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G^1 construction of parameterization with unstructured Bézier elements

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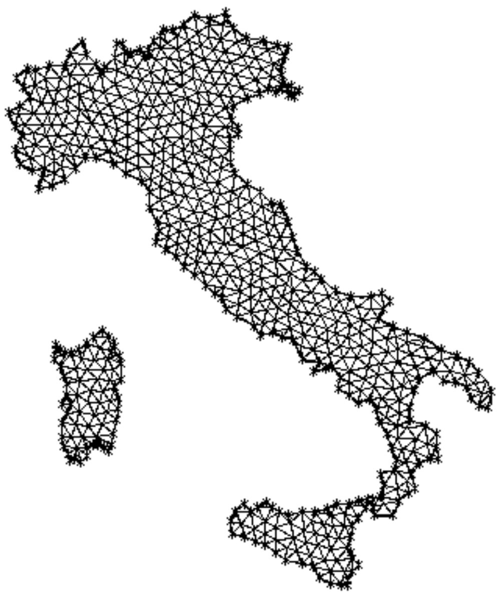
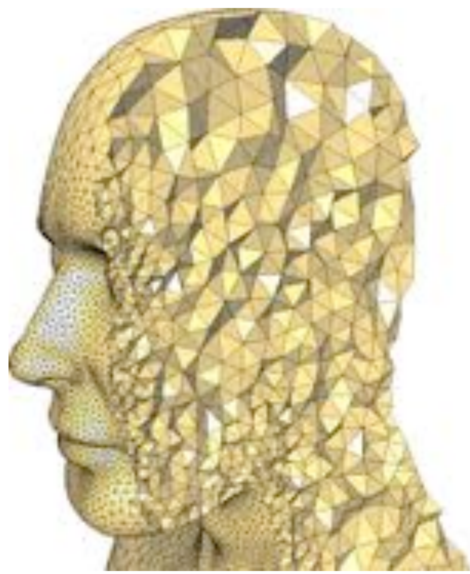
Outline

- Parameterization in isogeometric analysis
- Analysis-suitable G^1 planar parameterization from complex boundary

IGA-meshing

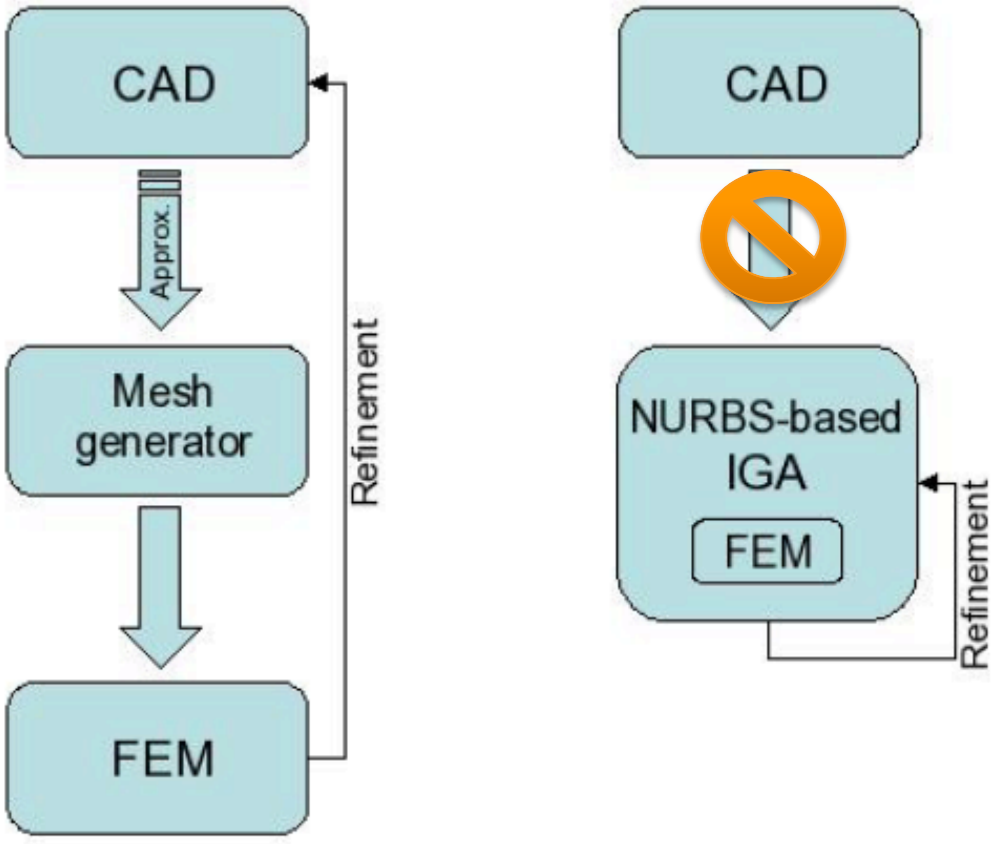
- IGA is a spline-version of FEA
- Mesh generation in FEA
- CAD models usually define only the boundary of a solid, but the application of isogeometric analysis requires a volumetric representation
- As it is pointed by Cotrell et al., the most significant challenge facing isogeometric analysis is developing three- dimensional spline parameterizations from boundary information



