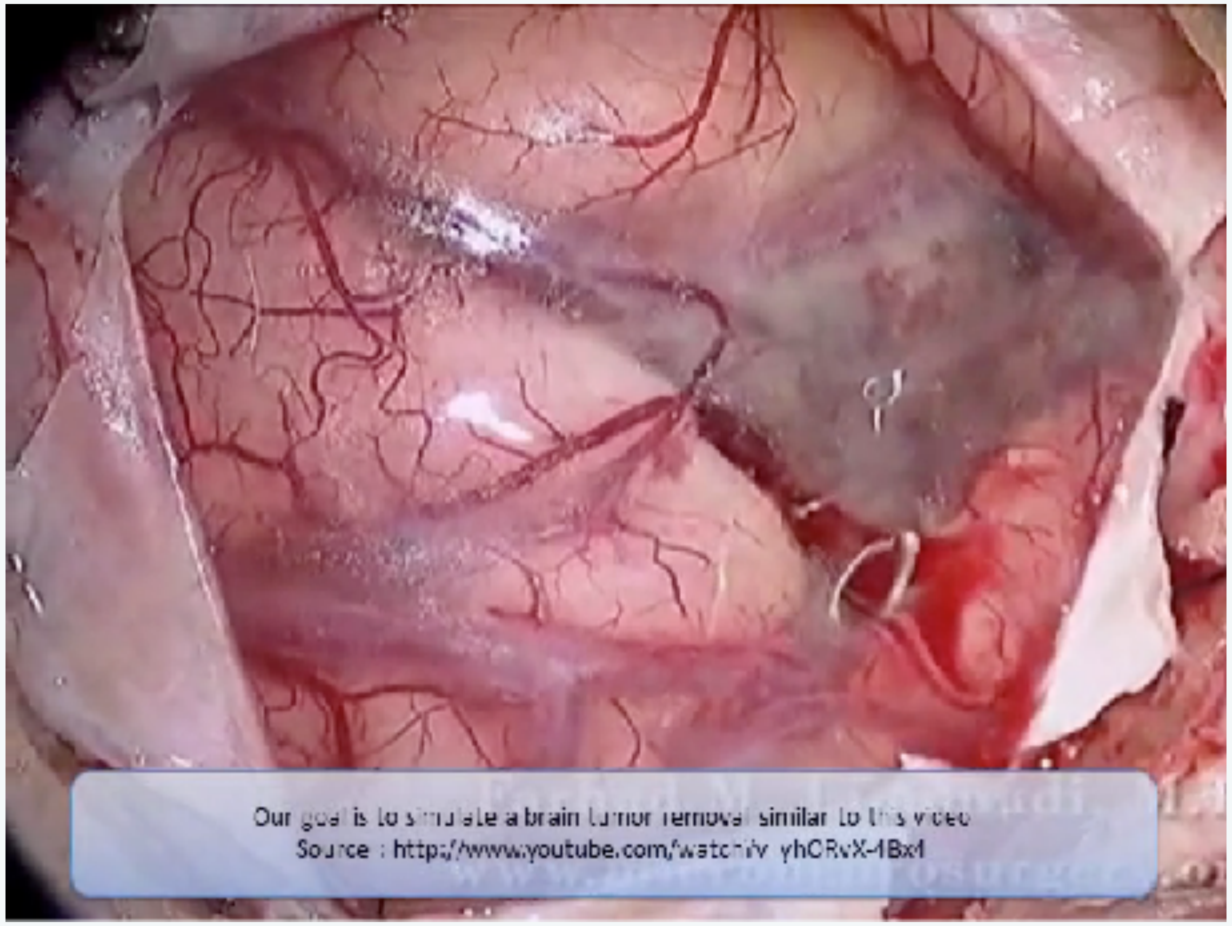
http://hdl.handle.net/10993/31487

Advances in enriched finite element formulations for fracture and cutting: *engineering and surgical simulation applications*

Stéphane P.A. Bordas

stephane.bordas@alum.northwestern.edu





Our goal is to simulate a brain tumor removal similar to this video

Source : http://www.youtube.com/watch?v=yhORvX-4Bxl

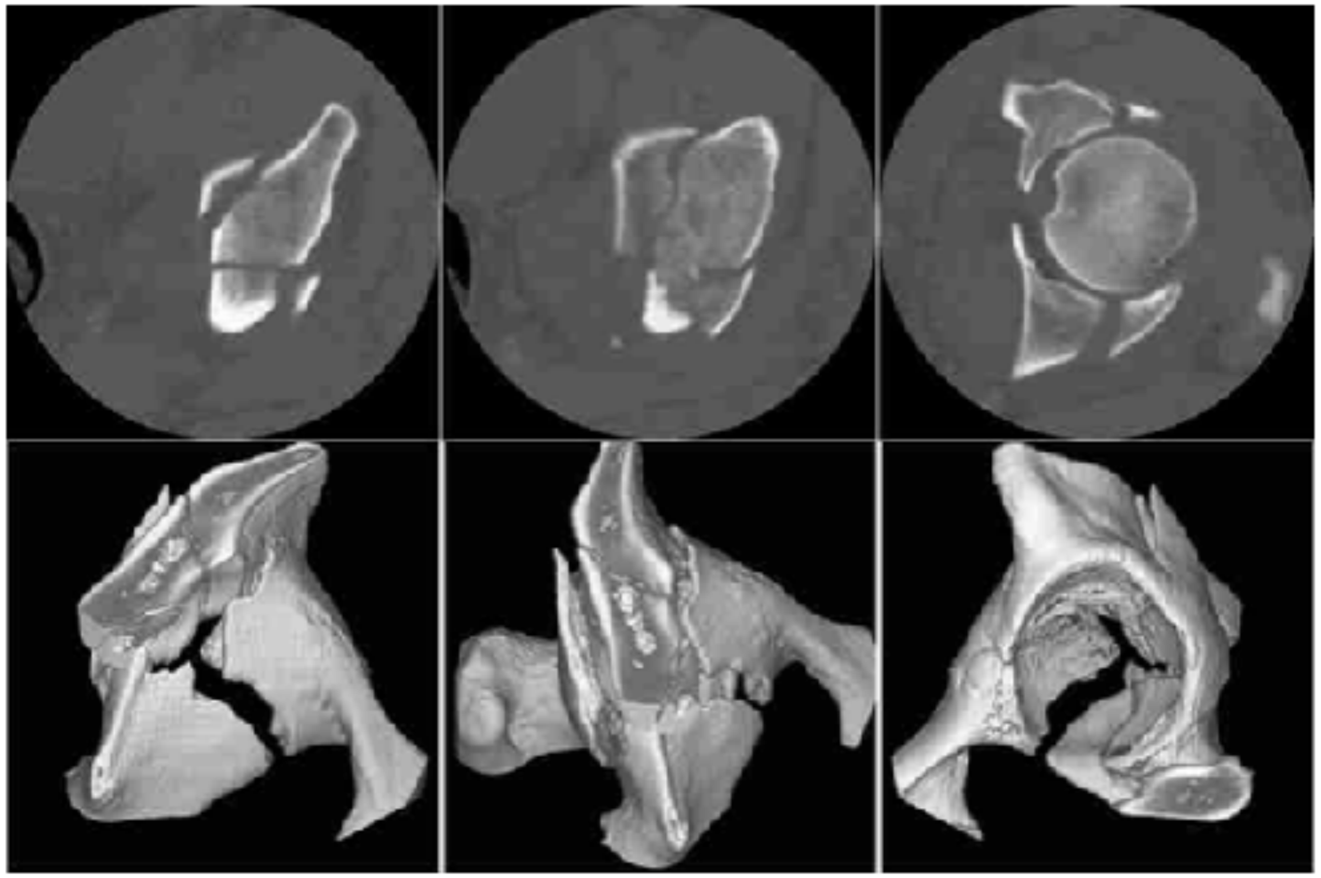
Motivation: fracture mechanics

Shuttle crash, 2003

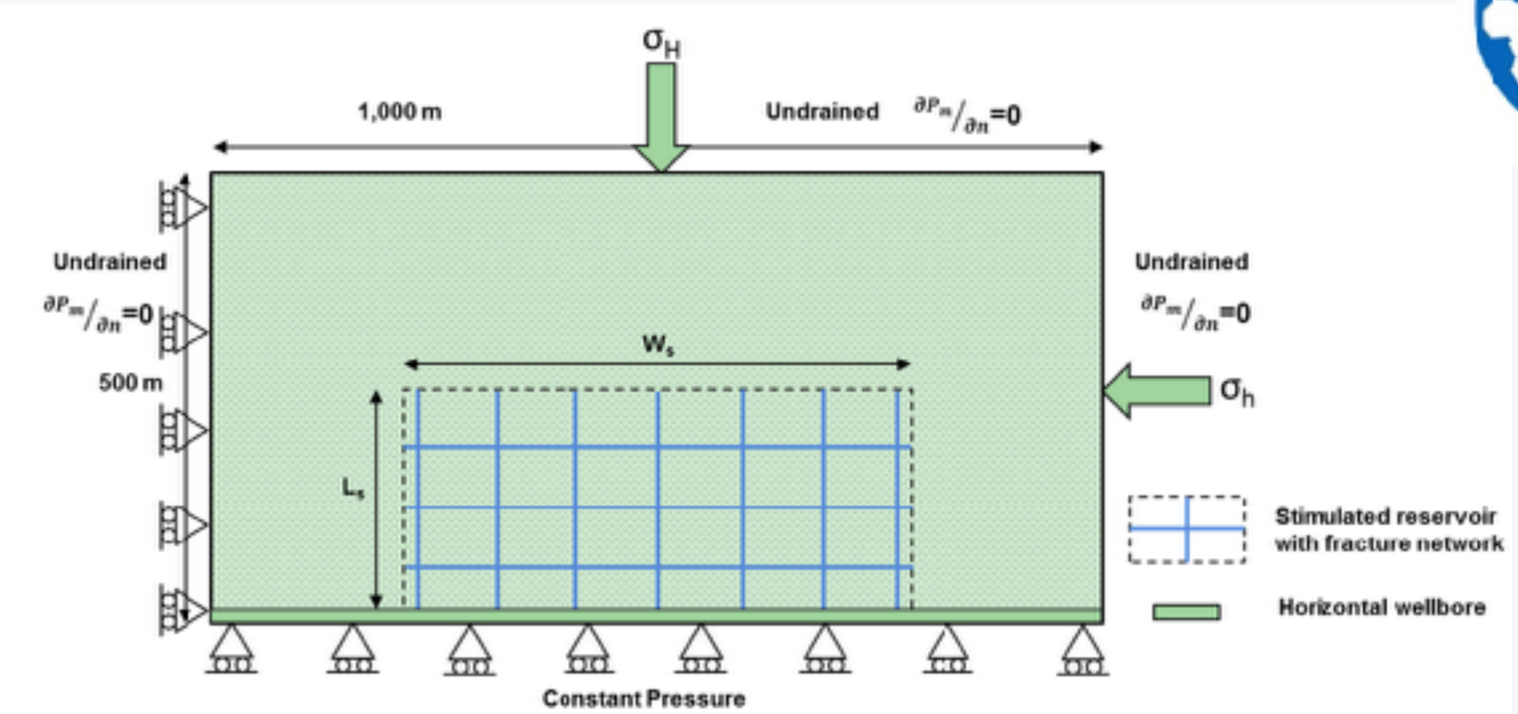
Landslide, Colorado

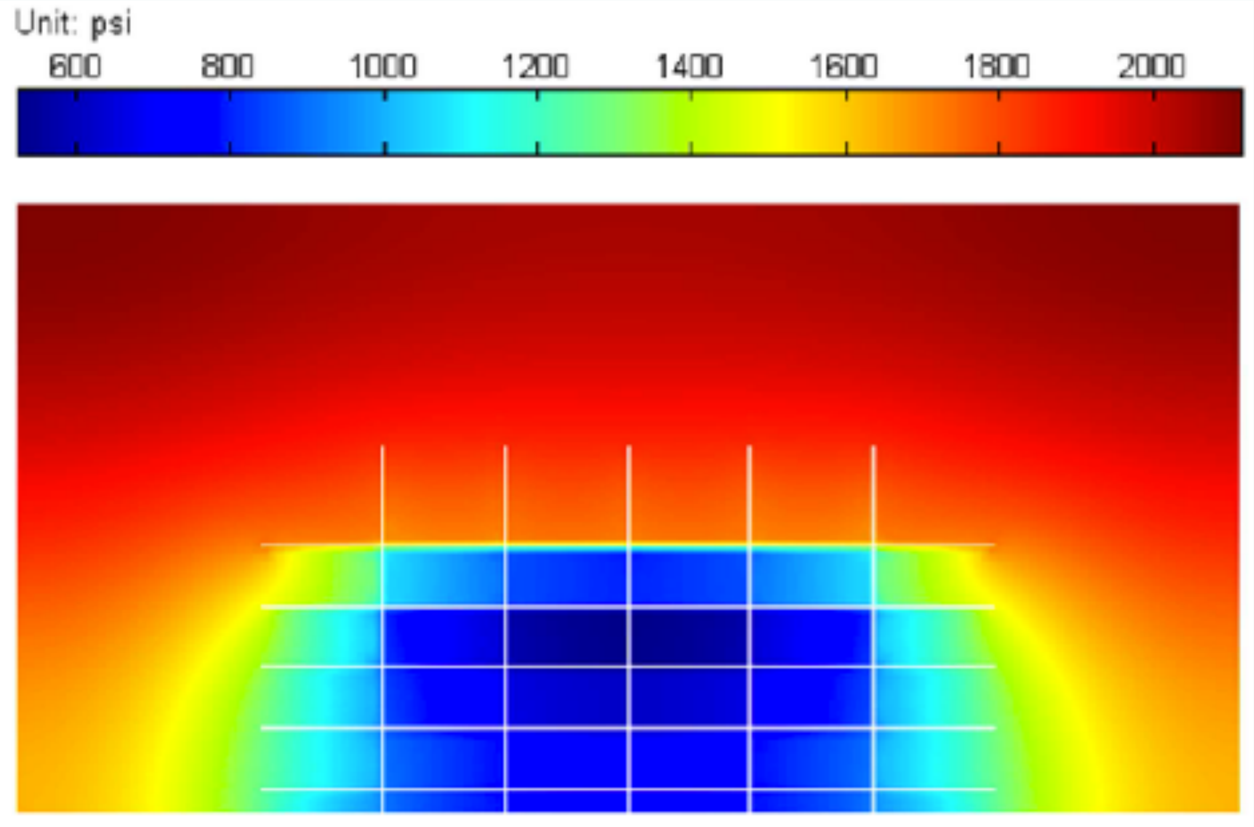
Taiwan earthquake, 2003

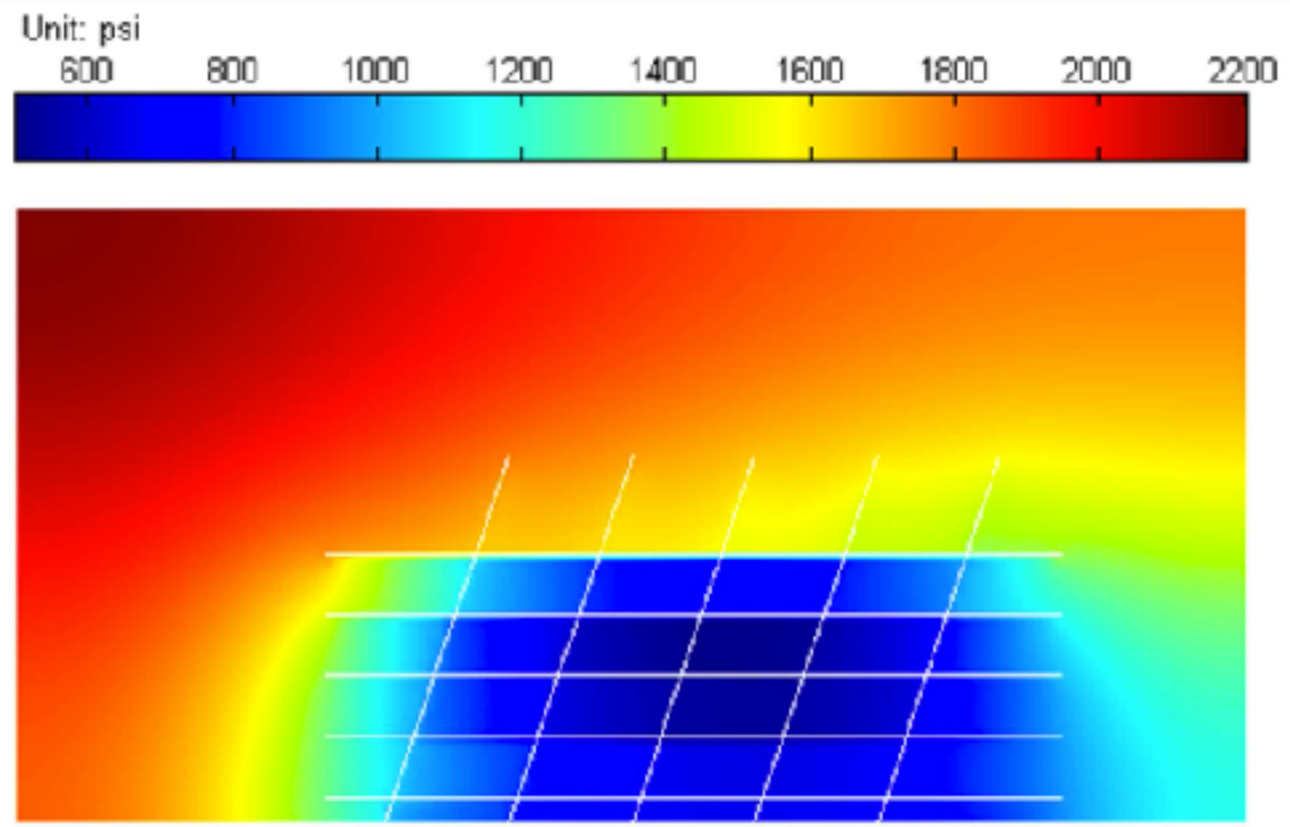
Fragmentation of concrete



Oil extraction from shale -



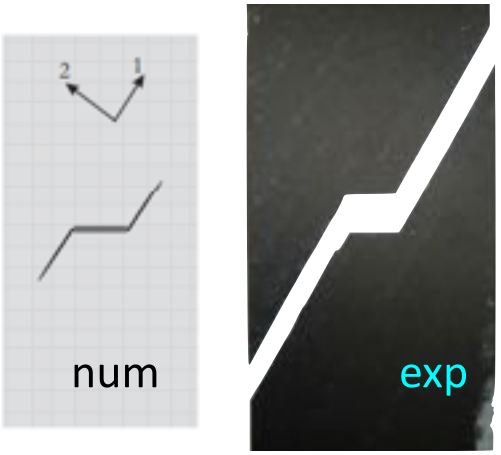




bordasS@cardiff.ac.uk

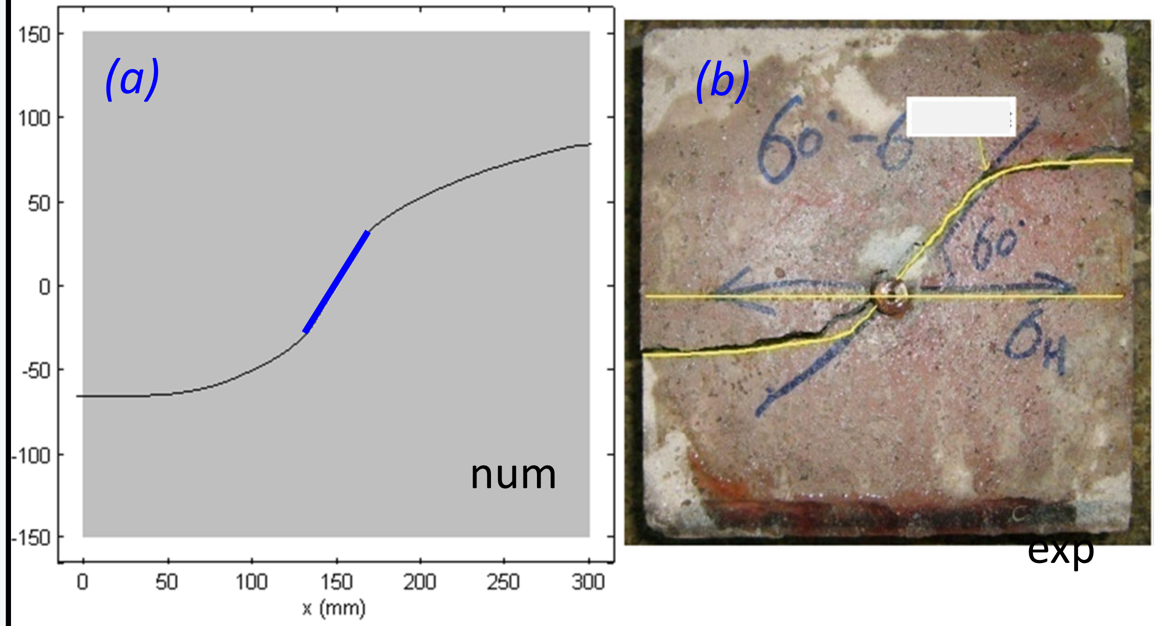
Motivation: fracture of engineering structures and materials

- **Limerick**: unidirectional composites

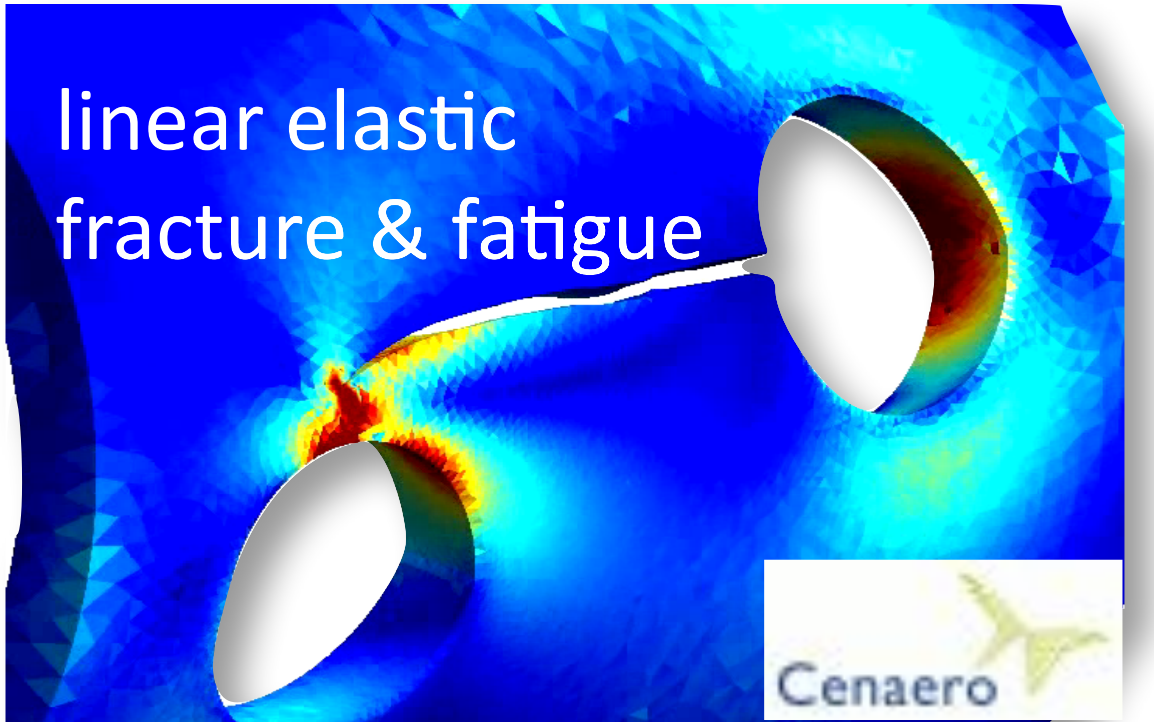


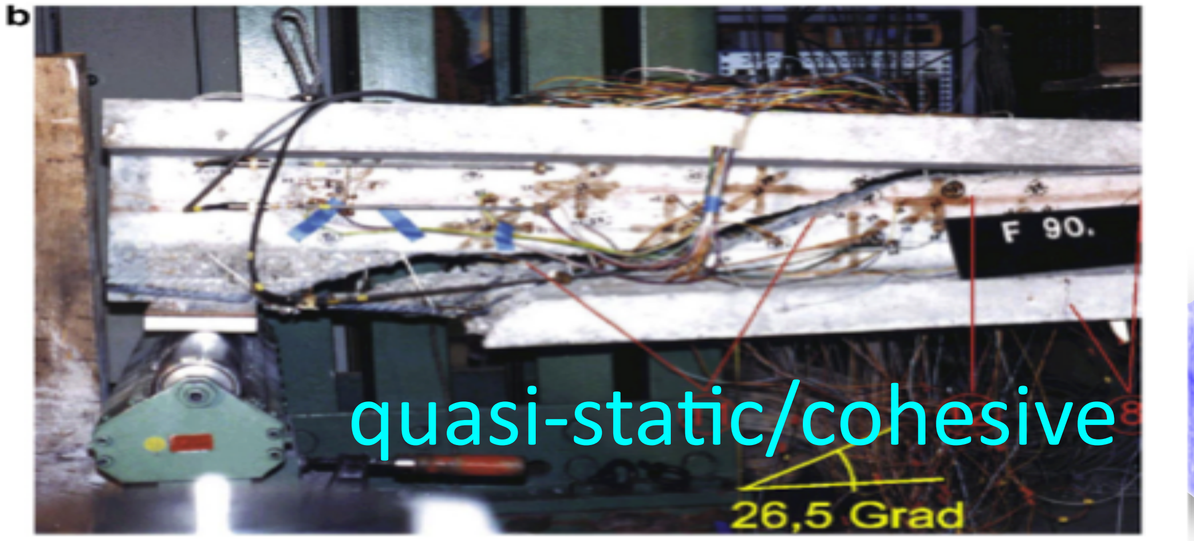
thesis L. Cahill, 2014

- **China/USA**: hydraulic fracturing (shale gas)

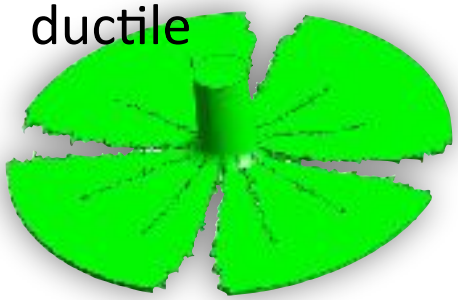


thesis M. Sheng, USA, China, 2016

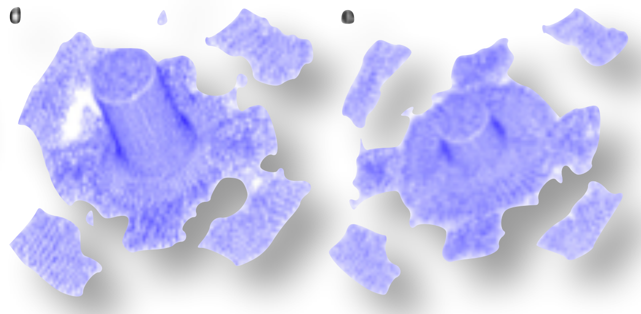




dynamics
ductile

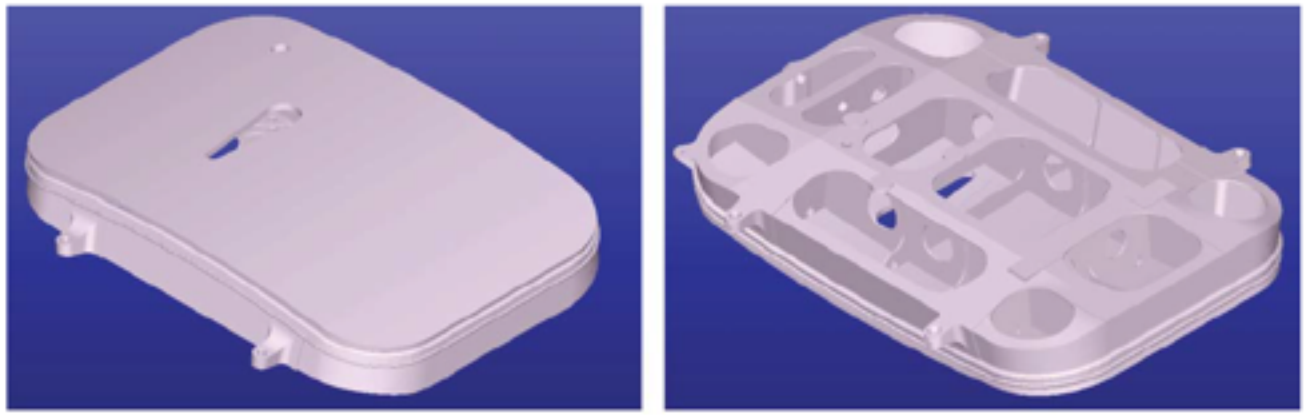


dynamics/brittle



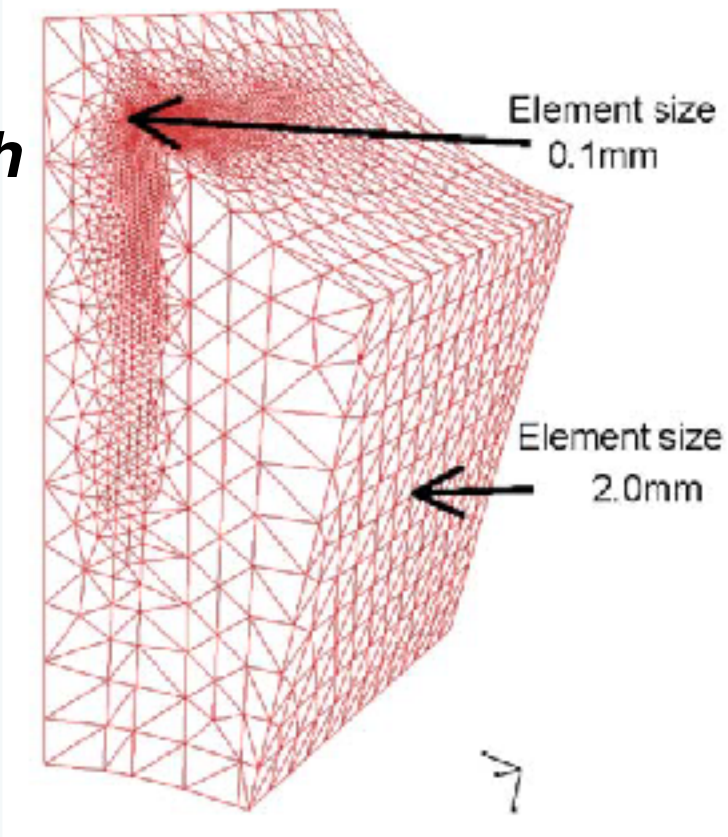
Fracture of 'homogeneous' materials

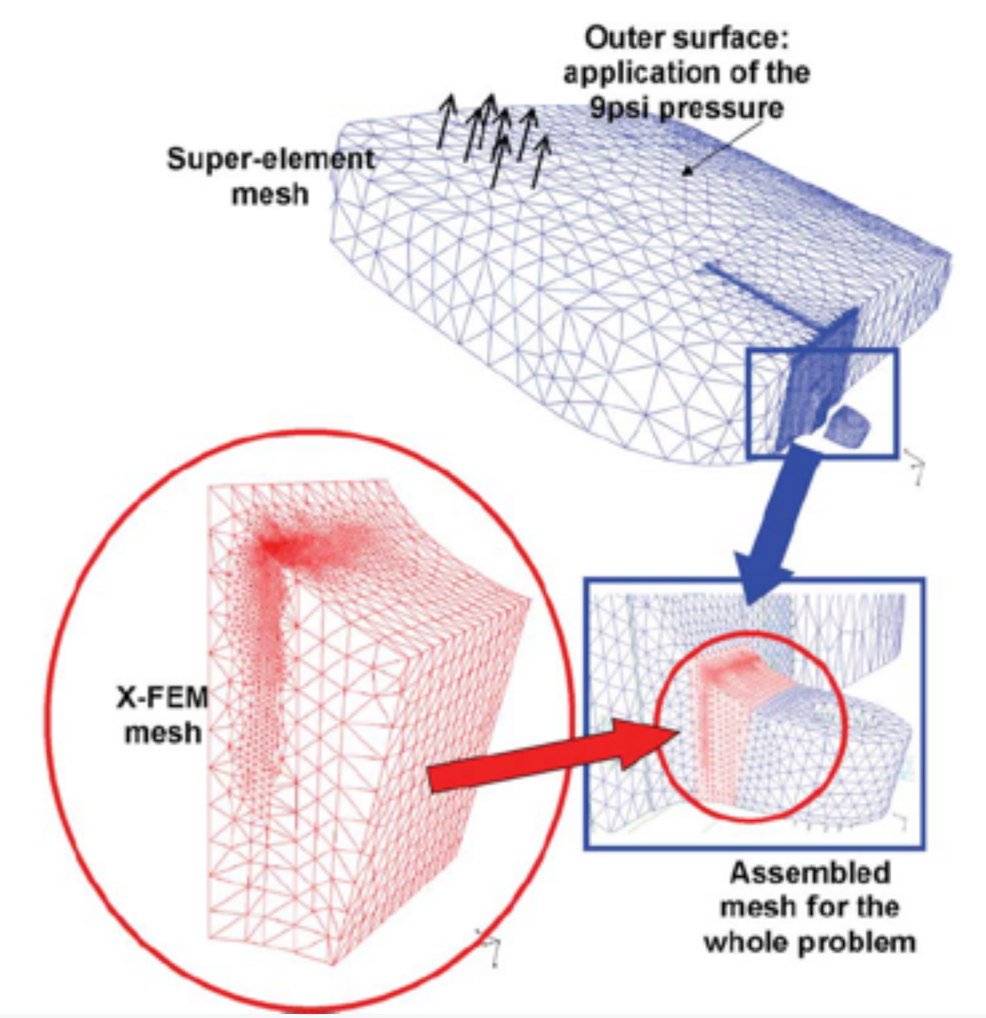
Question: when should a structure be inspected for flaws?



(a) Top view (b) Bottom view

ad hoc mesh refinement





SPAB and B. Moran, Engineering Fracture Mechanics, 2006
V.P. Nguyen et al. XFEM C++ Library IJNME, 2007
Industrial applications of extended finite element methods
See also E. Wyart et al, EFM, IJNME, 2008

Main issues in Computational Fracture

Choice of the Model

Choice of the Discretisation Scheme

Model Choice

Small scale yielding? Linear elastic fracture?

Elastic-Plastic fracture mechanics?

Damage models (local? non-local? gradient?)

Multi-scale? (concurrent? semi-concurrent?)

Discretisation Choice

Finite element method (remeshing?)

Boundary element method (non-linearities?)

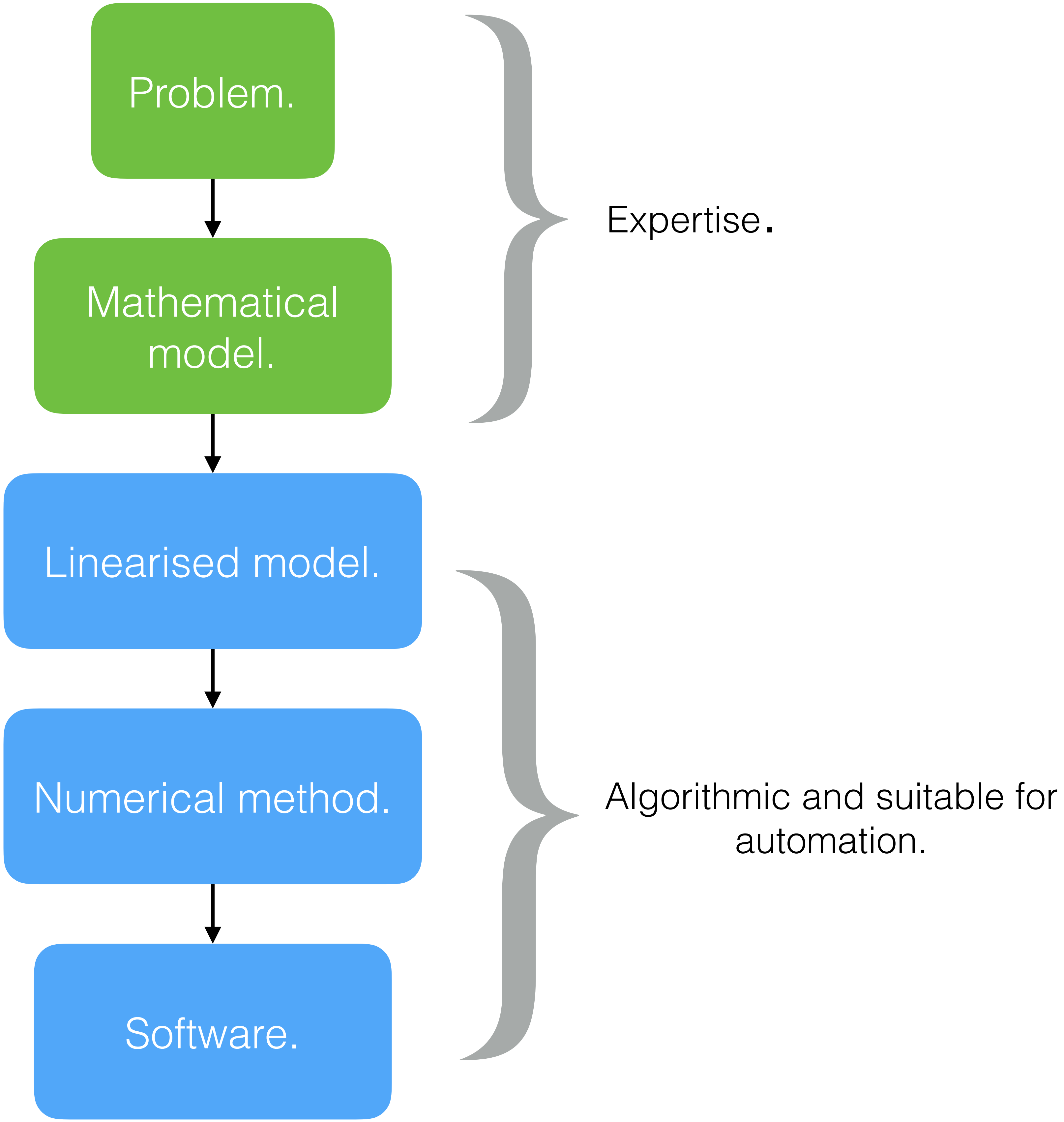
Extended finite element methods (multi-crack?)

Meshfree methods (cost? stability? robustess?)

FEniCS project

Steering council: Alnaes, Bletcha, ***Hale***, Logg, Richardson, Ring, Rognes and Wells.
Contributors: Too many to name!

- Key idea: implement high-level description of finite element models in the Unified Form Language.
- Let algorithms take over the tedious/difficult work of linearisation and transforming maths into lower-level languages.
- Not a toy; scales to huge problems with billions of unknowns on Top100 supercomputers.



Problem.

Mathematical model.

Linearised model.

Numerical method.

Software.

Expertise.

Algorithmic and suitable for automation.

Case study I

Linear elastic fracture mechanics (LEFM)

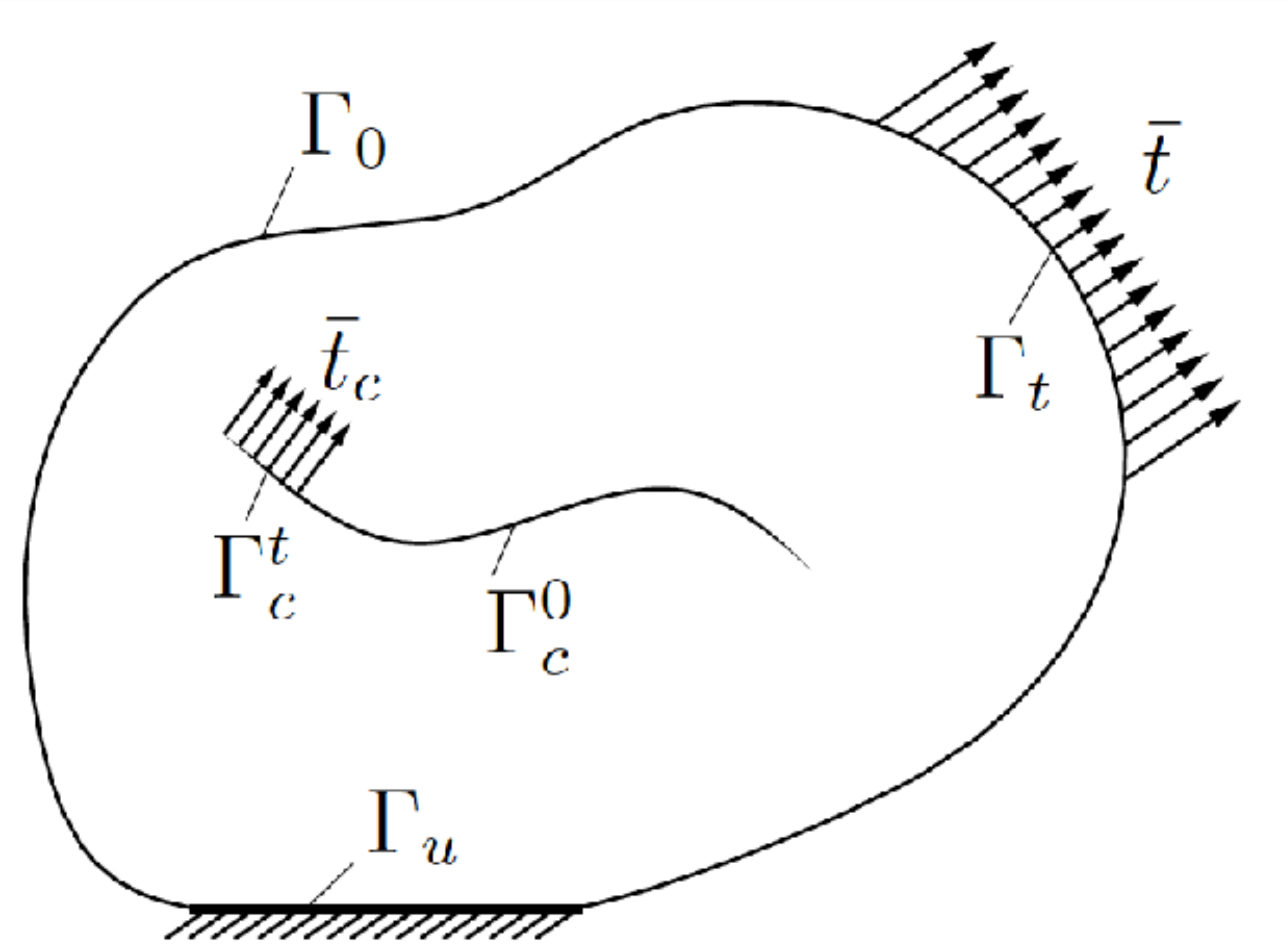
(Extended/Enriched) Finite element methods

(Extended/Enriched) Isogeometric Boundary Element Methods

What is a crack?

