Probability signatures of multistate systems made up of two-state components

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Outline of the talk

- A few results about a single system with binary components and binary output;
- Several systems with binary components and binary output;
- A single system with binary components and multistate output;
- Outlook and further questions

Semi-Coherent systems : notation

- $C = \{c_1, \ldots, c_n\}$: n binary components (two possible *states*);
- they are connected to form a system;
- Basic examples : series, parallel, bridge, k-out-of-n systems...

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- they are connected to form a system;
- Basic examples : series, parallel, bridge, k-out-of-n systems...
- With each component c_k, ($k \in [n] = \{1, \ldots, n\}$), we associate a *Boolean variable*

$$x_k = \begin{cases} 0 & \text{if } c_k \text{ is in a failed state} \\ 1 & \text{if } c_k \text{ is in function.} \end{cases}$$

  The Boolean vector $\mathbf{x} = (x_1, \ldots, x_n)$ encodes the states of all components.
- We can also consider the set A of components in function :
  $\mathbf{x} = (1, 0, 1, 0, 1)$ corresponds to $A = \{1, 3, 5\}$.
  So the states are represented by $\mathbf{x} \in \{0, 1\}^n$ or $A \subset [n] = \{1, \ldots, n\}$.
- The *structure function* defines the state of the system :

$$\phi : \{0, 1\}^n \to \{0, 1\} : \mathbf{x} = (x_1, \ldots, x_n) \mapsto x_S = \phi(x_1, \ldots, x_n).$$

Required properties of ϕ

- $\phi : \{0,1\}^n \to \{0,1\}$ or $\phi : \mathcal{P}([n]) \to \{0,1\}$;
- $\phi(0,\ldots,0) = \phi(\varnothing) = 0$;
- $\phi(1,\ldots,1) = \phi([n]) = 1$;
- ϕ is increasing (nondecreasing) :

$$A \subset B \Rightarrow \phi(A) \leqslant \phi(B).$$

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- This system is coherent if in addition, all the variables are *essential* in ϕ. Here in general, we do not require this property.

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- Every function with these properties is the structure function of a semi-coherent system.
- This system is coherent if in addition, all the variables are *essential* in ϕ. Here in general, we do not require this property.
- **Example :** a *k-out-of-n system* is a system that fails with the k-th failure :

$$\phi(A) = 1 \quad \text{iff} \quad |A| > n - k.$$

or $\phi(\mathbf{x}) = x_{k:n}$ (series are 1-out-of-n, parallel are n-out-of-n).

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Some notation concerning probability

1. T_k : random lifetime of component c_k.
2. For $t > 0$, $X_k(t)$: rand. state of comp. c_k at time t (Bernoulli var.).
3. T_S : system random lifetime.
4. $X_S(t)$: random state of the system at time t (Bernoulli var.).
5. Joint cumulative distribution of component lifetimes :

$$F(t_1, \ldots, t_n) = \Pr(T_1 \leqslant t_1, \ldots, T_n \leqslant t_n).$$

6. Order statistics $T_{1:n}, \ldots, T_{n:n}$, such that (when there are no ties)

$$T_{1:n} < \cdots < T_{n:n}.$$

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Classical hypotheses :

- F is absolutely continuous; the lifetimes are i.i.d.
- or the lifetimes are exchangeable;
- or ties have null probability (no ties) :

$$\Pr(T_k = T_\ell) = 0, \quad \text{when } k \neq \ell.$$

Structure signatures

Definition (Samaniego (1985))

Consider a system $\mathcal{S} = (n, \phi, F)$, where F is absolutely continuous i.i.d. The *structure signature* of $\mathcal{S}$ is the n-tuple $\boldsymbol{s} = (s_1, \ldots, s_n)$, where

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Theorem: $\boldsymbol{s}$ does not depend on F. It is a combinatorial object.

Proposition (Boland (2001))

If the components have continuous i.i.d. lifetimes, we have for $k \leqslant n$

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{|A|=n-k+1} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{|A|=n-k} \phi(A)$$

where for $A \subset [n]$, $|A|$ is the cardinality of A.