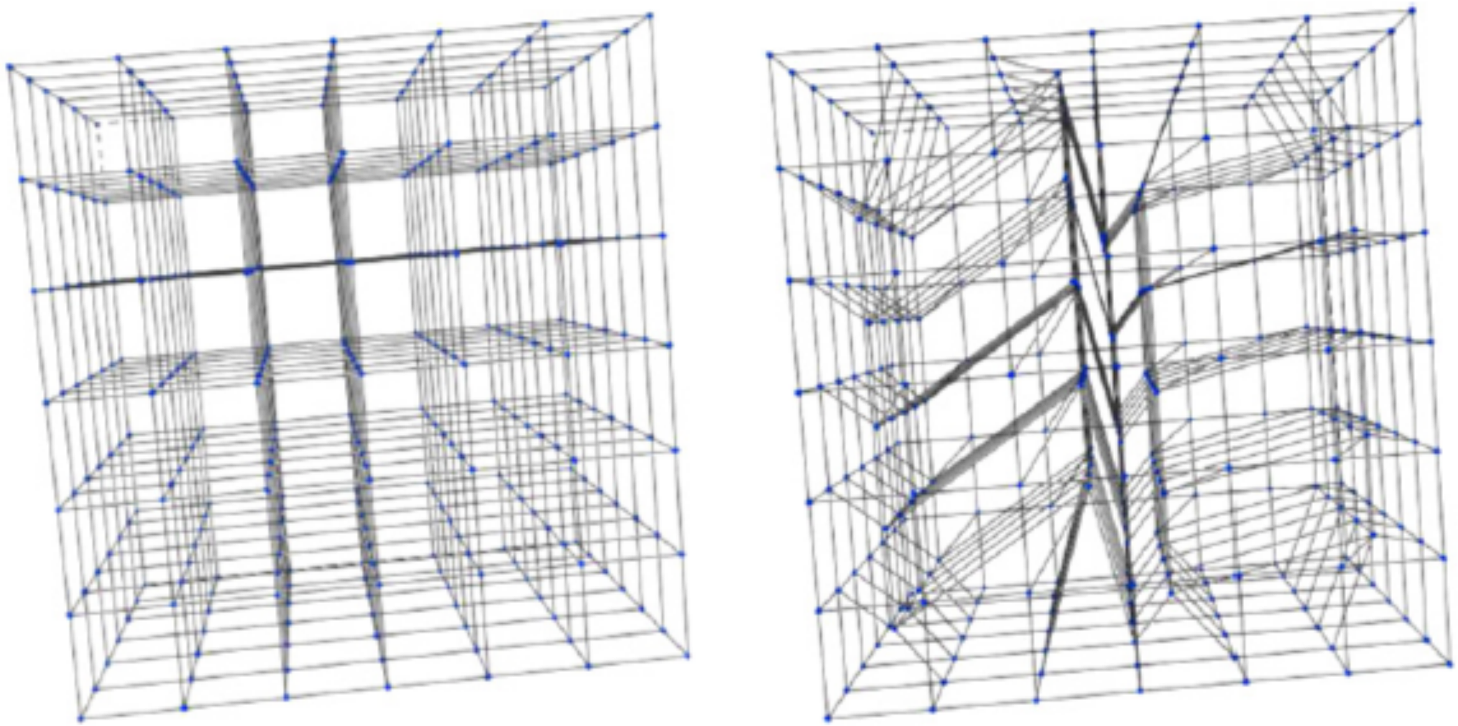


Parameterization of computational domain

- Open problem

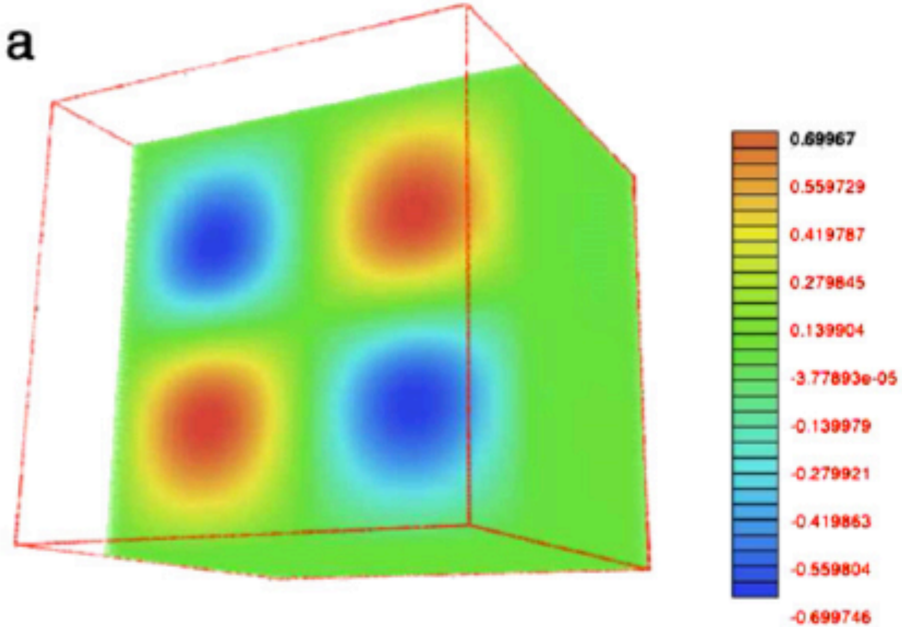


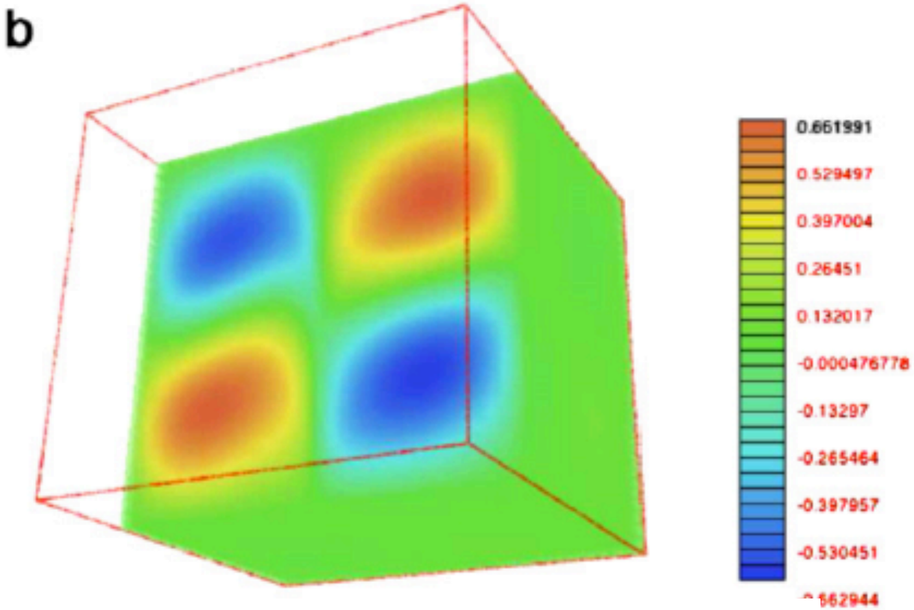
计算域参数化质量对分析结果影响 CAD2013

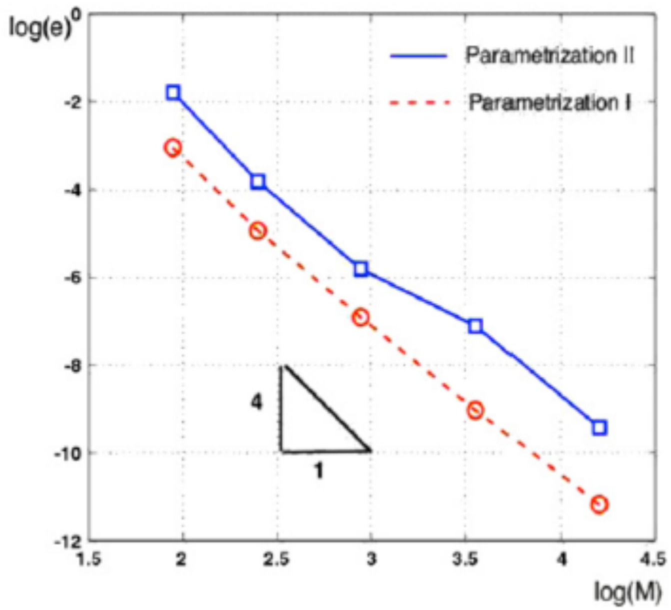


(a) Control point placement I.　　(b) Control point placement II.

计算域参数化质量对分析结果影响 CAD2013

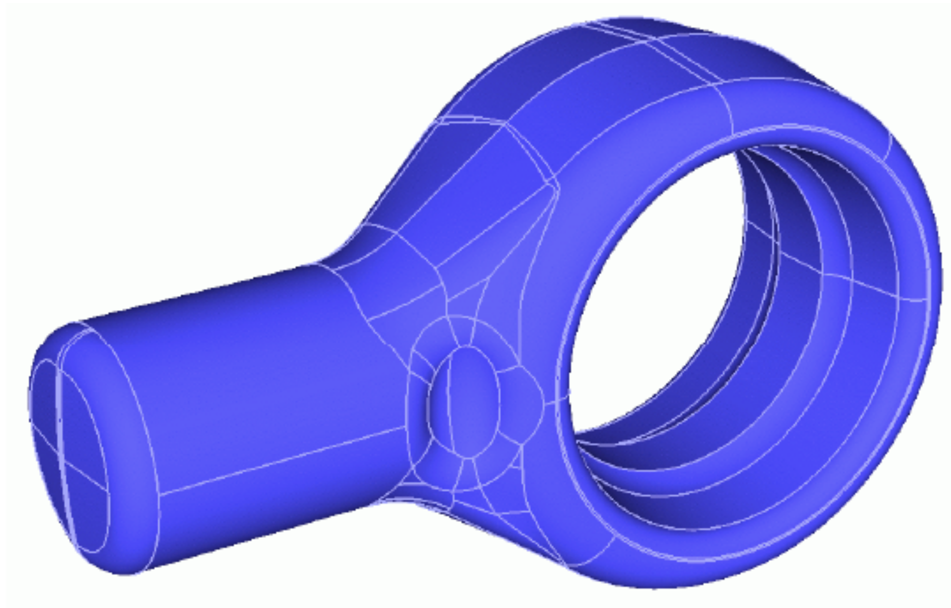






Main difficulties

- Trimmed surface
- Complex topology
- Analysis-suitable



Related work on parameterization for IGA

- Analysis-aware optimal parameterization

  E. Cohen et al.(CMAME, 2010) , Xu et al.(CMAME,2011), Pilgerstorfer et al (CMAME, 2013)