a 1D line in 2D space

a 2D surface in 3D space



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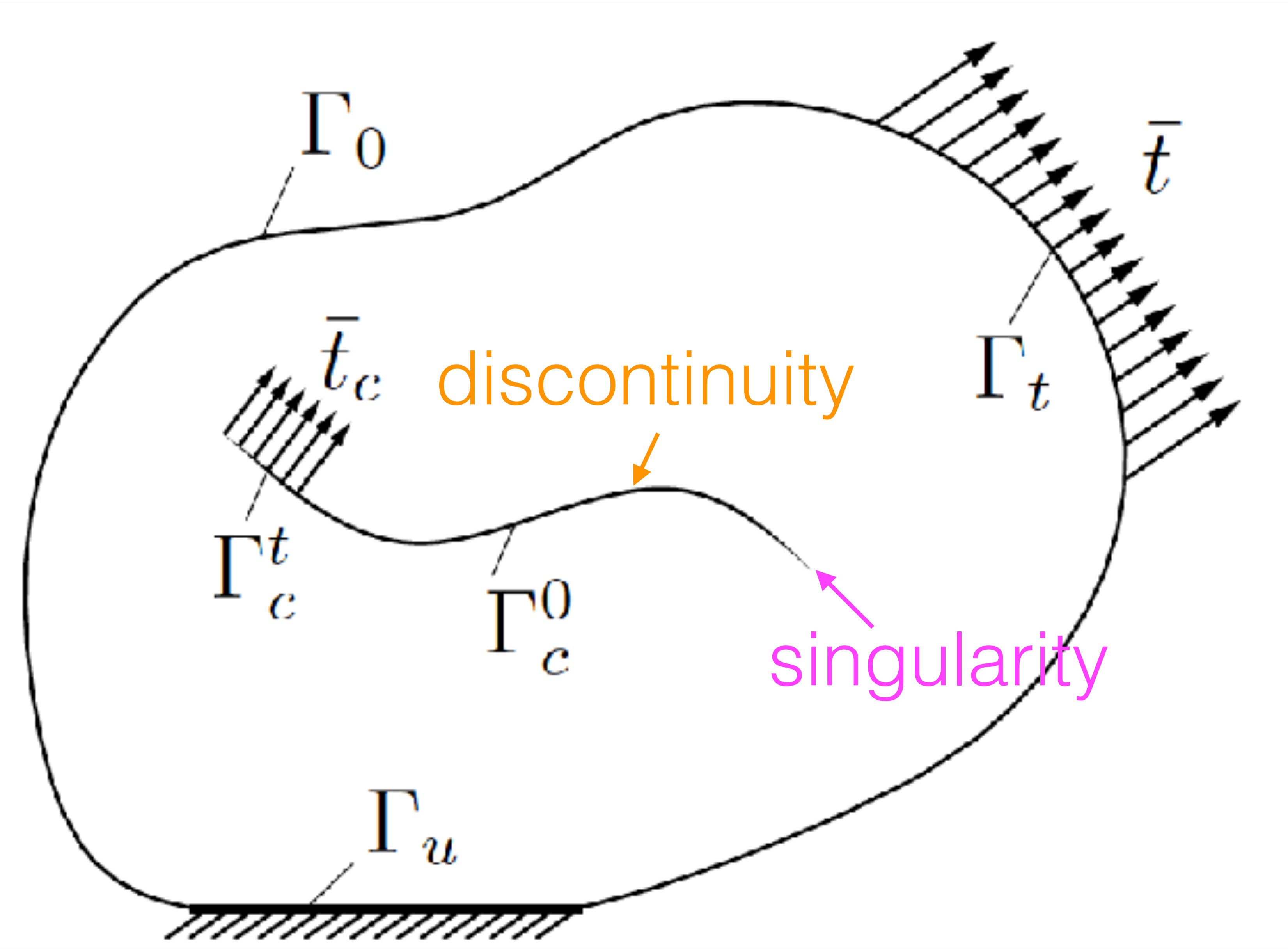
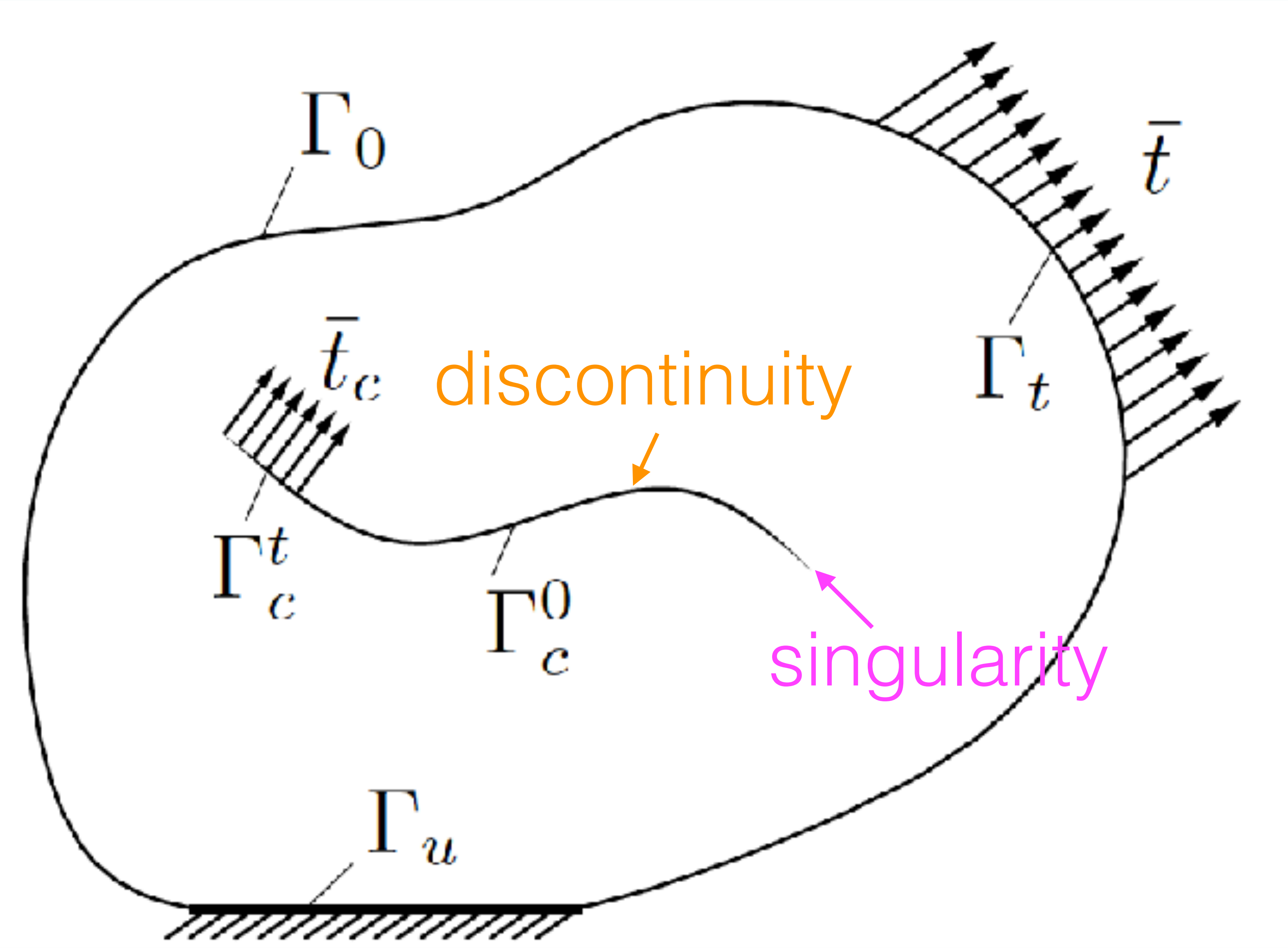
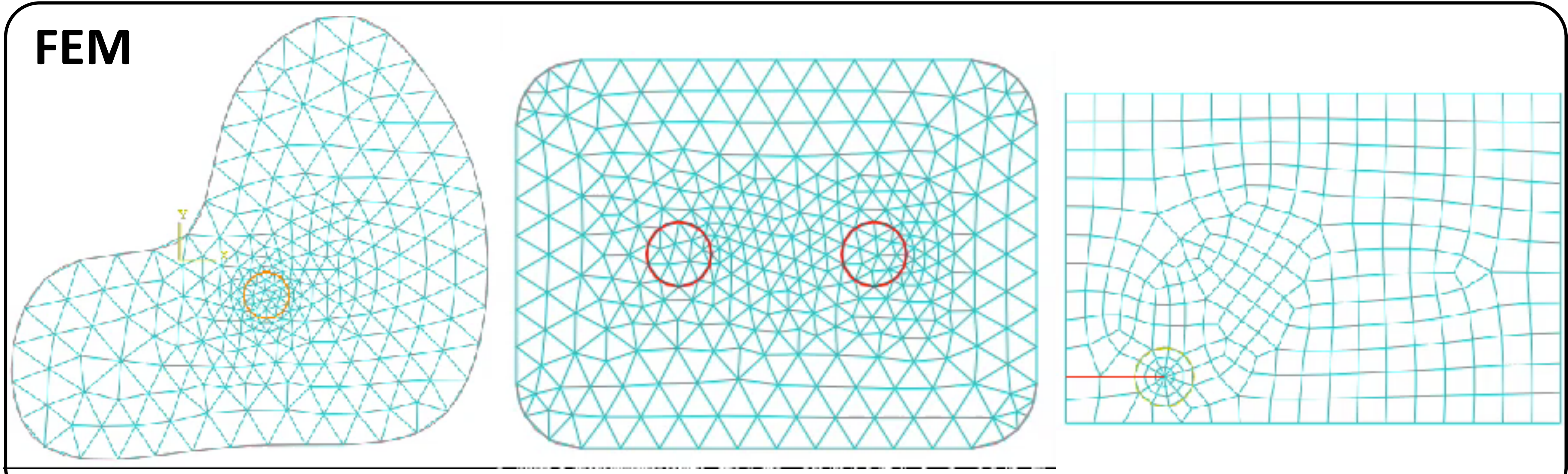


Finite elements for evolving discontinuities & singularities

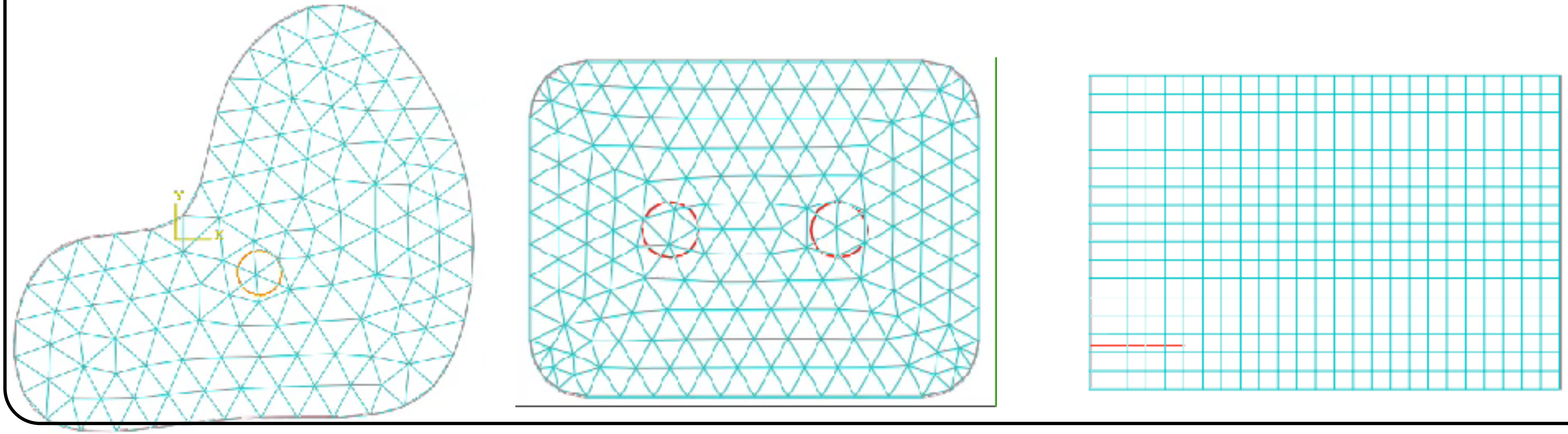


Free boundary problems

FEM



XFEM



One can show . if $v_h = I_h u$ Lagrange approximation of u.

Assume $u \in H^2(\Omega)$ twice weakly differentiable

$$\|\underbrace{u-u_h}_{e_h}\|_{H^1(\Omega)} \leq \frac{1}{c} \|v - \underbrace{I_h u}_{v_h}\|_{H^1(\Omega)}$$

Cea's Constant

$$\boxed{\|\text{error}\|_{H^1(\Omega)} \leq C \, h \, \boxed{\|u\|_{H^2(\Omega)}}}$$

error of FE approx. in $H_1(\Omega)$ $\|\cdot\|$.

- Dependence
  - Physical Constants in Ω
  - geometry of Ω
  - Quality of elements in $\mathcal{T}_h$ (mesh)
  - Degree of polynomial approx.

largest element in $\mathcal{T}_h$

- V_h: is as good as the best approximant in V_h:

Babuška, 1994. Partition of Unity.
1995.

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Assume $u \in H^2(\Omega)$ twice weakly differentiable

$$\|\underbrace{u-u_h}_{e_h}\|_{H^1(\Omega)} \leq \frac{1}{\underset{\text{Cea's Constant}}{\boxed{c}}} \|v - \underbrace{I_h u}_{v_h}\|_{H^1(\Omega)}$$

$$\boxed{\|\text{error}\|_{H^1(\Omega)} \leq \boxed{C}\,\boxed{h}\,\boxed{\|u\|_{H^{\boxed{2}}(\Omega)}}}$$

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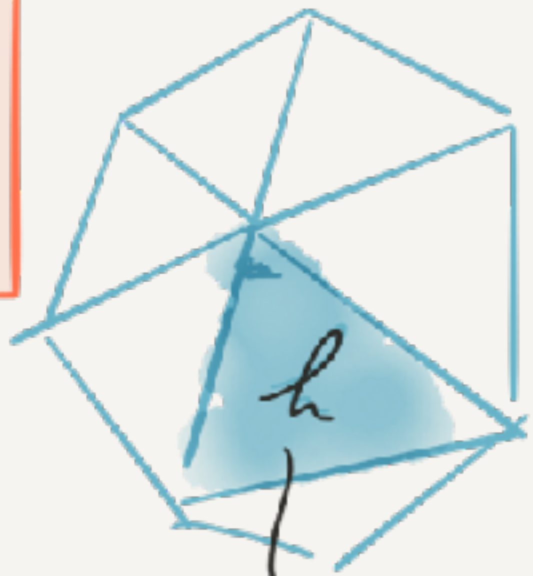

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  - Degree of polynomial approx.

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Babuška, 1994. Partition of Unity.
1995.

Why PUM ?

Babuška 1994 → 1996 ...

- A priori error estimate:

polynomial order

$$\|e_h\|_{\underbrace{m}_{H^m(\Omega)}} \leq \underbrace{C}_{\text{see ⑥}} \; \underbrace{h}_{\text{see ⑥}}^{\min(p+1-m,\; r-m)} \|u\|_{H^r(\Omega)}$$

measure of the error

r : smoothness of u

Elasticity . $m = 1$

Fracture r : small ⇒ u "not so smooth" → only a "few" ∂. of u are smooth

$$\|e_h\|_1 \leq C\, h^{\min(\underbrace{p+1-1}_{p},\; \underbrace{r-1}_{r-1})} \underline{\|u\|_{H^r(\Omega)}}$$

LARGE.

$$\leq C \underbrace{h}_{\text{mesh refinement}}^{\min(p,\, \underbrace{r-1}_{})} \text{"small"} \cdot \|u\|_{H^r(\Omega)}$$

Why PUM?

Babuška 1994 → 1996 ...

- A priori error estimate:

polynomial order

$$\boxed{\underbrace{\|e_h\|_m}_{\substack{H^m(\Omega)\\ \text{measure of the error}}} \leq \underbrace{C\, h}_{\text{see ⑥}}^{\min(p+1-m,\; r-m)} \|u\|_{H^r(\Omega)}}$$

r : smoothness of u

Elasticity . $m = 1$

Fracture r : small → only a "few" ∂. of u are smooth

⇒ u "not so smooth".

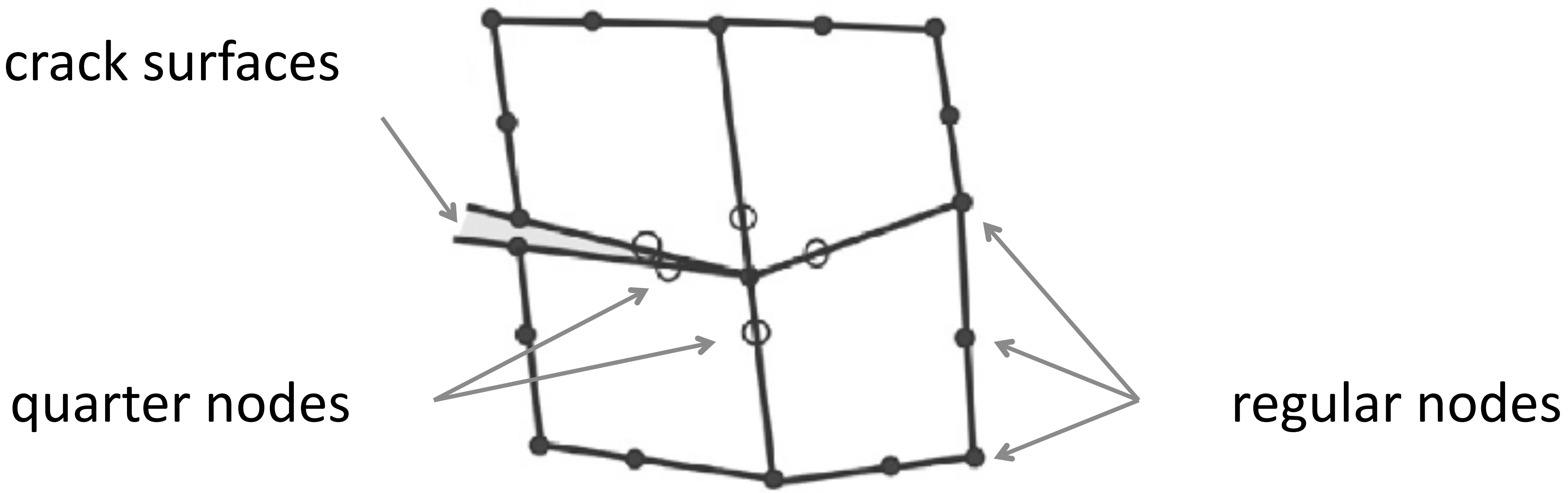
$$\|e_h\|_1 \leq C\, h^{\min(\underbrace{p+1-1}_{p},\; \underbrace{r-1}_{r-1})} \underbrace{\|u\|_{H^r(\Omega)}}_{\text{LARGE.}}$$

$$\leq C\, \underbrace{h}_{\text{mesh refinement}}^{\min(p,\; \underbrace{r-1}_{\text{"small"}})} \|u\|_{H^r(\Omega)}$$

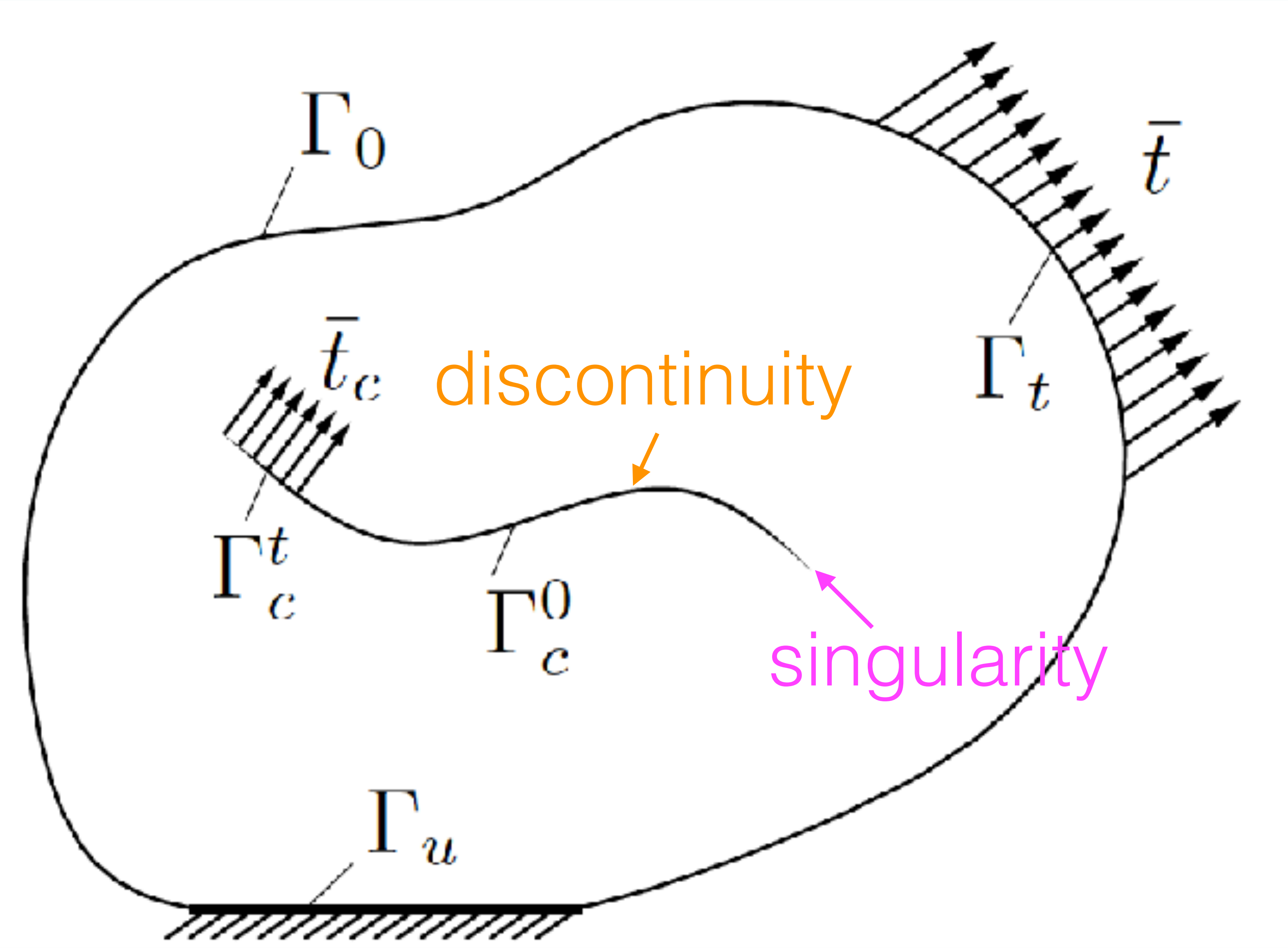
Singular elements (Barsoum, 1974)

For simulating the crack tip singular field in LEFM

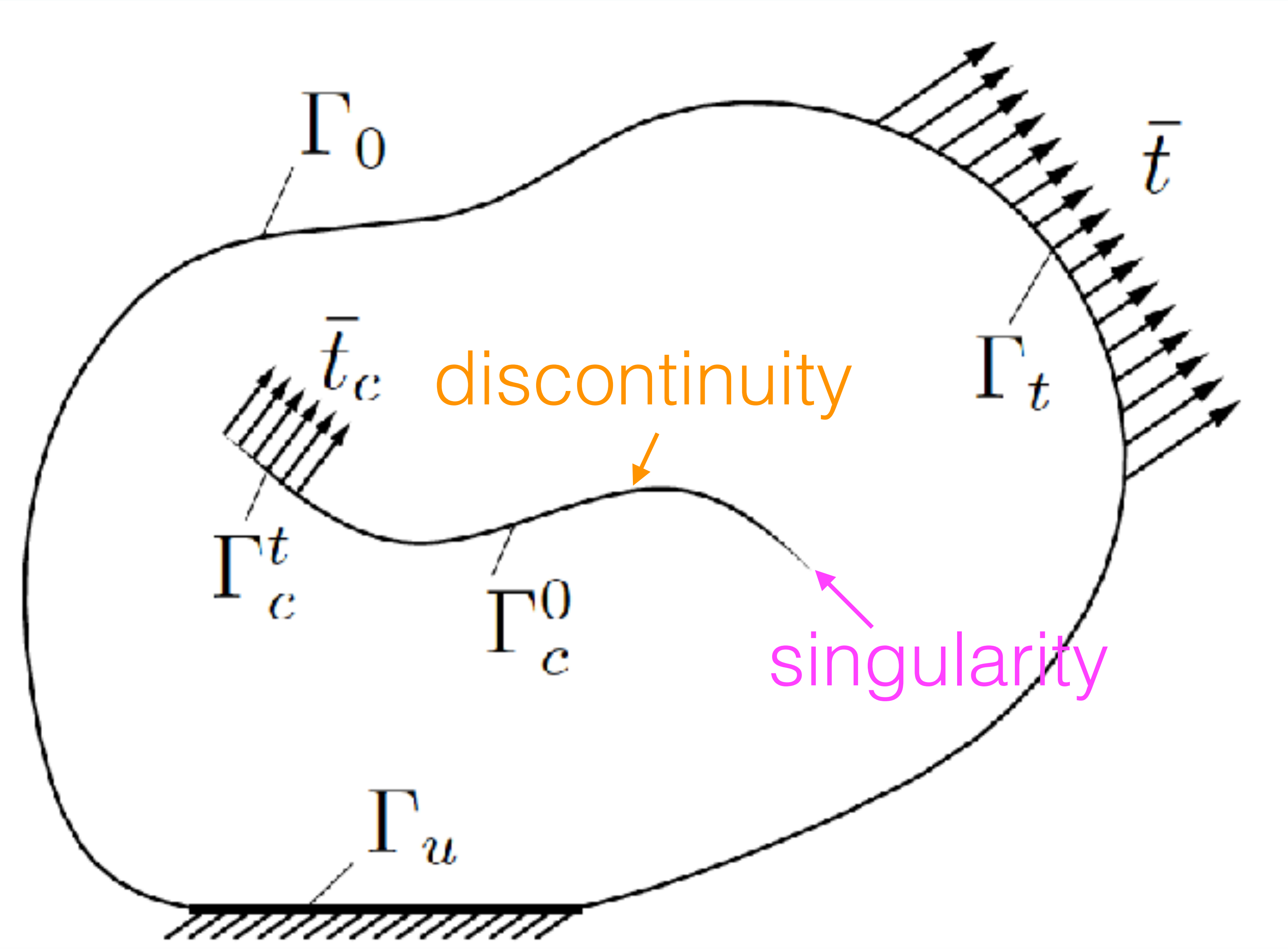
- A simple way how to introduce a singularity of $1/\sqrt{r}$ in isoperimetric finite elements is by displacing the mid-side nodes of two adjacent edges to one quarter of the element edge length from the node where the singularity is desired.



Finite elements are intrinsically limited for problems involving discontinuities & singularities such as cracks

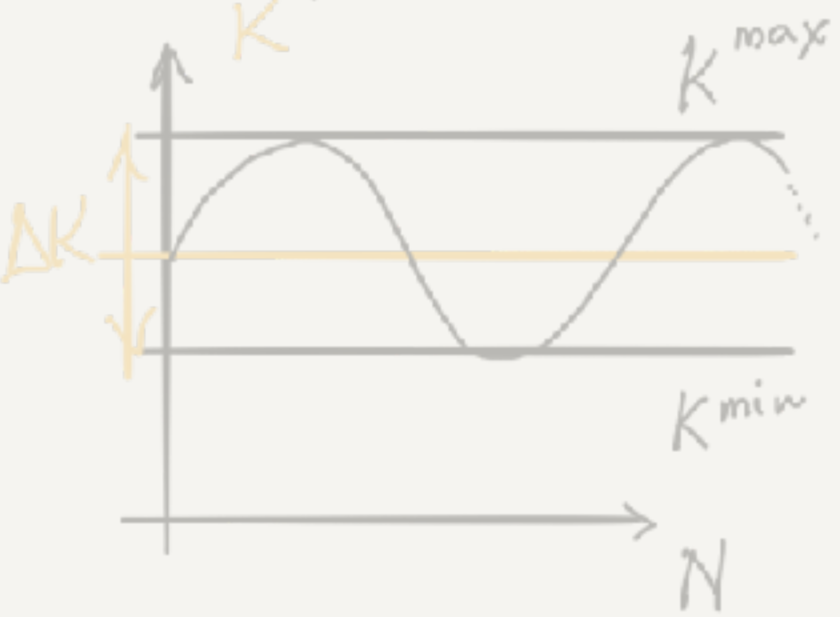


More over, computational fracture (LEFM) requires highly accurate solutions… why?



COMPUTATIONAL FRACTURE

- Aerospace applications typically assume Linear Elastic Fracture.
- Empirical crack growth laws, e.g. Paris law & generalizations.



$$\underbrace{\Delta a}_{\text{amount of crack growth for } \Delta N \text{ cycles}} = C\,(\Delta K)^m\,\underbrace{\Delta N}_{\text{number of cycles}}$$

$\boxed{\text{Stress Intensity factor amplitude}}$

C, m are empirical coefficients

$m \in [3,5]$ typically.

SIF ~ Amount of energy released for a unit increment in crack growth.

$$[SIF] = \text{Stress}\sqrt{\text{length}} = \sigma\sqrt{\ell} = \frac{N}{m^2}\sqrt{m}$$

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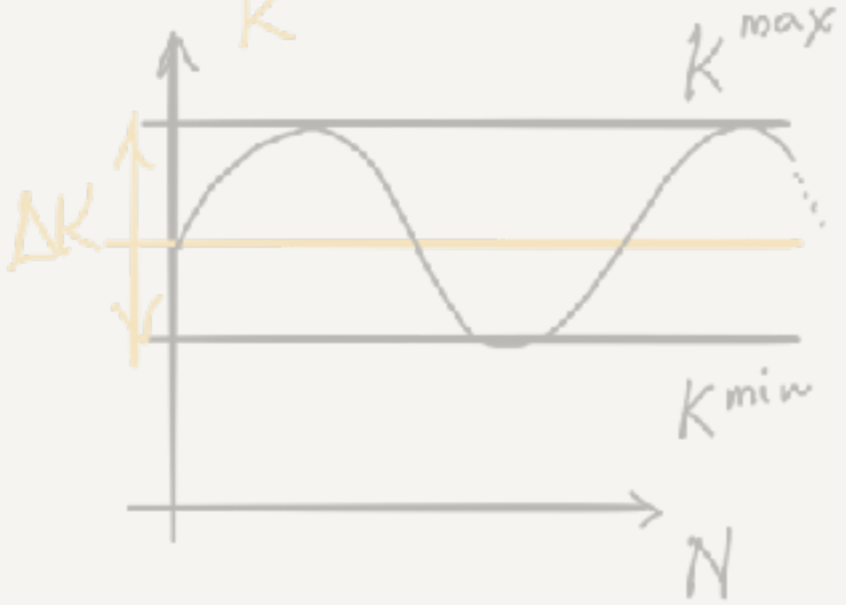

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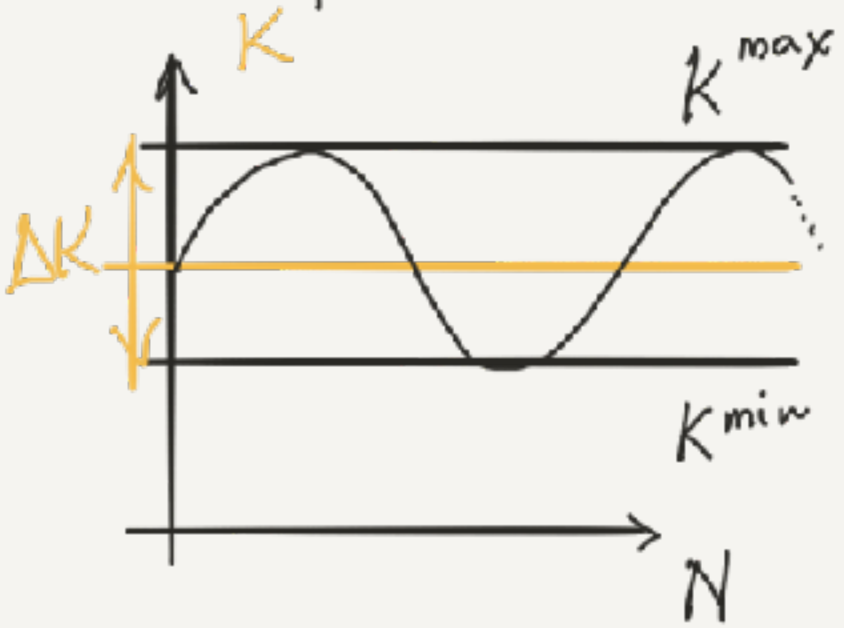

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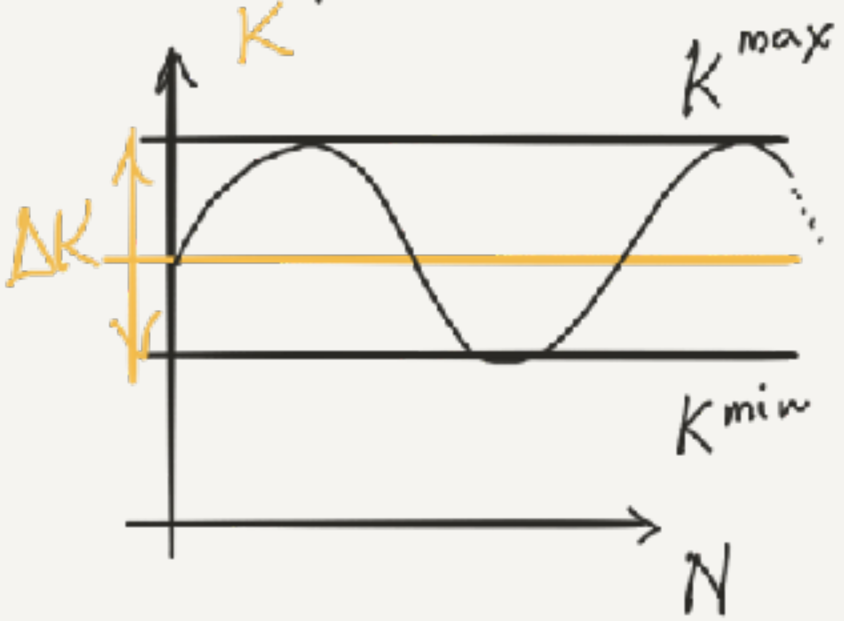

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Stress Intensity factor amplitude

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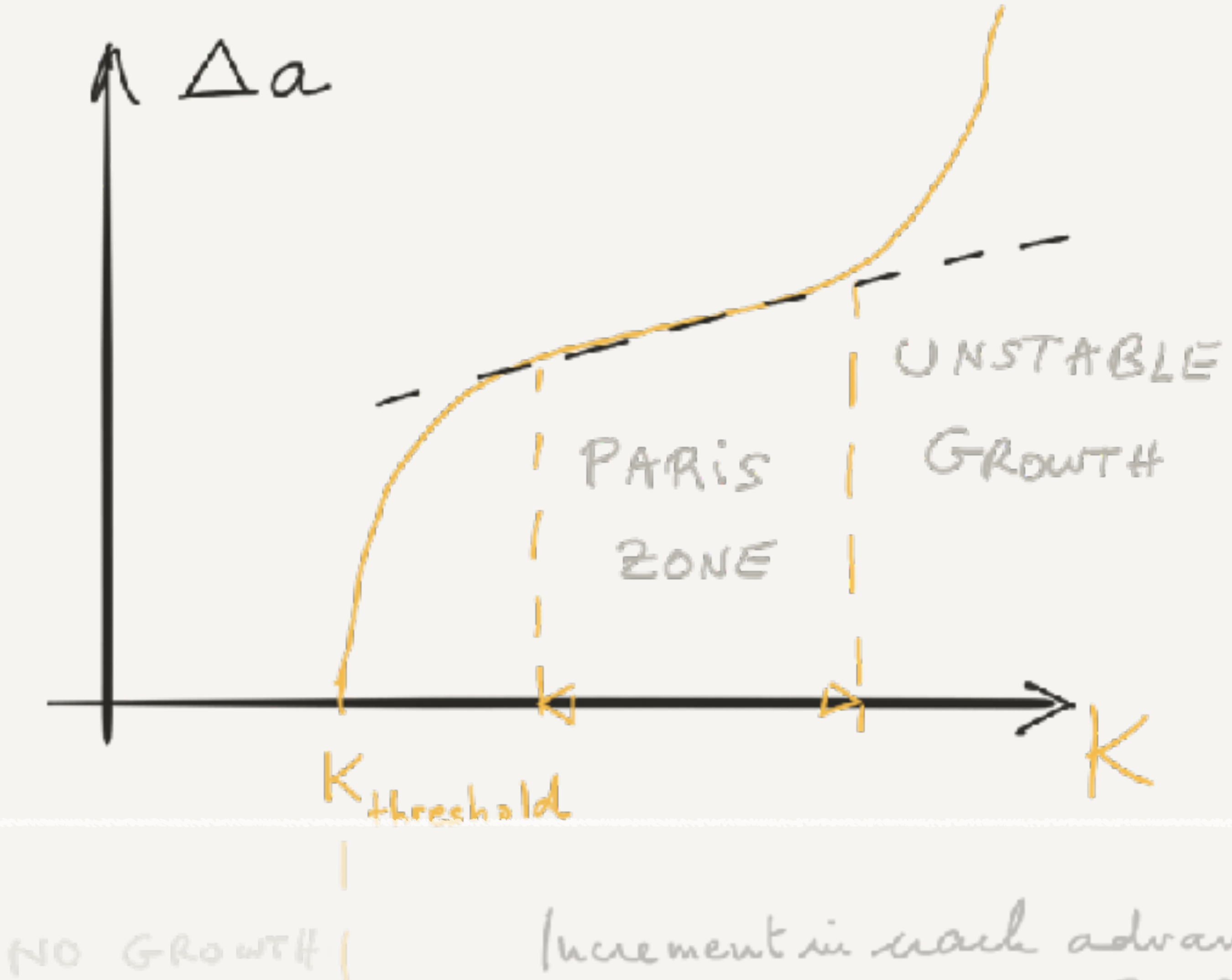
Fracture ②



Numerically Computed

$$\boxed{\Delta a = C\,(K)^m\,\Delta N}$$

$3 \leq m \leq 5$ Paris Exponent

Assume error ε_K on K

$\frac{\varepsilon_K}{K} << 1$ — ε

Increment in crack advance assuming an error ε_K is committed

$$\boxed{\Delta a^{\varepsilon}} = C\,(K+\varepsilon_K)^m\,\Delta N$$

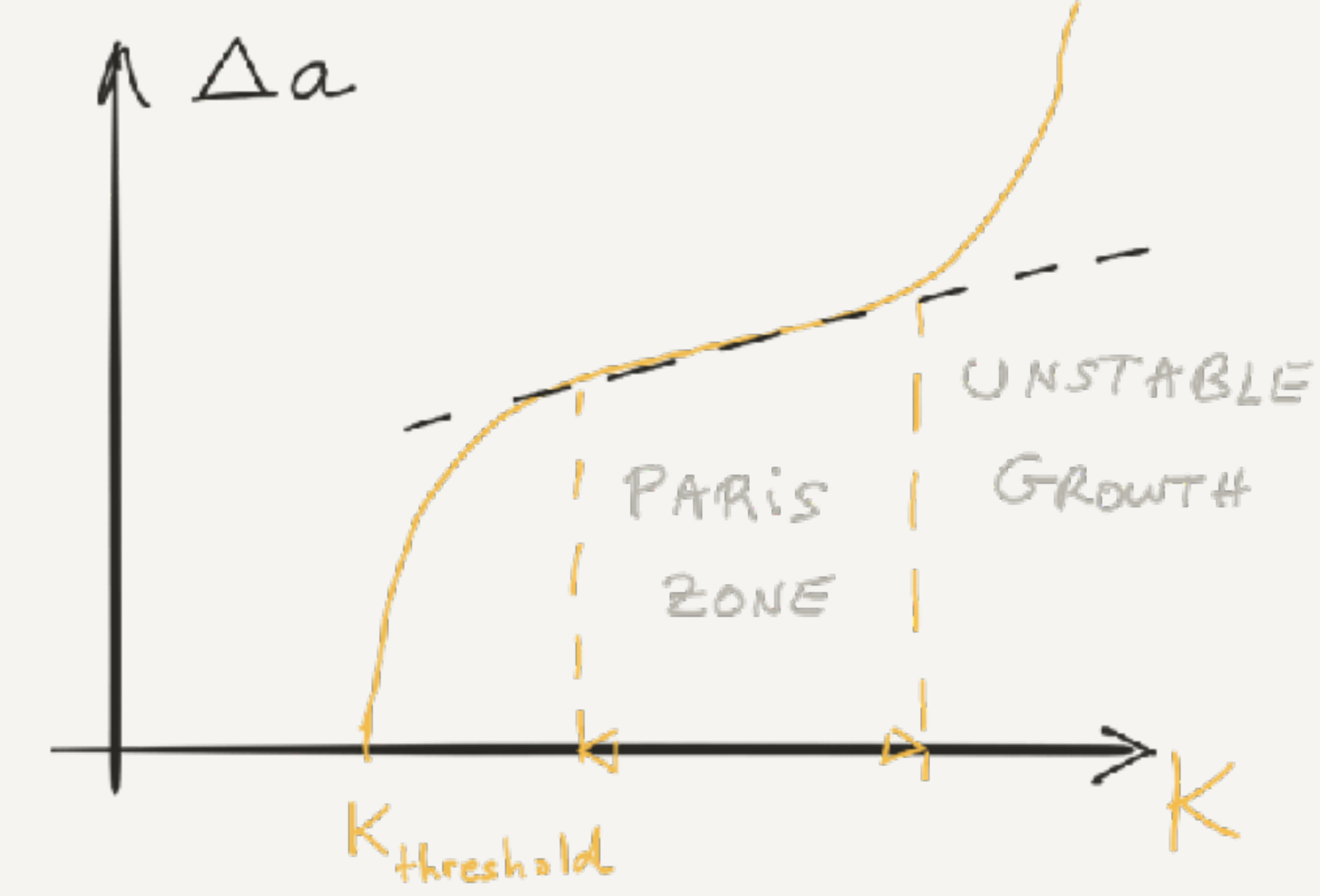
$$\Delta a^{\varepsilon} = C\;K\;\left(1+\frac{\varepsilon_K}{K}\right)^m \Delta N$$

$$\Delta a^{\varepsilon} \simeq C\;K^m(1+m\,\varepsilon)\,\Delta N$$

$$\boxed{\Delta a^{\varepsilon} \simeq \Delta a\,(1+m\varepsilon)}$$

error of $\varepsilon\%$ on K leads to ⟹ an error of $m\,\varepsilon\%$ on Δa !!

Fracture ②



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$$\frac{\varepsilon_K}{K} \ll 1$$

ε

Increment in crack advance assuming an error ε_K is committed

$$\boxed{\Delta a^{\varepsilon}} = C\left(K+\varepsilon_K\right)^m \Delta N$$

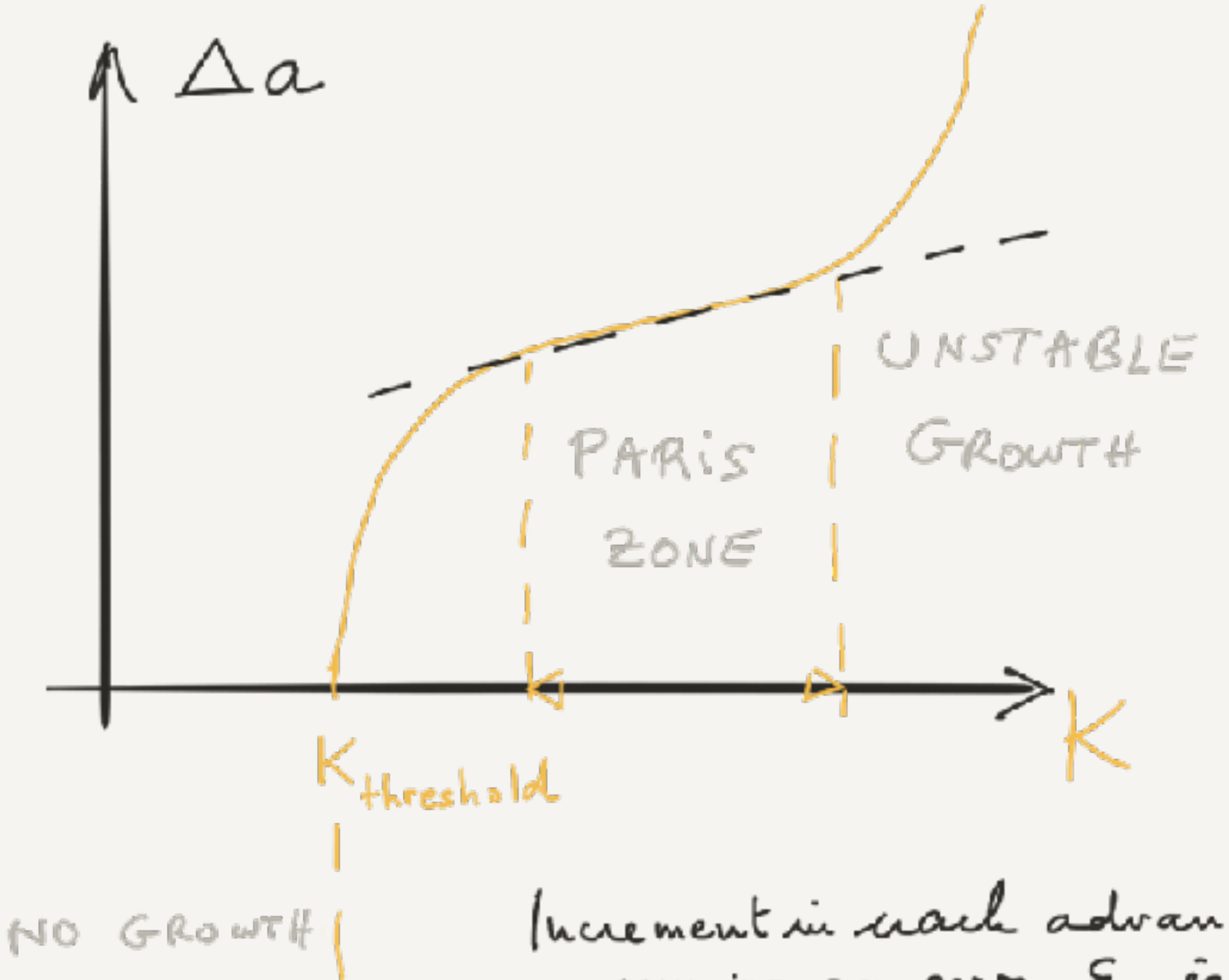
$$\Delta a^{\varepsilon} = C\;K\;\left(1+\frac{\varepsilon_K}{K}\right)^m \Delta N$$

$$\Delta a^{\varepsilon} \simeq C\;K^m\left(1+m\,\varepsilon\right)\Delta N$$

$$\boxed{\Delta a^{\varepsilon} \simeq \Delta a\left(1+m\varepsilon\right)}$$

$\Rightarrow$ error of $\varepsilon\%$ on K leads to an error of $m\,\varepsilon\%$ on Δa !!

Fracture ②



Numerically Computed

$$\boxed{\Delta a = C\,(K)^{m}\,\Delta N}$$

$3 \leq m \leq 5$ Paris Exponent

Assume error ε_K on K

$$\underbrace{\frac{\varepsilon_K}{K}}_{\varepsilon} << 1$$

Increment in crack advance assuming an error ε_K is committed

$$\boxed{\Delta a^{\varepsilon}} = C\,(K+\varepsilon_K)^{m}\,\Delta N$$

$$\Delta a^{\varepsilon} = C\;K\;\left(1+\frac{\varepsilon_K}{K}\right)^{m}\Delta N$$

$$\Delta a^{\varepsilon} \simeq C\;K^{m}(1+m\,\varepsilon)\,\Delta N$$

$$\boxed{\Delta a^{\varepsilon} \simeq \Delta a\,(1+m\varepsilon)}$$

$\Longrightarrow$ error of ε% on K leads to an error of $m\,\varepsilon$% on Δa !!

Conclusion

. It is critical to compute SIFs as accurately as possible

① $\text{error}(\Delta a) = m\,\text{error}(K)$ $m \in [3,5]$

② Over 10'000 increments are typically needed to estimate the fatigue life. $\Rightarrow$ $\text{error}(\text{path}) \simeq 10^4\,\text{error}(\Delta a)$

$$\simeq m\,10^4\,\text{error}(K)$$

③ Fracture is history dependent.
Cracks cannot heal.
Errors cannot be corrected during growth.

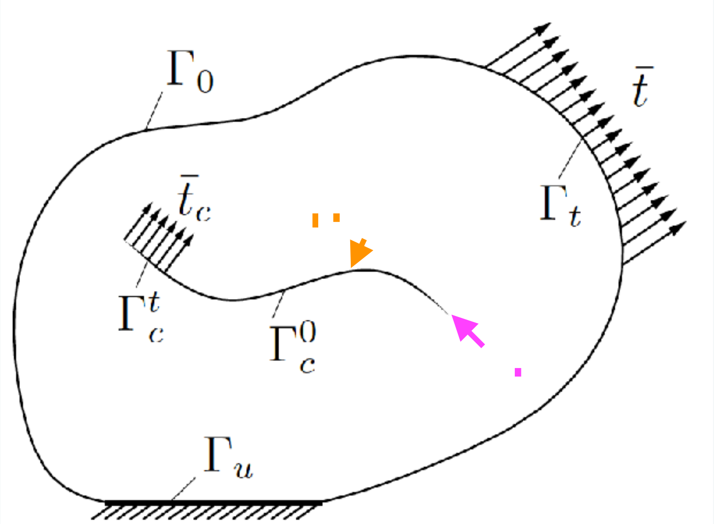
The idea of Partition of Unity Enrichment (PUFEM, GFEM, XFEM, hp clouds, enriched IGA, enriched mesfhree methods, enriched BEM…)

add what you know about the solution to the (finite element) basis

Singularities?