Signatures

Both terms that appear in the formula have a meaning :

$$\overline{S}_k = \frac{1}{\binom{n}{n-k}} \sum_{|A|=n-k} \phi(A) \;=\; \sum_{i=k+1}^{n} s_i = \Pr(T_S > T_{k:n}).$$

It is the kth component of the *tail structure signature*.
For convenience we set $\overline{S}_0 = 1$ and $\overline{S}_n = 0$ and we get

$$s_k = \overline{S}_{k-1} - \overline{S}_k, \quad \forall\, k : 1 \leqslant k \leqslant n.$$

For a system $\mathcal{S} = (n, \phi, F)$ such that F has no ties, we can define

- The structure signature $\boldsymbol{s}$ (through Boland's formula, or replacing F by an i.i.d. F_0);
- The probability signature $\boldsymbol{p} = (p_1, \ldots, p_n)$, defined (Navarro et al. 2010) by:

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$$p_k = \Pr(T_S = T_{k:n}).$$

- The probability signature may depend both on F and ϕ.
- So with $\mathcal{S}$ we can associate two objects $\boldsymbol{s}$ and $\boldsymbol{p}$.

The relative quality function

The *relative quality function* is defined by

$$q : \mathcal{P}([n]) \to \mathbb{R} : A \mapsto q(A) = \Pr\Big(\max_{k \notin A} T_k < \min_{j \in A} T_j\Big),$$

and $q(\varnothing) = q([n]) = 1$. So $q(A)$ measures the quality of elements of A. It is related to the probability that, in the degradation process, the set of components in function "passes through A."

Proposition (Marichal, M. (2011))

If F has no ties, then the probability signature is given by

$$p_k = \sum_{|A|=n-k+1} q(A)\,\phi(A) - \sum_{|A|=n-k} q(A)\,\phi(A).$$

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$$p_k = \sum_{|A|=n-k+1} q(A)\,\phi(A) - \sum_{|A|=n-k} q(A)\,\phi(A).$$

Here again both terms have a direct meaning

$$\overline{P}_k = \sum_{|A|=n-k} q(A)\,\phi(A) \;=\; \sum_{i=k+1}^{n} p_i \;=\; \Pr(T_S > T_{k:n})$$

is the k-th coordinate of the *tail probability signature*.

Decomposition of reliability

The reliability is $\overline{F}_{\mathcal{S}}(t) = \Pr(T_S > t)$. Set $\overline{F}_{k:n}(t) = \Pr(T_{k:n} > t)$.

Proposition (Samaniego (1985))

If F is absolutely continuous and i.i.d. , then

$$\overline{F}_{\mathcal{S}}(t) = \sum_{k=1}^{n} s_k \, \overline{F}_{k:n}(t), \tag{1}$$

for all $t > 0$, and every coherent system $\mathcal{S} = ([n], \phi, F)$.

- First extensions : Navarro-Rychlik (2007) for exchangeable components.

Proposition (Marichal, M., Waldhauser (2011))

This decomposition holds at time t for every (semi-)coherent structure ϕ if and only if the state variables $X_1(t), \ldots, X_n(t)$ are exchangeable.

Rmk : Needs a combinatorial proof. Same kind of results for decomp. w.r.t. $\boldsymbol{p}$.

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Several binary systems

- Given (C, F), we may consider several systems $\mathcal{S}_1 = (C, \phi_1, F)$,...,$\mathcal{S}_m = (C, \phi_m, F)$.
- Navarro et al (2010, 2013) proposed to analyze the joint behavior of these systems. They obtained a signature based decomp. of reliability in the i.i.d. continuous setting.
- Set $m = 2$ for simplicity. Assume that F has no ties.

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Definition

- The *joint probability signature* of two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ is the square matrix $\mathbf{p}$ whose (k, l)-entry is the probability

$$p_{k,l} = \Pr(T_{\mathcal{S}_1} = T_{k:n} \text{ and } T_{\mathcal{S}_2} = T_{l:n}), \qquad k, l = 1, \ldots, n.$$

- The *joint structure signature* of systems $\mathcal{S}_1$ and $\mathcal{S}_2$ is $\mathbf{s}$ defined by

$$s_{k,l} = \Pr(T_{\mathcal{S}_1} = T_{k:n} \text{ and } T_{\mathcal{S}_2} = T_{l:n}), \qquad k, l = 1, \ldots, n,$$

when F is replaced by some i.i.d continuous distribution.

- The joint reliability is $\overline{F}_{\mathcal{S}_1,\mathcal{S}_2}(t_1, t_2) = \Pr(T_{\mathcal{S}_1} > t_1 \text{ and } T_{\mathcal{S}_2} > t_2), \qquad t_1, t_2 \geqslant 0,$

Signature matrices II

We use the tail version of signatures, and concentrate on $\boldsymbol{p}$.