- Volumetric spline parameterization from boundary triangulation

  T. Martin et al.(CMAME, 2009), Zhang et al.(CMAME, 2012).

- Analysis-suitable planar parameterization from spline boundary

  Xu et al.(CAD 2013), Gravessen et al.(CMAME, 2014）, Xu et al. (CMAME 2015)

- Analysis-suitable volume parameterization from spline boundary

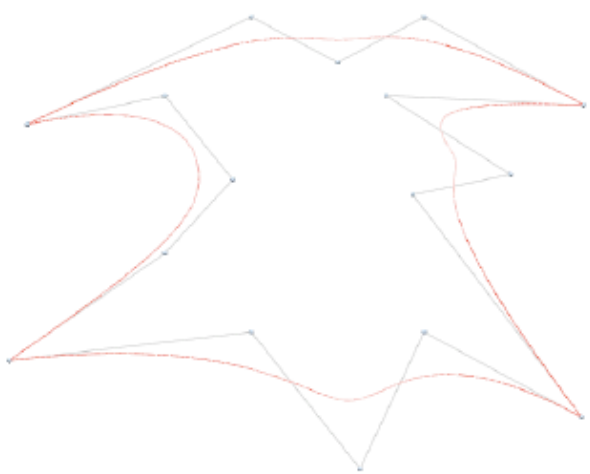
  Xu et al.(JCP2013), , Zhang et al.(CM, 2012), Chan et al (CAD 2017)

Most of the current methods with spline boundary on ly focus on the computational domain with simple bo undaries.