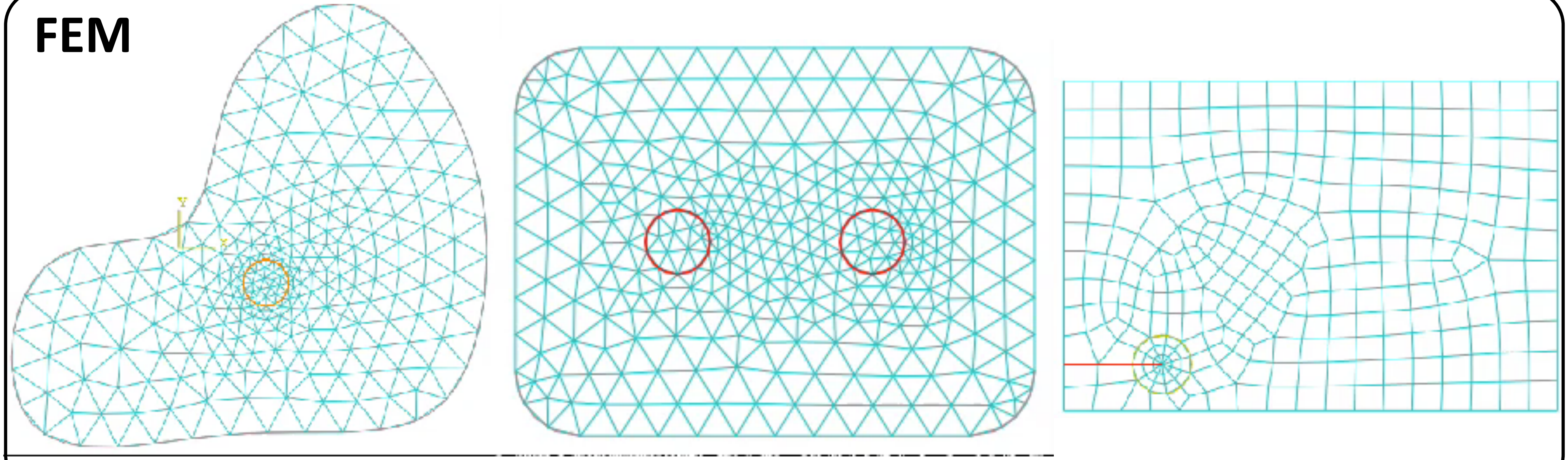
Discontinuities?

Boundary layers?

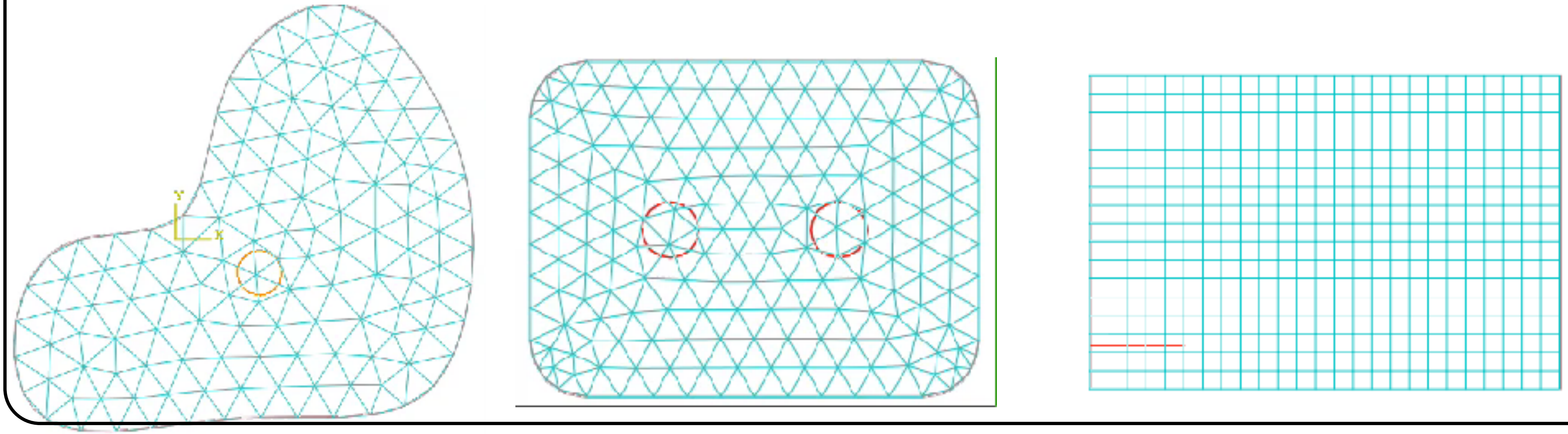


Free boundary problems

FEM



XFEM



Classification of enrichments

Global enrichment

- The enrichment is employed on the global level, over the **entire domain**.
- Useful for problems that can be considered as **globally non-smooth** e.g. high-frequency solutions (Helmholtz equation)

Local enrichment

- This enrichment scheme is adopted locally, over a **local subdomain**.
- Useful for problems that only involve **locally non-smooth** phenomena, e.g. solutions with discontinuities.

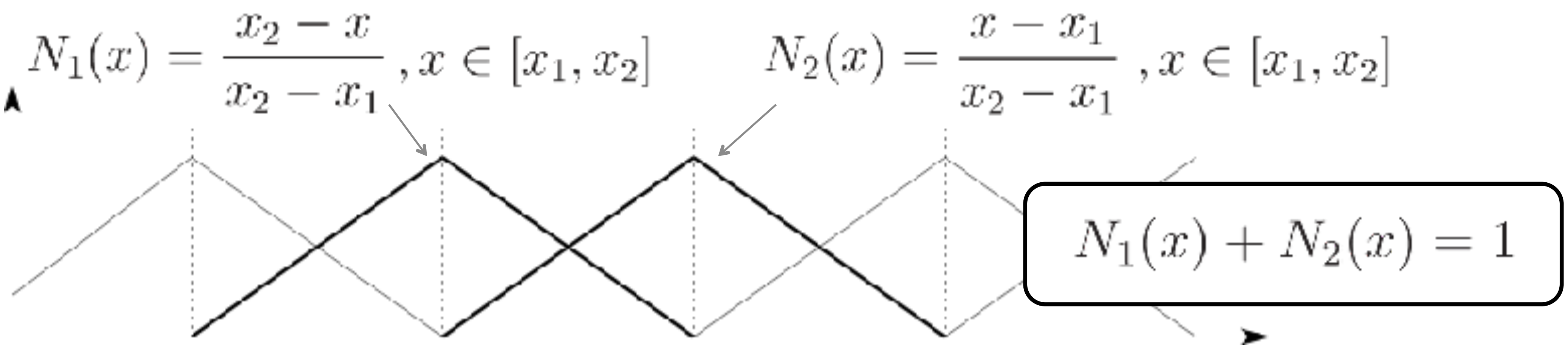
Partition of unity finite element method (PUFEM)

Partition of unity (PU)

- A set of functions ϕ_i whose sum at any point $\boldsymbol{x}$ inside a domain Ω is equal to unity:

$$\forall \mathbf{x} \in \Omega, \mathbf{x} : \sum_{I=1} \phi_I(\mathbf{x}) = 1$$

- Example PU functions are the finite element "hat" functions:



$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}, x \in [x_1, x_2] \qquad N_2(x) = \frac{x - x_1}{x_2 - x_1}, x \in [x_1, x_2]$$

$$\boxed{N_1(x) + N_2(x) = 1}$$

Partition of unity finite element method (PUFEM)

Reproducibility of PU

- Any function $\boldsymbol{p}(\boldsymbol{x})$ can be reproduced by a product of that function and the partition of unity functions:

$$\sum_{I=1} \phi_I(\mathbf{x}) p(\mathbf{x}) = p(\mathbf{x})$$

- The function can be adjusted if the sum is modified by introducing parameters $\boldsymbol{q_I}$:

$$\sum_{I=1} \phi_I(\mathbf{x}) p(\mathbf{x}) q_I = \bar{p}(\mathbf{x})$$

- Reproducibility of $\boldsymbol{p}(\boldsymbol{x})$ can be controlled and localised to arbitrary regions where $\boldsymbol{q_I} \neq \mathbf{0}$

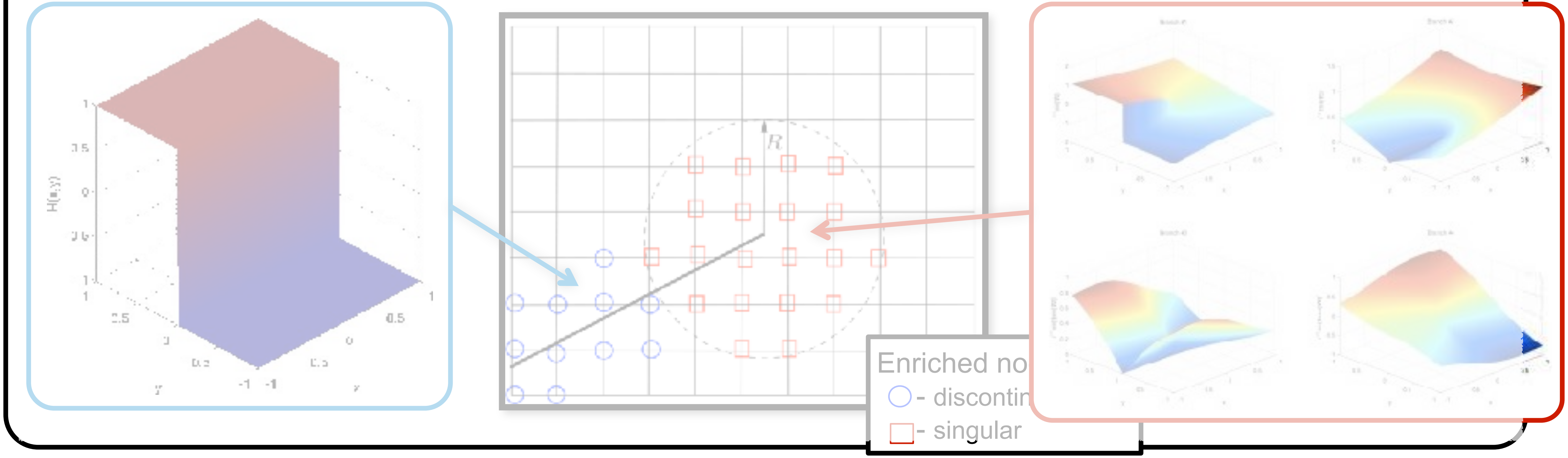
GFEM/XFEM

Formulation for crack growth:

$$\mathbf{u}^h(\mathbf{x}) = \underbrace{\sum_{I\in\mathcal{N}_I} N_I(\mathbf{x})\mathbf{u}^I}_{\text{standard part}} + \underbrace{\sum_{J\in\mathcal{N}_J} N_J(\mathbf{x})H(\mathbf{x})\mathbf{a}^J}_{\text{discontinuous enrichment}} + \underbrace{\sum_{K\in\mathcal{N}_K} N_K(\mathbf{x})\sum_{\alpha=1}^{4} f_\alpha(\mathbf{x})\mathbf{b}^{K\alpha}}_{\text{singular tip enrichment}}$$

$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } \mathbf{x} \text{ above crack} \\ -1 & \text{if } \mathbf{x} \text{ below crack} \end{cases}$$

$$\{f_\alpha(r,\theta), \alpha = 1,4\} = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\}$$



GFEM/XFEM

Formulation for crack growth:

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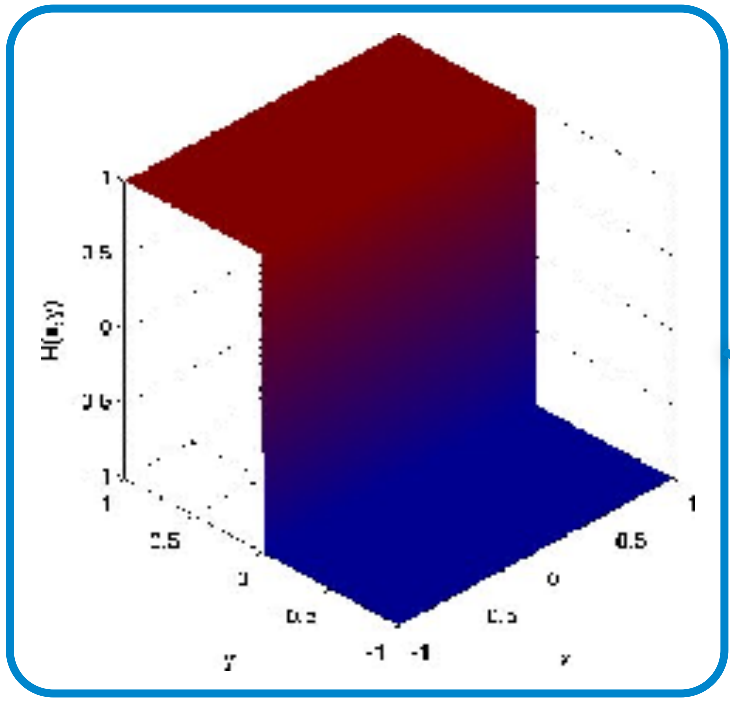
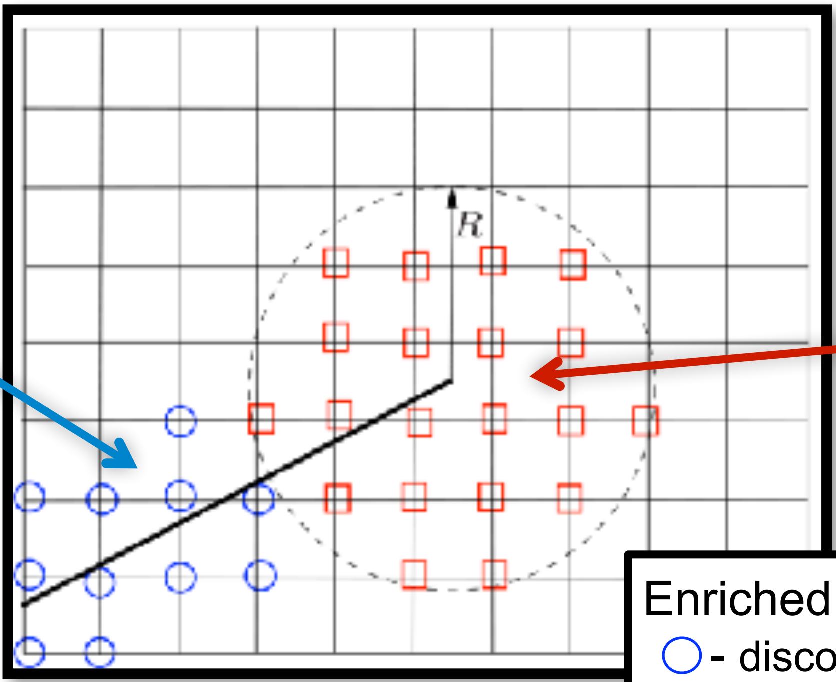
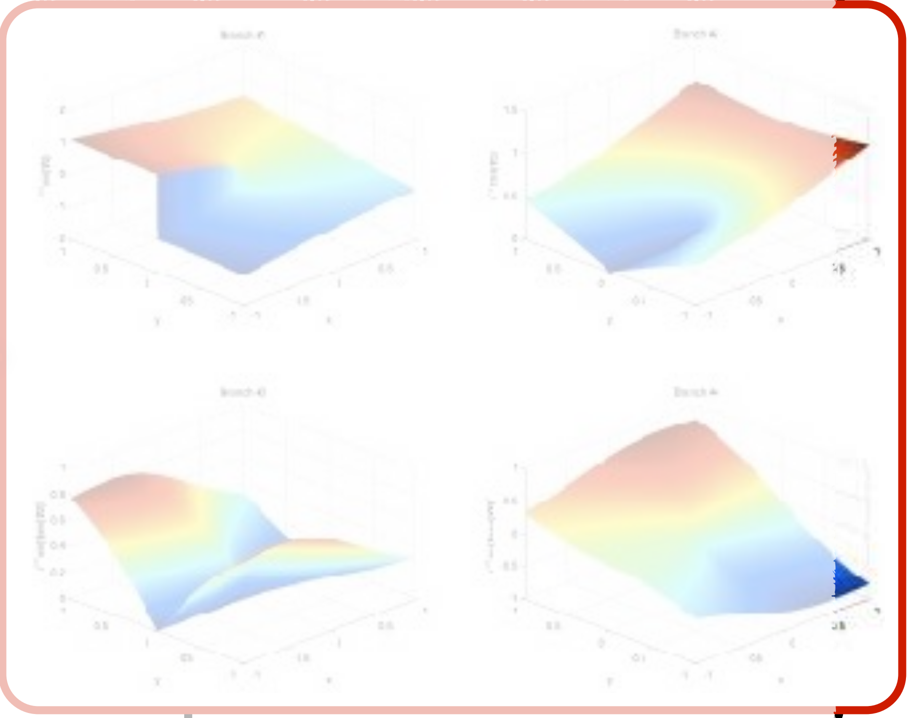




Enriched no

○- discontin

□- singular



GFEM/XFEM

Formulation for crack growth:

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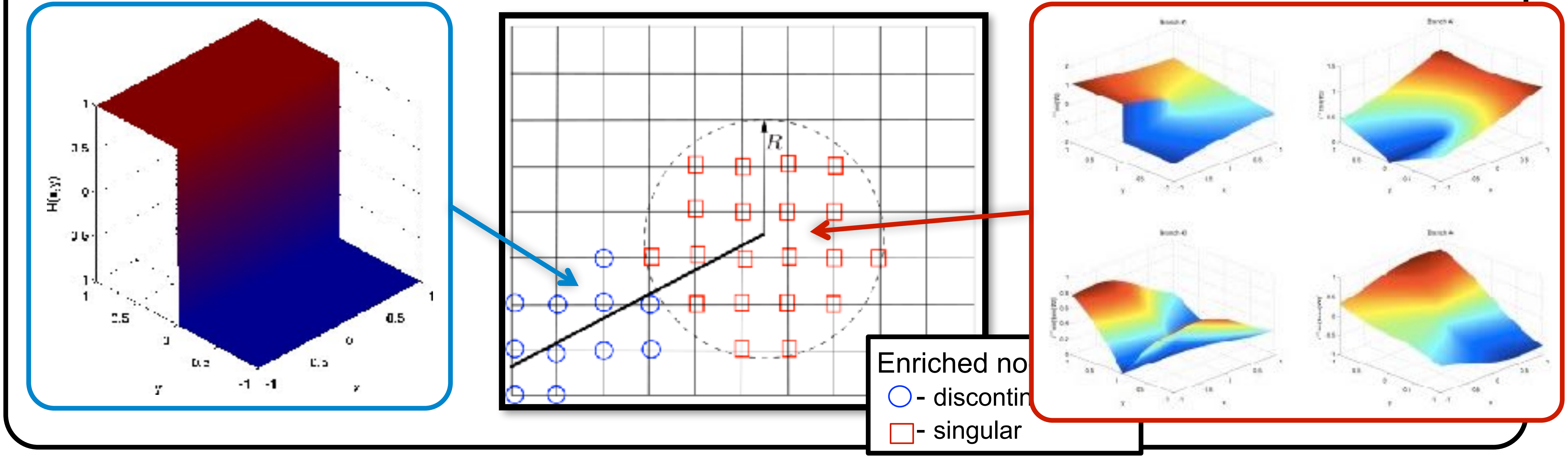
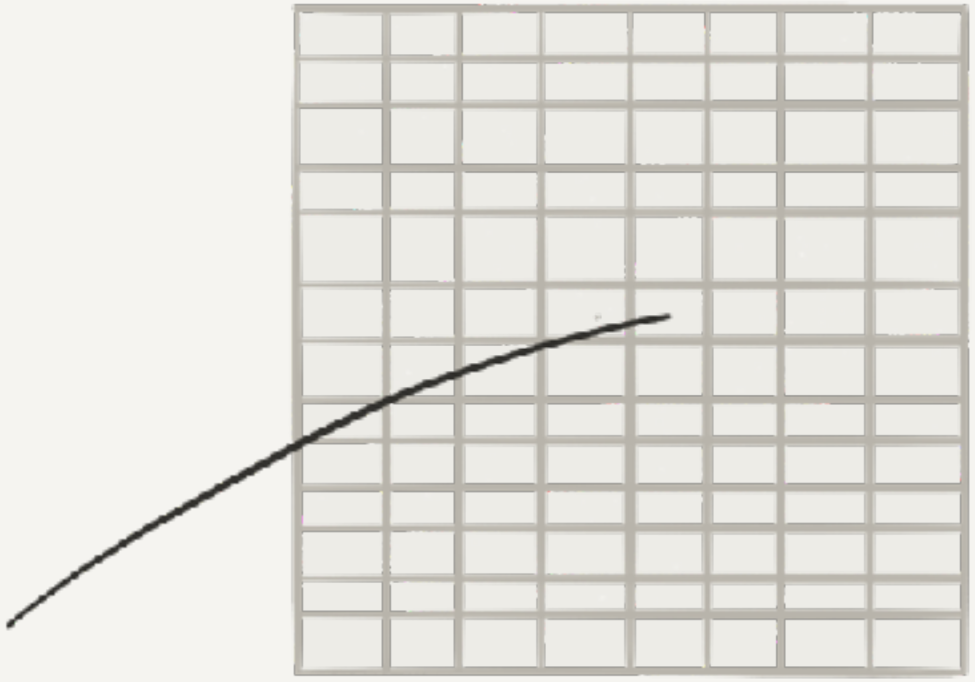
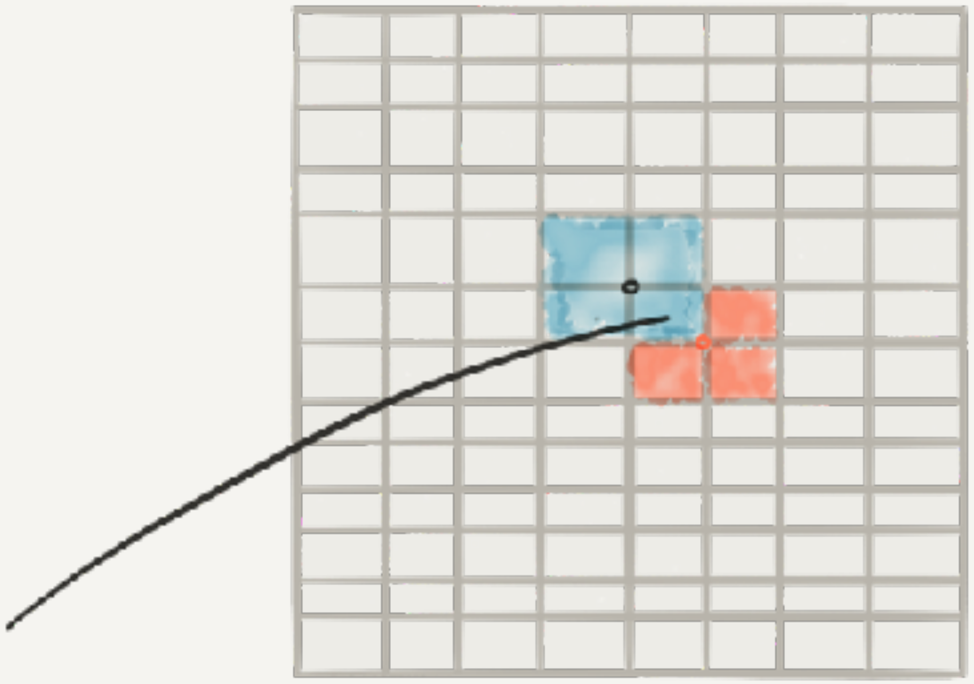
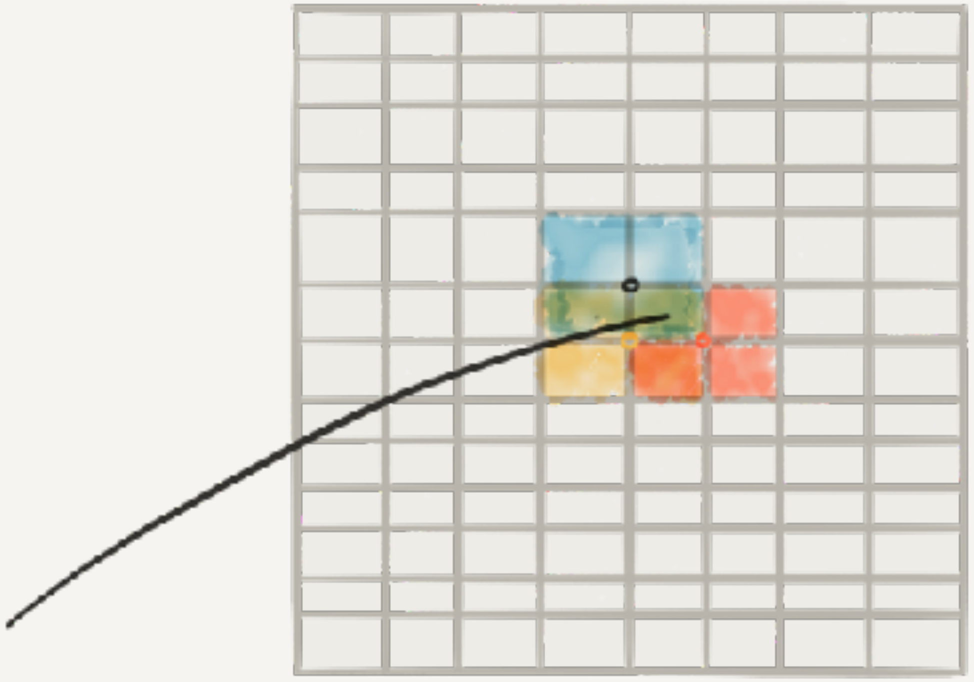
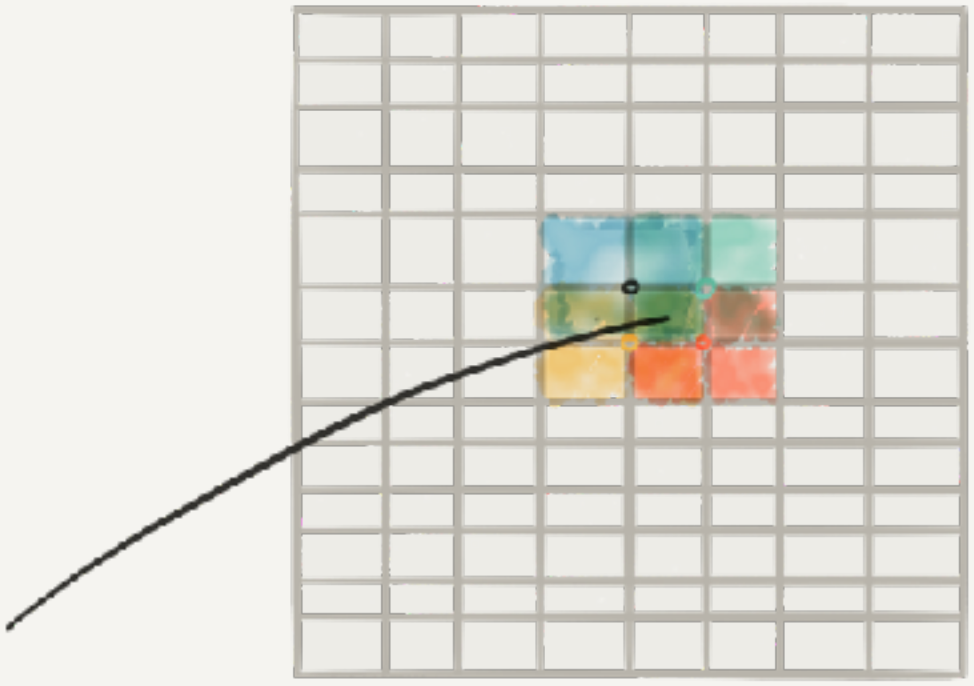
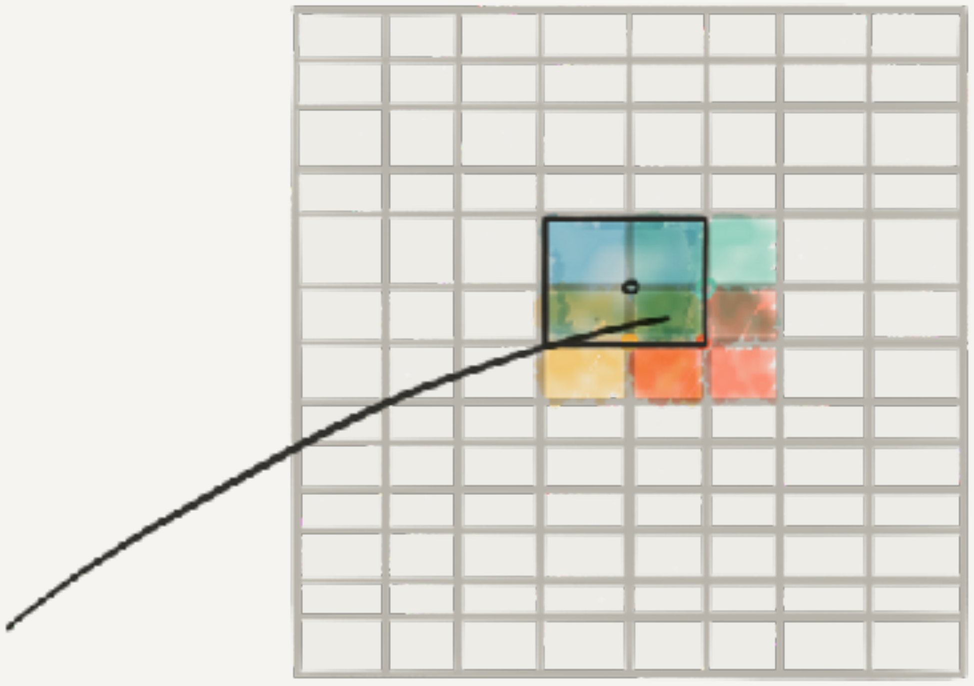


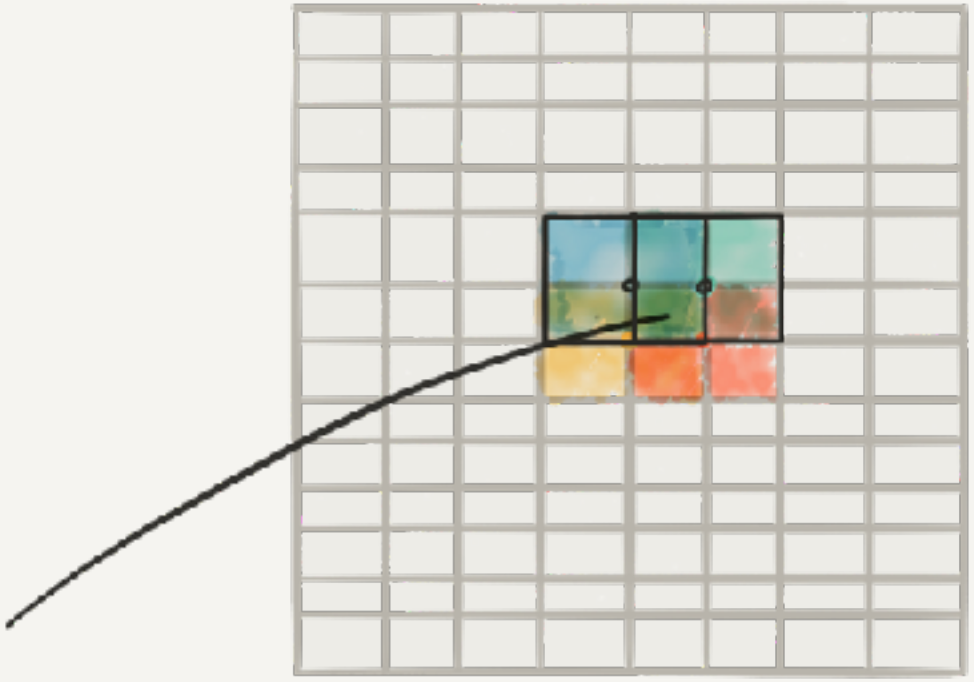


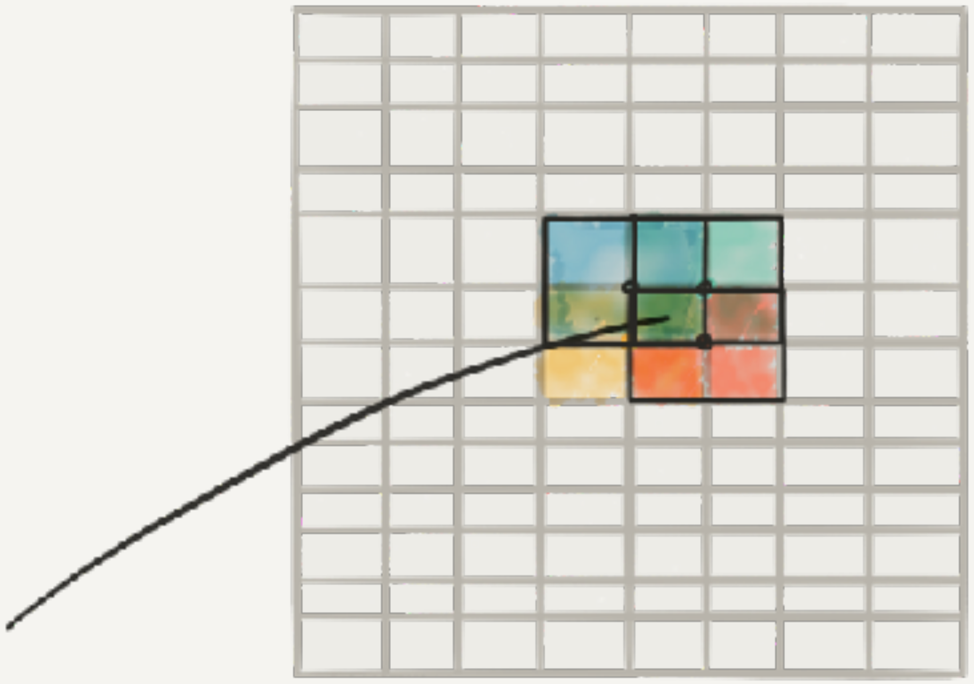


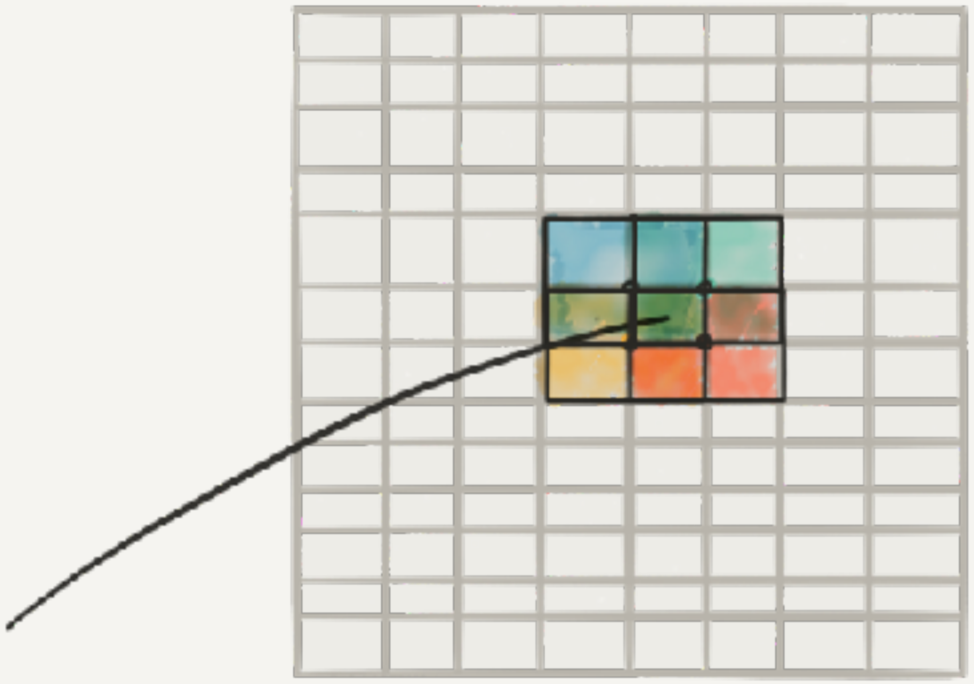


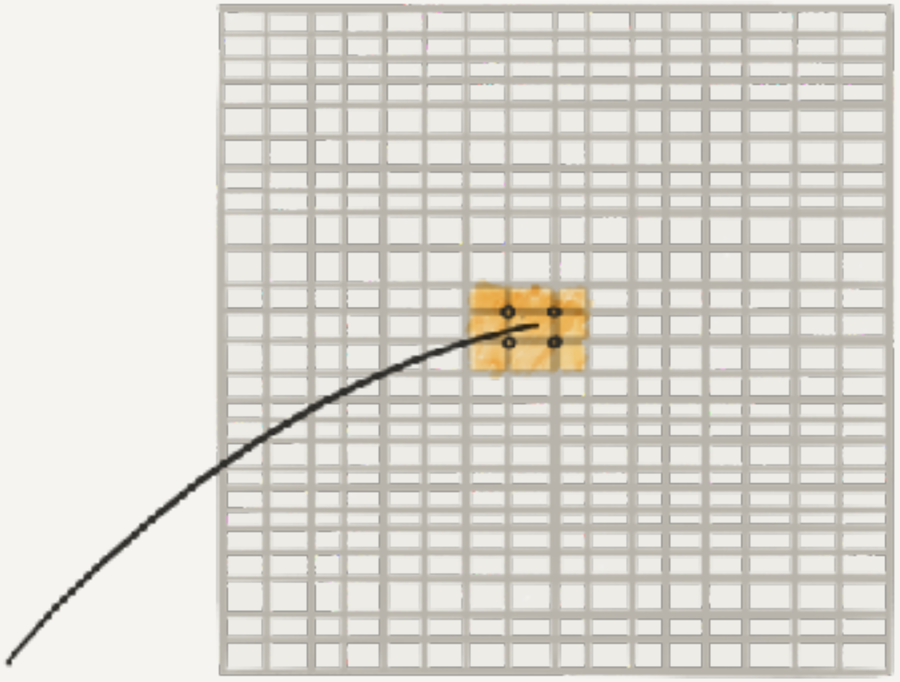


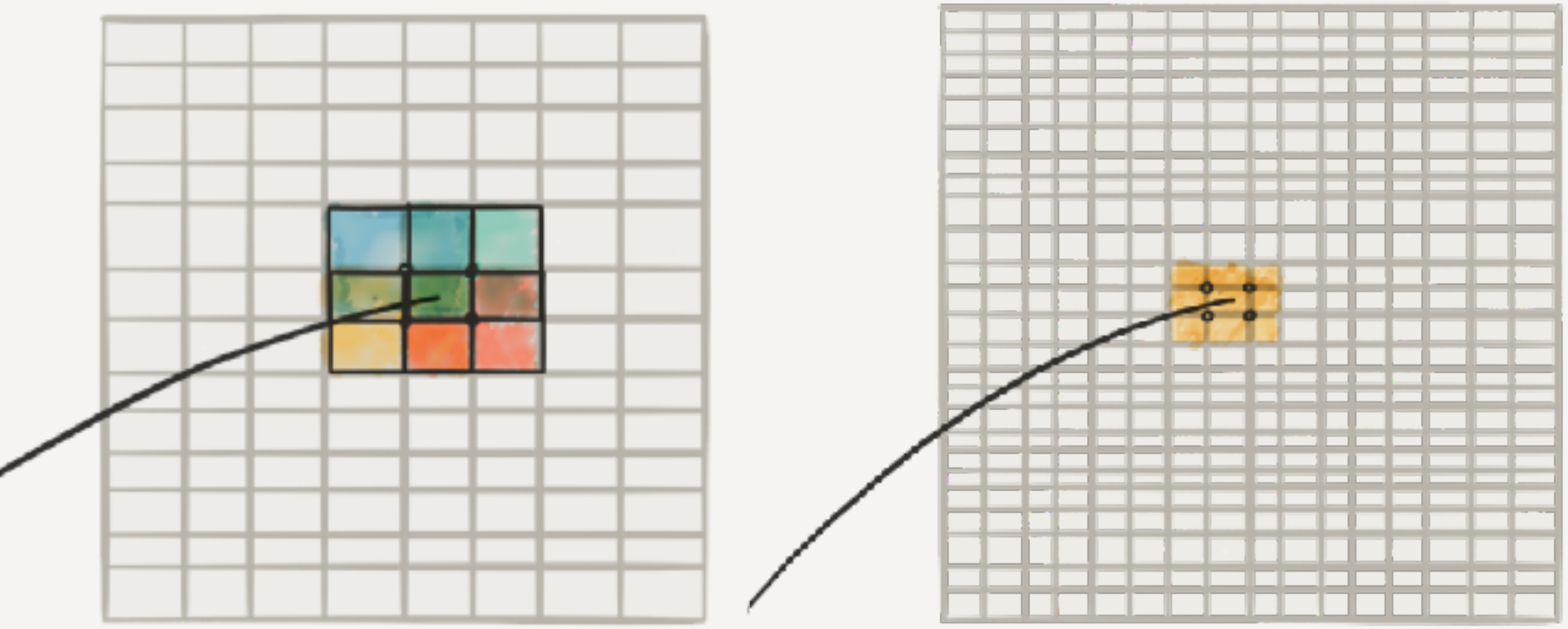






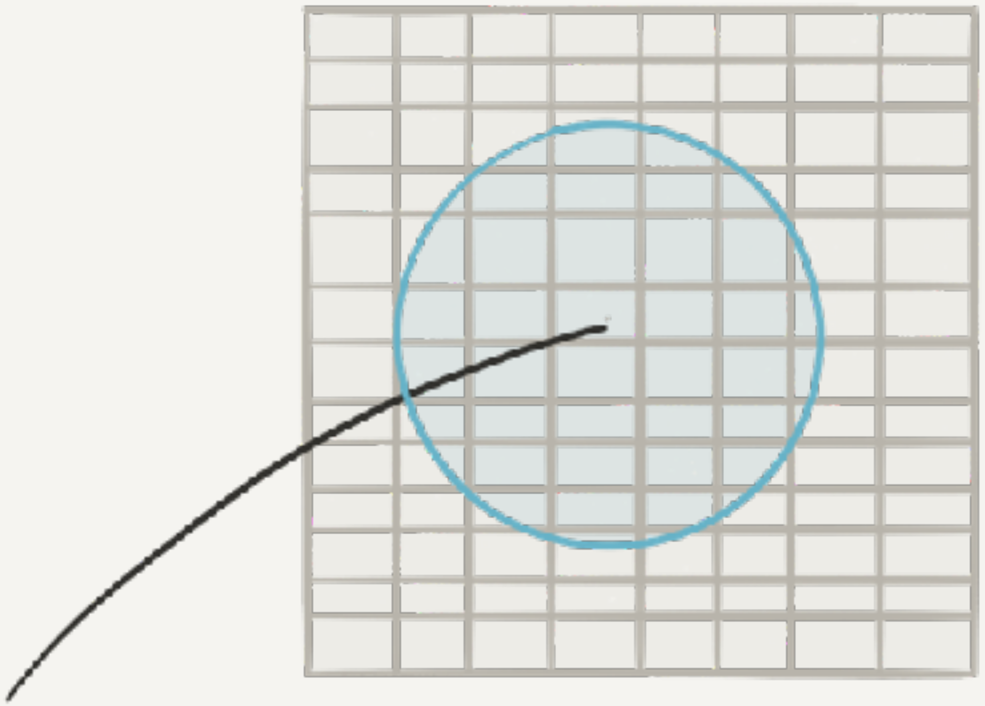




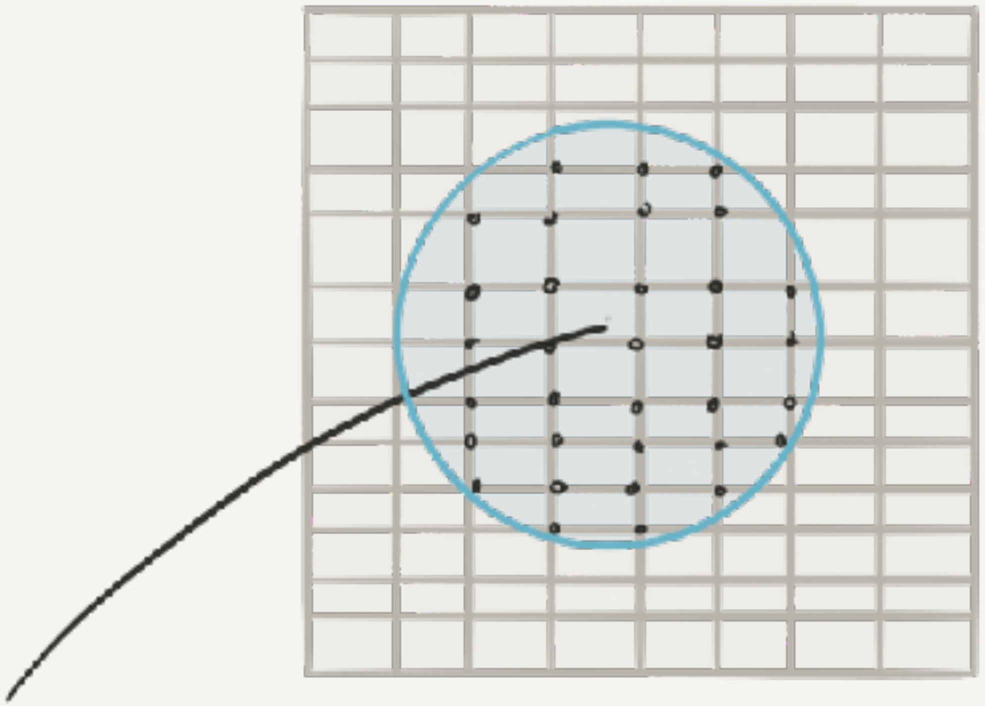


By refining the mesh, the influence of the enrichment zone on the convergence of the method tends to zero

We lose the benefit of enrichment



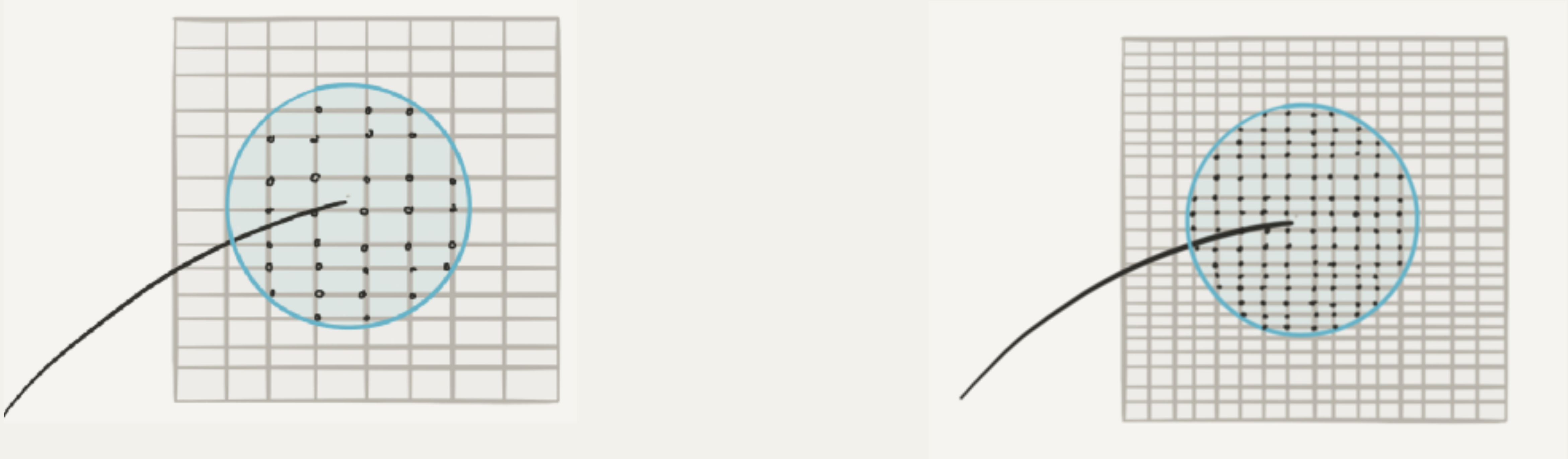
Enriching an area independent of the mesh size



ensures that as the mesh is refined, more and more nodes become enriched

the optimal convergence rate is preserved

Conditioning issues can be so severe that the set of equations is unsolvable



- ☑ Large enrichment zones (see stable GFEM, Banerjee, Babuška + Agathos)
- ☑ For arbitrary enrichment schemes
    - ☑ T-stress - 2nd order terms in Westergaard expansion
    - ☑ Multiple enrichments due to multiple cracks

Conclusion: difficult to set up robust and automatic enrichment schemes without specific tricks (preconditioner, e.g. Béchet or Menk)

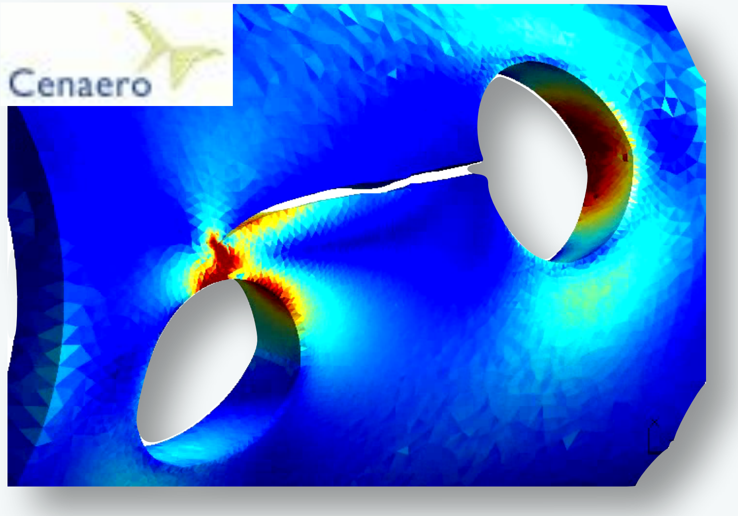
Summary

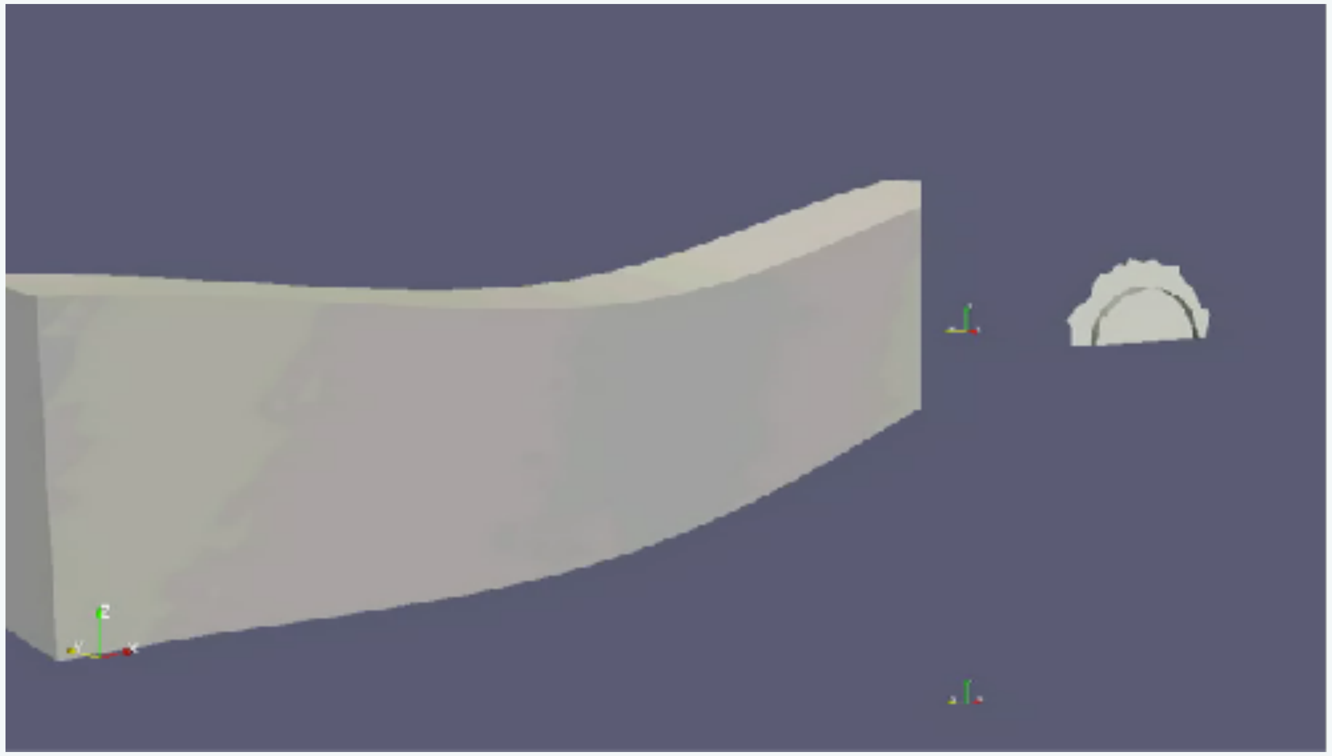
Fracture of homogeneous materials

Question: How to control accuracy and simplify/avoid meshing?