Definition

The *joint tail probability signature* is the square matrix $\overline{\mathbf{P}}$ of order $n+1$ whose (k,l)-entry is the probability

$$\overline{P}_{k,l} \;=\; \Pr(T_{\mathcal{S}_1} > T_{k:n} \text{ and } T_{\mathcal{S}_2} > T_{l:n}), \qquad k,l = 0,\ldots,n.$$

We have standard conversion formulas:

- $\overline{P}_{k,l} = \sum_{i=k+1}^{n} \sum_{j=l+1}^{n} p_{i,j}$, for $k,l = 0,\ldots,n$,
- $p_{k,l} \;=\; \overline{P}_{k-1,l-1} - \overline{P}_{k,l-1} - \overline{P}_{k-1,l} + \overline{P}_{k,l}$, for $k,l = 1,\ldots,n$.

We first want to compute $\overline{\mathbf{P}}$. We first generalize q to the case of several systems.

The joint relative quality function

Definition

The *joint relative quality function* associated with the joint c.d.f. F is the symmetric function $q\colon 2^{[n]} \times 2^{[n]} \to [0,1]$ defined by

$$q(A,B) \;=\; \Pr\Big(\max_{i\in C\setminus A} T_i < \min_{j\in A} T_j \;\text{ and }\; \max_{i\in C\setminus B} T_i < \min_{j\in B} T_j\Big),$$

Here again, we can interpret q as

- The simultaneous quality of components in A and in B.
- A probability that in the degradation process, the set of components in function passes through A and through B.

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In many situations (including i.i.d. or exchangeability), q, reduces to q_0 defined by

$$q_0(A,B) \;=\; \begin{cases} \frac{(n-|A|)!\,(|A|-|B|)!\,|B|!}{n!} & \text{if } B\subseteq A,\\ \frac{(n-|B|)!\,(|B|-|A|)!\,|A|!}{n!} & \text{if } A\subseteq B,\\ 0 & \text{otherwise.}\end{cases}$$

Computation of signatures

Theorem

For every $k, l \in \{0, \ldots, n\}$ *we have*

$$\overline{P}_{k,l} = \sum_{|A|=n-k} \sum_{|B|=n-l} q(A,B)\phi_1(A)\phi_2(B).$$

In particular

Theorem

For every $k, l \in \{0, \ldots, n\}$ *we have*

$$\overline{S}_{k,l} = \sum_{|A|=n-k} \sum_{|B|=n-l} q_0(A,B)\phi_1(A)\phi_2(B).$$

Decomposition of reliability

- We analyze the decomposition of $\overline{F}_{\mathcal{S}_1,\mathcal{S}_2}(t_1,t_2)$ with respect to $\boldsymbol{s}$ and

$$\overline{F}_{k:n,l:n}(t_1,t_2) \;=\; \Pr(T_{k:n} > t_1 \text{ and } T_{l:n} > t_2).$$

- The result depends on the state vectors $\mathbf{X}(t_1) = (X_1(t_1), \ldots, X_n(t_1))$ and $\mathbf{X}(t_2) = (X_1(t_2), \ldots, X_n(t_2))$ at times $t_1 \geqslant 0$ and $t_2 \geqslant 0$.
- It is related to the exchangeability of the columns of $\binom{\mathbf{X}(t_1)}{\mathbf{X}(t_2)}$, i.e.

$$\Pr(\begin{pmatrix}\mathbf{X}(t_1)\\ \mathbf{X}(t_2)\end{pmatrix} = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}) \;=\; \Pr(\begin{pmatrix}\mathbf{X}(t_1)\\ \mathbf{X}(t_2)\end{pmatrix} = \begin{pmatrix}\sigma(\mathbf{x})\\ \sigma(\mathbf{y})\end{pmatrix}) \qquad (2)$$

for any $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ and any permutation σ of $\{1,\ldots,n\}$.

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for any $\mathbf{x},\mathbf{y} \in \{0,1\}^n$ and any permutation σ of $\{1,\ldots,n\}$.

Theorem

Let $t_1, t_2 \geqslant 0$ be fixed. If the joint c.d.f. F satisfies condition (2) for any nonzero $\mathbf{x},\mathbf{y} \in \{0,1\}^n$, then we have

$$\overline{F}_{\mathcal{S}_1,\mathcal{S}_2}(t_1,t_2) \;=\; \sum_{k=1}^{n}\sum_{l=1}^{n} s_{k,l}\,\overline{F}_{k:n,l:n}(t_1,t_2) \tag{3}$$

for any semicoherent systems $\mathcal{S}_1$ and $\mathcal{S}_2$.