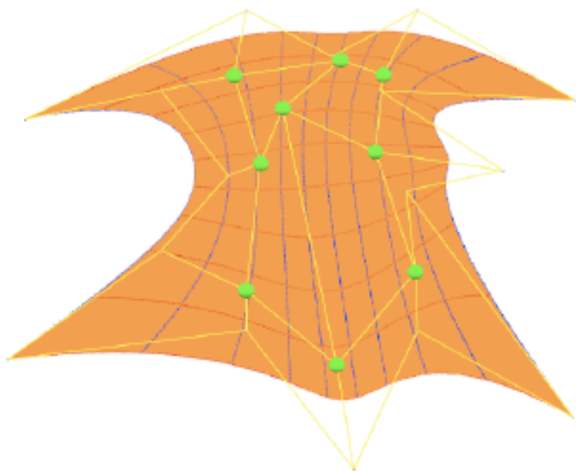
Problem statement

Construction of computational domain from boundary

given boundary control points of computational domain, construct the inner control points to generate analysis-suitable parameterization of computational domain



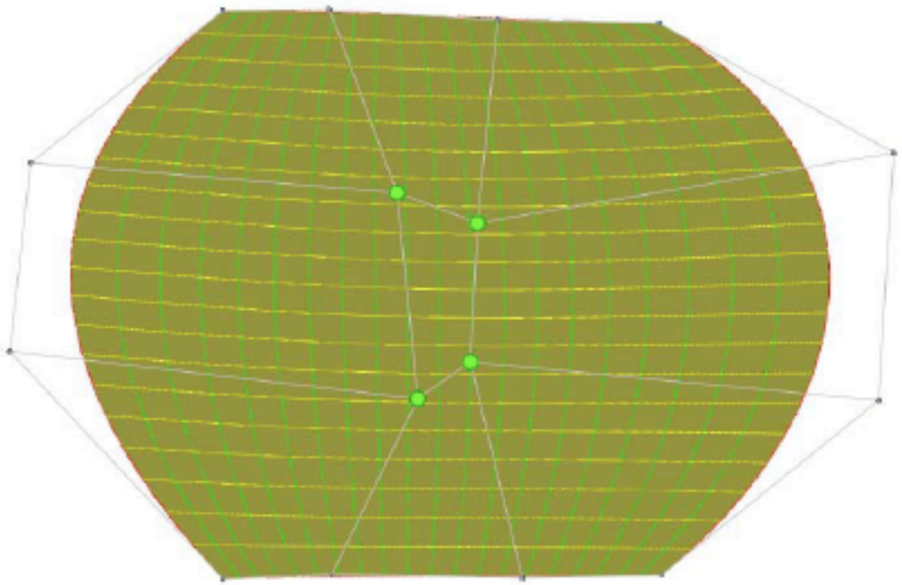
boundary curves



computational domain

Analysis-suitable parameterization

- injective (no self-intersections)
- as uniform as possible
- orthogonal isoparametric curves



CMAME 2013, CAD 2013

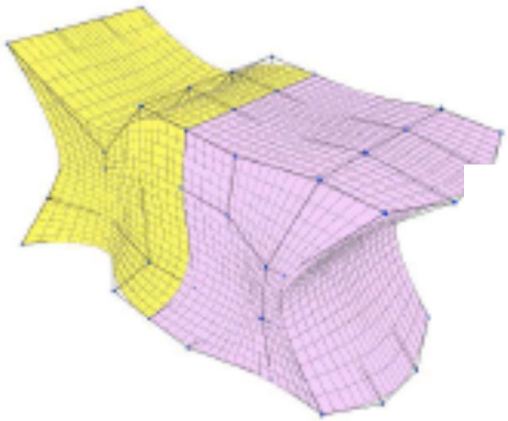
Input: six boundary B-spline surfaces
Output: inner control points and the corresponding B-spline volume parameterization

- Construct the initial inner control points by discrete Coons method;
- Construct the constraint condition from boundary B-spline surfaces;
- Solve the following constraint optimization problem by using sequential quadratic programming (SQP for short) method

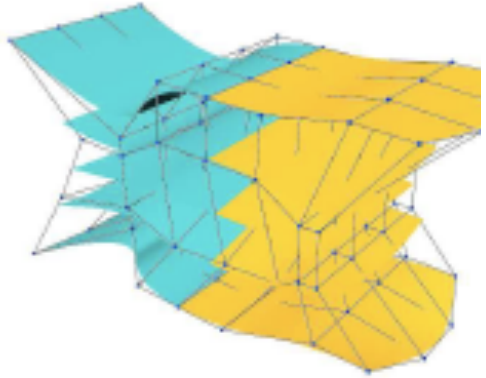
$$
\begin{aligned}
&\mathit{min} \iiint (\| \sigma_\xi \|^2 + \| \sigma_\eta \|^2 + \| \sigma_\zeta \|^2) \\
&+\omega(\| \sigma_{\xi\xi} \|^2 + \| \sigma_{\eta\eta} \|^2 + \| \sigma_{\zeta\zeta} \|^2 \\
&+2 \| \sigma_{\xi\eta} \|^2 + 2 \| \sigma_{\xi\zeta} \|^2 + 2 \| \sigma_{\eta\zeta} \|^2) d\xi d\eta d\zeta. \\
&s.t. \quad G_{ijk} > 0
\end{aligned}
$$

- Generate the corresponding B-spline volume parameterization $\sigma(\xi, \eta, \zeta)$ as computational domain.