- Partition of Unity - eXtended/Generalized Finite Element Methods
  - ☑ Discretisation error governed by the worst approximant
  - ☑ Local enrichment of approximations
  - ☑ Requires enrichment volumes independent of the mesh
  - ☑ Conditioning issues for large enrichment zones or arbitrary enrichment (see stable GFEM, Banerjee, Babuška + Agathos)
- 3D fracture requires **accurate** stress intensity factors (SIFs)
  - ☐ Error at each step ~ (Error on SIF)^4
  - ☐ Standard enrichment => oscillations along the front

Question: How to control accuracy and simplify/avoid meshing?





K. Agathos et al. IJNME 2016, CMAME 2016, IJNME 2017, CMAME 2017 with Eleni Chatzi and Giulio Ventura

How can we use large enrichment radii?
How can we control conditioning in large-scale enriched FEM?
How can we use higher order terms in the expansion?

X. Peng et al. IJNME 2016, CMAME 2017
Enriched Isogeometric Boundary Elements

How to avoid meshing completely for crack propagation simulations?

Don’t worry…

You can get a gradual introduction to the method in the following papers

Agathos K, Ventura G, Chatzi E, Bordas S. Stable 3D XFEM/vector-level sets for non-planar 3D crack propagation and comparison of enrichment schemes. International Journal for Numerical Methods in Engineering. Computational Mechanics, 2017.

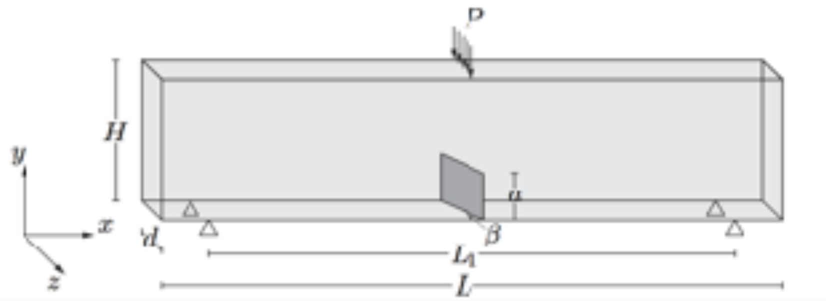
Agathos K, Chatzi E, Bordas S, Talaslidis D. A well–conditioned and optimally convergent XFEM for 3D linear elastic fracture. International Journal for Numerical Methods in Engineering. 2016 Mar 2;105(9):643-77.

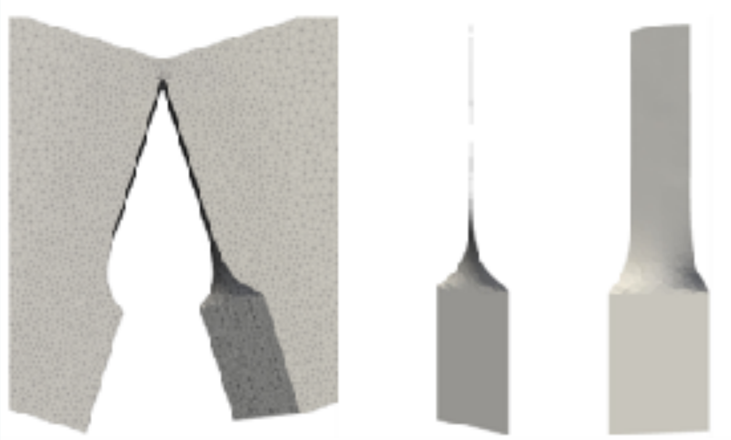
Agathos, K., E. Chatzi, and SPA Bordas. "Stable 3D extended finite elements with higher order enrichment for accurate non planar fracture." *Computer Methods in Applied Mechanics and Engineering* 306 (2016): 19-46.

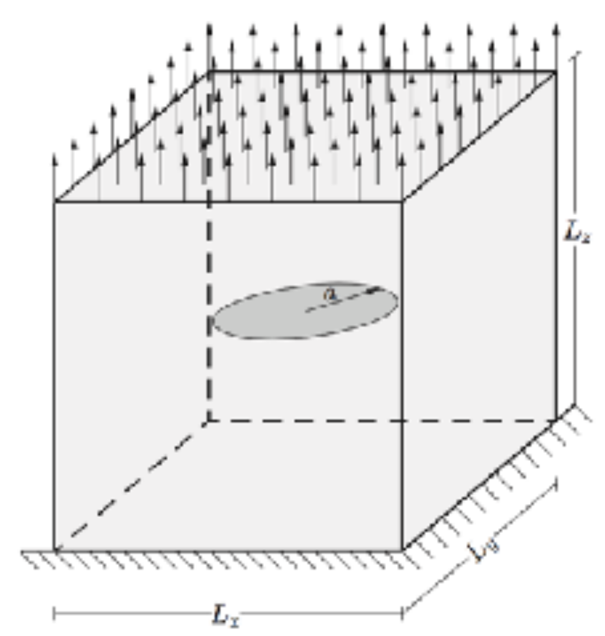
https://orbilu.uni.lu/bitstream/10993/22331/2/paper.pdf

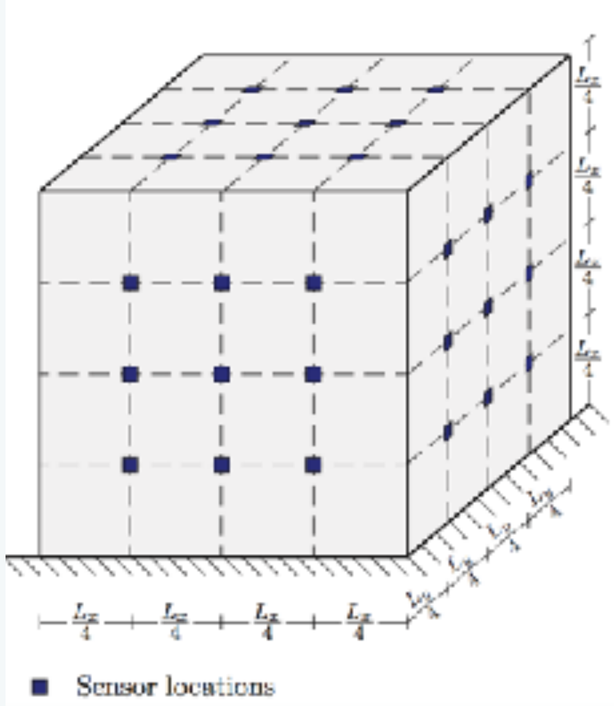
http://orbilu.uni.lu/bitstream/10993/22420/1/presentation.pdf

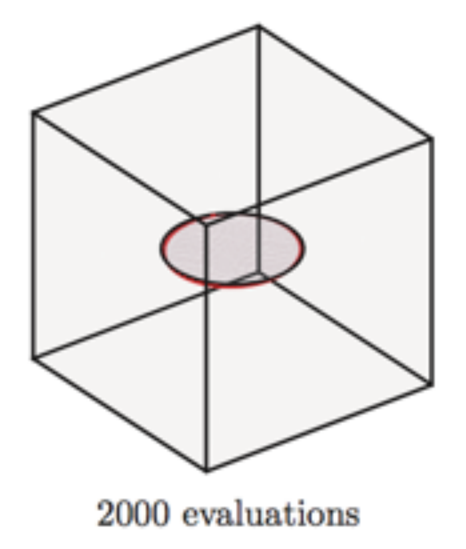
Conclusions from Kostas Agathos' work











2000 evaluations

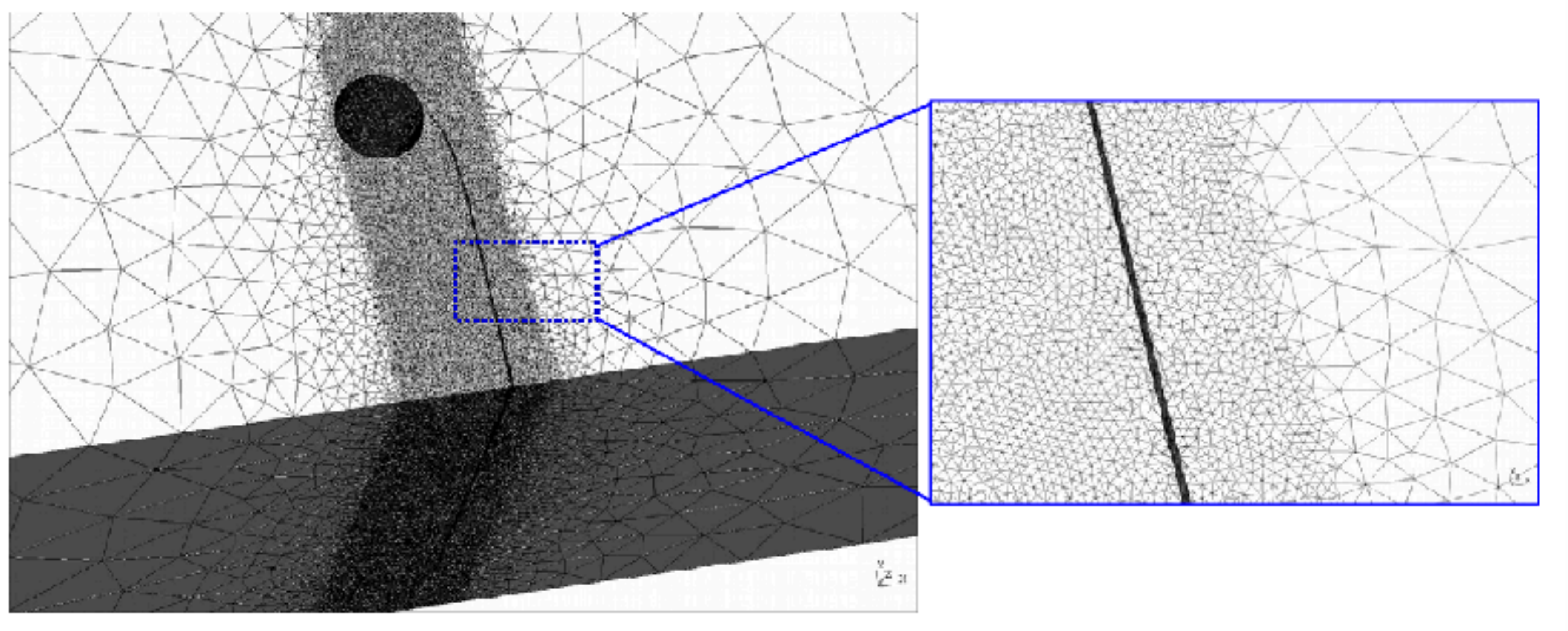
- ☑ Introduces a novel form of enrichment.
- ☑ Provides improved conditioning.
- ☑ Enables the use of geometrical enrichment.
- ☑ Enables the use of higher order terms in fracture mechanics
- ☑ Was combined to vector level sets to solve crack propagation problems.
- ☑ Was applied to inverse problems.
- ☑ Provides high accuracy and optimal convergence.

Conclusion: we can now add arbitrary numbers of enrichments and enrich over 'large' volumes of the domain.

What if you can’t add new functions or you don’t want to increase the enrichment radius?

Motivation

(Goal oriented) adaptive computational fracture use h-refinement



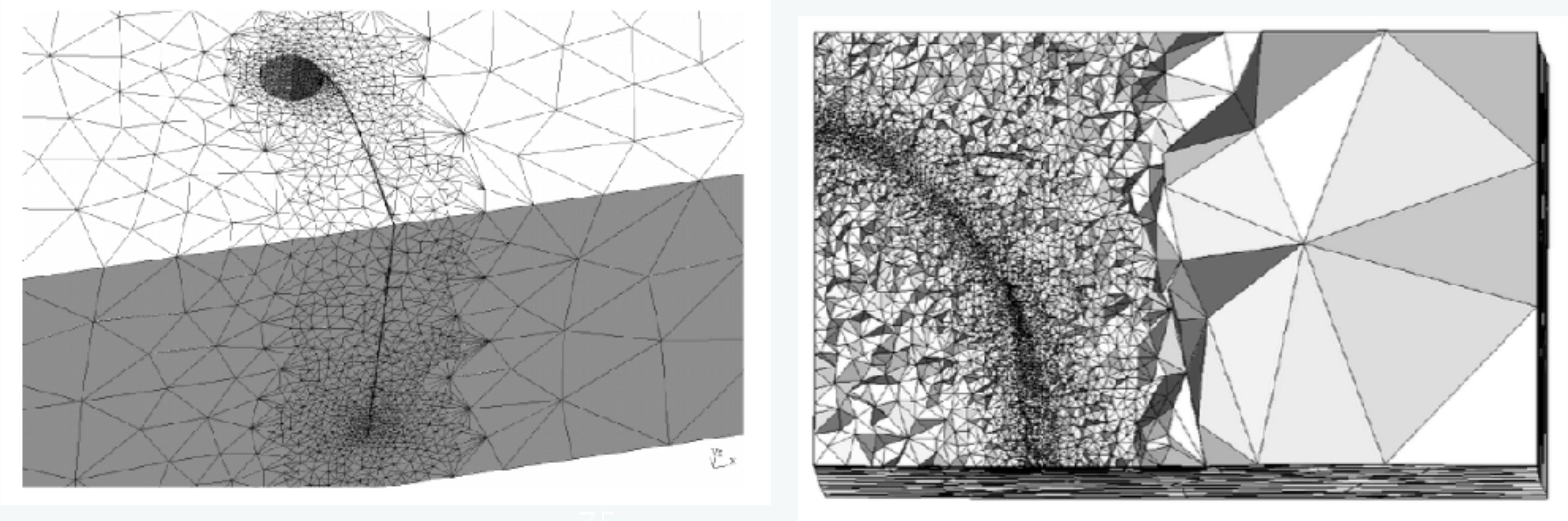
Before: mesh "finely" in the region where the crack is "expected" to propagate

Y. Jin, O. Pierard, et al. Comput. Methods Appl. Mech. Engrg. 318 (2017) 319–348
O.A. González-Estrada et al. Computers and Structures 152 (2015) 1–10
O.A. González-Estrada et al. Comput Mech (2014) 53:957–976
C. Prange et al. IJNME 91.13 (2012): 1459-1474.
M. Duflot, SPAB, IJNME 2007, CNME 2007, IJNME 2008.
J-J. Ródenas Garcia, IJNME 2007
F.B. Barros, et allJNME 60.14 (2004): 2373-2398.

M. Rüter CMECH (2013) 1;52(2):361-76.
J. Panetier IJNME 81.6 (2010): 671-700.
P. Hild, CMECH (2010): 1-28.

Motivation

Fracture of homogeneous materials: error estimation and adaptivity



After: determine mesh refinement adaptively using a (goal-oriented) error estimate

Y. Jin, O. Pierard, et al. Error-controlled adaptive extended finite element method for 3D linear elastic crack propagation Comput. Methods Appl. Mech. Engrg. 318 (2017) 319–348
M. Duflot, SPAB, IJNME 2007, CNME 2007, IJNME 2008.

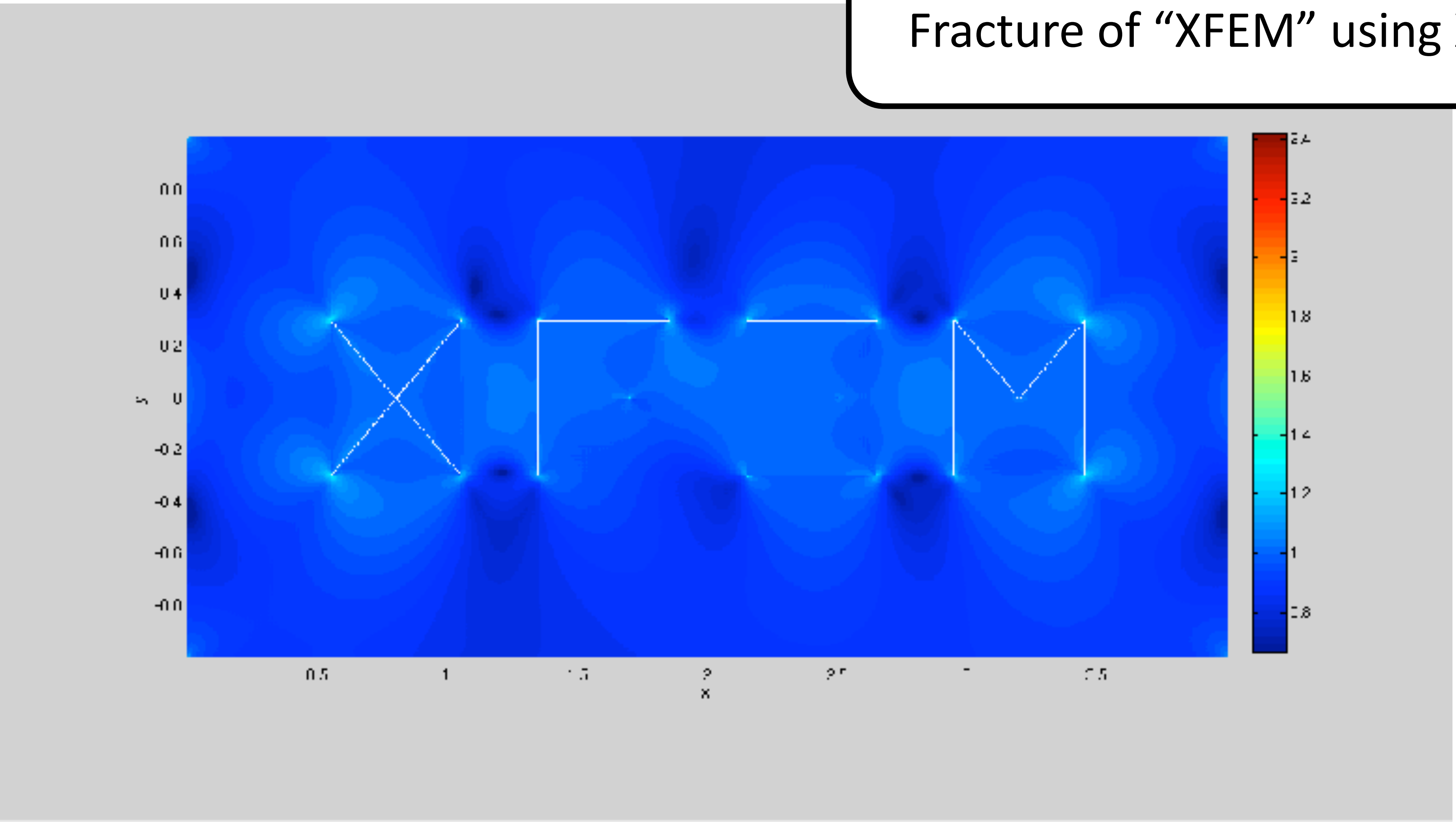
Partial Conclusions

- FEM has intrinsic difficulties with singularities and discontinuities
- Enrichment helps to decrease but not eliminate remeshing
- This remeshing can be driven by error estimates
- Arbitrary enrichment functions can be chosen
- (almost) arbitrary enrichment zones
- **Question**: what are the limitations of these enrichment approaches?

What if we have to deal with more cracks….

Discretization: XFEM

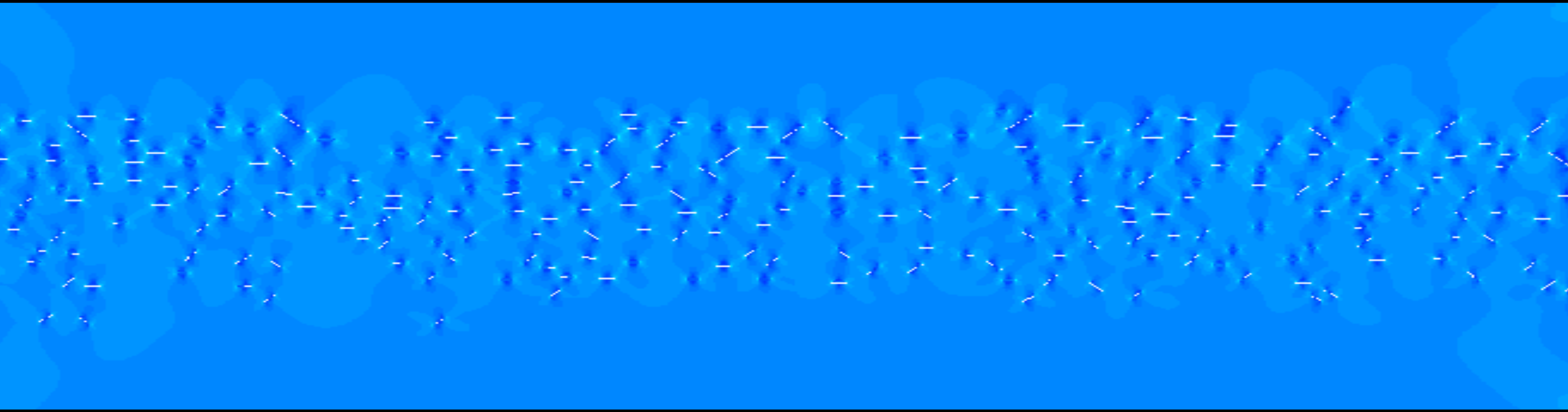
Extended Finite Element Method (XFEM)



Fracture of “XFEM” using XFEM

Case study II. Plate with 300 cracks vertical extension BCs

Energy-minimal crack growth using XFEM

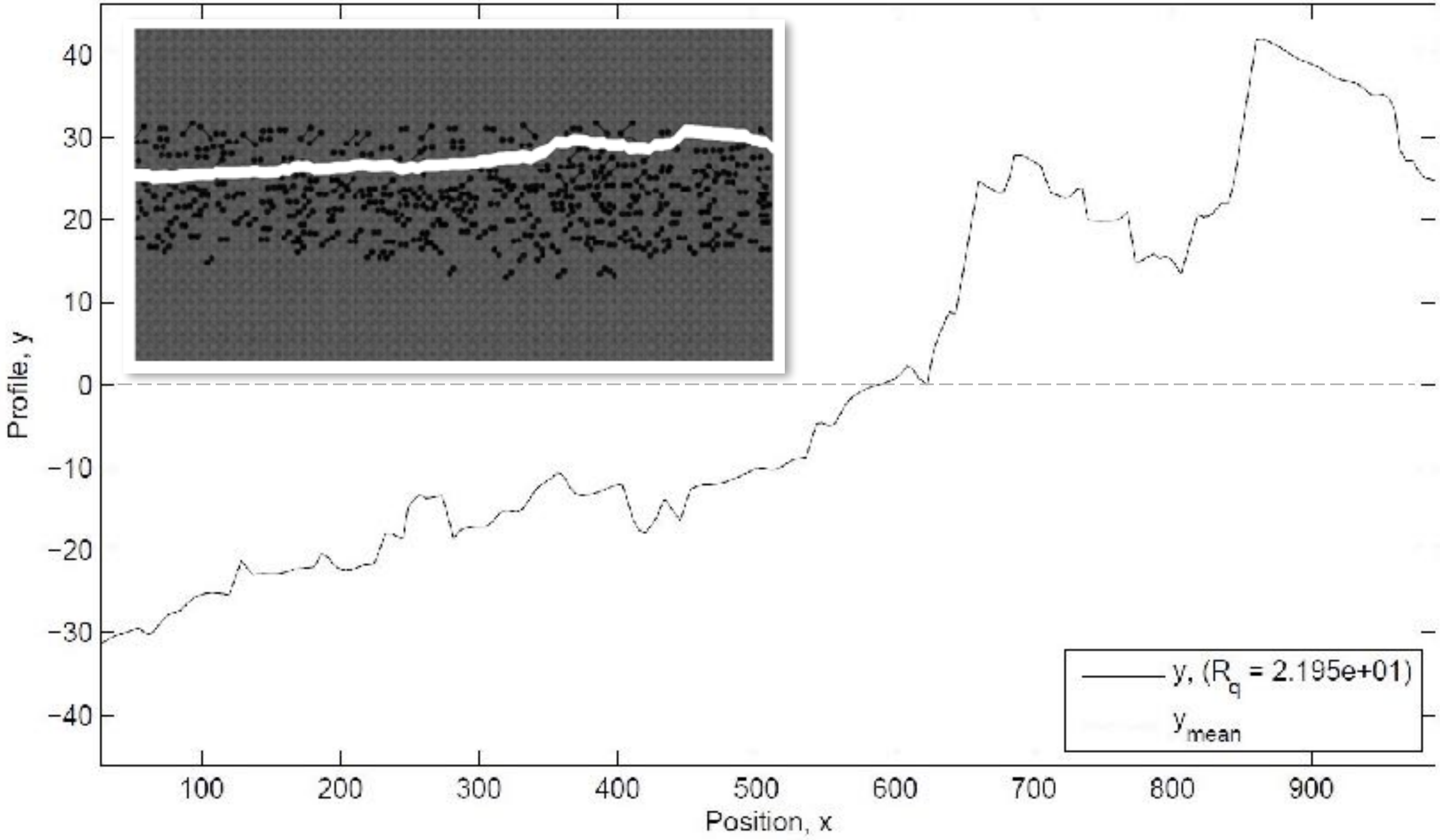


Sutula et al. Preprint of three part EFM paper at http://hdl.handle.net/10993/29414

Example #1

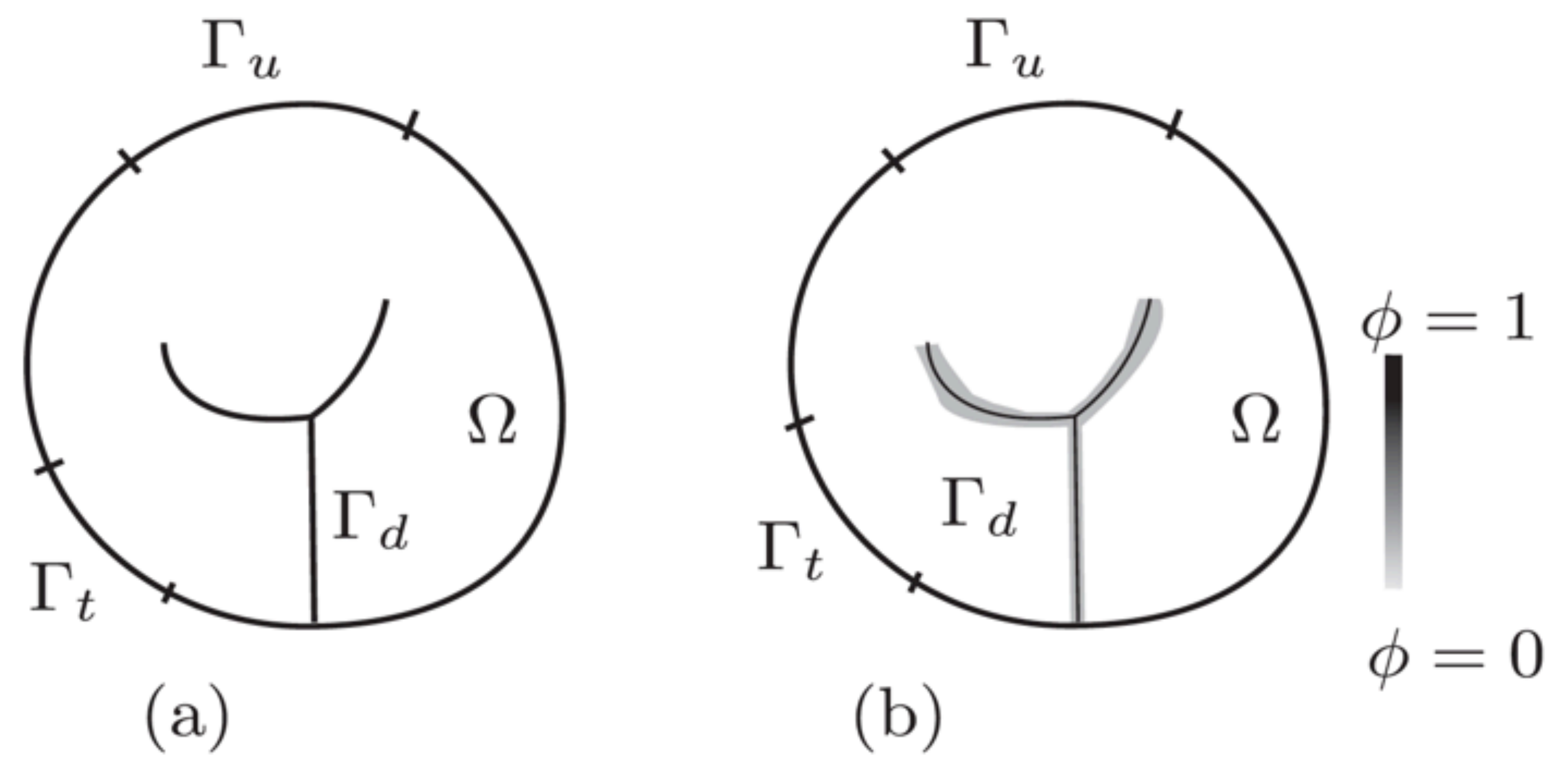
Vertical extension of a plate with 300 cracks

Post-split roughness



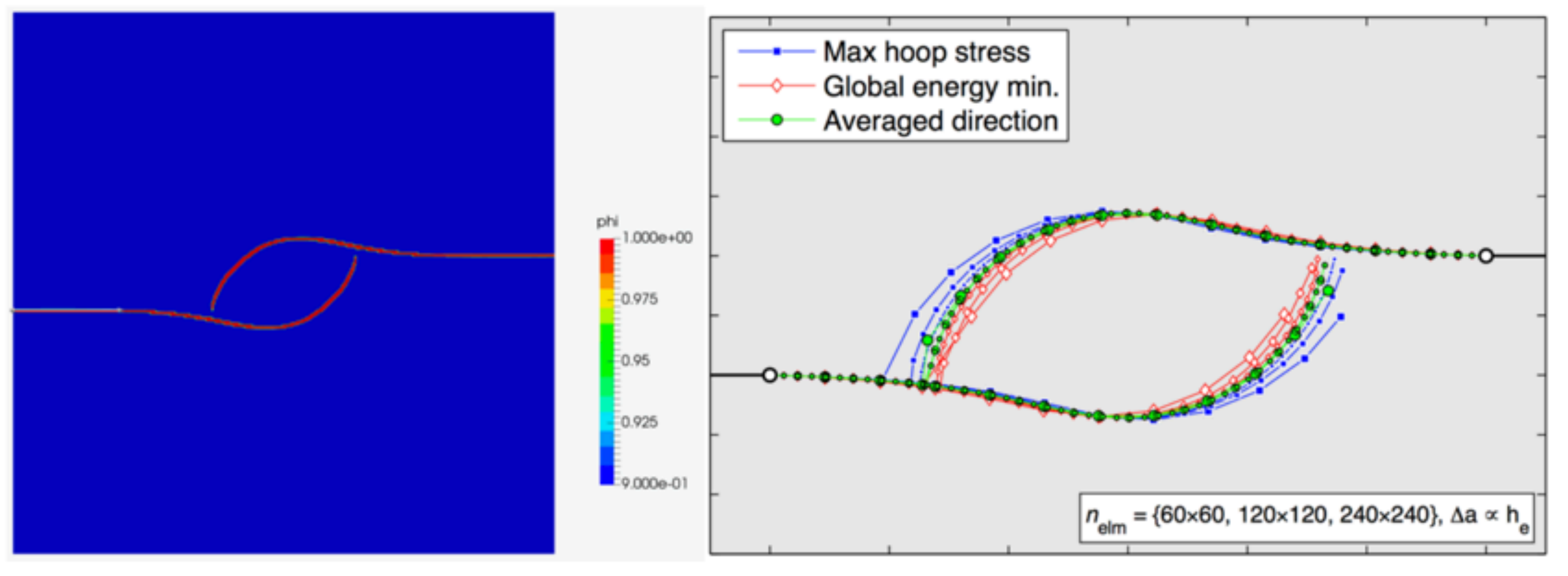
More cracks?… 3D? …

Phase field/thick level sets



With Danas Sutula and Nguyen Vinh Phu (Monash)
9TH Australasian Congress on Applied Mechanics (ACAM9)
27 - 29 November 2017
phu.nguyen@monash.edu

Energy minimal XFEM vs. Phase field



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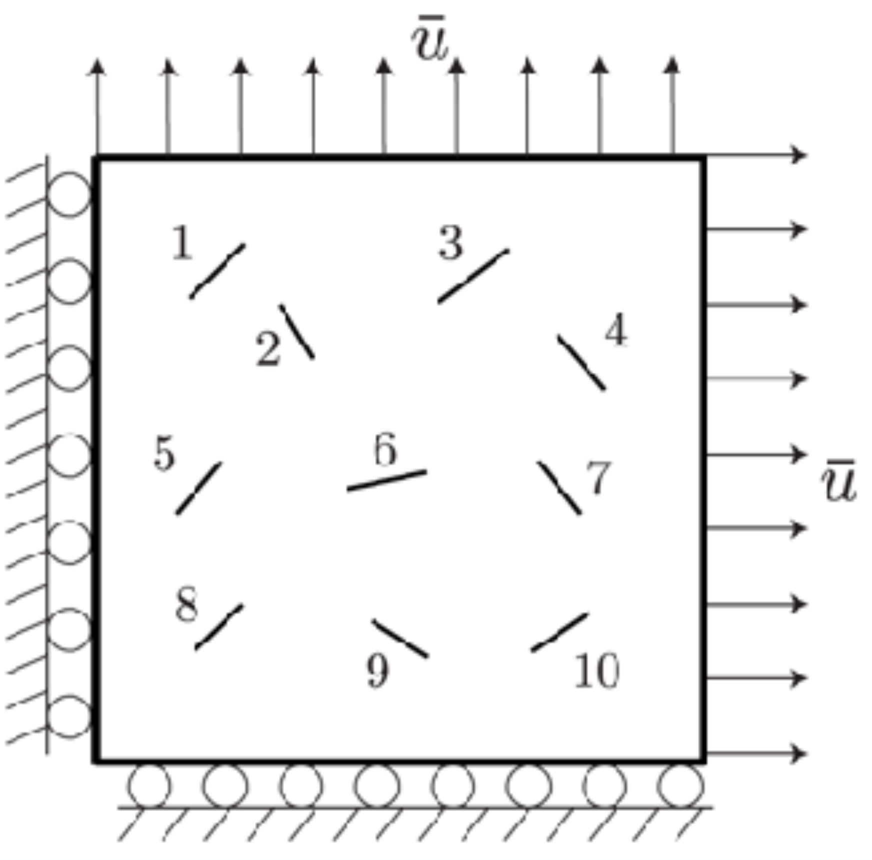
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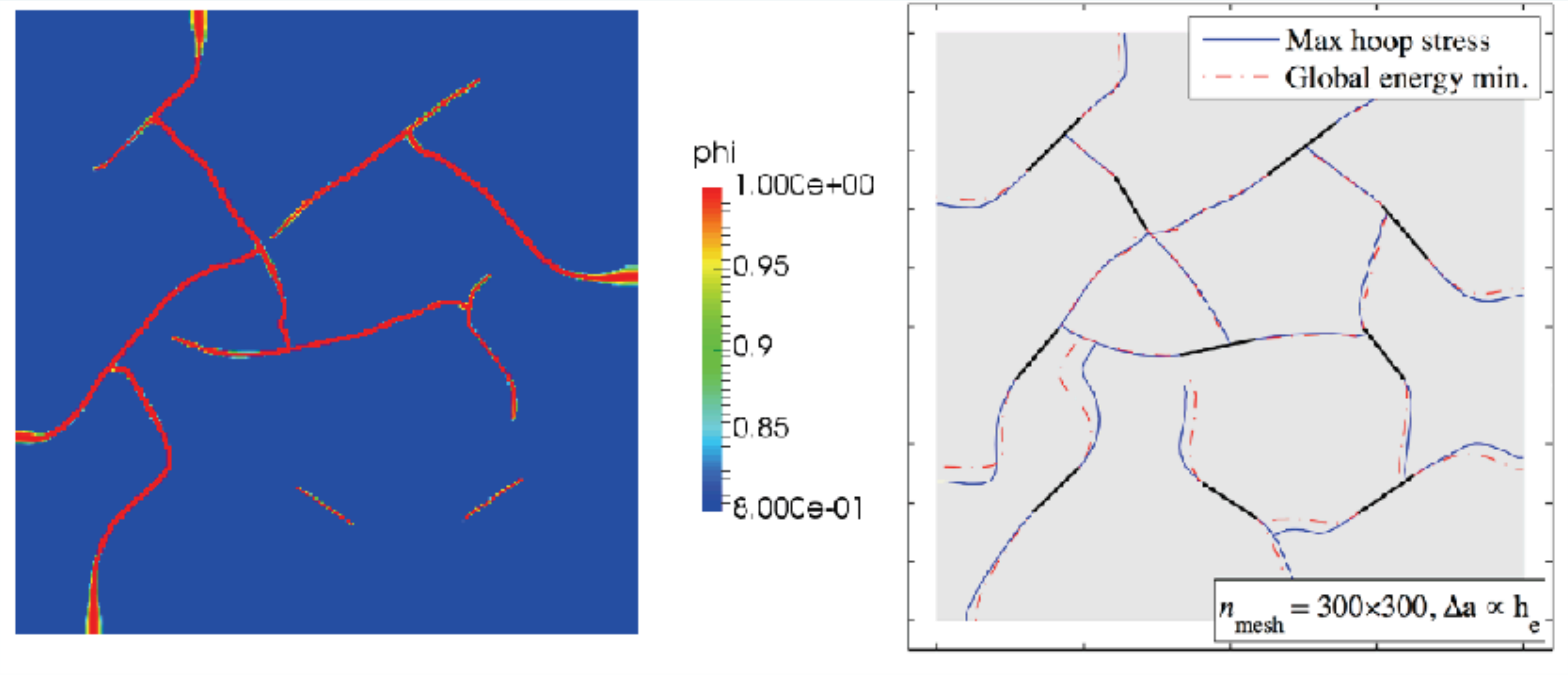
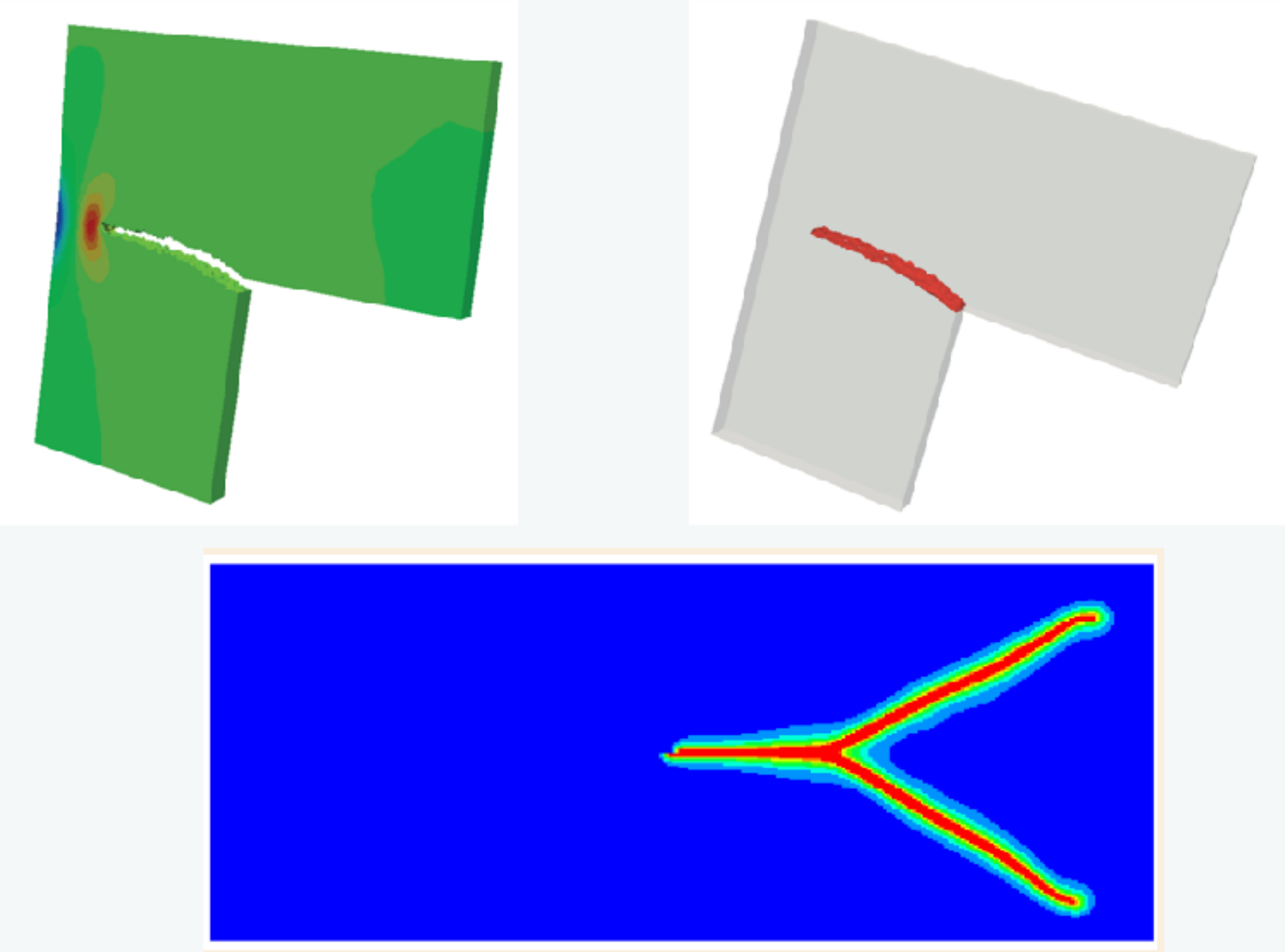

phi
1.000e+00
0.95
0.9
0.85
8.000e-01
Max hoop stress
Global energy min.
n_{mesh} = 300×300, Δa ∝ h_e



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phu.nguyen@monash.edu

Partial conclusions on fracture of homogeneous materials using enriched FEM

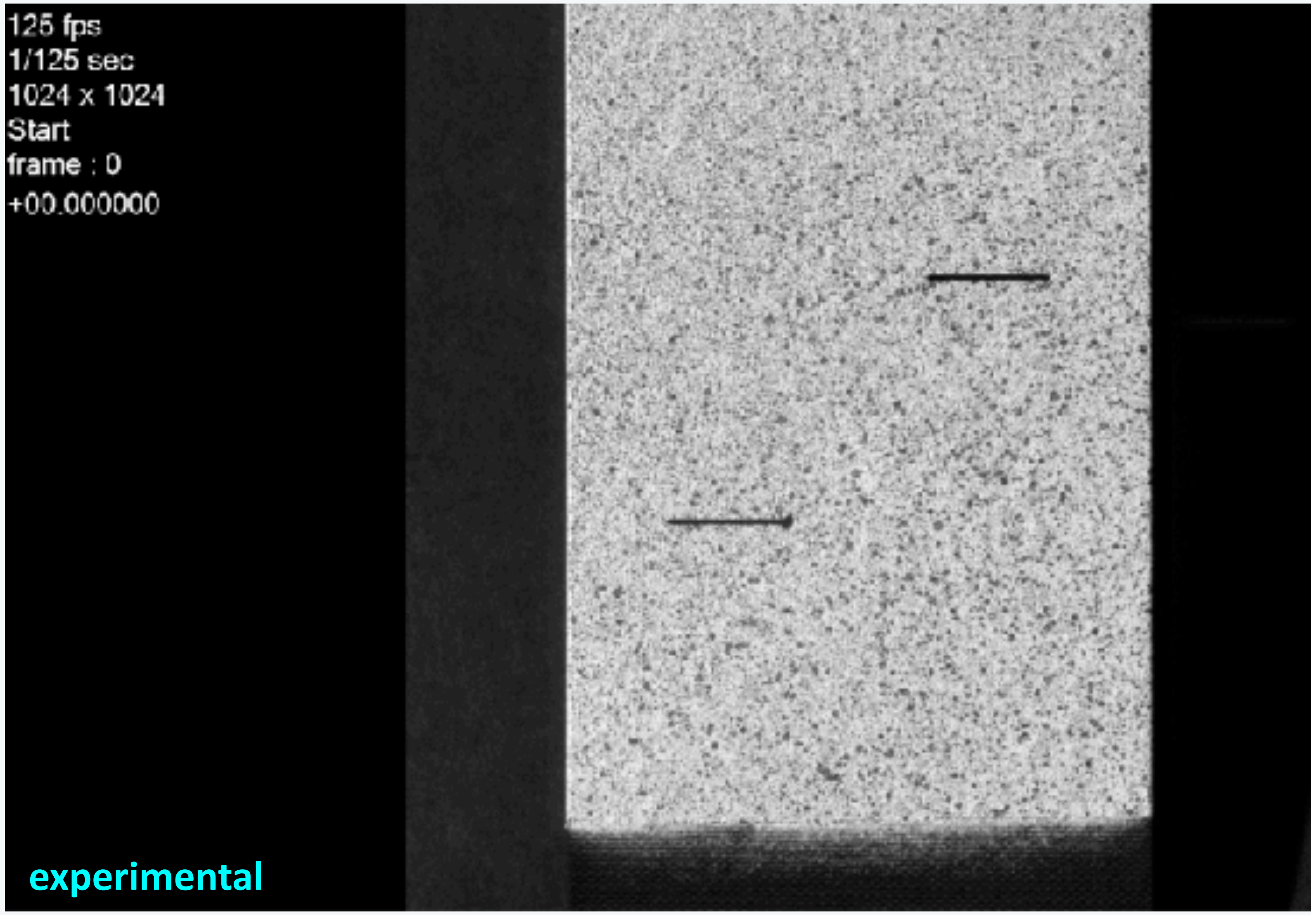
- More than a few cracks in 3D may warrant using phase fields models as opposed to discrete cracks
- Meshfree methods are possible alternatives (See the work of Rabczuk, Belytschko, Zi, SPAB)

Question: how to handle heterogeneities over the scales in computational fracture ?