More results

Theorem

If we have (3) for any coherent systems $\mathcal{S}_1$ and $\mathcal{S}_2$, then the joint c.d.f. F satisfies condition (2) for any nonzero $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$.

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Proposition

(a) *If the component lifetimes $T_1, \ldots, T_n$ are exchangeable, then condition (2) holds for any $t_1, t_2 \geqslant 0$ and any $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$.*

(b) *If condition (2) holds for some $0 \leqslant t_1 < t_2$ and any nonzero $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$, then the component states $X_1(t_2)$, ..., $X_n(t_2)$ at time t_2 are exchangeable.*

(c) *Condition (2) holds for every $t_1, t_2 > 0$ and every nonzero $\mathbf{x}, \mathbf{y}$ if and only if it holds for every $t_1, t_2 > 0$ and every $\mathbf{x}, \mathbf{y}$. Moreover in this case, the component states $X_1(t)$, ..., $X_n(t)$ are exchangeable at every time $t > 0$.*

The same kind of results is obtained for a decomposition with $\boldsymbol{p}$.

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Multistate systems: Definitions

Definition

- An $(m+1)$-state system made of binary comp. is a triple $\mathcal{S} = (C, \phi, F)$, where C and F are as usual and where $\phi\colon \{0,1\}^n \to \{0,\ldots,m\}$ is nondecreasing in each variable and satisfies the boundary conditions $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = m$.
- The system state at time t is given by $X_{\mathcal{S}}(t) = \phi(\mathbf{X}(t))$.
- We define he lifetimes at different levels $T_{\mathcal{S}}^{\geqslant 1}, \ldots, T_{\mathcal{S}}^{\geqslant m}$, by

$$T_{\mathcal{S}}^{\geqslant k} \; > \; t \quad \Leftrightarrow \quad \phi(\mathbf{X}(t)) \; \geqslant \; k\,, \qquad k = 1, \ldots, m.$$

- We have a reliability at states $\geqslant k$:

$$\overline{F}_{\mathcal{S}}^{\geqslant k}(t) \; = \; \Pr(T_{\mathcal{S}}^{\geqslant k} > t), \quad t \geqslant 0.$$

- And an overall reliability

$$\overline{F}_{\mathcal{S}}(t_1, \ldots, t_m) \; = \; \Pr(T_{\mathcal{S}}^{\geqslant 1} > t_1 \;, \ldots, \; T_{\mathcal{S}}^{\geqslant m} > t_m), \quad t_1, \ldots, t_m \geqslant 0.$$

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Signatures of MSS with binary comp.

Set $m = 2$ for simplicity.

Definition

The *probability signature* of a 3-state system $\mathcal{S} = (C, \phi, F)$ is the matrix $\mathfrak{p}$ defined by

$$\mathfrak{p}_{k,l} \;=\; \Pr(T_{\mathcal{S}}^{\geqslant 1} = T_{k:n} \text{ and } T_{\mathcal{S}}^{\geqslant 2} = T_{l:n}), \qquad k, l = 1, \ldots, n, \qquad (4)$$

The *tail probability signature* of a 3-state system $\mathcal{S} = (C, \phi, F)$ is the matrix $\overline{\mathfrak{P}}$ defined by

$$\overline{\mathfrak{P}}_{k,l} \;=\; \Pr(T_{\mathcal{S}}^{\geqslant 1} > T_{k:n} \text{ and } T_{\mathcal{S}}^{\geqslant 2} > T_{l:n}), \qquad k, l = 0, \ldots, n. \qquad (5)$$

These concepts were used in Gertsbakh et al. (2012) and Da and Hu (2013). They are called "bivariate signatures".
We want to link these concepts with the previous ones. This is done via the "decomposition principle".

Decomposition principle I

This is a simple observation. See for instance Block and Savits (1982), or Natvig (1982, 2011).

Proposition

Any semicoherent structure function $\phi\colon \{0,1\}^n \to \{0,\ldots,m\}$ *decomposes in a unique way as a sum*

$$\phi = \sum_{k=1}^{m} \phi_{\langle k\rangle}, \qquad (6)$$

where $\phi_{\langle k\rangle}\colon \{0,1\}^n \to \{0,1\}$ $(k=1,\ldots,m)$ *are semicoherent structure functions such that* $\phi_{\langle 1\rangle}(\mathbf{x}) \geqslant \cdots \geqslant \phi_{\langle m\rangle}(\mathbf{x})$ *for all* $\mathbf{x} \in \{0,1\}^n$.