Multi-block case



(a) C^1 B-spline blocks



(b) Isoparametric surfaces and control lattices in C^1 B-spline blocks

$$\frac{\partial}{\partial \xi}\sigma_1(\xi,\eta,\zeta)|_{\xi=\xi_1} = \frac{\partial}{\partial \xi}\sigma_2(\xi,\eta,\zeta)|_{\xi=\xi_1}$$

$$\sum_{\substack{0\le j\le m\\ 0\le k\le n}} \omega^{1,1}_{i,j,k}\Delta^{1,1}_{i,j,k}N^q_j N^r_k = \sum_{\substack{0\le j\le m\\ 0\le k\le n}} \omega^{1,2}_{i,j,k}\Delta^{1,2}_{i,j,k}N^q_j N^r_k.$$

$$\omega^{1,1}_{i,j,k}\Delta^{1,1}_{i,j,k} = \omega^{1,2}_{i,j,k}\Delta^{1,2}_{i,j,k}, i = 0, ..., l,$$

Variational harmonic method
(Journal of Computational Physics, 2013)

- Given: computational domain $\mathcal{S}$, parametric domain $\mathcal{P}$,

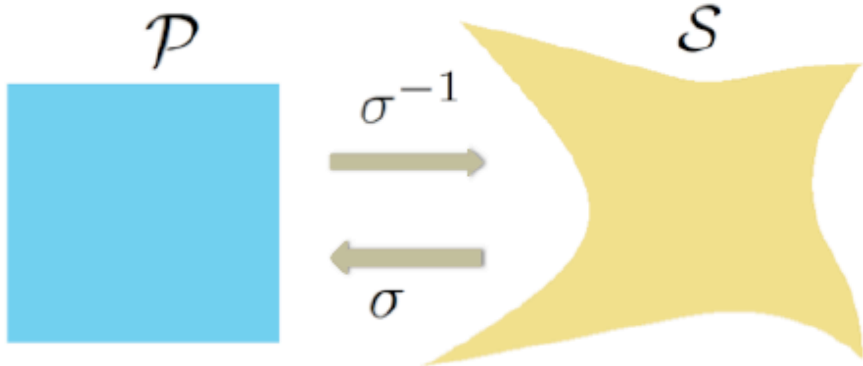
$$S(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta)) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_i^p(\xi)N_j^q(\eta)p_{i,j}$$

- Harmonic mapping: $\sigma : \mathcal{S} \mapsto \mathcal{P}$

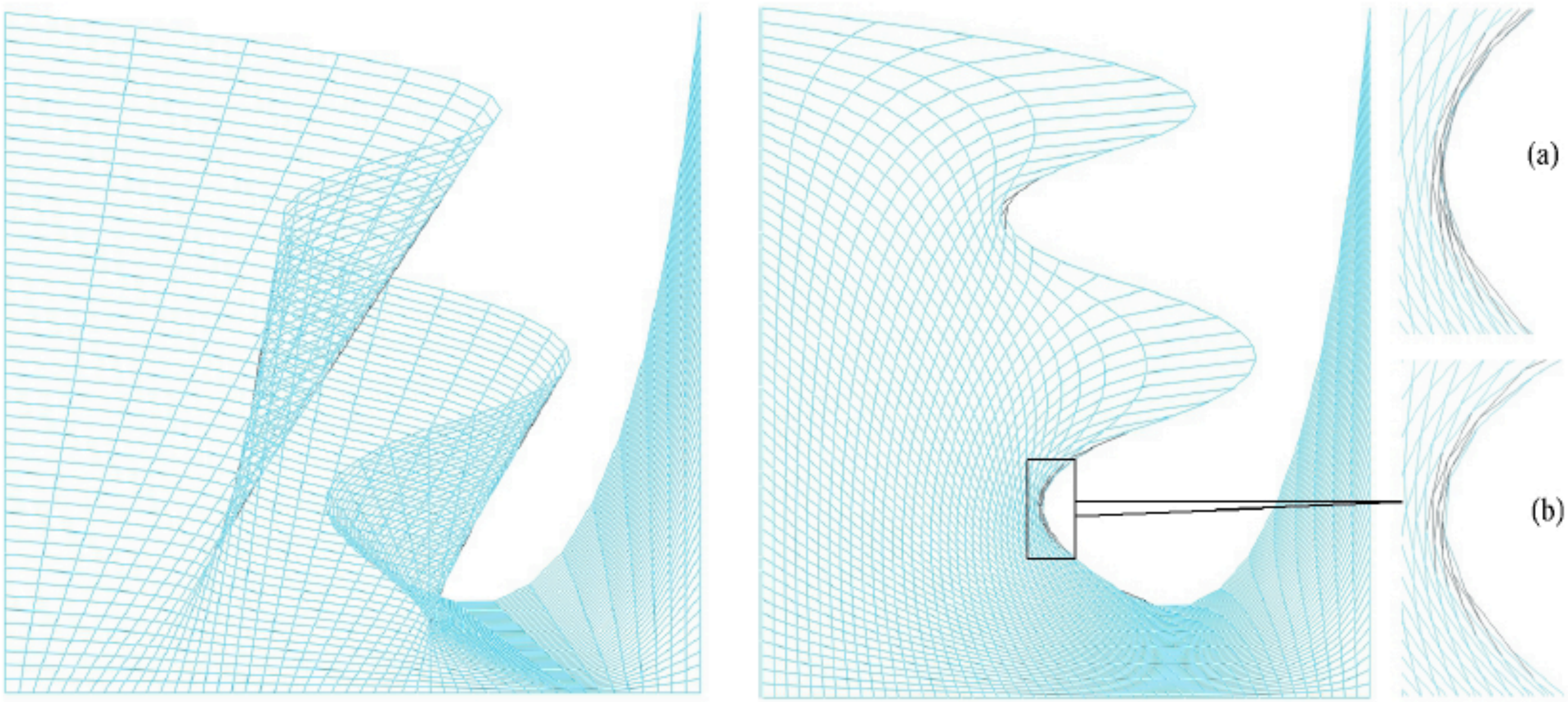
$$\Delta\xi(x,y) = \xi_{xx} + \xi_{yy} = 0$$