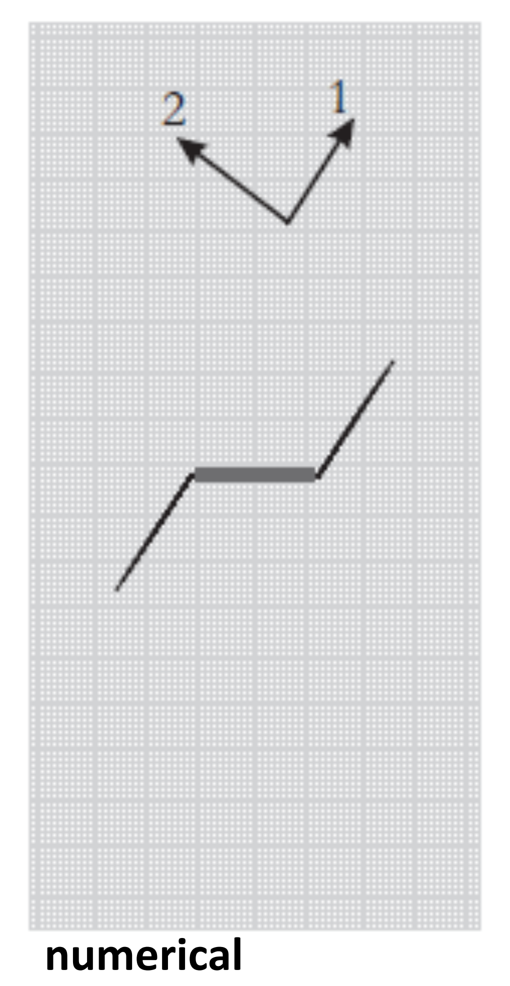
Case study III: Fracture of heterogeneous materials

Mild heterogeneities/anisotropy
Homogenized models are sufficient

Question: what main factors govern crack growth in composite laminates?



experimental



numerical

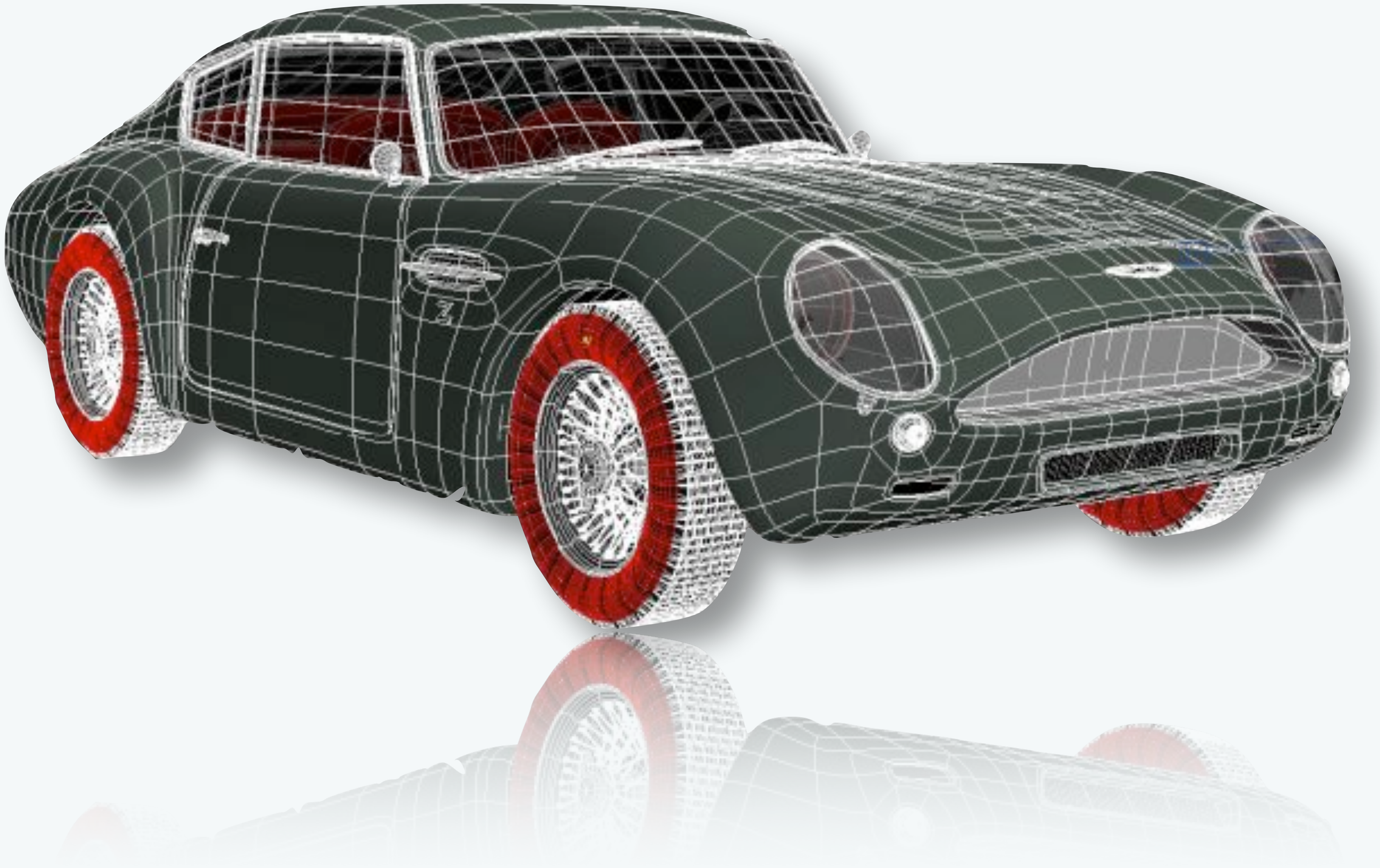
experimental

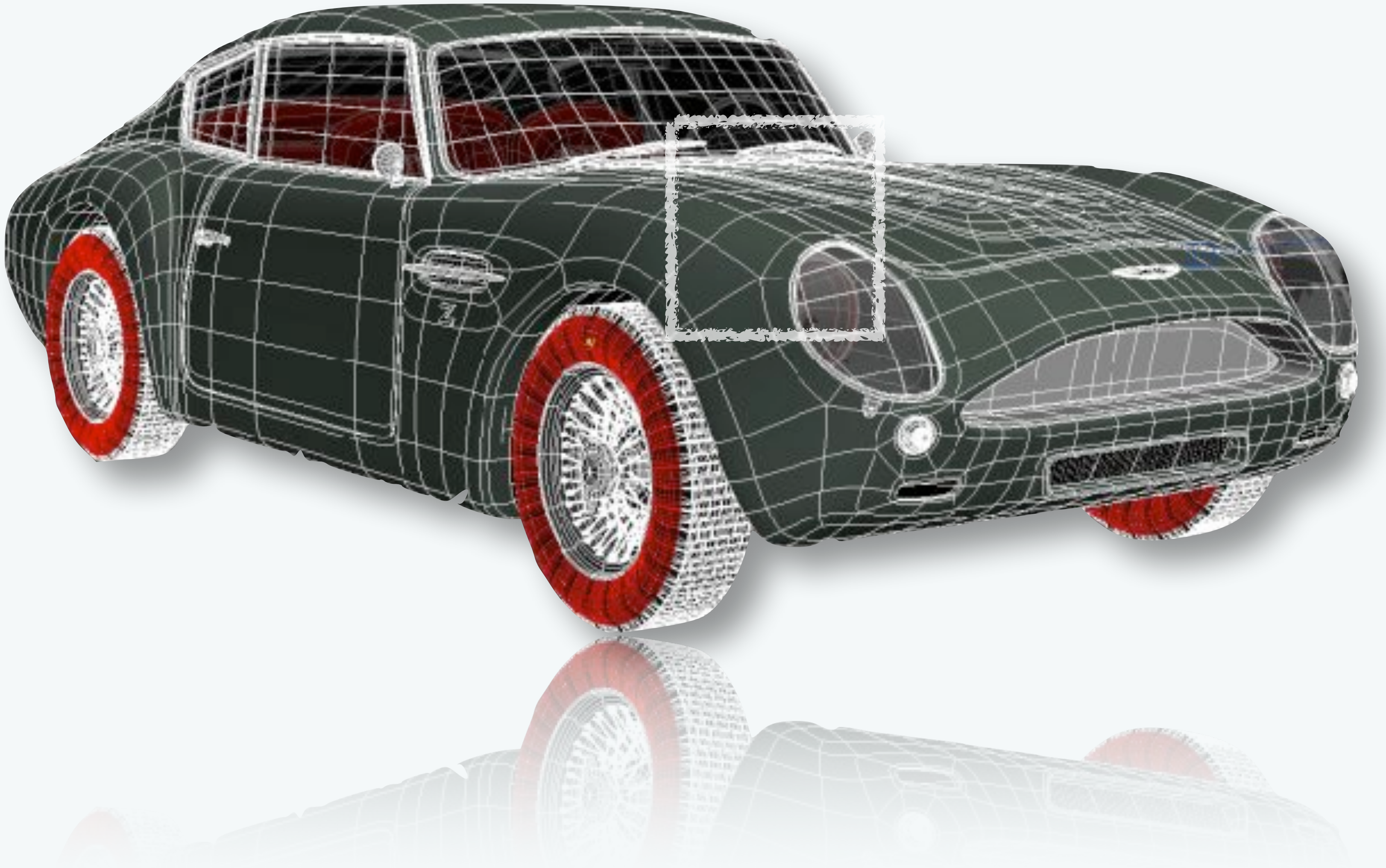
L. Cahill et al. Composite Structures, 2014

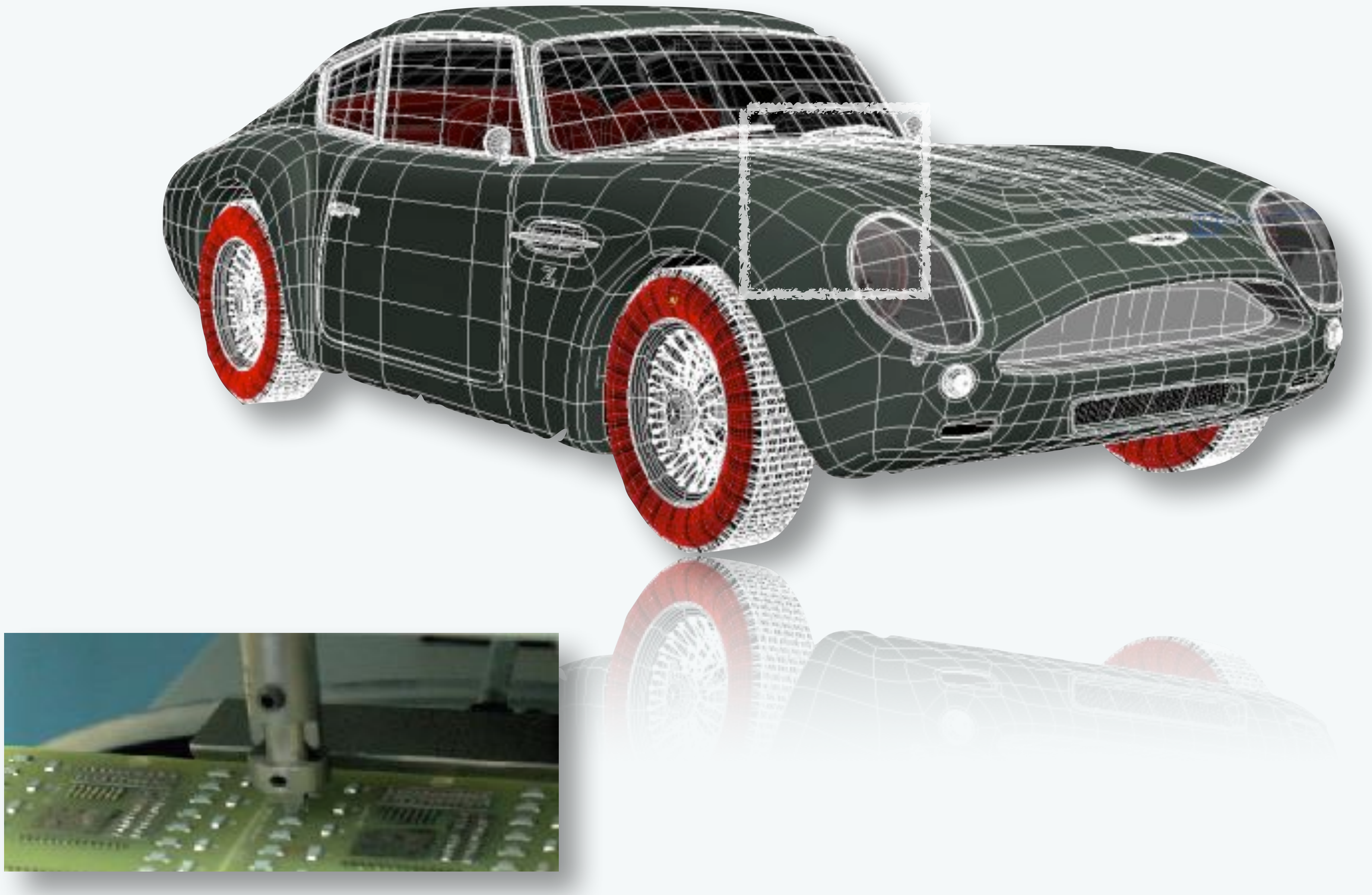
Experimental/Numerical approach to determining the driving force for fracture in composites

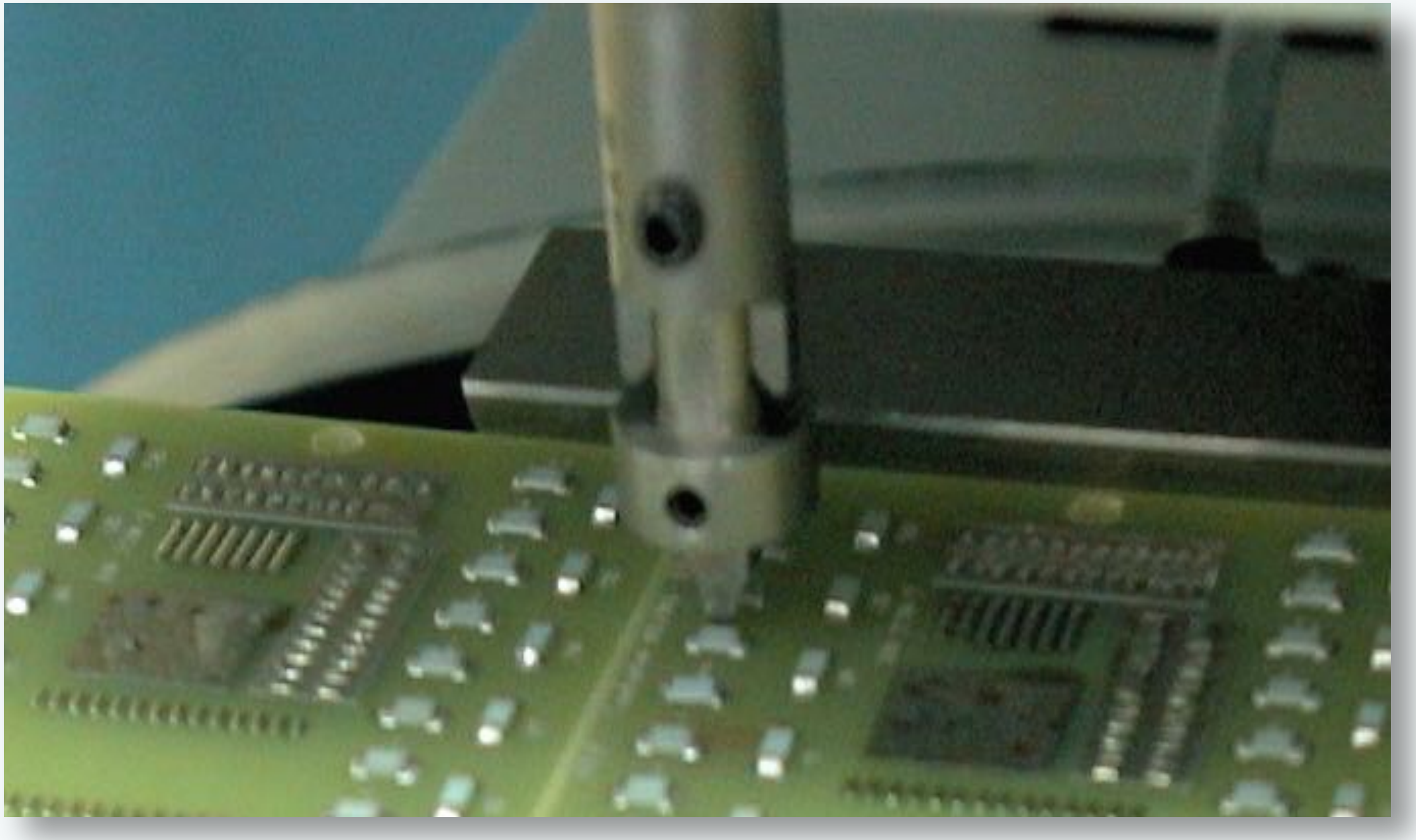
XFEM can effectively deal with orthotropic fracture

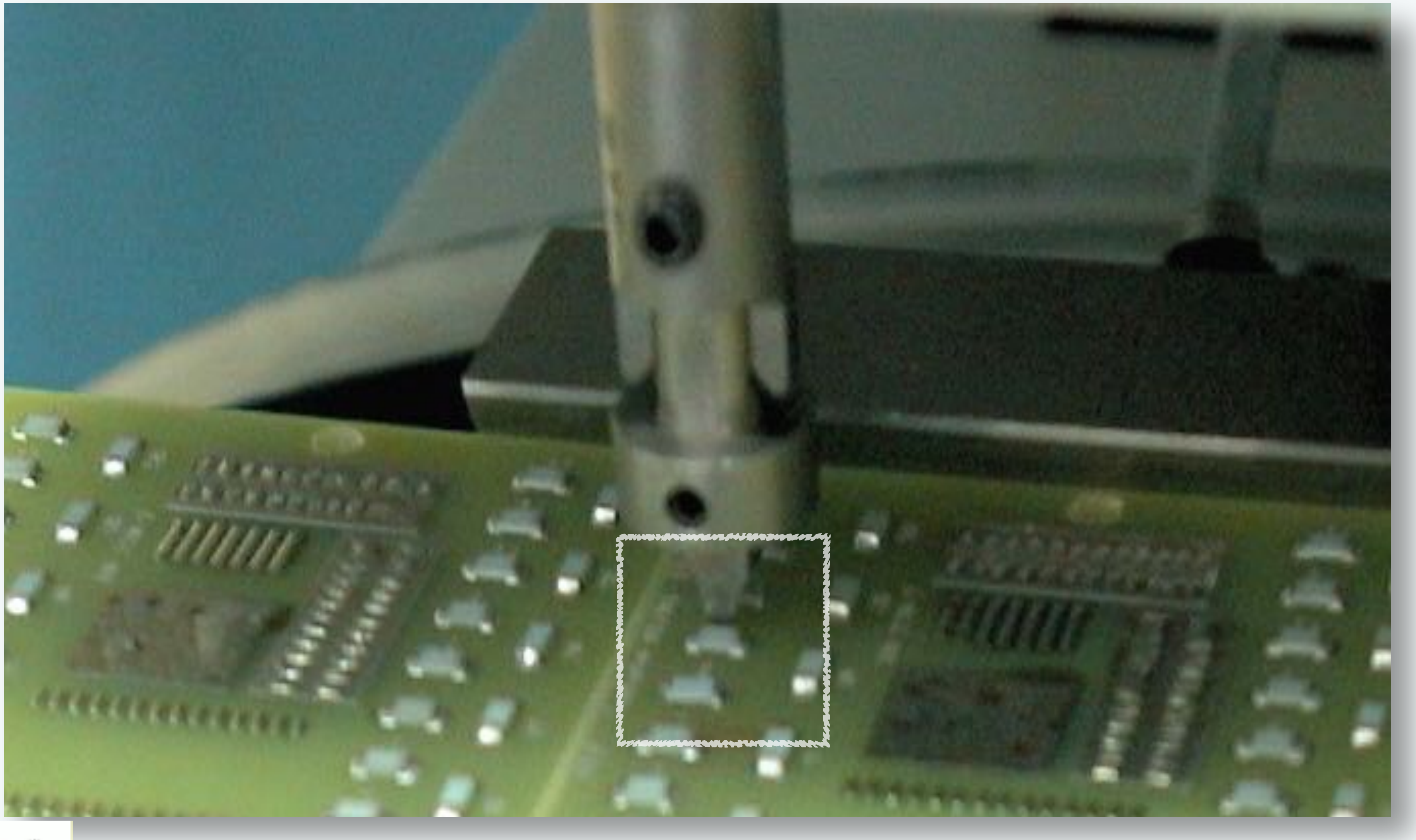
Strong heterogeneities
Homogenized models are insufficient

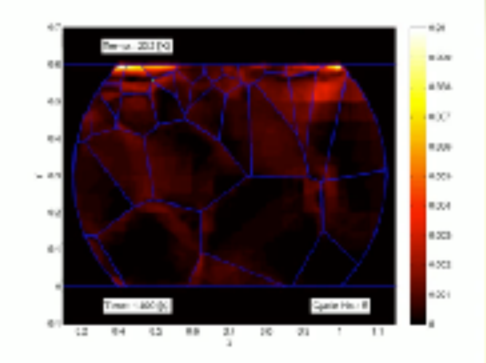


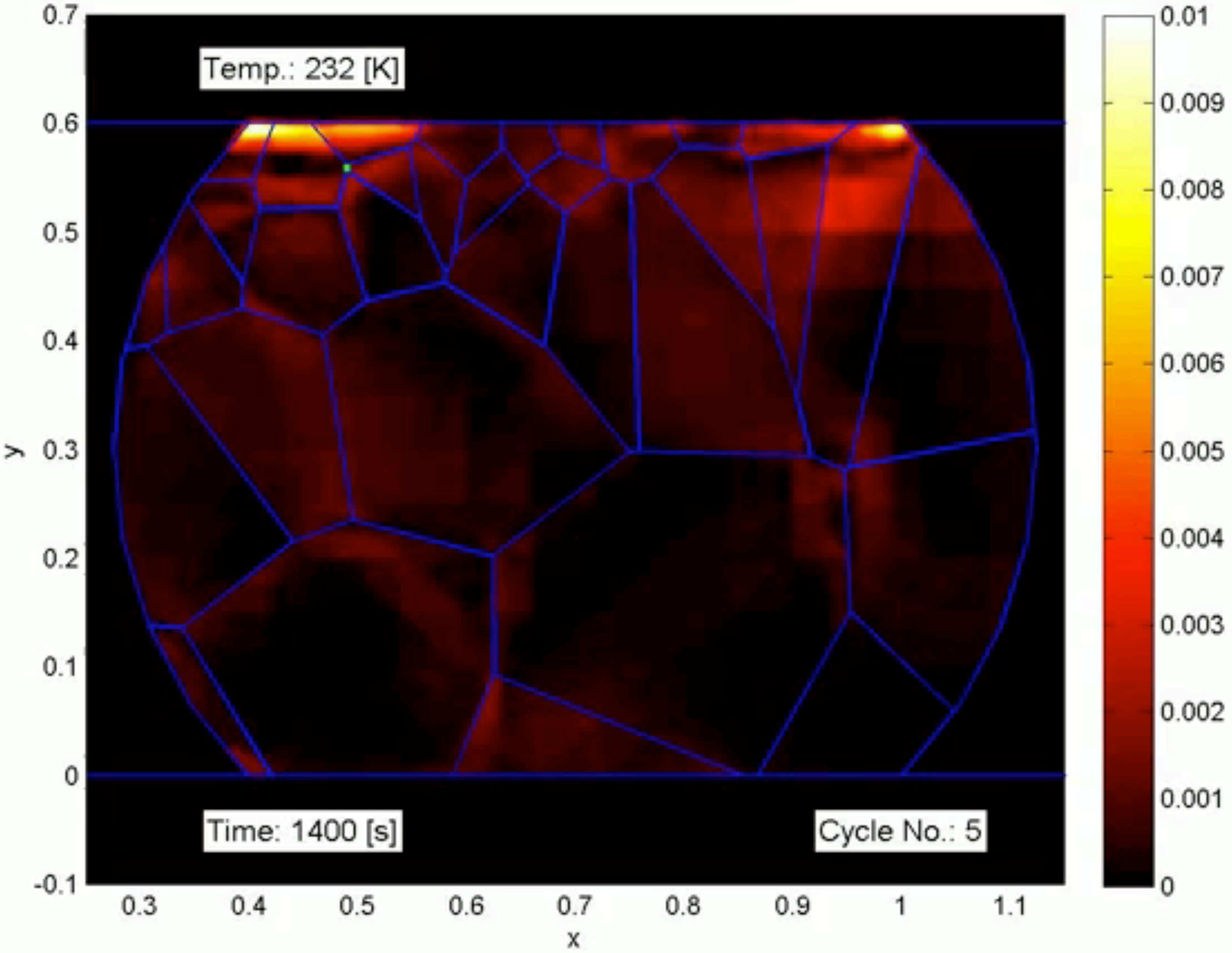






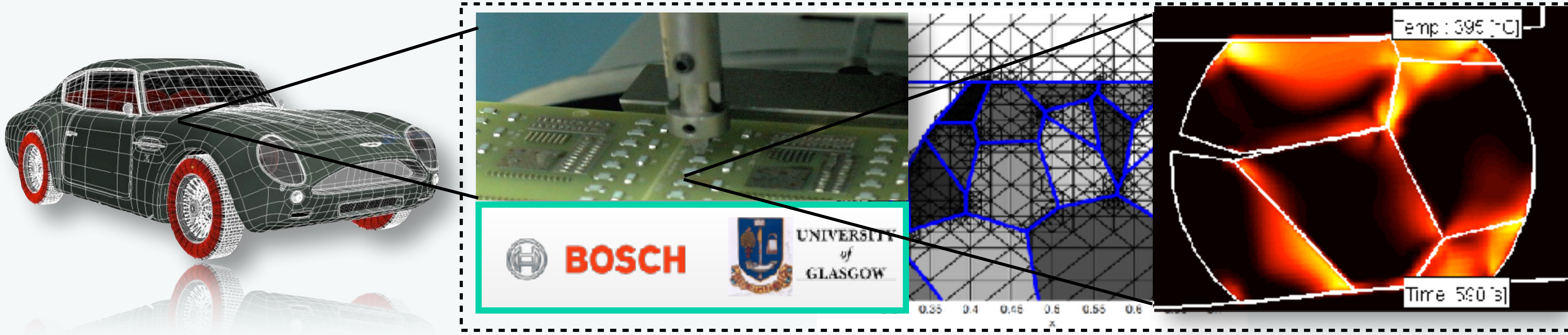





Temp.: 232 [K]
Time: 1400 [s]
Cycle No.: 5
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
y
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
x
0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0

Fracture over the scales

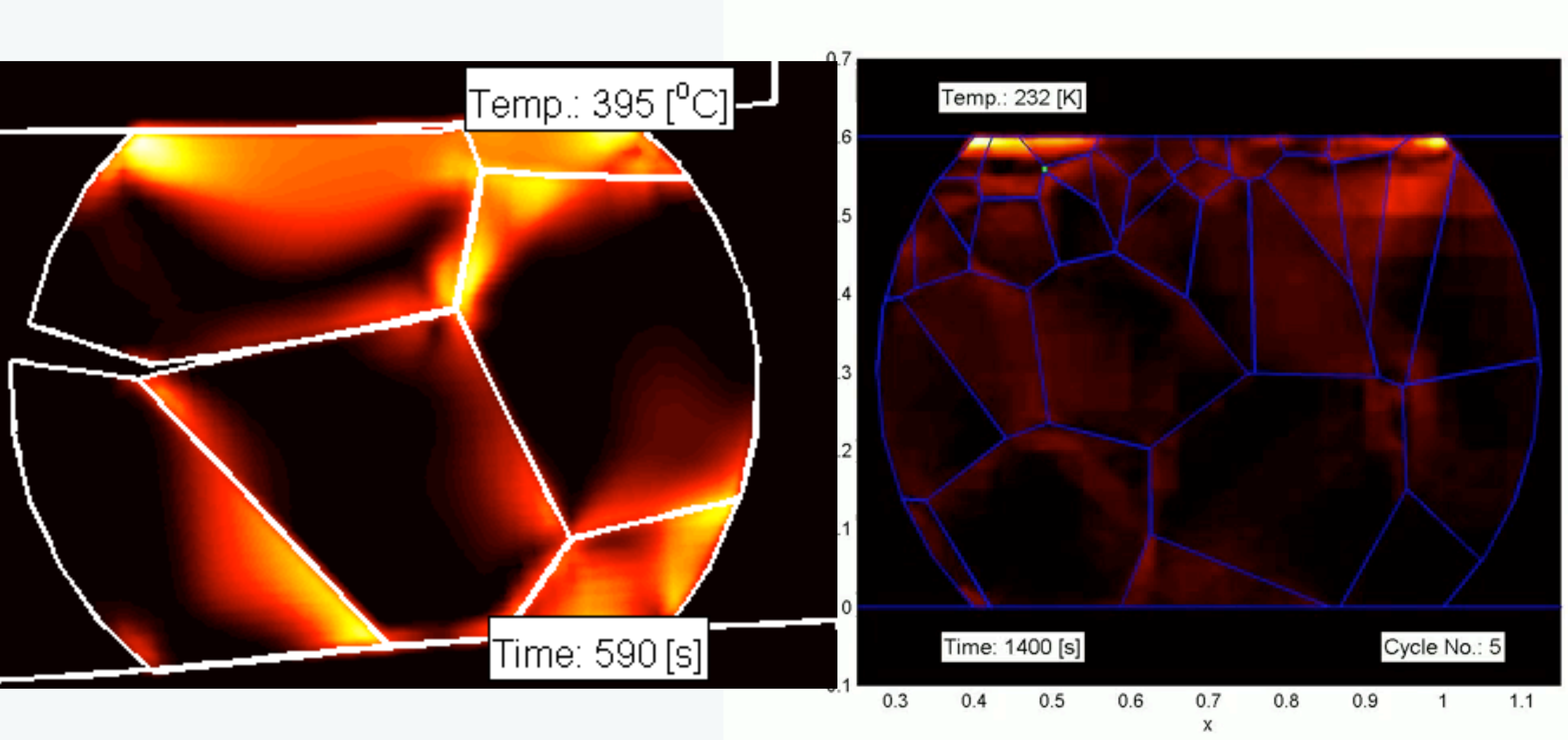
Solder joint durability (microelectronics), Bosch GmbH



Question: what is the role of Pb in thermo-mechanical reliability of solder joints?

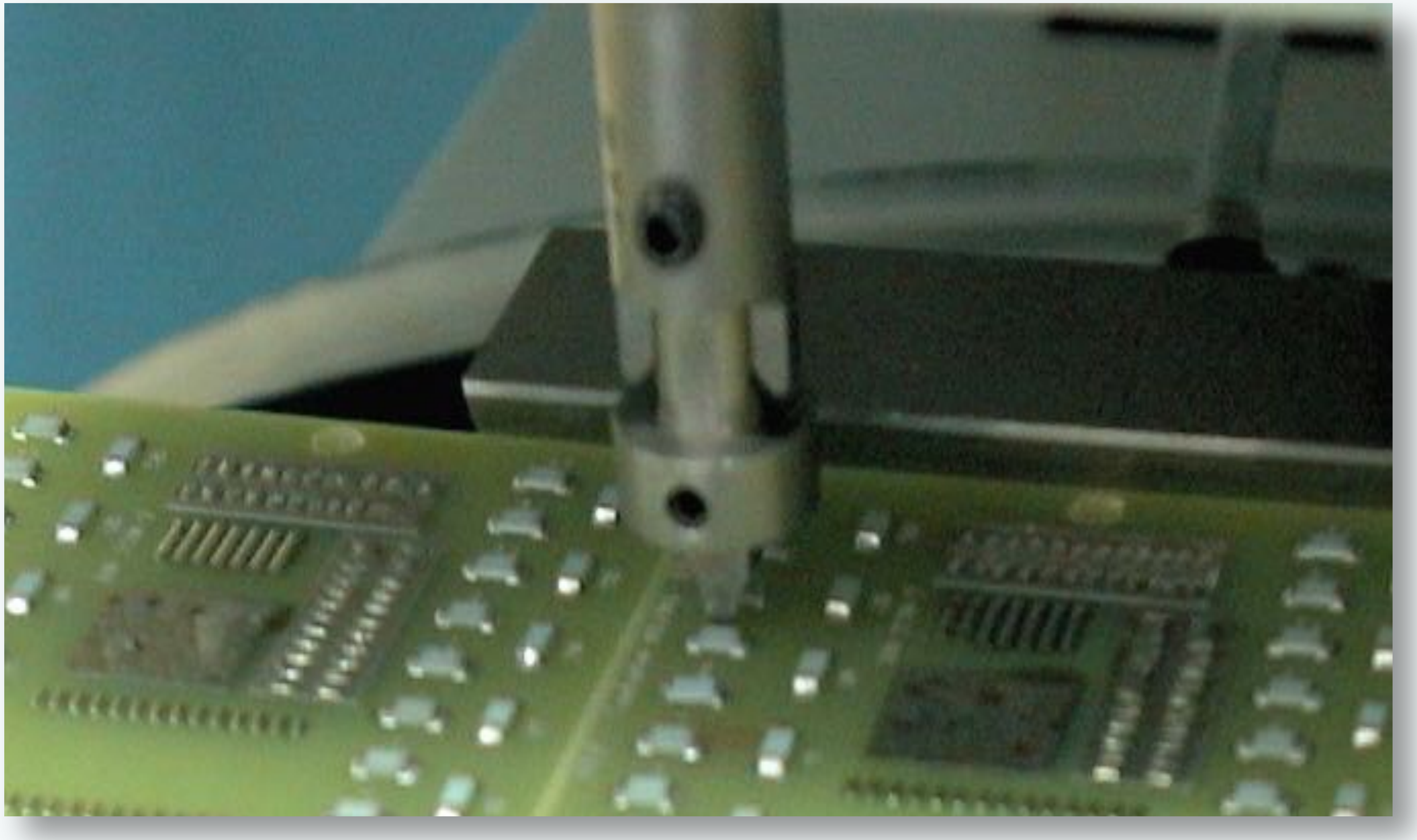
A. Menk and SPAB, IJNME 2011, Comp. Mat. Sci. 2012
XFEM Preconditioning and application to polycrystalline fracture

D. A. Paladim et al. Int. J. Numer. Meth. Engng 2017; 110:103–132
P. Kerfriden et al. Int. J. Numer. Meth. Engng 2014; 97:395–422
P. Kerfriden et al. Int. J. Numer. Meth. Engng 2012; 89:154–179
P. Kerfriden et al. Comput. Methods Appl. Mech. Engrg. 200 (2011) 850–866
K. C. Hoang et al. Num Meth PDEs DOI 10.1002/num.21932

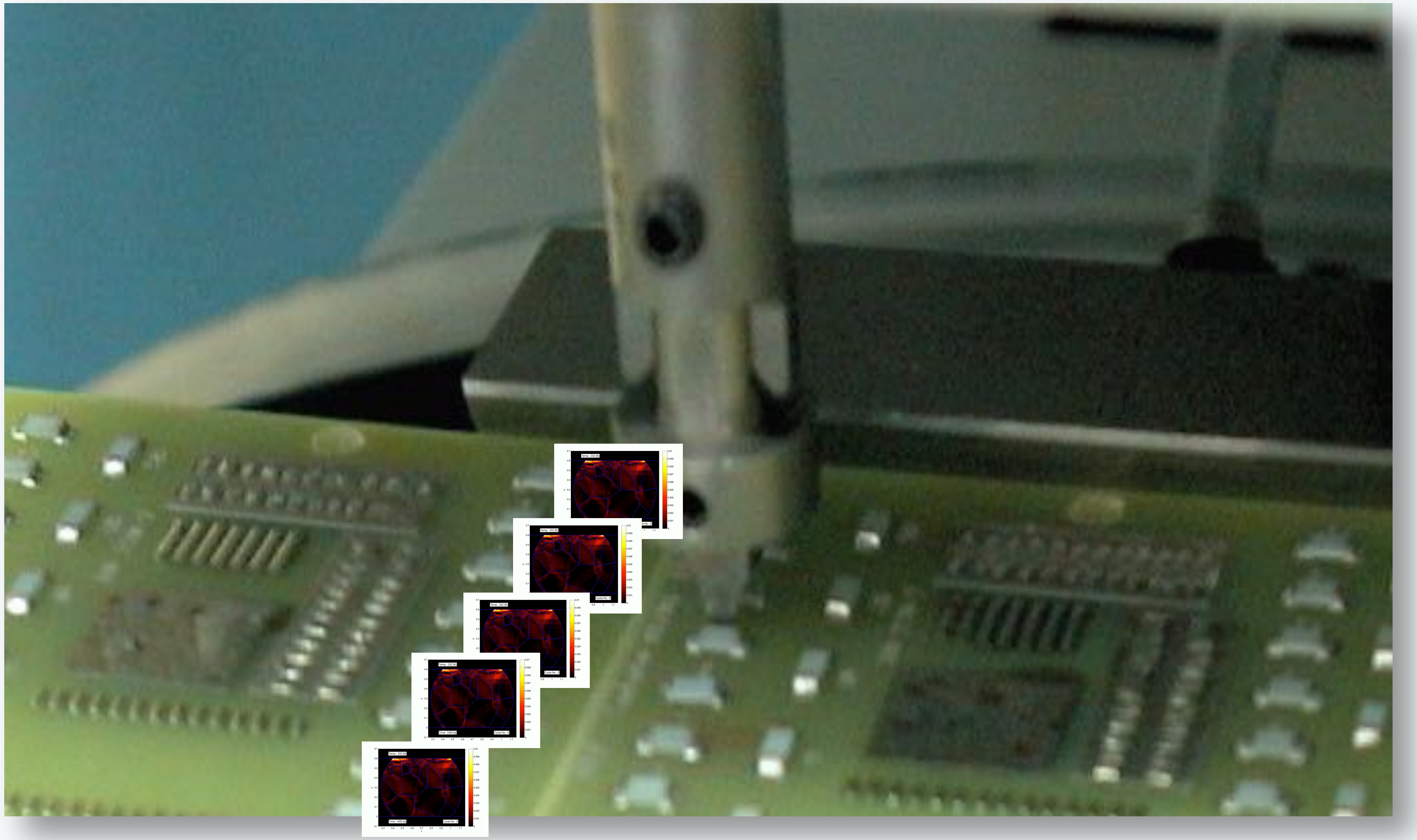


Microstructure plays a major role in thermomechanical durability in Pb-free solders

Microstructures have a critical effect on the durability of structures at the engineering scale



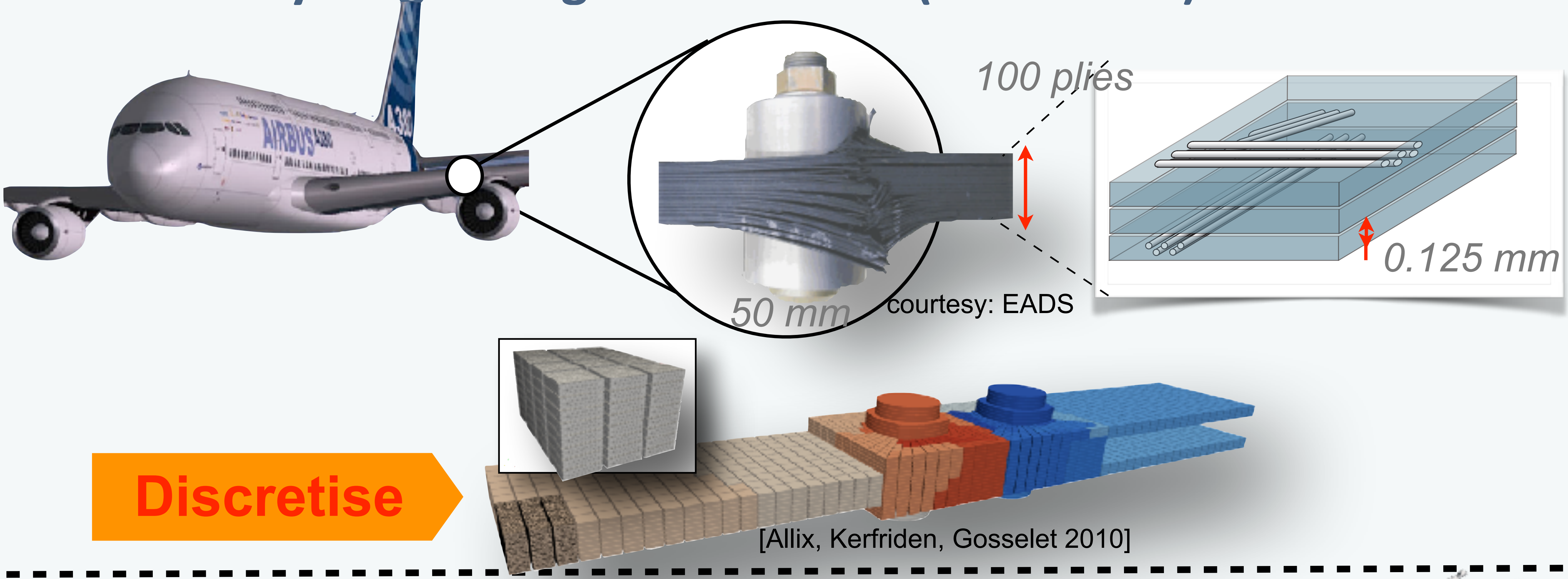
What can be done to account for microstructures for structures of engineering relevance?