Actually, we have

$$\phi_{\langle k\rangle}(\mathbf{x}) = 1 \Leftrightarrow \phi(\mathbf{x}) \geqslant k.$$

So the functions $\phi_{\langle k\rangle}$ can be seen as "layers" of ϕ.

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Decomposition principle II

Definition

Given a semicoherent $(m+1)$-state system $\mathcal{S} = (C, \phi, F)$, with Boolean decomposition $\phi = \sum_k \phi_{\langle k \rangle}$, we define the semicoherent systems $\mathcal{S}_k = (C, \phi_{\langle k \rangle}, F)$ for $k = 1, \ldots, m$.

Proposition (Decomposition)

Any semicoherent $(m+1)$-state system $\mathcal{S}$ made up of two-state components can be additively decomposed into m semicoherent two-state systems $\mathcal{S}_1, \ldots, \mathcal{S}_m$ constructed on the same set of components, with the property that for any $k \in \{1, \ldots, m\}$ the lifetime of $\mathcal{S}_k$ is the time at which $\mathcal{S}$ deteriorates from a state $\geqslant k$ to a state $< k$.

Theorem

We have $T_{\mathcal{S}}^{\geqslant k} = T_{\mathcal{S}_k}$ and $\overline{F}_{\mathcal{S}}^{\geqslant k} = \overline{F}_{\mathcal{S}_k}$ for $k = 1, \ldots, m$. Moreover we have $\mathfrak{p} = \mathbf{p}$, $\overline{\mathfrak{P}} = \overline{\mathbf{P}}$, and $\overline{F}_{\mathcal{S}} = \overline{F}_{\mathcal{S}_1, \mathcal{S}_2}$.

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A short example

Consider a 3-state system with 3 components given by ϕ satisfying

$$\phi(1,1,1) = 2, \quad \phi(1,1,0) = 1, \quad \phi(1,0,1) = 1$$

and $\phi(\mathbf{x}) = 0$ for all other $\mathbf{x}$.

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and $\phi(\mathbf{x}) = 0$ for all other $\mathbf{x}$.
Then $\phi_{\langle 2\rangle}(\mathbf{x}) = 1$ iff $\mathbf{x} = (1,1,1)$, i.e,

$$\phi_{\langle 2\rangle}(x_1, x_2, x_3) \;=\; x_1 \wedge x_2 \wedge x_3.$$

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Moreover $\phi_{\langle 1\rangle}(\mathbf{x}) = 1$ iff $\mathbf{x} \in \{(1,1,1),(1,1,0),(1,0,1)\}$ so

$$\phi_{\langle 1\rangle}(x_1, x_2, x_3) \;=\; x_1 \wedge (x_2 \vee x_3).$$

A short example

Consider a 3-state system with 3 components given by ϕ satisfying

$$\phi(1,1,1) = 2, \quad \phi(1,1,0) = 1, \quad \phi(1,0,1) = 1$$

and $\phi(\mathbf{x}) = 0$ for all other $\mathbf{x}$.
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One can check that $\phi = \phi_{\langle 1\rangle} + \phi_{\langle 2\rangle}$.
Note that the systems goes from state 2 to state 0 when $\mathbf{x}$ goes from $(1,1,1)$ to $(0,1,1)$

A possible use

Theorem

For every $k, l \in \{0, \ldots, n\}$ we have

$$\overline{\mathfrak{P}}_{k,l} \;=\; \sum_{\substack{A\subseteq C\\|A|=n-k}} \; \sum_{\substack{B\subseteq C\\|B|=n-l}} q(A,B)\,\phi_{\langle 1\rangle}(A)\phi_{\langle 2\rangle}(B).$$

Proposition

If, for any $t_1, t_2 \geqslant 0$, the joint c.d.f. F satisfies condition (2) for any nonzero $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$ and any permutation σ on $[n]$, then we have

$$\overline{F}_{\mathcal{S}}(t_1, t_2) \;=\; \sum_{k=1}^{n} \sum_{l=1}^{n} s_{k,l}\,\overline{F}_{k:n,l:n}(t_1, t_2), \qquad (7)$$

where the coefficients $s_{k,l}$ correspond to the structure signature of the pair of systems $\mathcal{S}_1$ and $\mathcal{S}_2$.

Questions

- Are signatures of multistate systems related to least squres approximations ?
- Can we generalize these nice formulas to multistate systems with multistate components ?
- Can we generalize the results with modular decomposition, in these geenarl settings ?

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Thank you for your attention

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