$$\Delta\eta(x,y) = \eta_{xx} + \eta_{yy} = 0$$

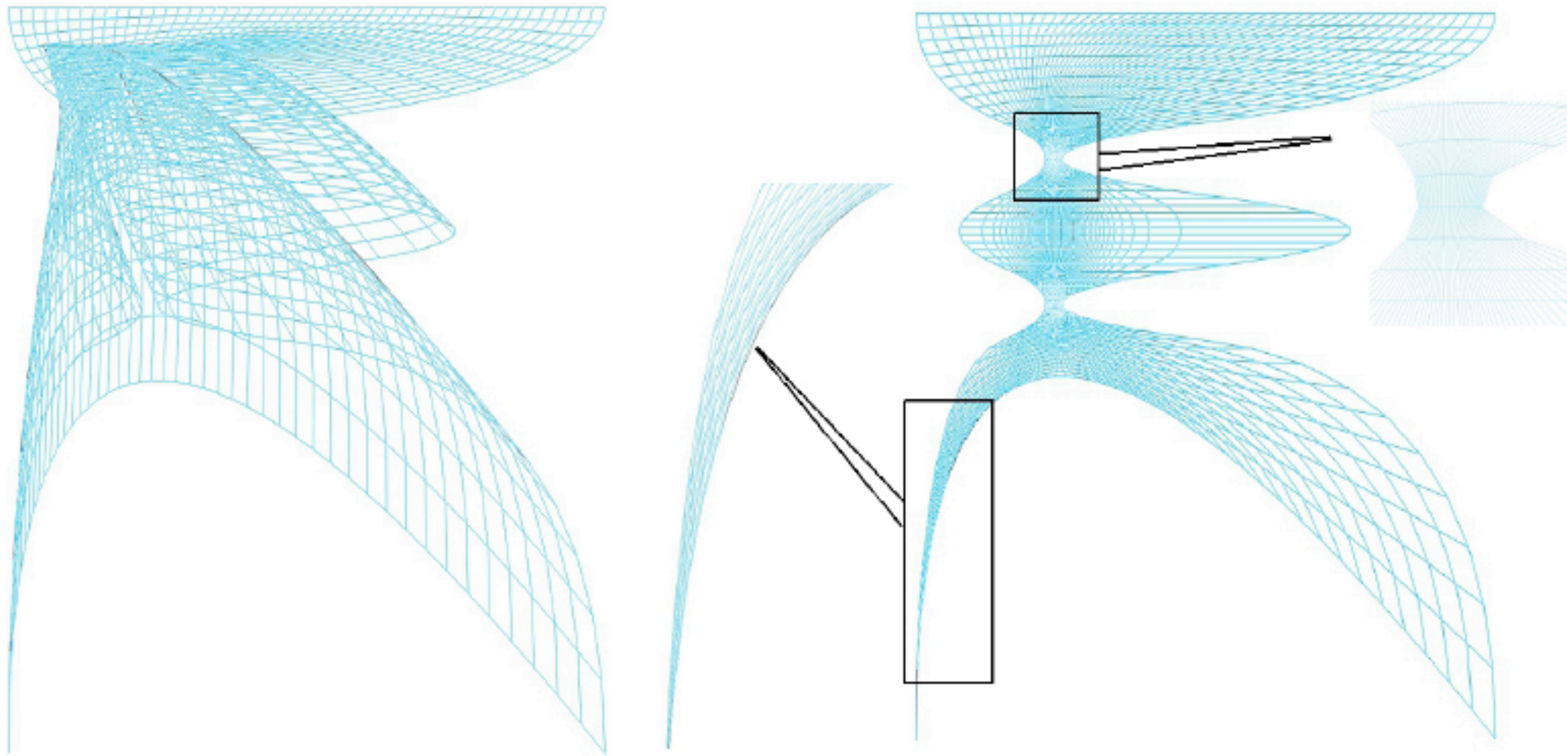
- $\sigma^{-1} : \mathcal{P} \mapsto \mathcal{S}$ is one-to-one