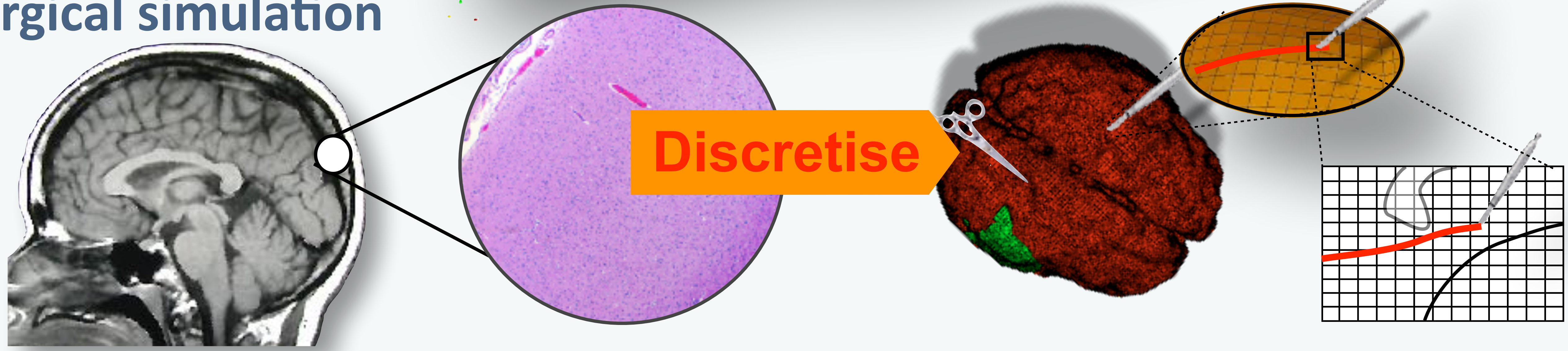
All is fine as long as the microstructures simulations are localised or few in number

Interfaces in engineering and biomechanics

Practical early-stage design simulations (interactive)

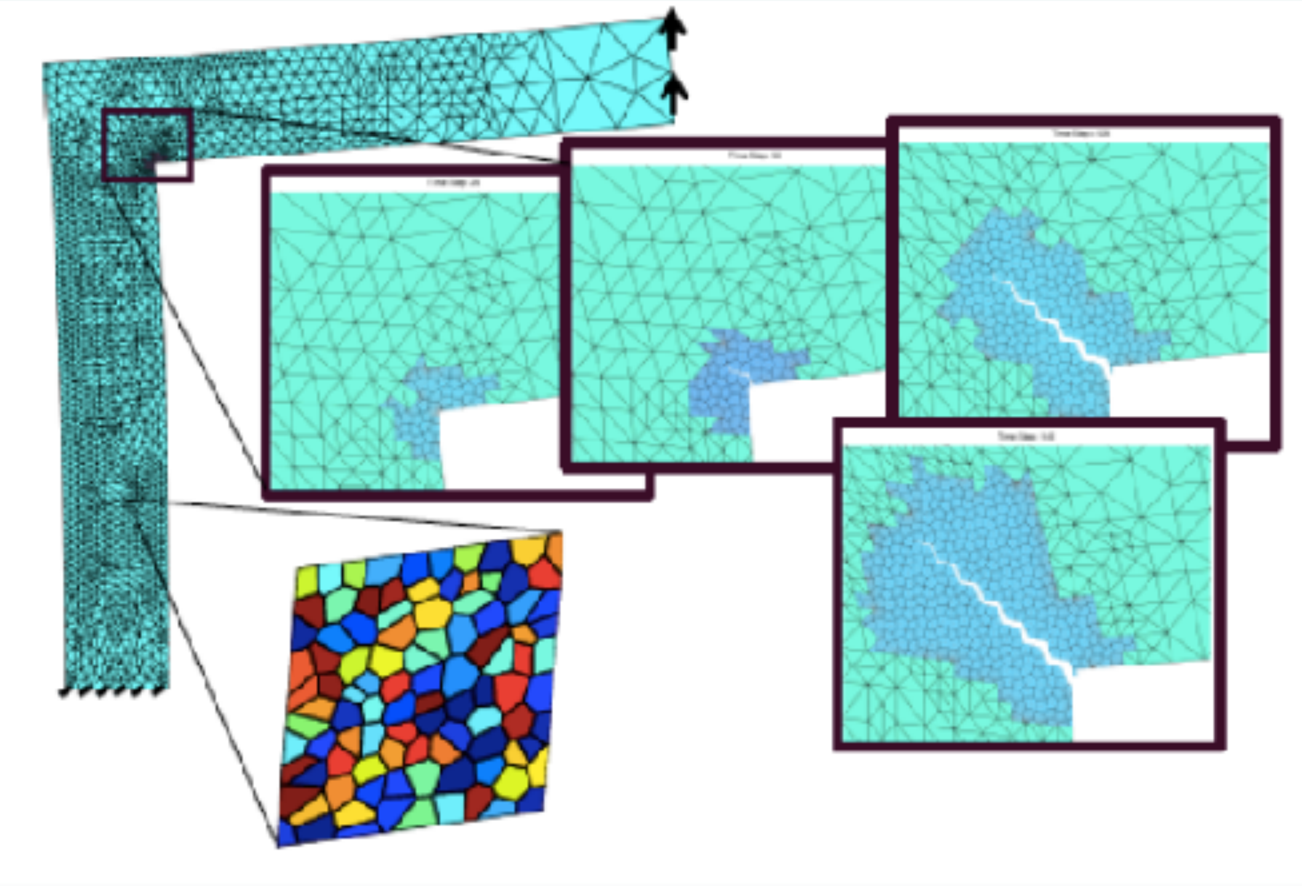


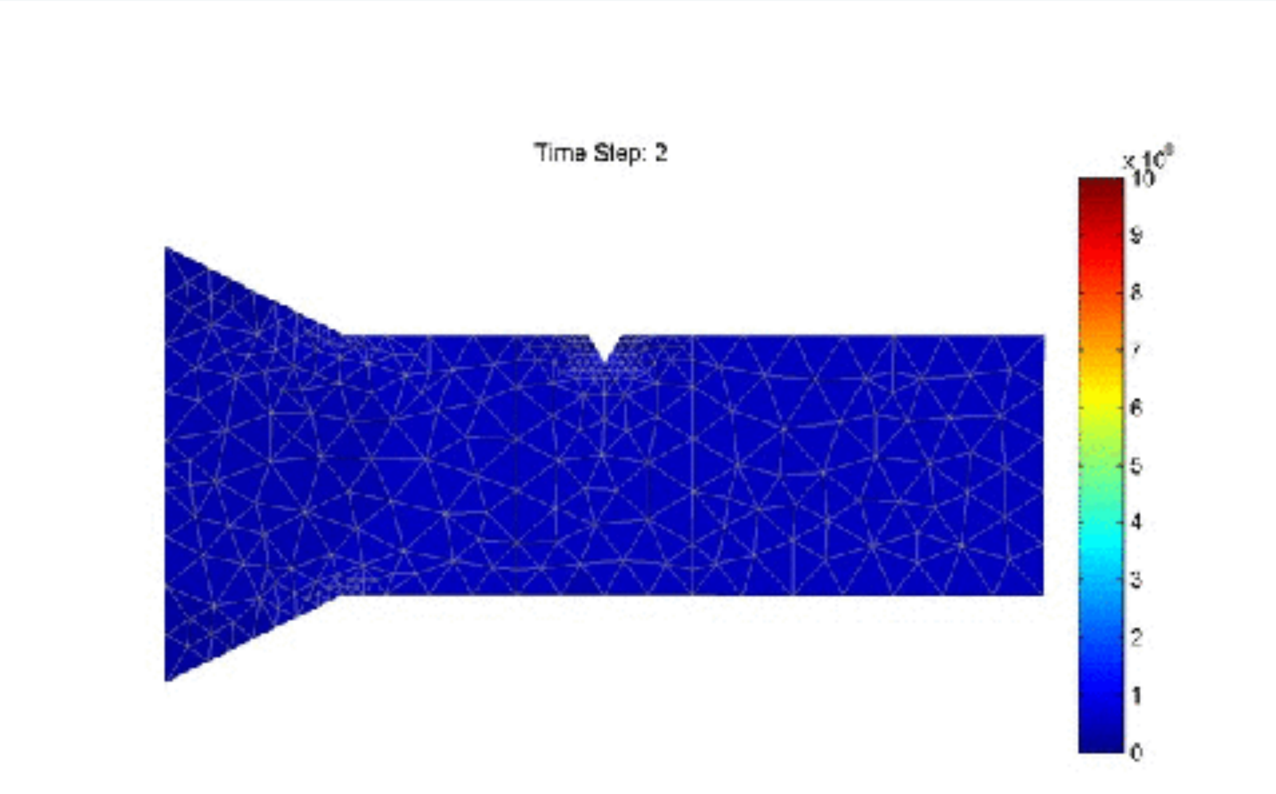
Surgical simulation



- Reduce the problem size while controlling the error (in QoI) when solving very large (multiscale) mechanics problems

Fracture over the scales, adaptivity model reduction and selection





Question: how can we account for microstructures in a computationally tractable way?

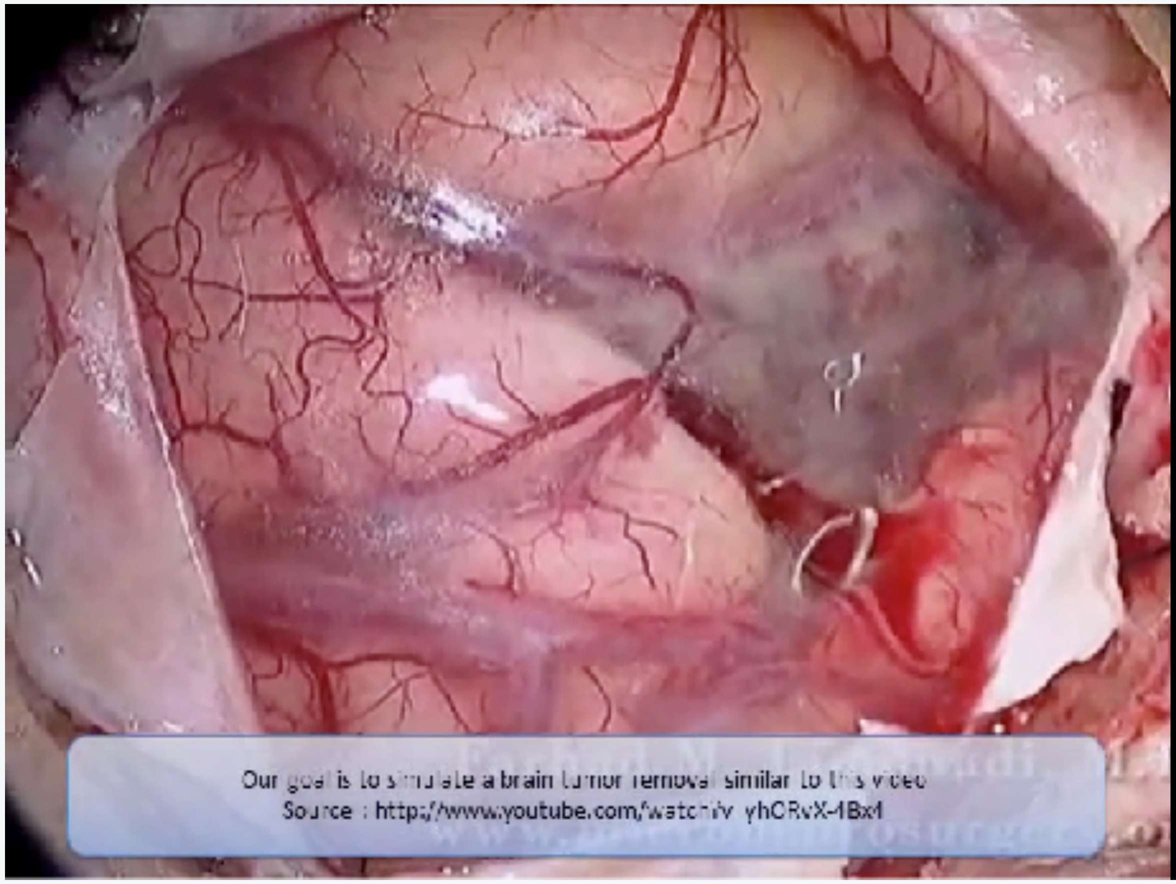
O. Goury, P. Kerfriden et al. CMAME, 2016, CMECH (**2017**) DOI 10.1007/s00466-016-1290-2 - Model reduction for fracture
C. Hoang et al. Comput. Methods Appl. Mech. Engrg. 298 (**2016**) 121–158 - Model reduction for elastodynamics
A. Akbari, P. Kerfriden and SPAB, Philosophical Magazine, (**2015**) http://dx.doi.org/10.1080/14786435.2015.1061716
P. Kerfriden et al. Comput. Methods Appl. Mech. Engrg. 256 (**2013**) 169–188 - Model reduction methods for fracture

Partial conclusions on fracture of heterogeneous materials

- Model + mesh adaptivity for adaptive fracture mechanics simulations: expensive + implementation must be done carefully
- Model order reduction, e.g. POD, PGD are ineffective for problems lacking separation of scales (see Kerfriden, Goury and others)
    - Domain-wise model selection
    - Adaptive model selection
    - Machine learning…

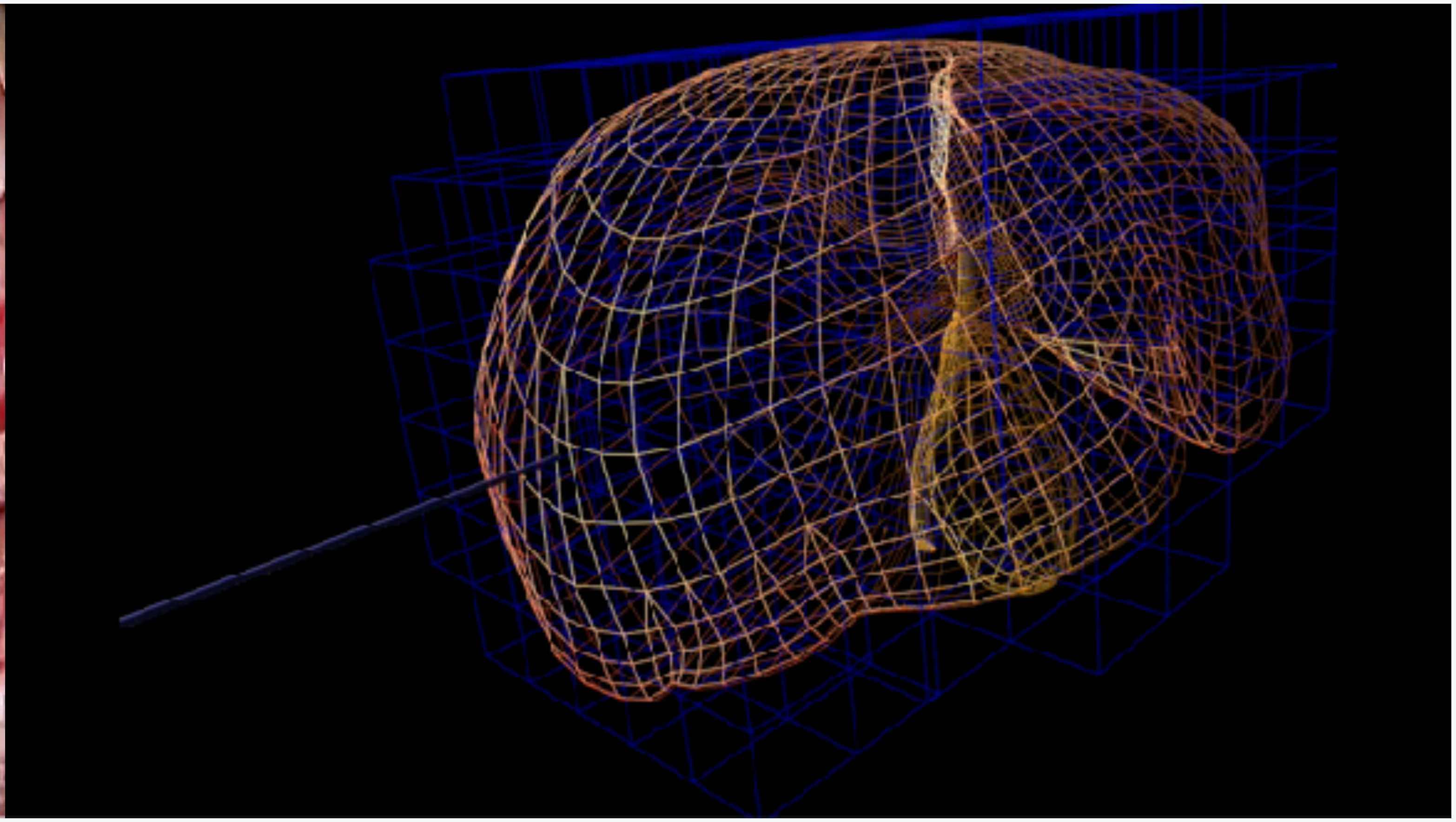
Topological changes in surgical simulation

Cutting and Needle Insertion



Our goal is to simulate a brain tumor removal similar to this video
Source : http://www.youtube.com/watch?v=yhORvX-1Bxl

H. Courtecuisse et al. Medical Image Analysis, 2014



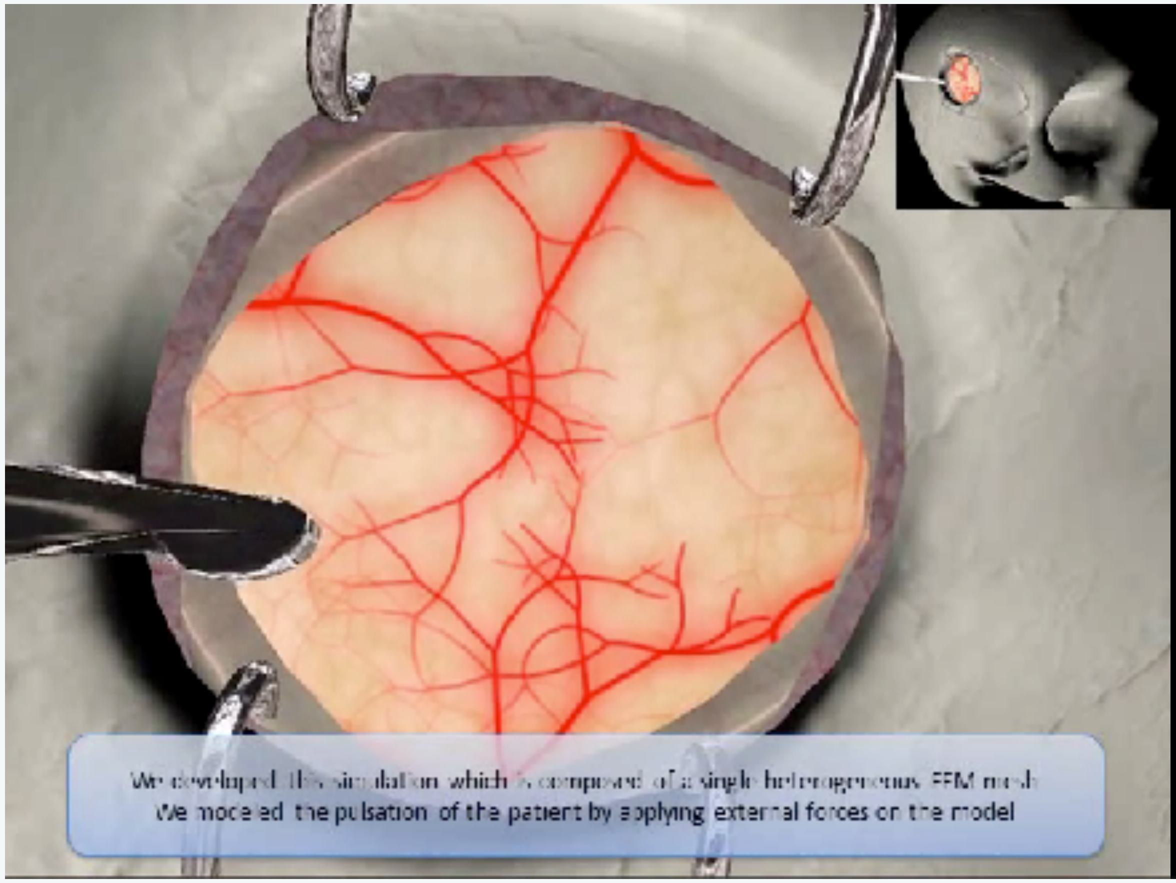
P.H. Bui et al. IEEE T. Biomed Eng. 2017 & Frontiers in Surgery, 2017

http://orbilu.uni.lu/handle/10993/30937

http://orbilu.uni.lu/handle/10993/29846

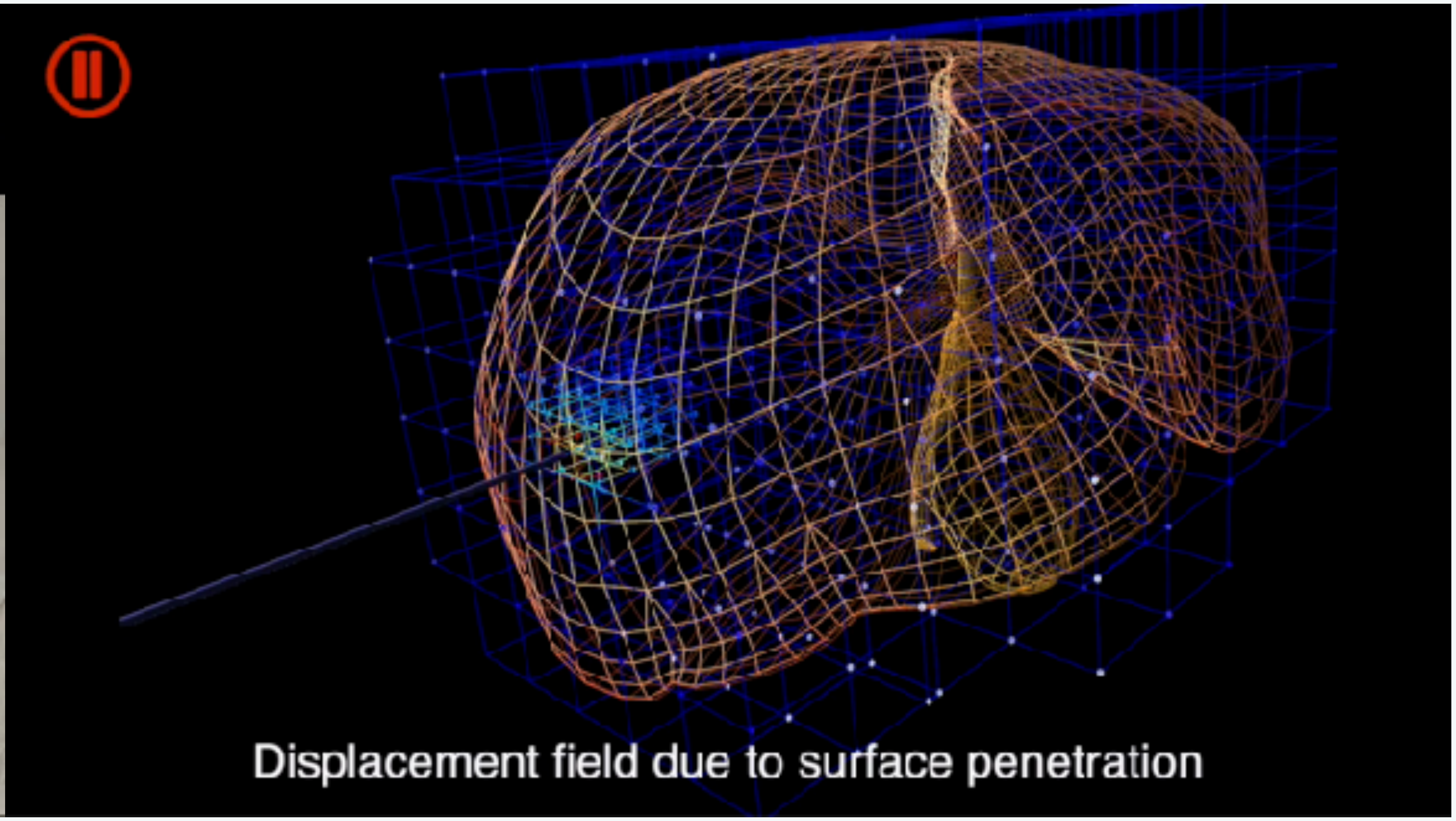
Topological changes in surgical simulation

Cutting and Needle Insertion



H. Courtecuisse et al. Medical Image Analysis, 2014

Question: how can we simulate cutting/fracture in real time using implicit time stepping?



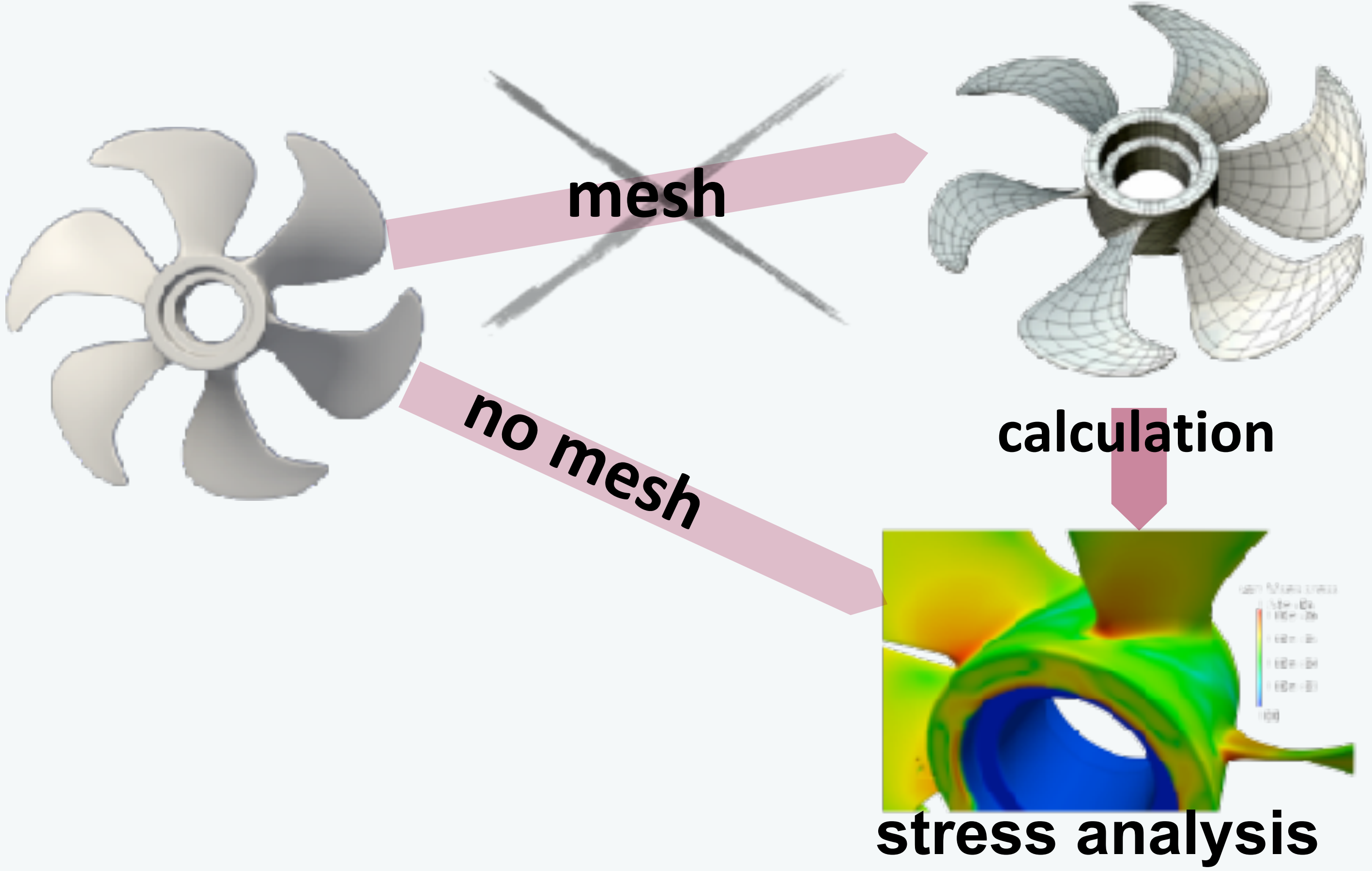
P.H. Bui et al. IEEE T. Biomed Eng. 2017 & Frontiers in Surgery, 2017

Question: how can we adapt the mesh in real time using a posteriori error estimates?

http://orbilu.uni.lu/handle/10993/30937 http://orbilu.uni.lu/handle/10993/29846

Handling (complex) interfaces numerically

Coupling, or decoupling?

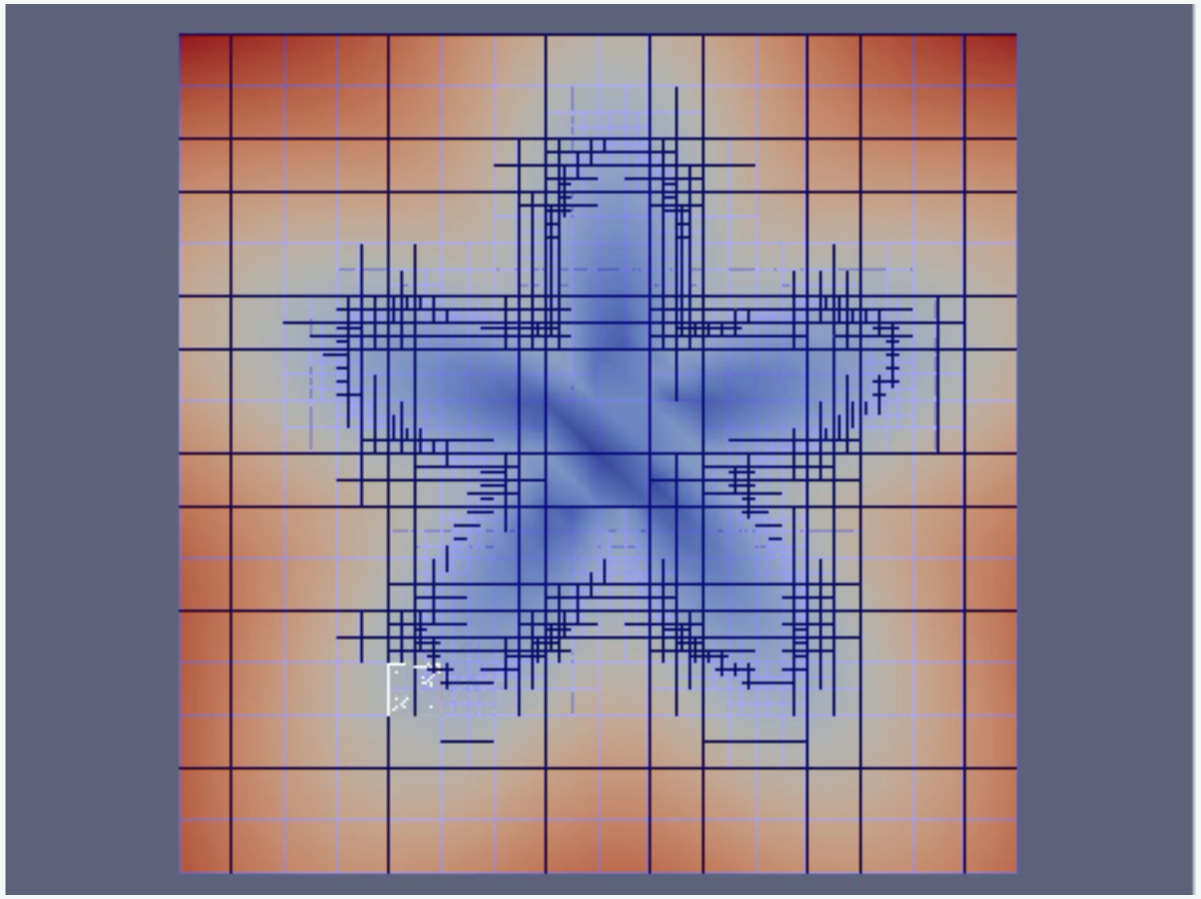


mesh

no mesh

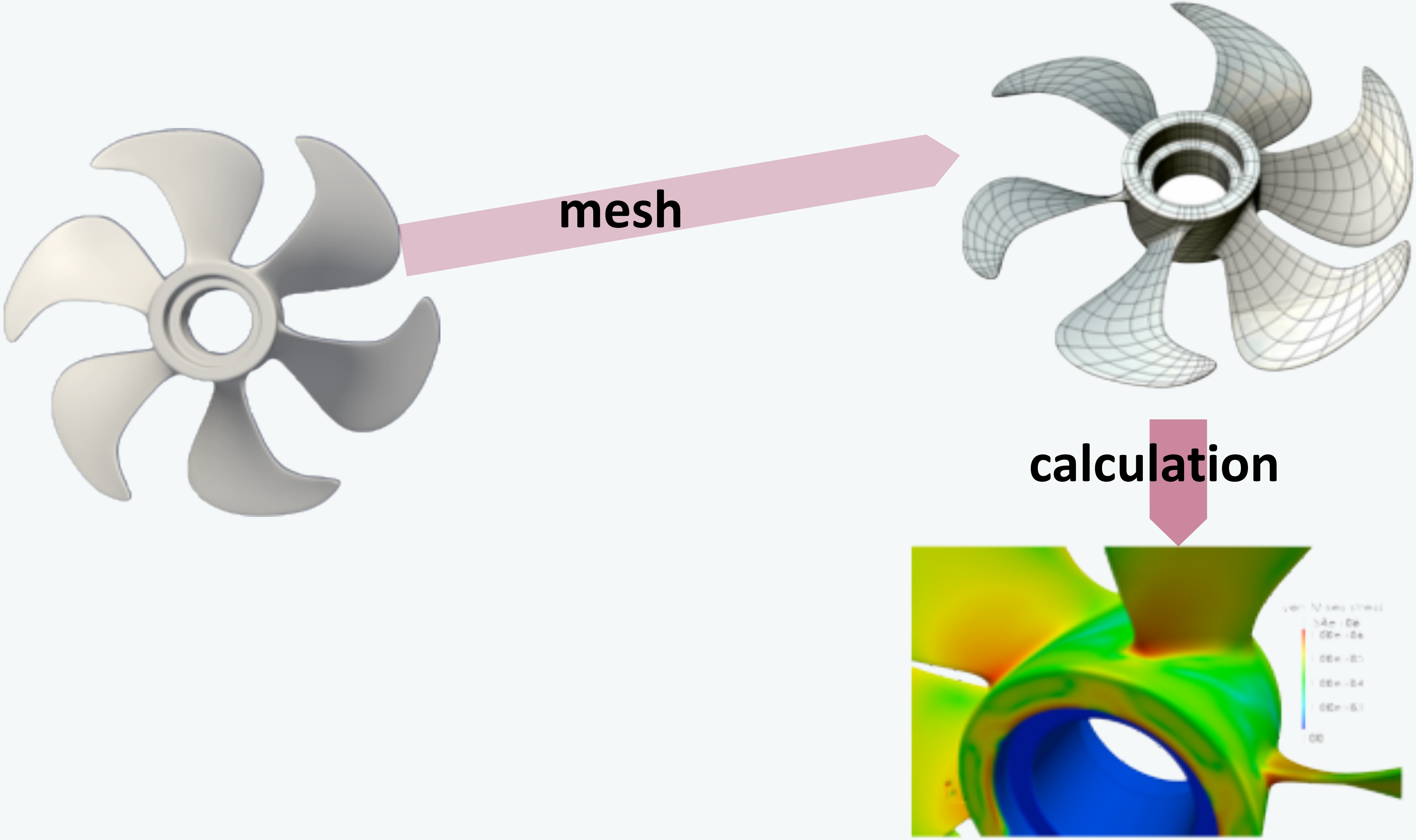
calculation

stress analysis



Question: When are we better off coupling/decoupling the geometry from the field approximation?

Isogeometric analysis

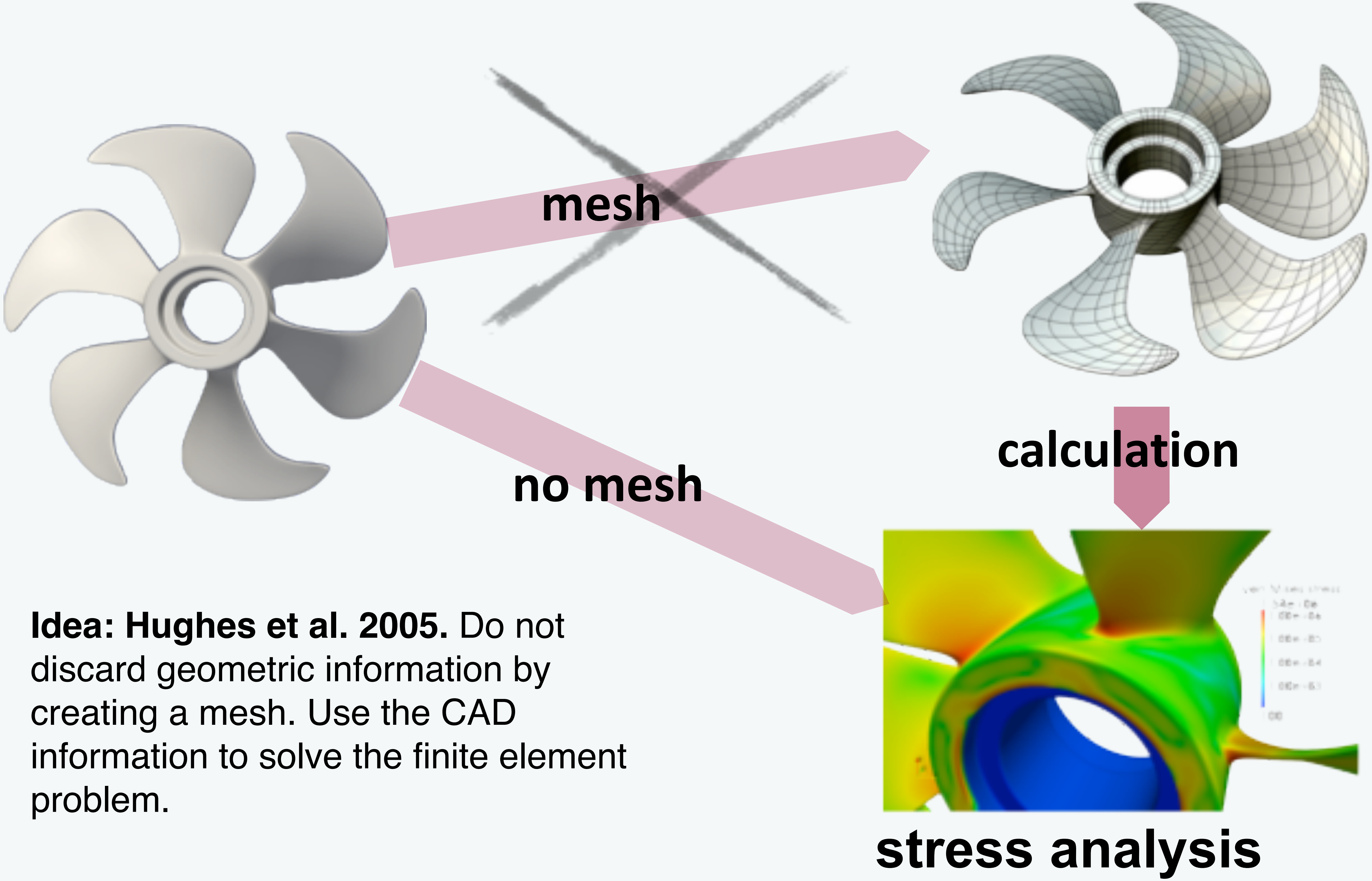


mesh

calculation

stress analysis

Isogeometric analysis



mesh

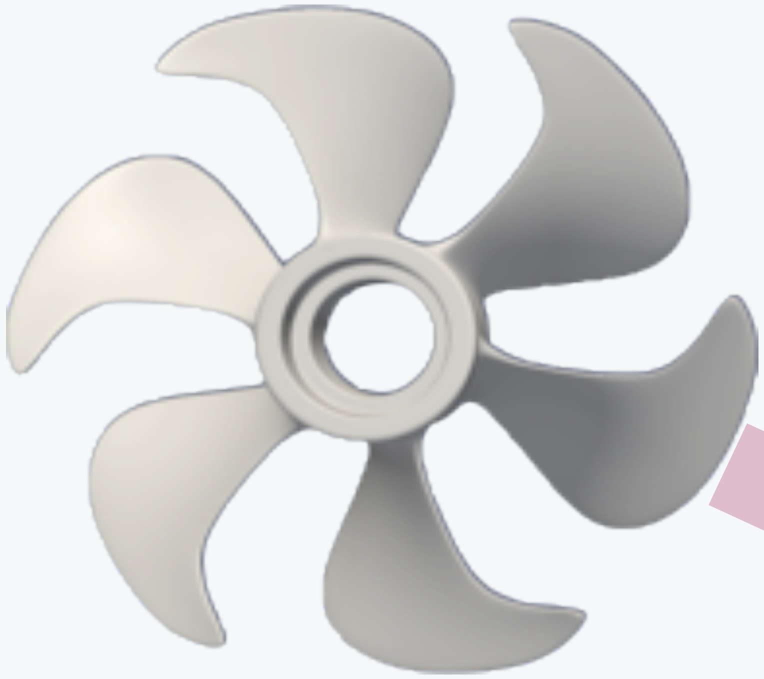
calculation

no mesh

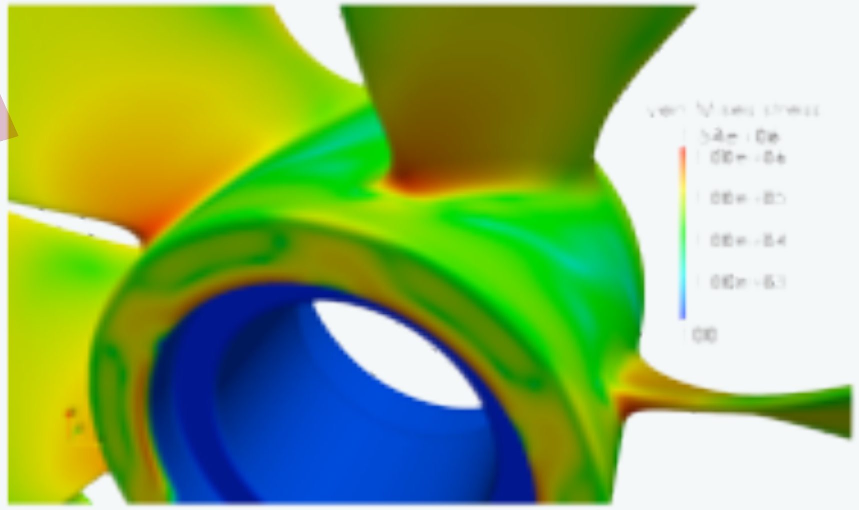
stress analysis

Idea: Hughes et al. 2005. Do not discard geometric information by creating a mesh. Use the CAD information to solve the finite element problem.

Isogeometric analysis



direct calculation

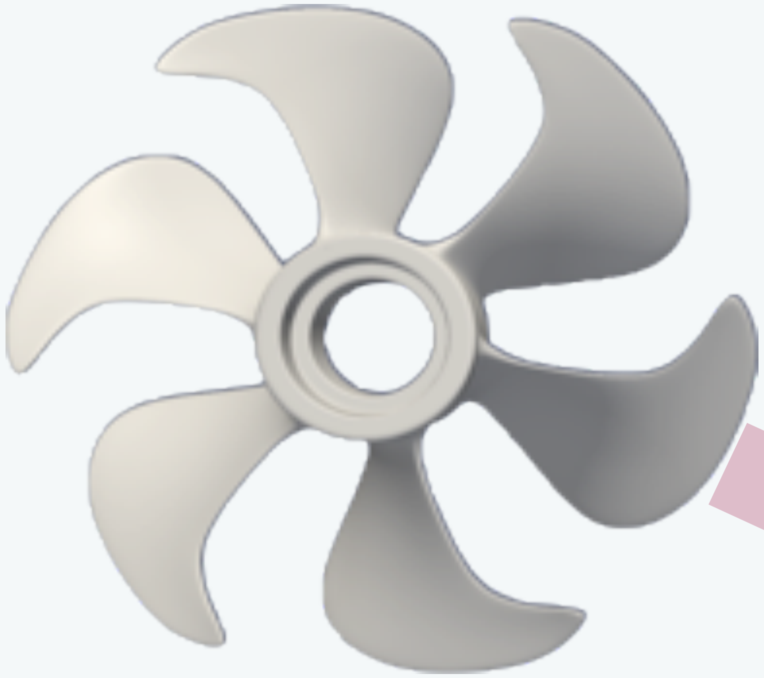


stress analysis

Idea: Hughes et al. 2005. Do not discard geometric information by creating a mesh. Use the CAD information to solve the finite element problem.

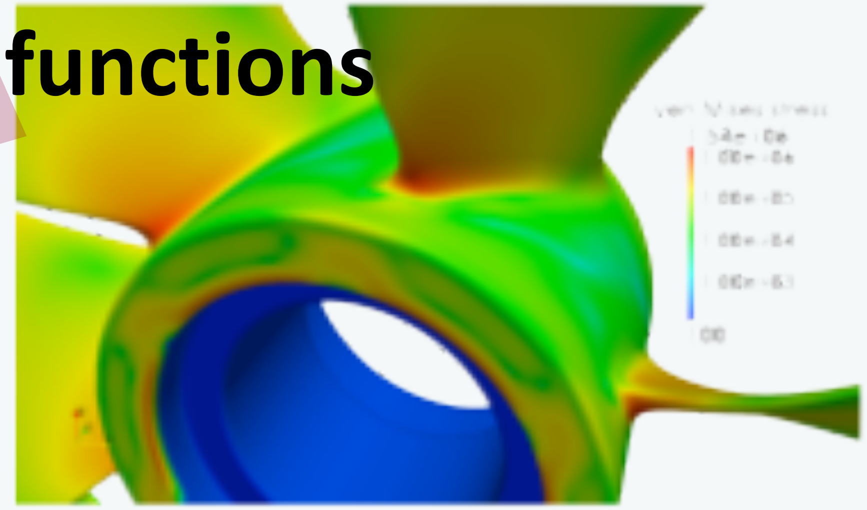
Isogeometric analysis

CAD: described by NURBS



Idea: Hughes et al. 2005. Do not discard geometric information by creating a mesh. Use the CAD information to solve the finite element problem.

Use NURBS as FE basis functions



stress analysis

Isogeometric analysis: limitations

Geometry

☑ For shell-like domains

☐ For volumes (needs volume parameterisation)

☐ Coupling between multiple patches (Nitsche, Mortar…)

Adaptivity

☑ Global refinement - cannot refine field without refining geo…

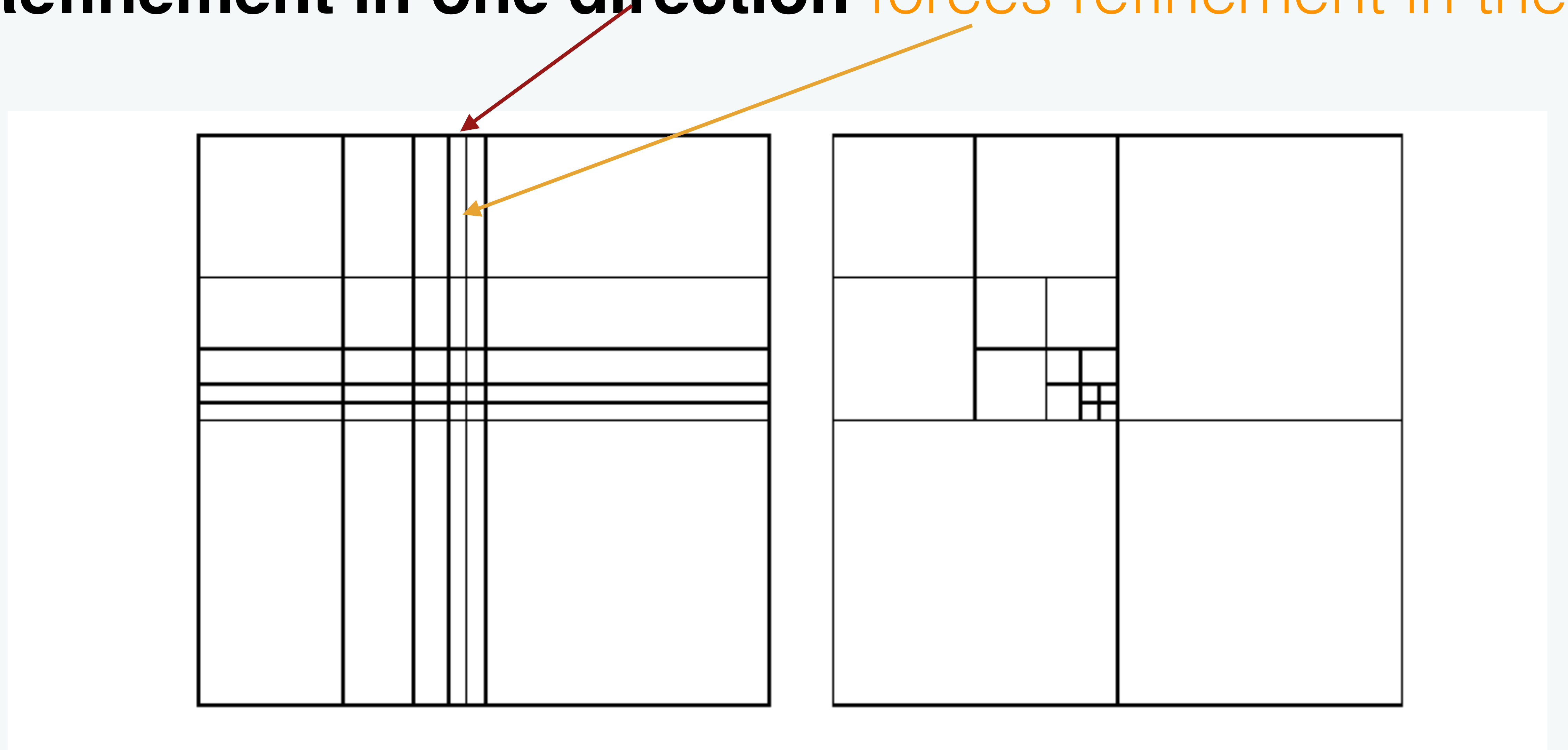
☐ Local refinement (not with NURBS)… (PH)T-splines…

☐ Geometry independent refinement for the field variables?