Two examples（1/2）

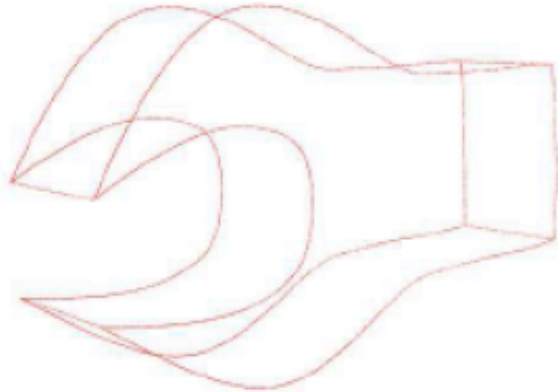


Two examples（2/2）

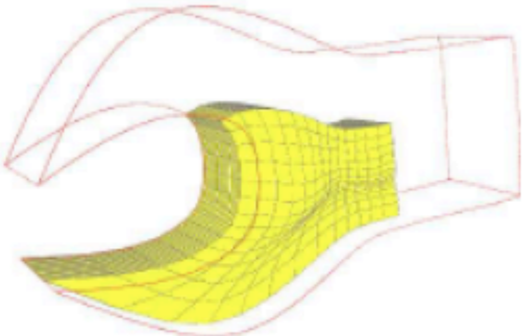


3D example I

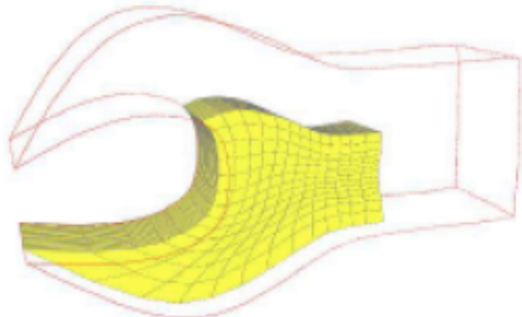
(a) boundary surfaces



(b) boundary curves



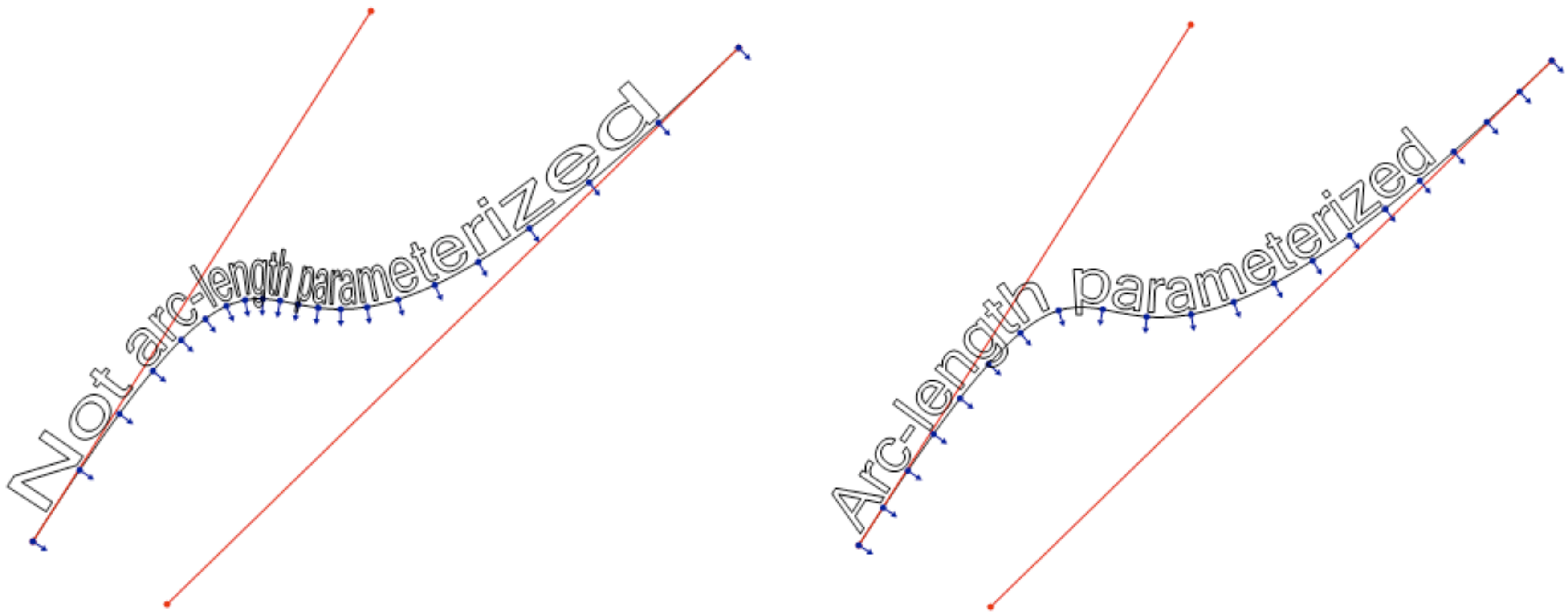
(c) Coons volume



(d) final volume parameterization

Boundary reparameterization for volumetric parameterization (Computational Mechanics, 2014)

Goal: construct optimal Möbius reparameterization of boundary surfaces to achieve high-quality isoparametric structure without changing the boundary geometry



Möbius reparameterization

$$\boldsymbol{R}(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m}\lambda_{i,j}\boldsymbol{C}_{i,j}N_i^p\boxed{(u)}N_j^q\boxed{(v)}}{\sum_{i=0}^{n}\sum_{j=0}^{m}\lambda_{i,j}N_i^p\boxed{(u)}N_j^q\boxed{(v)}},$$

$$\boxed{u = \frac{(1-\alpha)\xi}{\alpha(1-\xi)+(1-\alpha)\xi}}$$

$$\boxed{v = \frac{(1-\beta)\eta}{\beta(1-\eta)+(1-\beta)\eta}}$$

New NURBS surface with the same control points but different weights and knot vectors