Mesh refinement in IGA

Using NURBS,

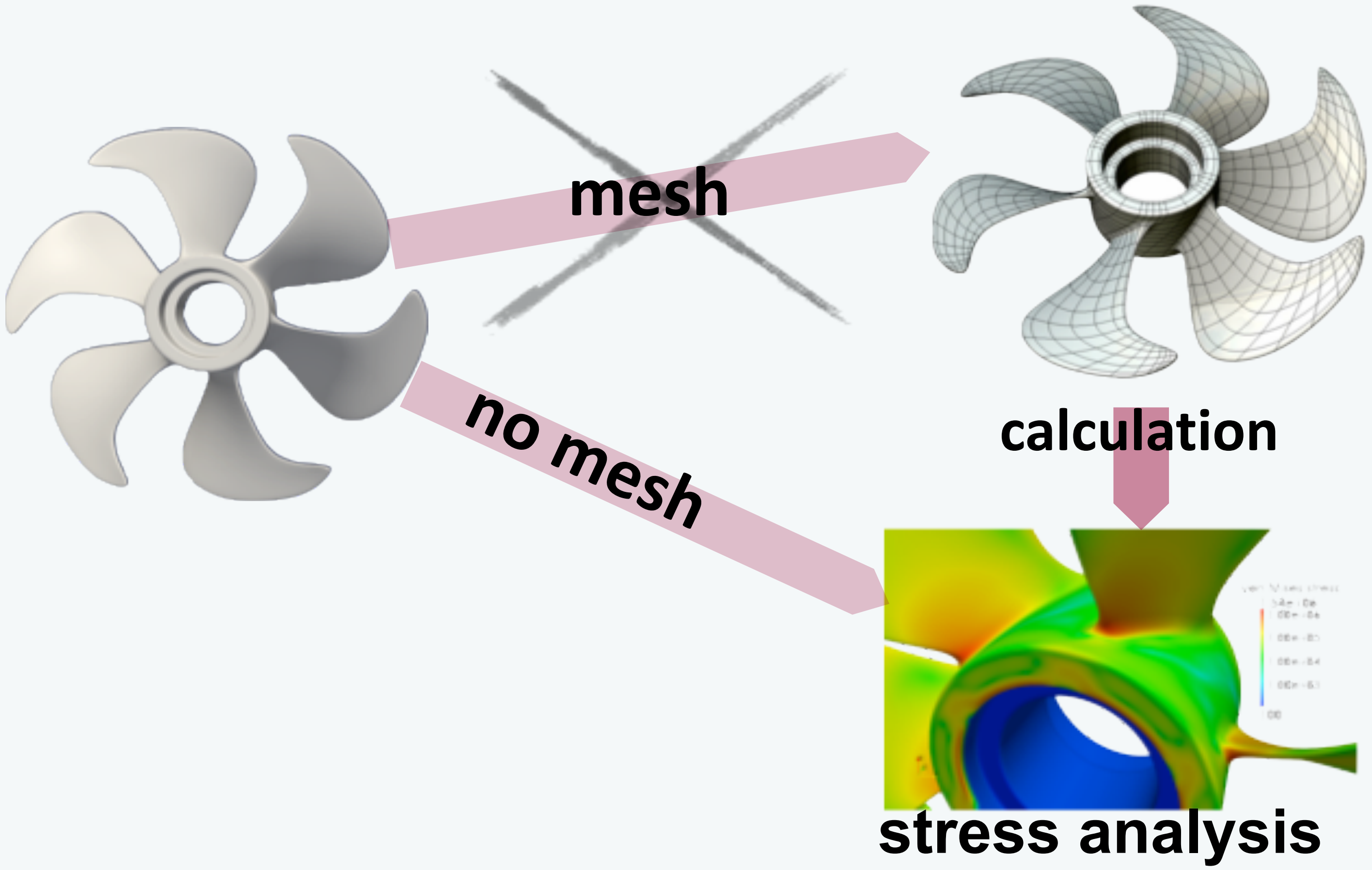
Refinement in one direction forces refinement in the other



Global refinement (tensor-product mesh) vs local refinement (T-mesh)

Handling (complex) interfaces numerically

Coupling



mesh

no mesh

calculation

stress analysis

Question: How can we fully benefit from the “IGA” concept?

☑ Refine the field independently from the geometry

☑ Suppress the mesh generation and regeneration completely

Handling (complex) interfaces numerically

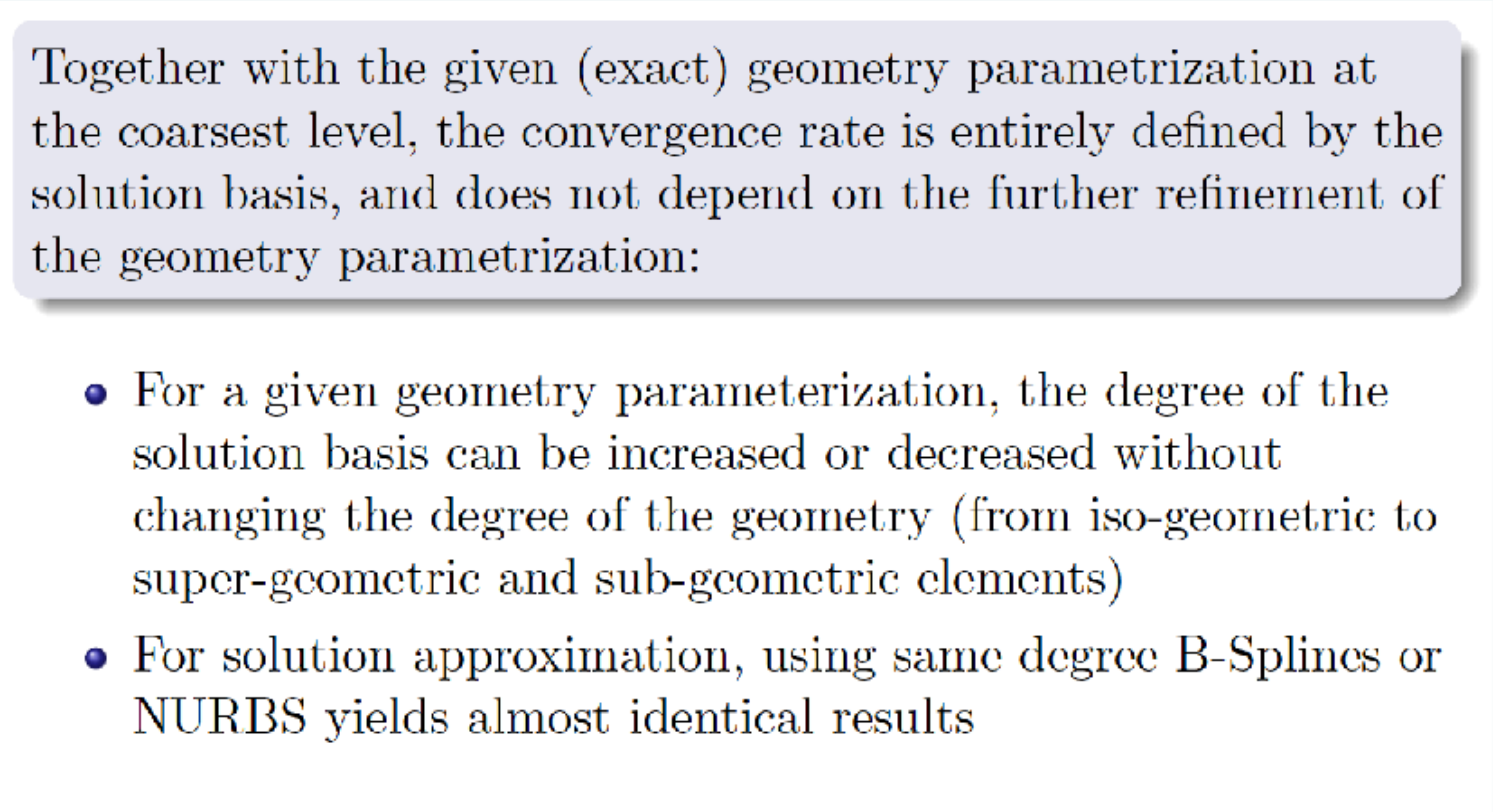
Coupling geometry and field approximation

Question: How can we fully benefit from the “IGA” concept?

☑ Refine the field independently from the geometry

[REF] Weakening the tight coupling between geometry and simulation in isogeometric analysis: from sub- and super- geometric analysis to Geometry Independent Field approximaTion (GIFT), IJNME, 2017, submitted [preprint available on arXiv]

Permalink: http://hdl.handle.net/10993/31469



Together with the given (exact) geometry parametrization at the coarsest level, the convergence rate is entirely defined by the solution basis, and does not depend on the further refinement of the geometry parametrization:

- For a given geometry parameterization, the degree of the solution basis can be increased or decreased without changing the degree of the geometry (from iso-geometric to super-geometric and sub-geometric elements)
- For solution approximation, using same degree B-Splines or NURBS yields almost identical results

Geometry Independent Field approximaTion (GIFT)

Conclusions

- ☑ Tight link between CAD and analysis
- ☑ The same basis functions, which are used in CAD to represent the geometry, are used in the IGA as shape functions to approximation the unknown solution
- ☑ Geometry is exact at any stage of the solution refinement process
- ☑ Better accuracy per DOF in comparison with standard FEM but higher computational cost (bandwidth…)

Geometry Independent Field approximaTion (GIFT) *Conclusions*

☑ Retain the advantages of IGA but decouple the geometry and the field approximation

☑ Standard patch tests may not always pass, yet the convergence rates are optimal as long as the geometry is exactly represented by the geometry basis

☑ With geometry exactly represented by NURBS, using same degree B-splines or NURBS for the approximation of the solution field yields almost identical results

☑ With geometry exactly represented by NURBS, using PHT splines for the approximation of the solution gives additional advantage of local adaptive refinement

☑ Any other approximation field can be used for the field variables

Handling (complex) interfaces numerically

Coupling

Question: How can we fully benefit from the “IGA” concept?

☑ Suppress the mesh generation and regeneration completely

Isogeometric Finite Elements

☑ For shell-like domains

☐ For volumes (needs volume parameterisation)

Isogeometric Boundary Element Analysis

☑ For shell-like domains

☑ For volumes

Stress analysis and shape optimisation directly from CAD

H. Lian et al. (2017). CMAME*:* 317 (2017): 1-41.

H. Lian et al. (2015). IJNME

H. Lian et al. (2013). EACM:166(2):88-99.

M. Scott et al. (2013) CMAME 254: 197-221.

R. N. Simpson et al. (2013) CAS 118: 2-12.

R. N. Simpson et al. (2012) CMAME Feb 1;209:87-100.

Fracture mechanics directly from CAD

X. Peng, et al. (2017). *IJF*, 204(1), 55–78.

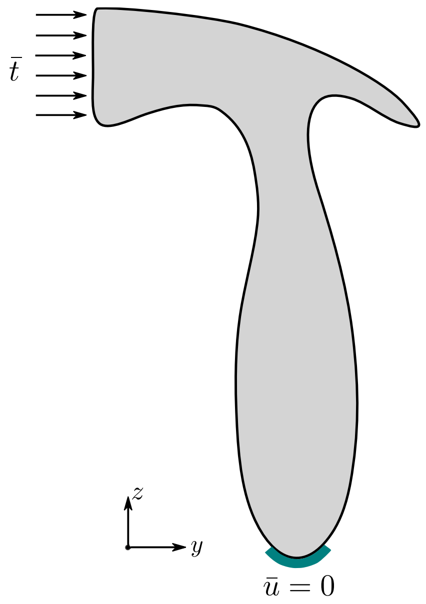
X. Peng, et al. (2017). *CMAME*, 316, 151–185.

Handling (complex) interfaces numerically

Example applications

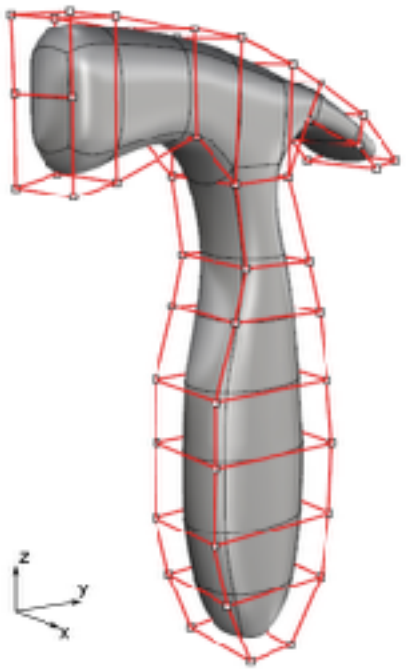
Isogeometric Boundary Element Analysis (IGABEM)

Shape optimisation



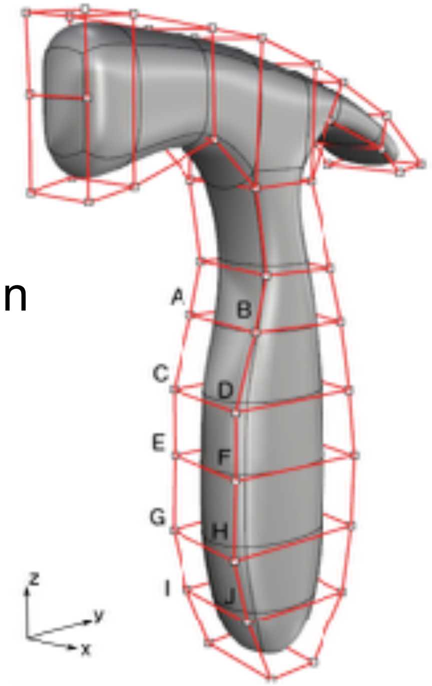
Problem definition

Model construction with CAD



Control points

Design points selection in control points



Design points

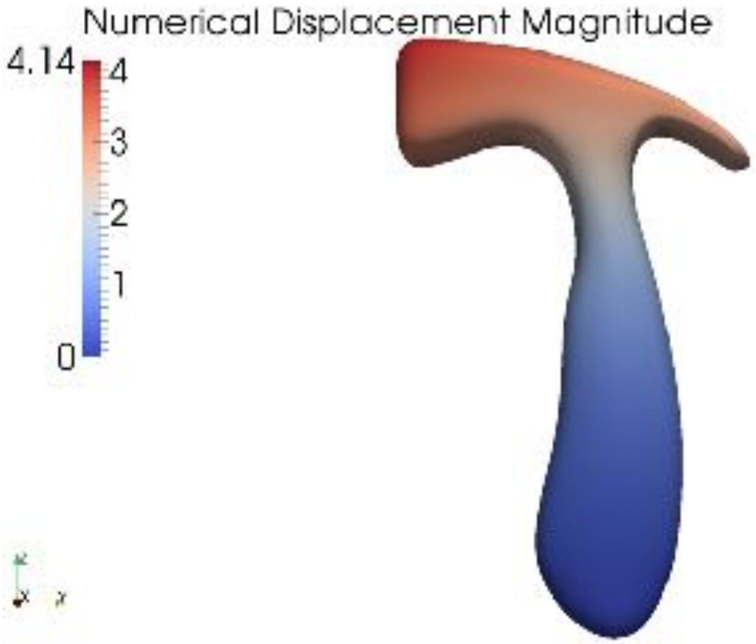
Objective function: $\int_S t_i u_i \mathrm{d}S$

Volume constraint: $V - V_0 \leq 1$

Side constraints:

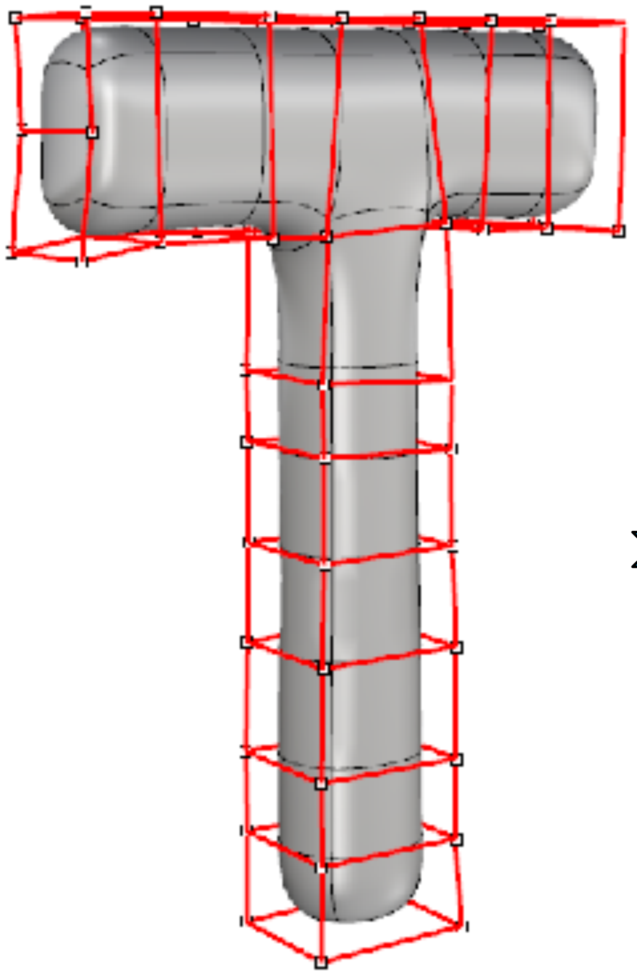
Design variable	Lower bound	Upper bound	Initial value
t_1	0	4	2.45
t_2	0	4	1.25
t_3	0	4	1.33
t_4	0	4	1.28
t_5	0	4	2.30

Structural analysis;
Sensitivity analysis;
gradient-based optimizer

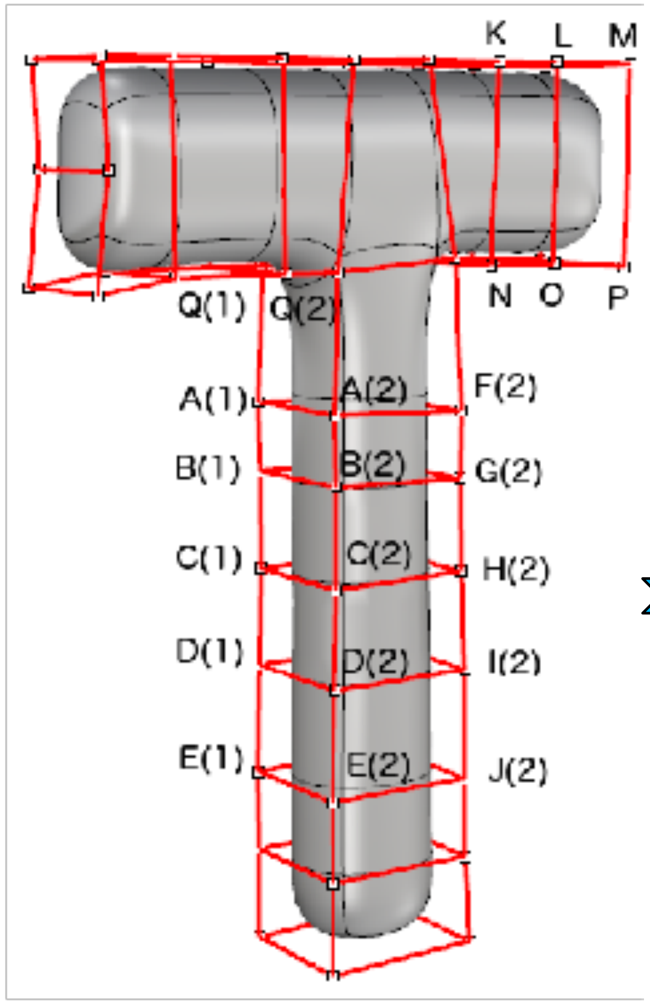


Optimized solution

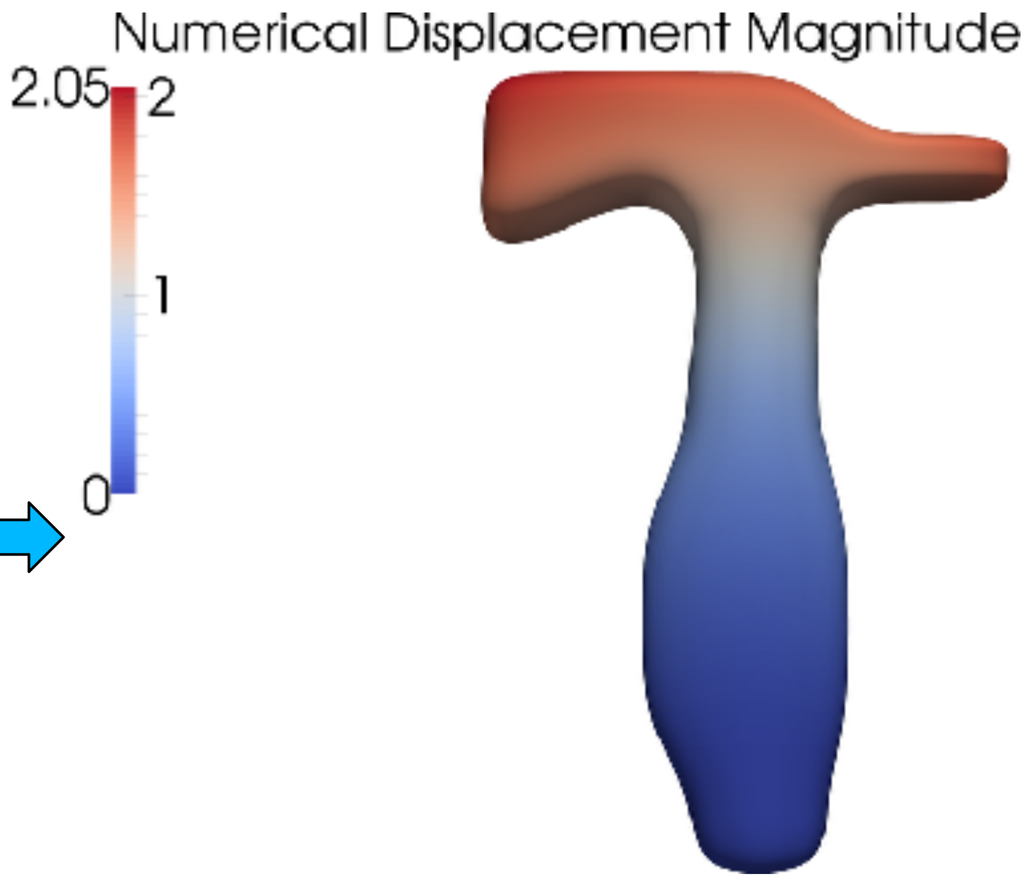
Shape optimisation



Construct the geometric model



Choose design points from the control points



Conduct sensitivity analysis to converge to the optimized solution

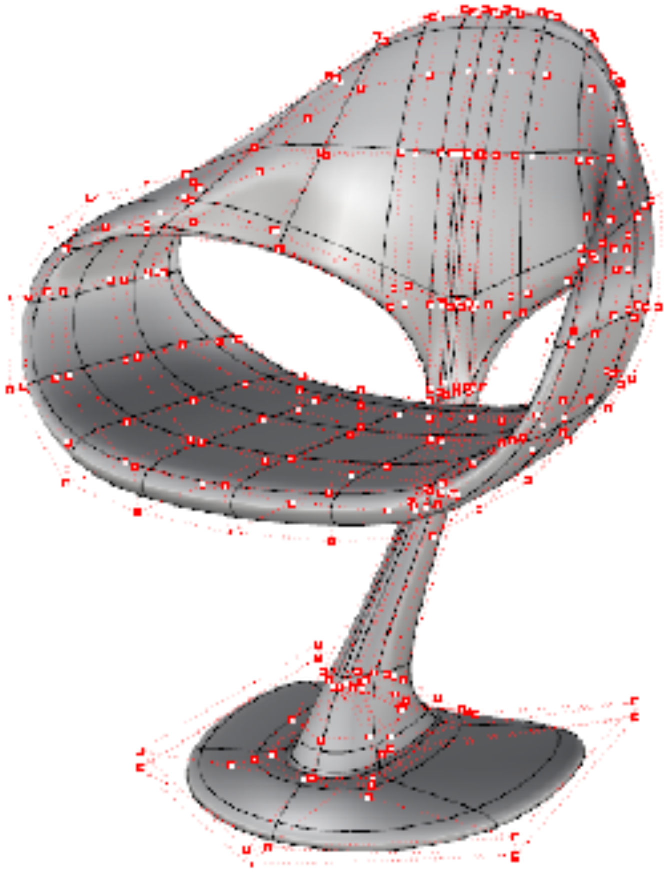
Stress analysis and shape optimisation directly from CAD

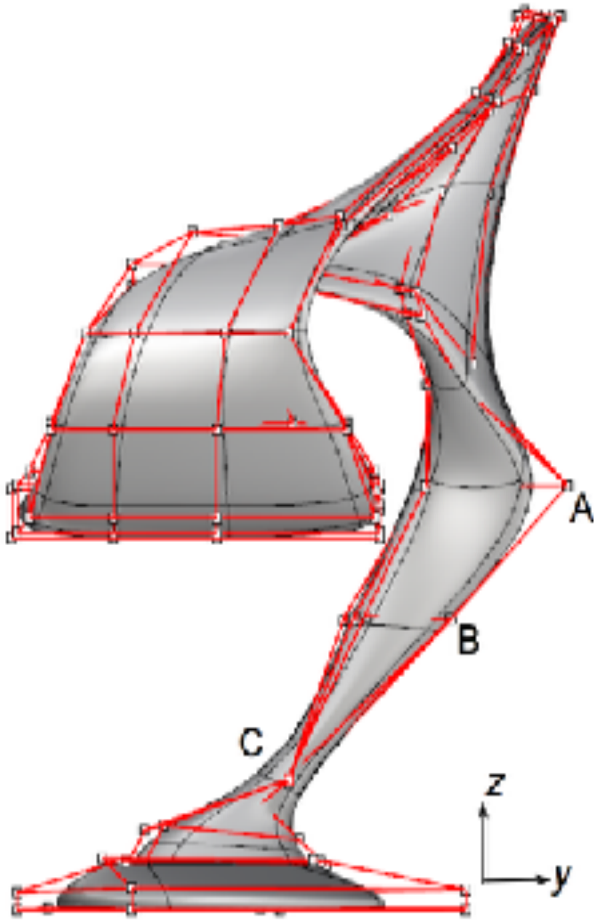
H. Lian et al. (2017). CMAME*:* 317 (2017): 1-41.
H. Lian et al. (2015). IJNME
H. Lian et al. (2013). EACM:166(2):88-99.

M. Scott et al. (2013) CMAME 254: 197-221.
R. N. Simpson et al. (2013) CAS 118: 2-12.
R. N. Simpson et al. (2012) CMAME Feb 1;209:87-100.

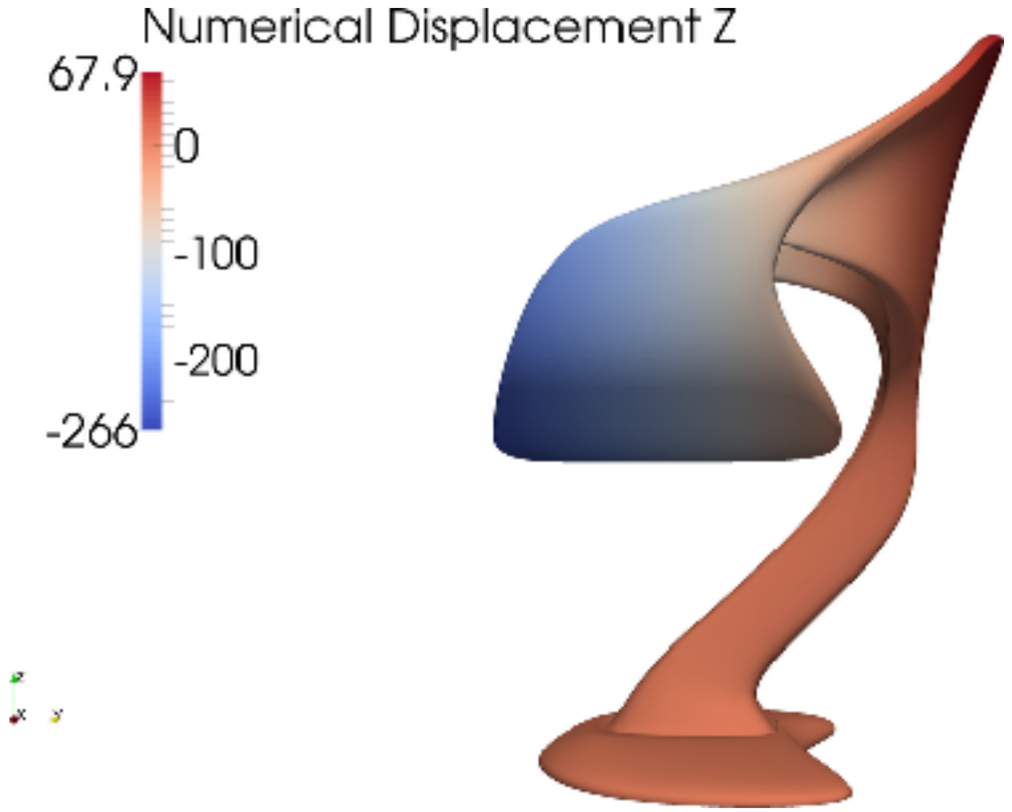
Shape optimisation

Construct the geometric model
(imported from Rhino)



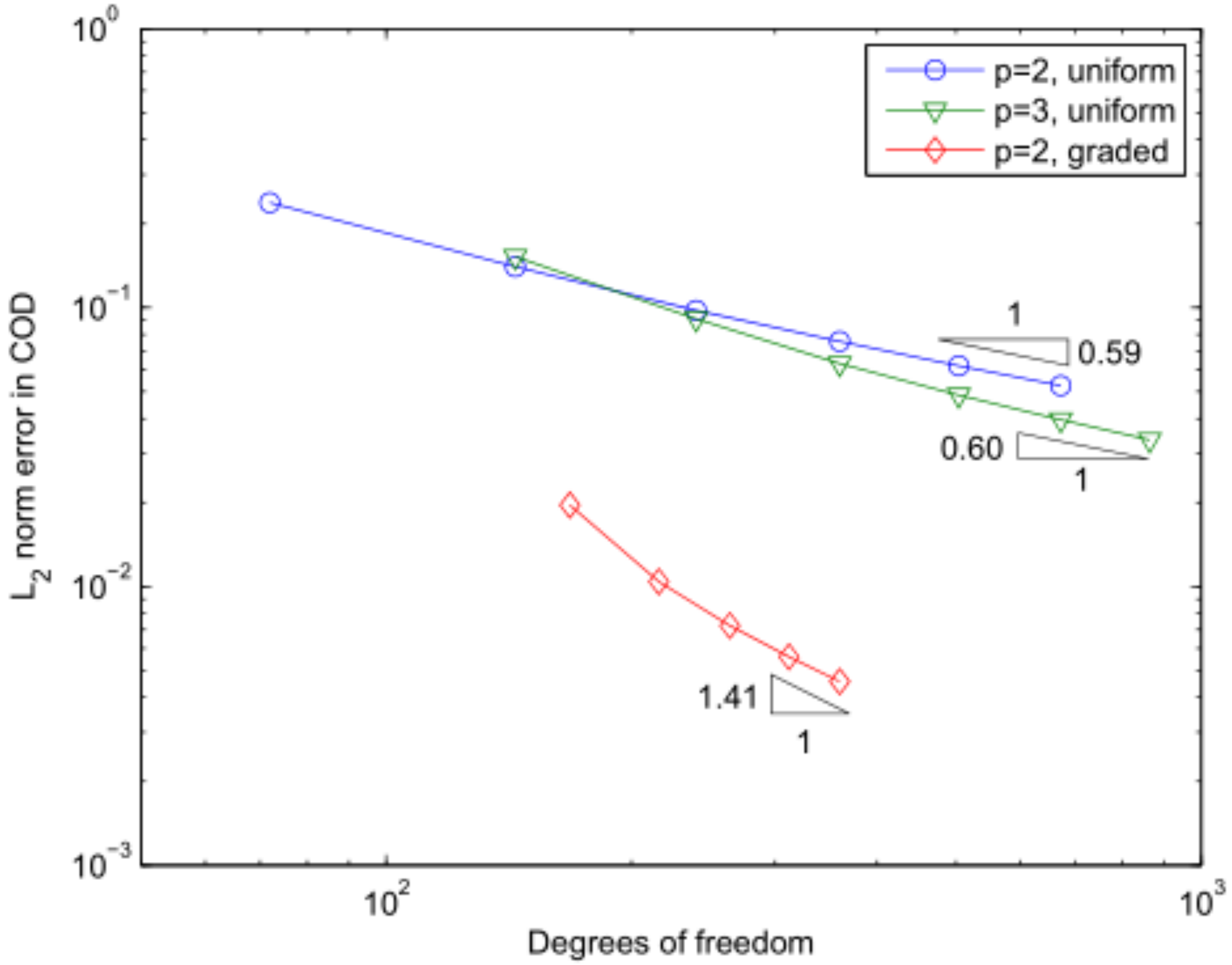


Select design points from control points

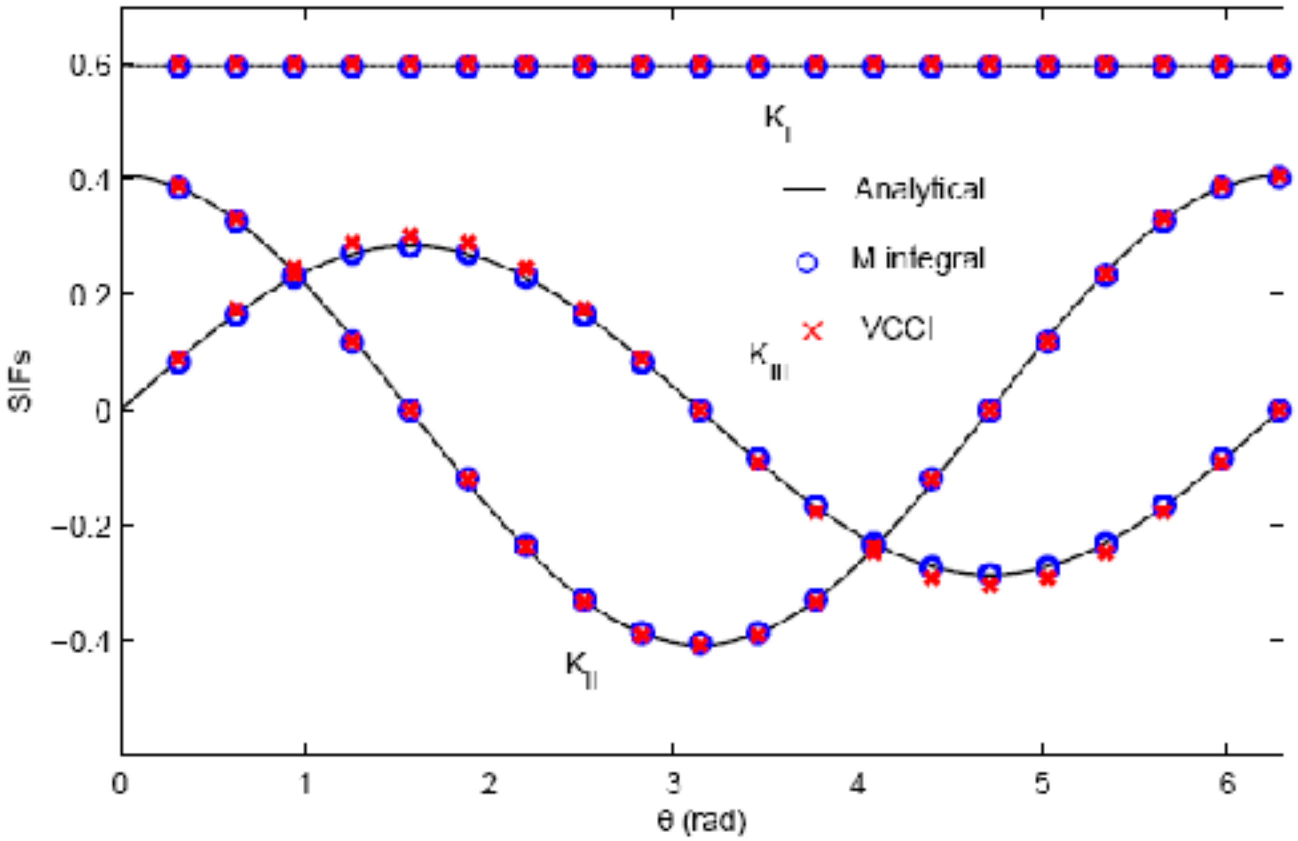


Find optimized solution

Penny-shaped crack under remote tension



L_2 norm error of COD for penny-shaped crack



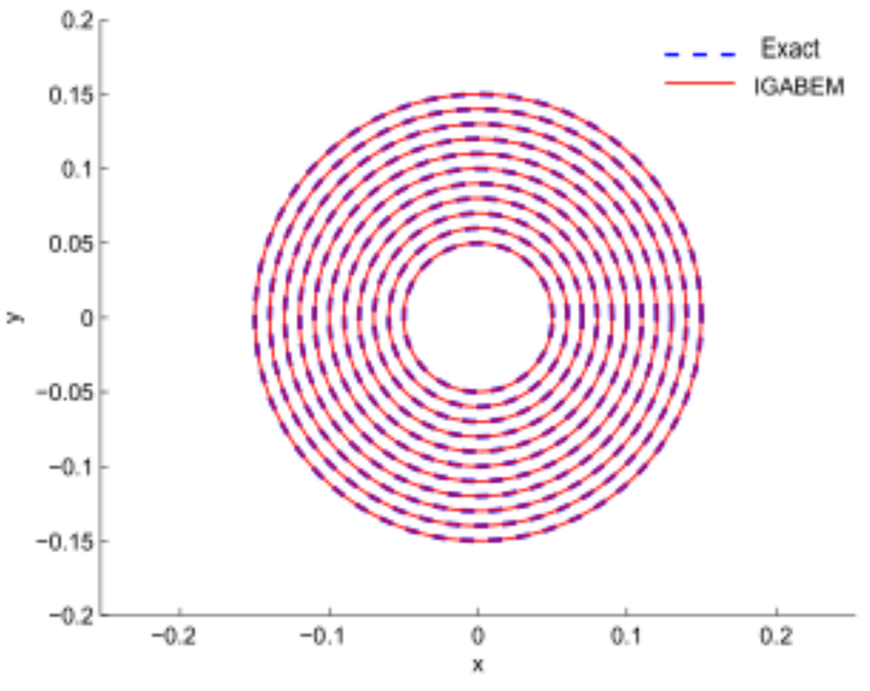
stress intensity factors for penny crack with $\varphi = \pi/6$

Fracture mechanics directly from CAD

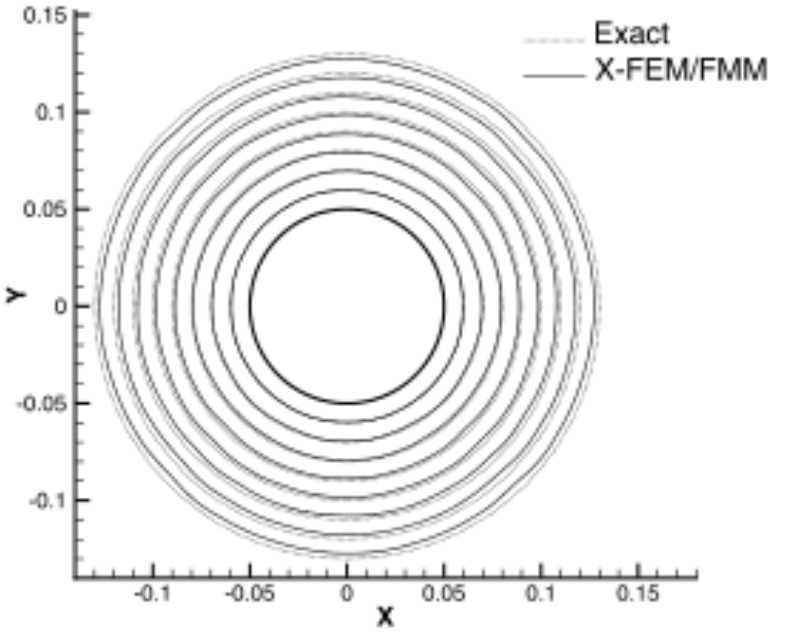
X. Peng, et al. (2017). *IJF*, 204(1), 55–78.

X. Peng, et al. (2017). *CMAME*, 316, 151–185.

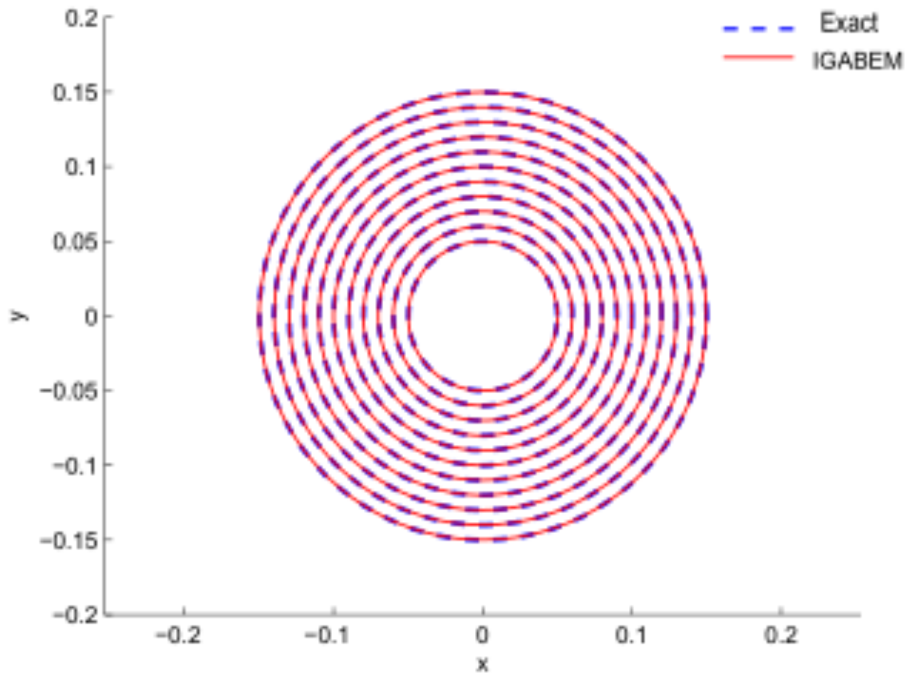
Numerical example of horizontal penny crack growth (first 10 steps)



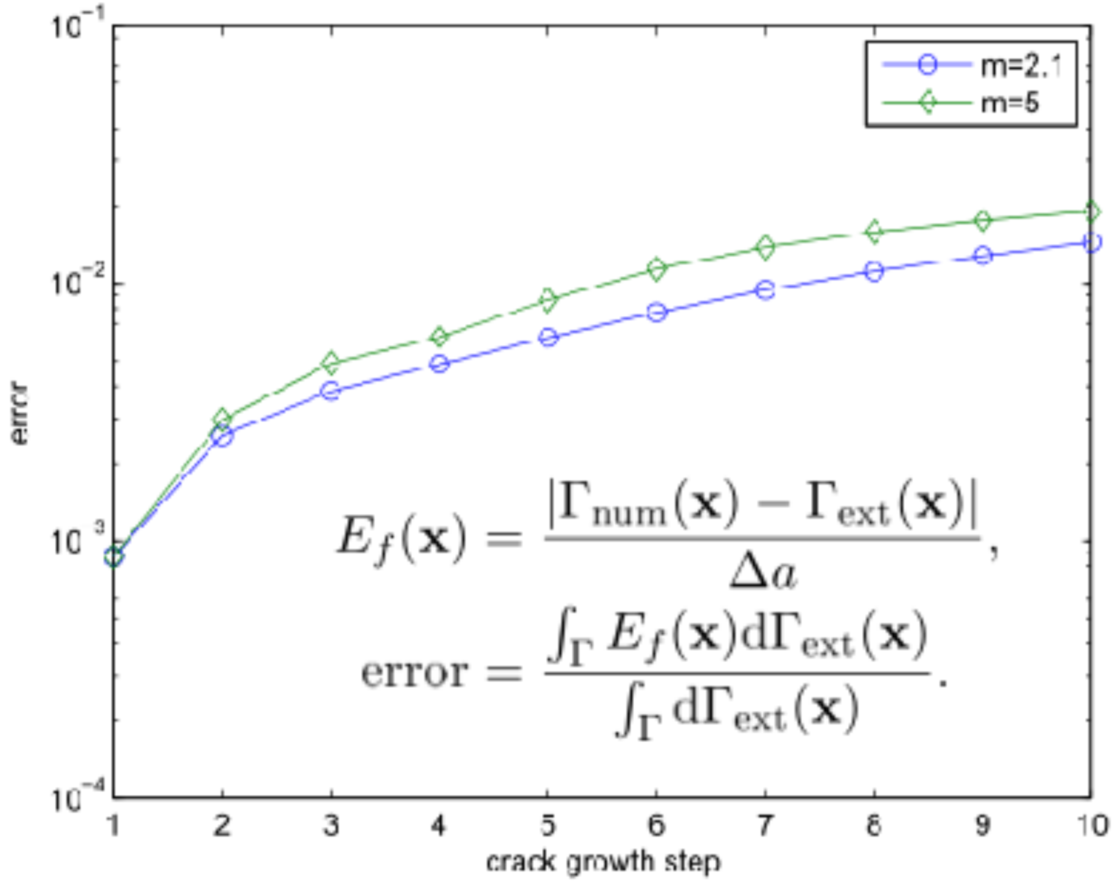
(a) IGABEM, $m = 2.1$



(b) XFEM/FMM, $m = 2.1$, Sukumar *et al* 2003



(c) IGABEM, $m = 5$

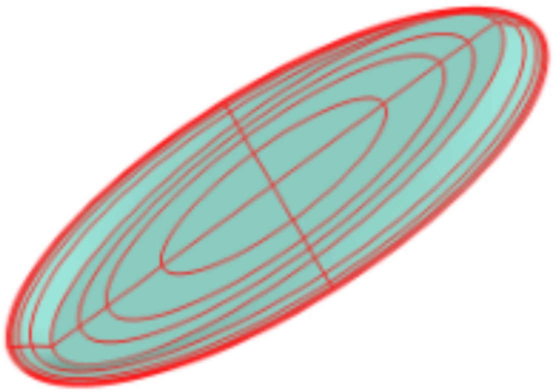


$$E_f(\mathbf{x}) = \frac{|\Gamma_{\text{num}}(\mathbf{x}) - \Gamma_{\text{ext}}(\mathbf{x})|}{\Delta a},$$

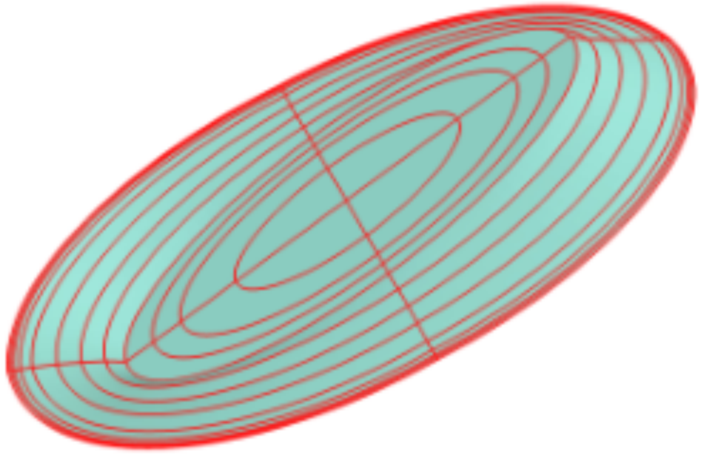
$$\text{error} = \frac{\int_\Gamma E_f(\mathbf{x}) d\Gamma_{\text{ext}}(\mathbf{x})}{\int_\Gamma d\Gamma_{\text{ext}}(\mathbf{x})}.$$

Relative error of the crack front for in each crack growth step by IGABEM

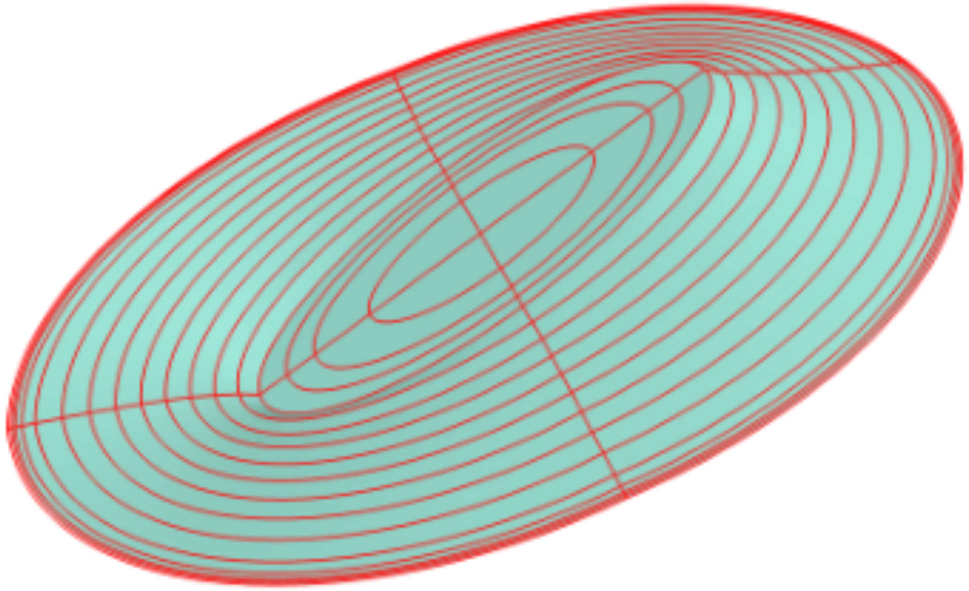
Numerical example of inclined elliptical crack growth (first 10 steps)



(a) Step 2



(b) Step 5



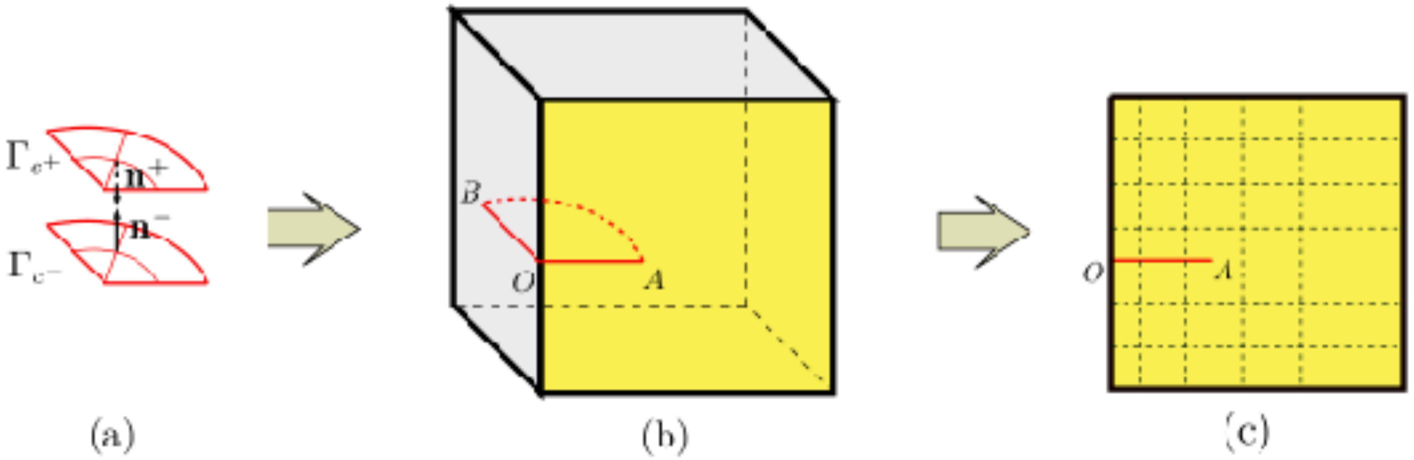
(c) Step 10

Fracture mechanics directly from CAD

X. Peng, et al. (2017). *IJF*, 204(1), 55–78.

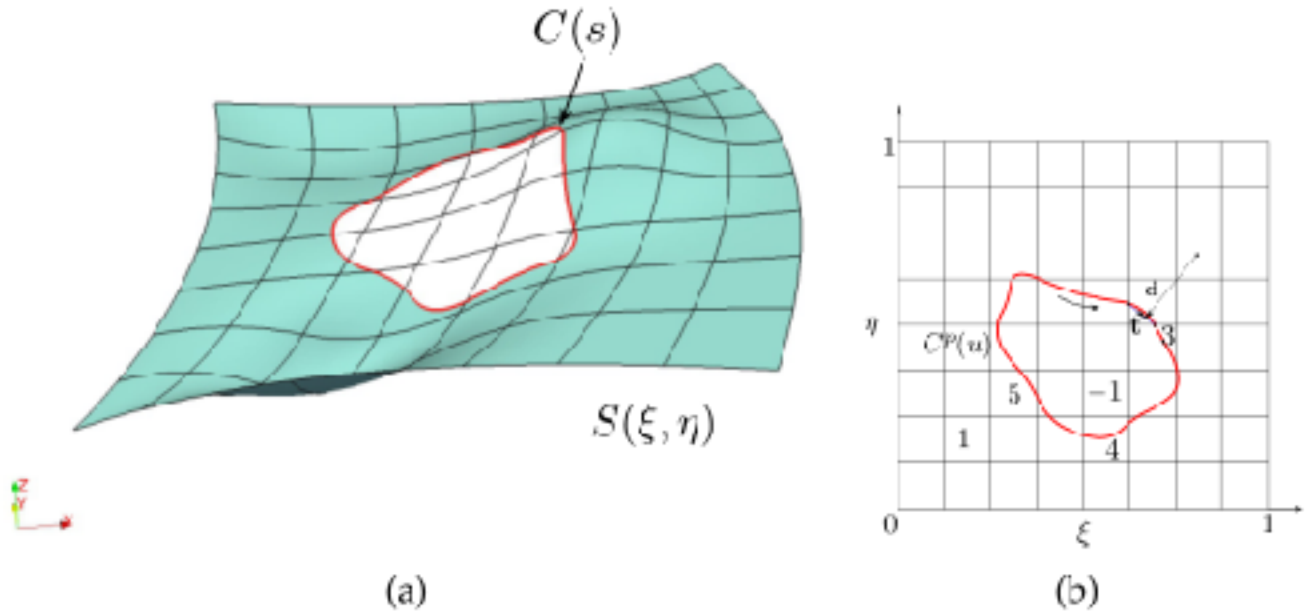
X. Peng, et al. (2017). *CMAME*, 316, 151–185.

Modeling techniques for surface cracks? Trimmed curves…



(a) (b) (c)

Surface discontinuity is introduced



(a) (b)

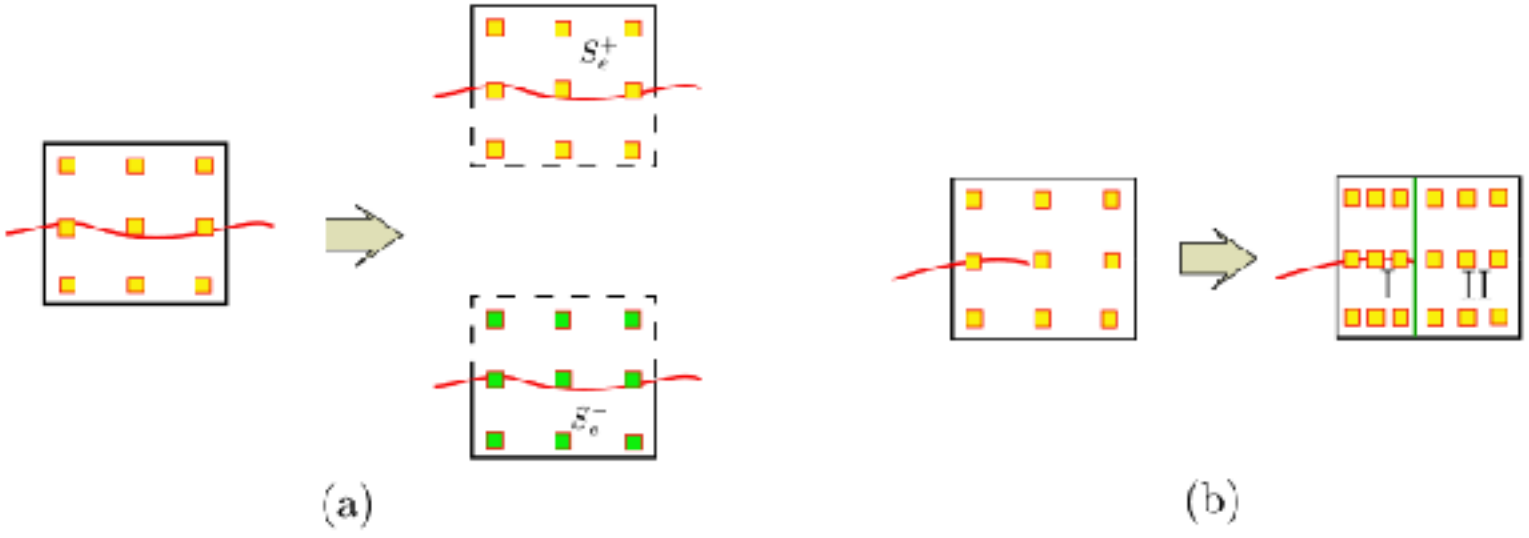
Trimmed NURBS technique

Crack = trimming curve

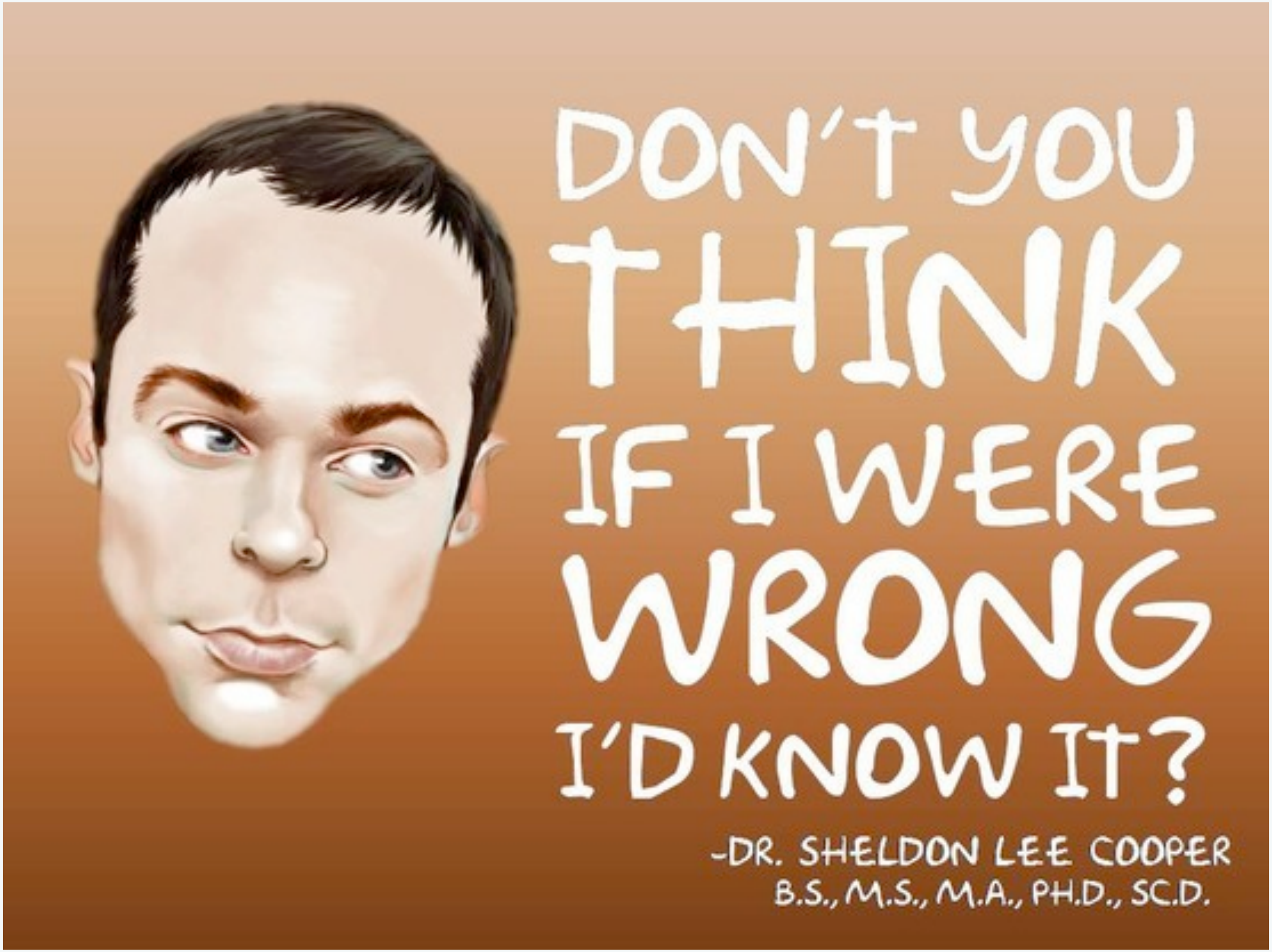
Phantom node method

$$\mathbf{u}^+(\mathbf{x}) = \sum_{j}^{N^c} \mathbf{R}_j(\mathbf{x})\mathbf{d}_j, \quad \mathbf{x} \in S_e^+,$$

$$\mathbf{u}^-(\mathbf{x}) = \sum_{k}^{N^c} \mathbf{R}_k(\mathbf{x})\mathbf{d}_k, \quad \mathbf{x} \in S_e^-$$

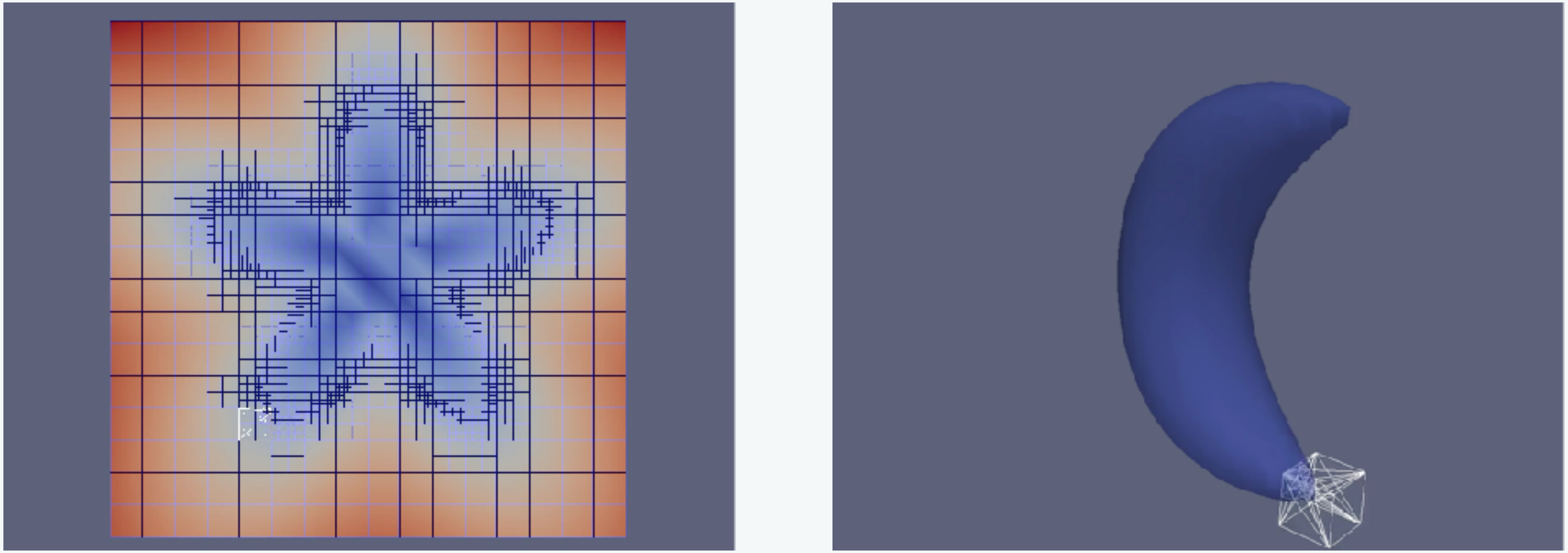


(a) (b)


DON'T YOU THINK IF I WERE WRONG I'D KNOW IT?
-DR. SHELDON LEE COOPER
B.S., M.S., M.A., PH.D., SC.D.

Handling (complex) interfaces numerically

Decoupling - Unfitted FEM?



Question: for which problems are we better off coupling/decoupling the geometry from the field approximation?

Implicit surfaces

T. Rüberg (2016) Advanced Modeling and Simulation in Engineering Sciences 3 (1), 22
M. Moumnassi (2011) CMAME *200*(5): 774-796. (CSG and multiple level sets)
N. Moës (2003) CMAME192.28 (2003): 3163-3177. (Single level set)
T. Belytschko IJNME 56.4 (2003): 609-635. (Structured XFEM)
…

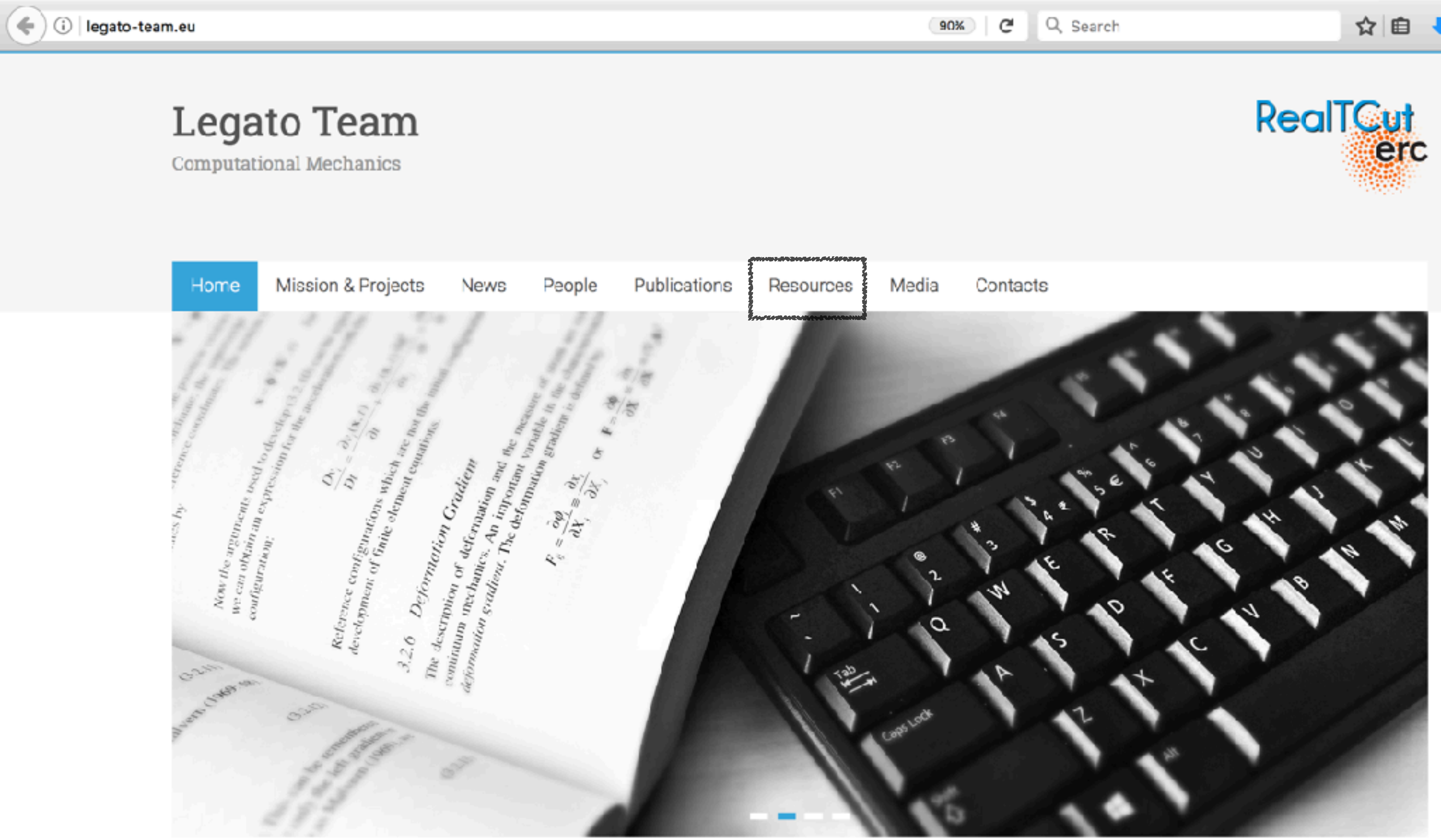
Partial conclusions on methods decoupling geometry and field approximations

- There are numerous alternatives (immersed, CutFEM, structured XFEM, collocation…)
- Discussions on higher order boundaries (see XDMS2017 book of abstracts!)
- Using CAD geometries within a structured mesh/grid is a versatile approach

Next: beyond discretisations… for a future lecture… how can we select the best model given (sparse) experimental data and quantify the uncertainties on the parameters of these models.

What’s next?

beyond discretisations… for a future lecture… how can we select the best model given (sparse) experimental data and quantify the uncertainties on the parameters of these models.



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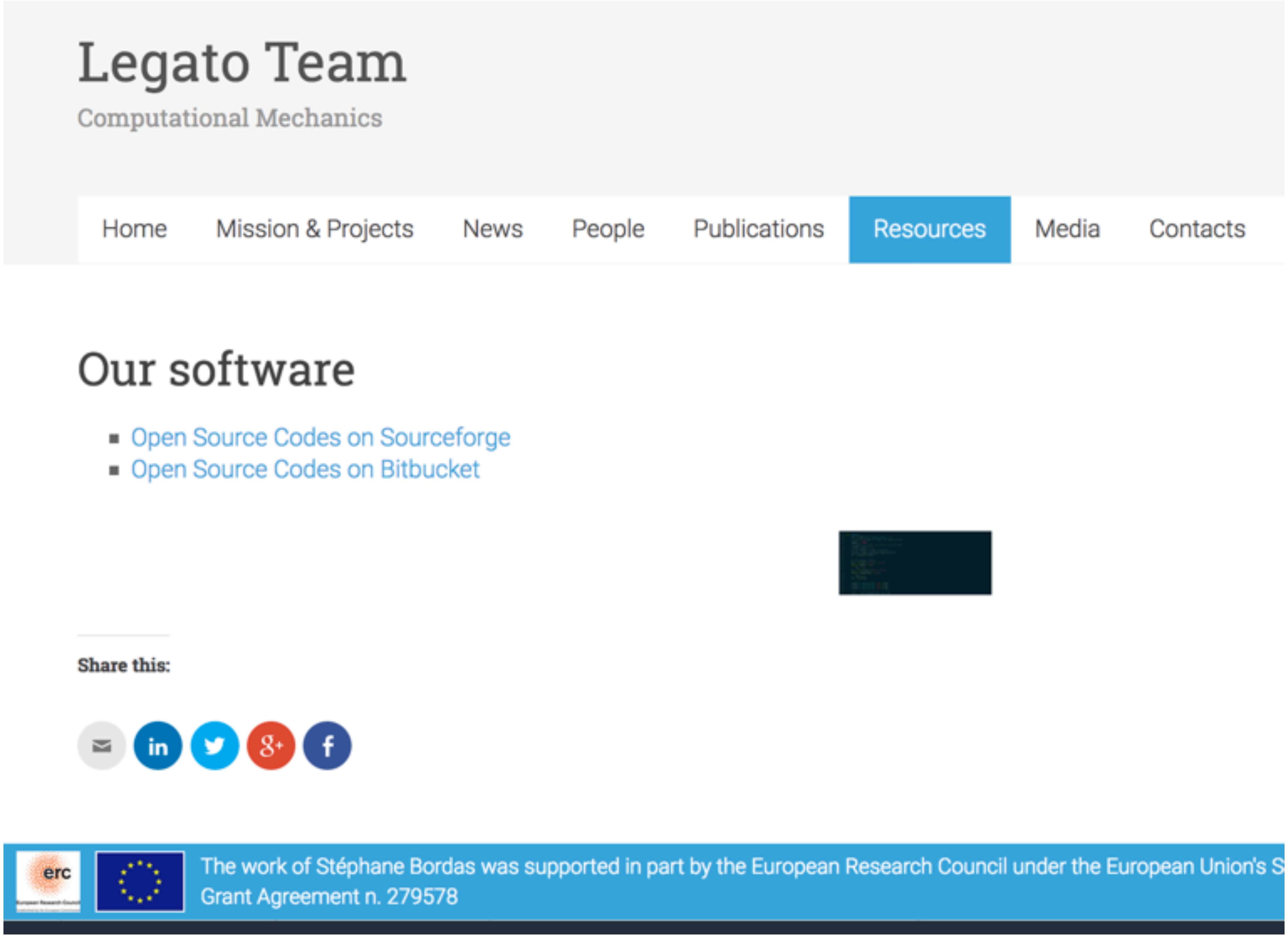
Computational Mechanics



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Our software

- Open Source Codes on Sourceforge
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The work of Stéphane Bordas was supported in part by the European Research Council under the European Union's S
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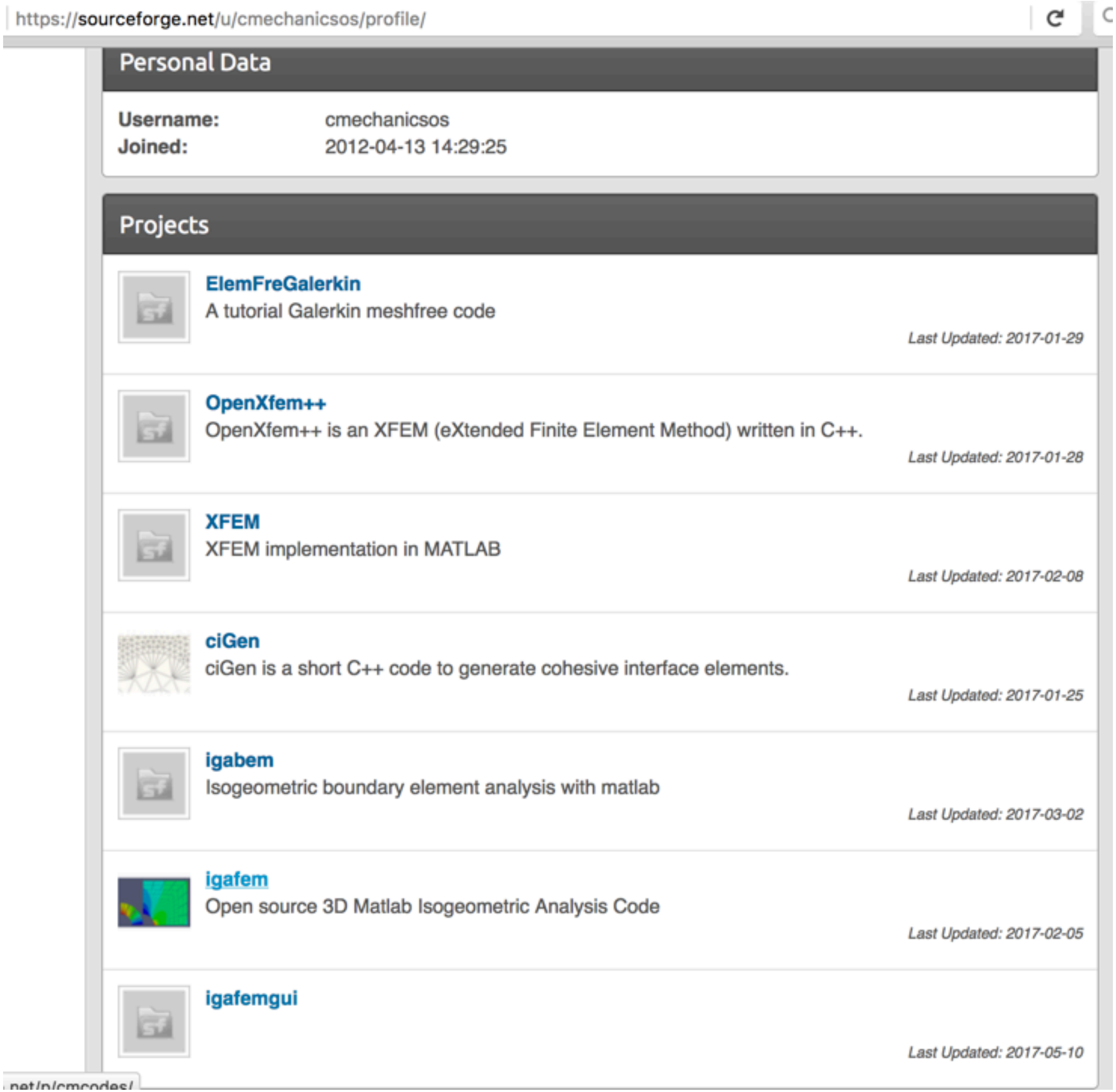


https://sourceforge.net/u/cmechanicsos/profile/

Personal Data

Username:	cmechanicsos
Joined:	2012-04-13 14:29:25

Projects

ElemFreGalerkin
A tutorial Galerkin meshfree code
Last Updated: 2017-01-29

OpenXfem++
OpenXfem++ is an XFEM (eXtended Finite Element Method) written in C++.
Last Updated: 2017-01-28

XFEM
XFEM implementation in MATLAB
Last Updated: 2017-02-08

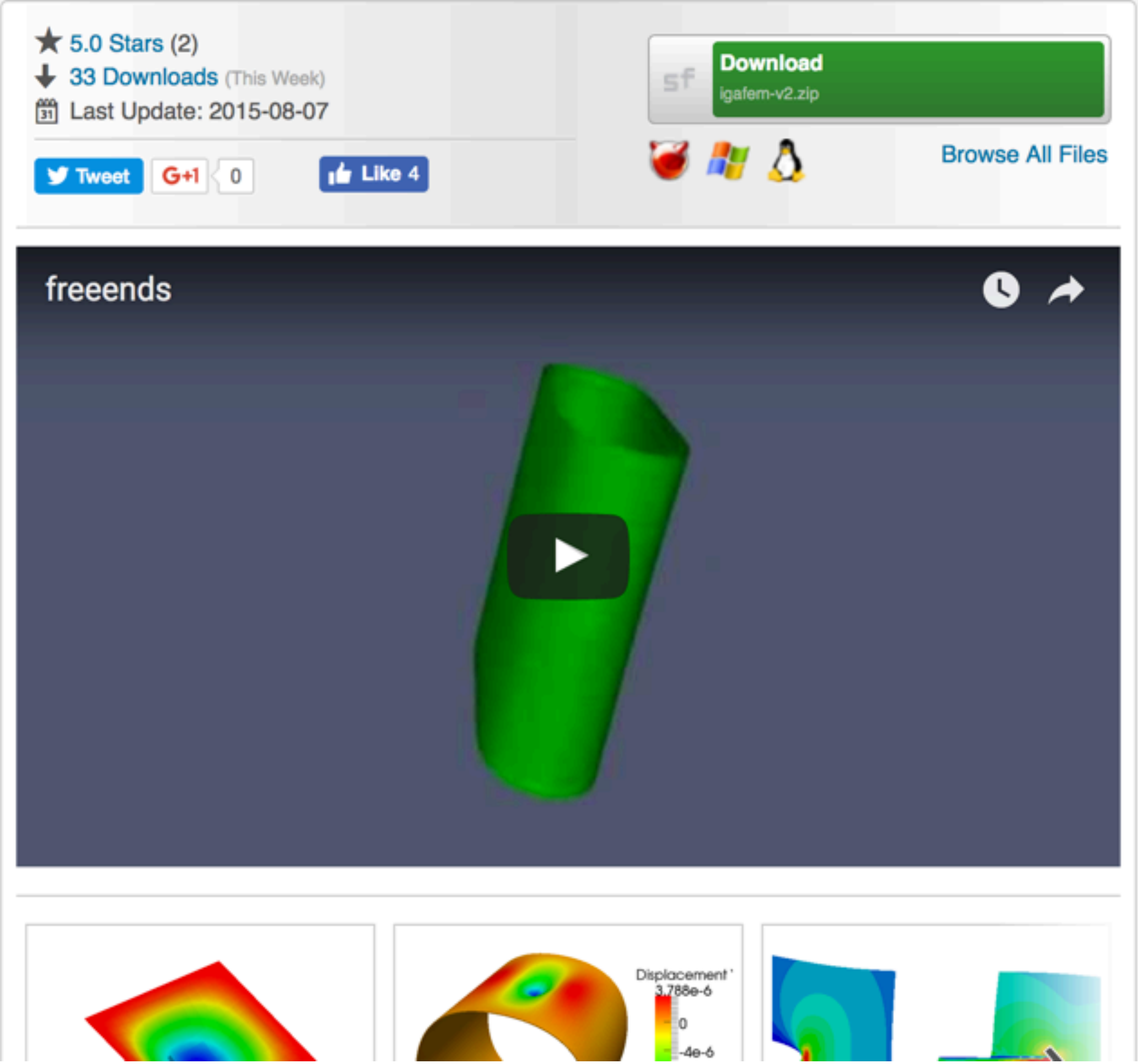
ciGen
ciGen is a short C++ code to generate cohesive interface elements.
Last Updated: 2017-01-25

igabem
Isogeometric boundary element analysis with matlab
Last Updated: 2017-03-02

igafem
Open source 3D Matlab Isogeometric Analysis Code
Last Updated: 2017-02-05

igafemgui
Last Updated: 2017-05-10

.net/p/cmcodes/



5.0 Stars (2)

33 Downloads (This Week)

Last Update: 2015-08-07

Thank you for your attention!

You can **download** these **slides** here

http://hdl.handle.net/10993/31487

or

http://orbilu.uni.lu/bitstream/10993/31487/1/XDMS_2017_Bordas.pdf

or

http://orbilu.uni.lu/bitstream/10993/31487/4/
XDMS_2017_Bordas_AverageResolution_withLinks.pdf

or/and email me stephane.bordas@alum.northwestern.edu

and check out legato-team.eu

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OPTIMIZATION
SIMULATION
Guidance
Design of Implants & Prosthetics
Diagnosis
Surgical Training
Prognosis
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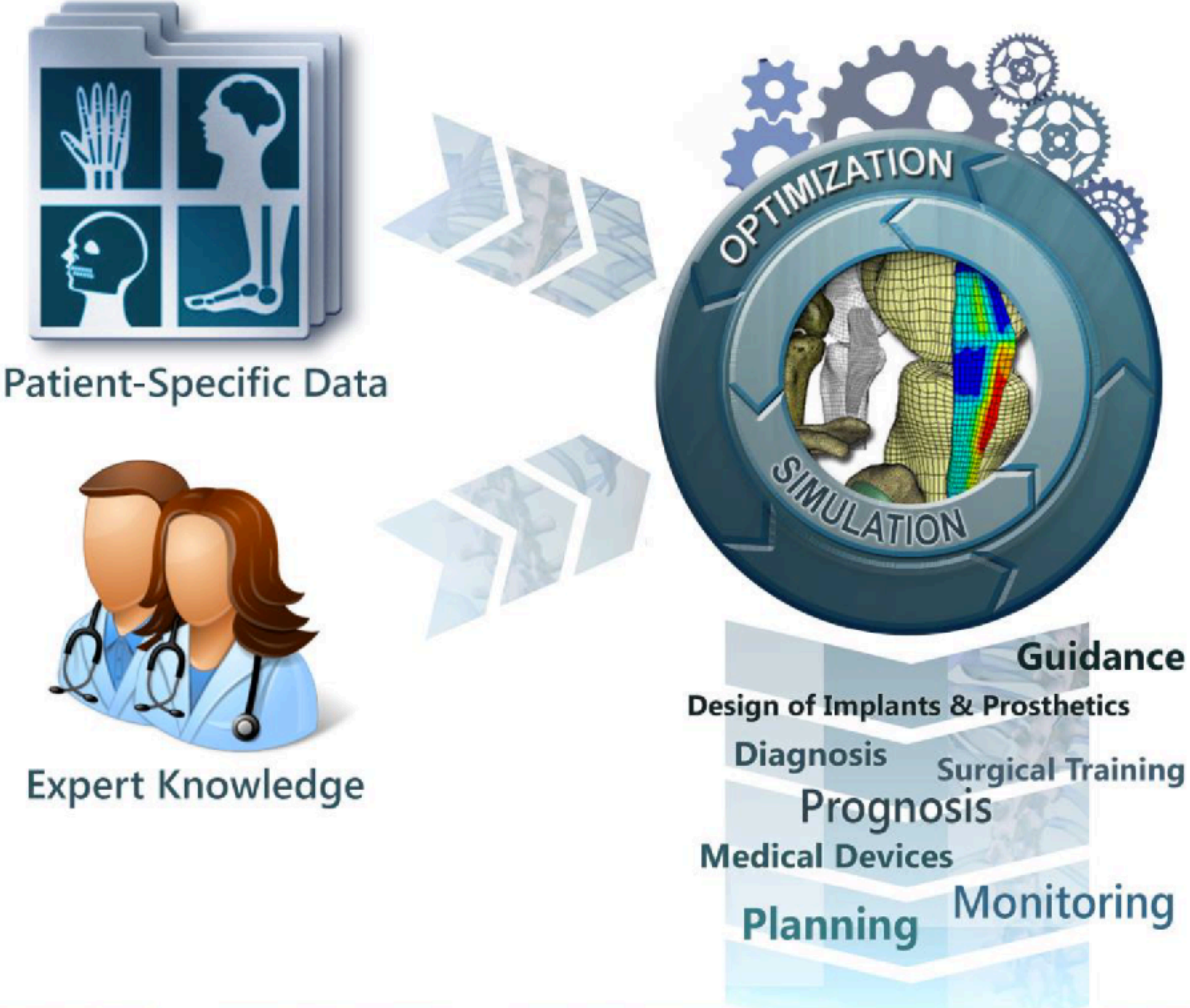
Cardiovascular Devices
Scoliosis
Eye surgery
Neurology
Spine Braces
Hip growth
Prostate Cancer
Intraoperative radiotherapy
Surgical guidance
Soft organ diagnosis
Dental prostheses
Breast Cancer
Surgical navigation
Surgical planning
Apps



Patient-Specific Data

Expert Knowledge

Guidance

Design of Implants & Prosthetics

Diagnosis

Surgical Training

Prognosis

Medical Devices

Monitoring

Planning

Eye surgery

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Neurology

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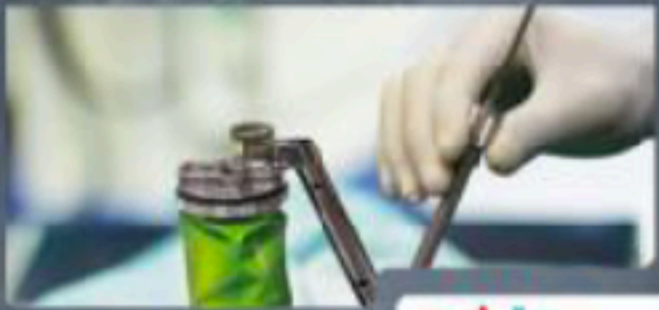
Spine Braces

AnatoScope

Intraoperative radiotherapy

gmv

Surgical guidance



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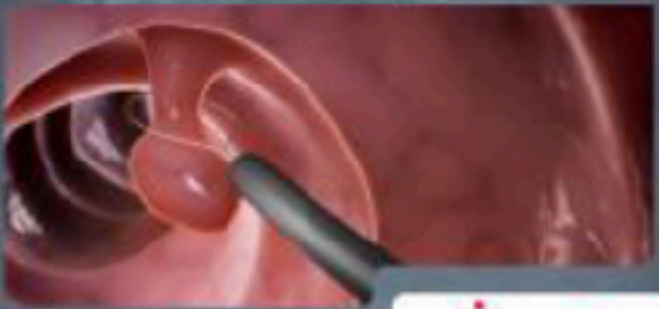
Soft organ diagnosis

Cardiff University simpleware

Surgical navigation



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Surgical planning

Cardiff University InSimo

Apps

Cardiovascular Devices

NEXT LIMIT TECHNOLOGIES

Scoliosis

AALBORG UNIVERSITET

Hvidovre Hospital

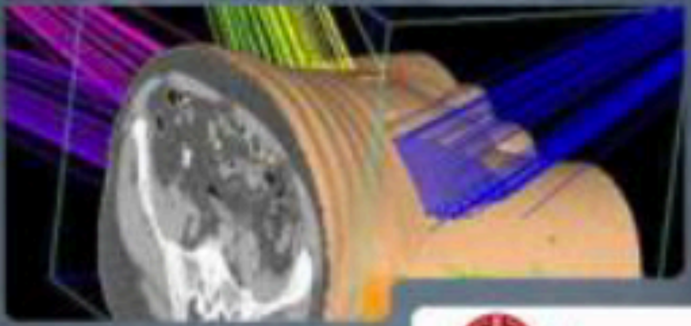
Eye surgery

InSimo

Hip growth

Hvidovre Hospital

Prostate Cancer



Kitware

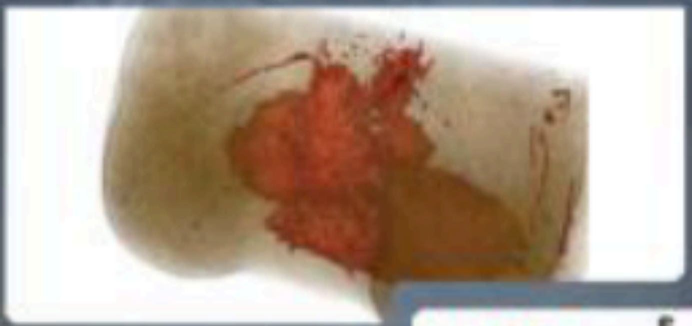
Intraoperative radiotherapy

gmv

Dental prostheses

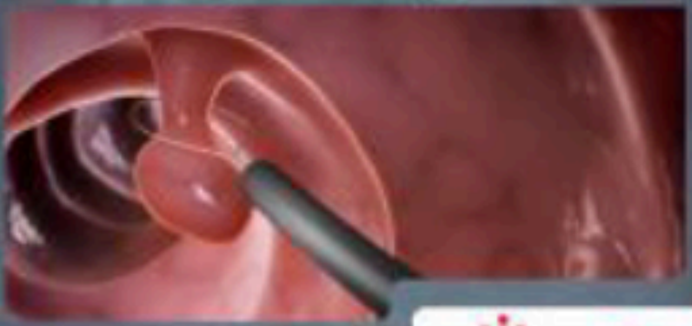
3shape

Breast Cancer



Anato Scope

Surgical navigation



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Multi-scale fracture and model order reduction

Pierre Kerfriden, Lars Beex, Jack Hale, Olivier Goury, Daniel Alves Paladim, Elisa Schenone, Davide Baroli, Thanh Tung Nguyen, Hoang Khac Chi, Timon Rabczuk

Advanced discretisation techniques

Elena Atroshchenko, Danas Sutula, Xuan Peng, Haojie Lian, Peng Yu, Qingyuan Hu, Sundararajan Natarajan, Nguyen-Vinh Phu

Error estimation

Pierre Kerfriden, Satyendra Tomar, Daniel Alves Paladim, Andrés Gonzalez Estrada

Biomechanics applications

Alexandre Bilger, Hadrien Courtecuisse, Bui Huu Phuoc

Related publications

Fracture of homogeneous materials

https://publications.uni.lu/bitstream/10993/10039/1/2013stressAnlaysisWithoutMeshingIGABEM3D_ICE.pdf

https://orbilu.uni.lu/bitstream/10993/14029/2/superGA_upd26_05_2014.pdf

https://arxiv.org/pdf/1210.8216

https://arxiv.org/pdf/1610.01694

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Uncertainty quantification for soft tissue biomechanics

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Real-time simulation and error control

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Bayesian parameter identification in mechanics

http://orbilu.uni.lu/bitstream/10993/29561/3/template.pdf

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Shape optimisation directly from CAD
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