Optimization method

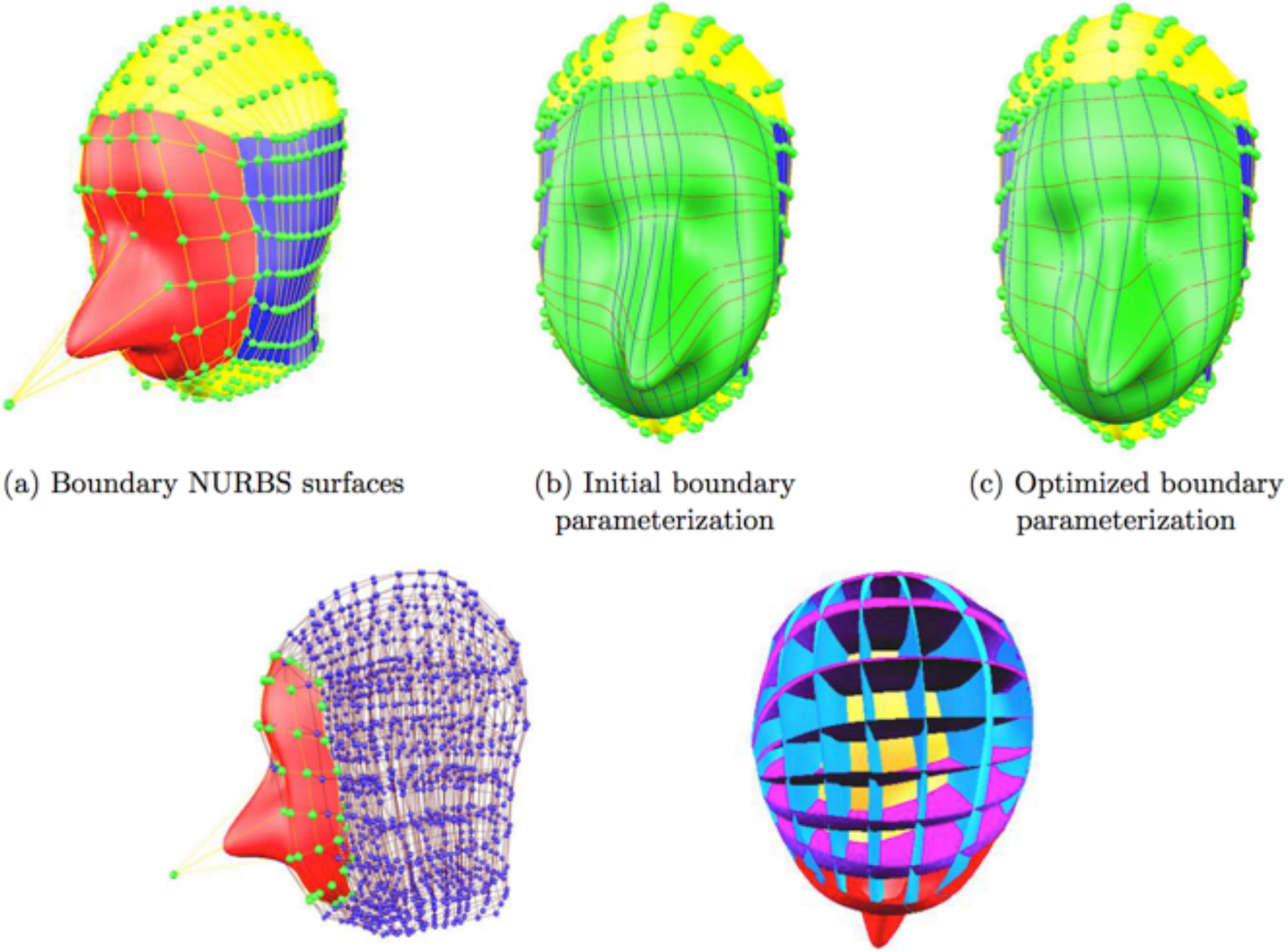
- Find the optimal

$$\boxed{\alpha, \beta}$$

such that the reparameterized NURBS surface minimizes the following objective function

$$\boxed{\int_{\mathcal{P}} (\mathbf{det}\, \widetilde{\boldsymbol{J}} - J_{avg})^2 + \omega_1 (\|\widetilde{\boldsymbol{R}}_{\xi\xi}\|^2 + \|\widetilde{\boldsymbol{R}}_{\eta\eta}\|^2)\, d\xi\, d\eta}$$

Reparameterization for VP problem



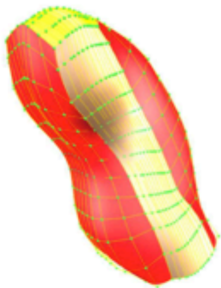
(a) Boundary NURBS surfaces

(b) Initial boundary parameterization

(c) Optimized boundary parameterization

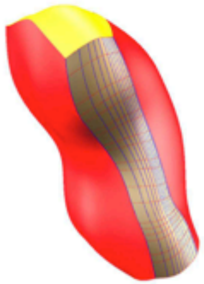
(d) Control lattice

(e) Final isoparametric structure (top view)

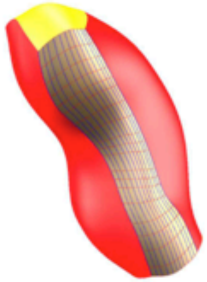


(a) Boundary NURBS surfaces

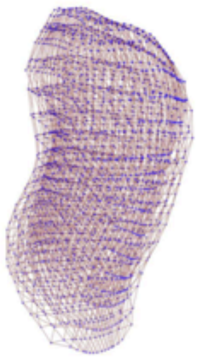
(b) Boundary NURBS curves



(c) Initial boundary parameterization

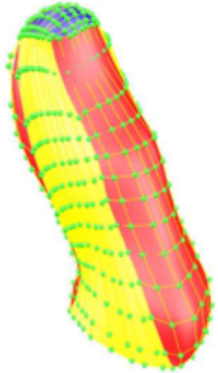


(d) Optimized boundary parameterization



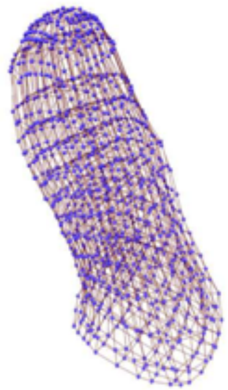
(e) Control lattice

(f) Final isoparametric structure

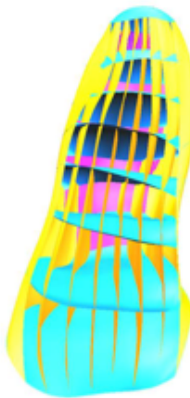


(a) Boundary NURBS surfaces and control mesh

(b) Boundary NURBS curves



(c) Resulting control lattice



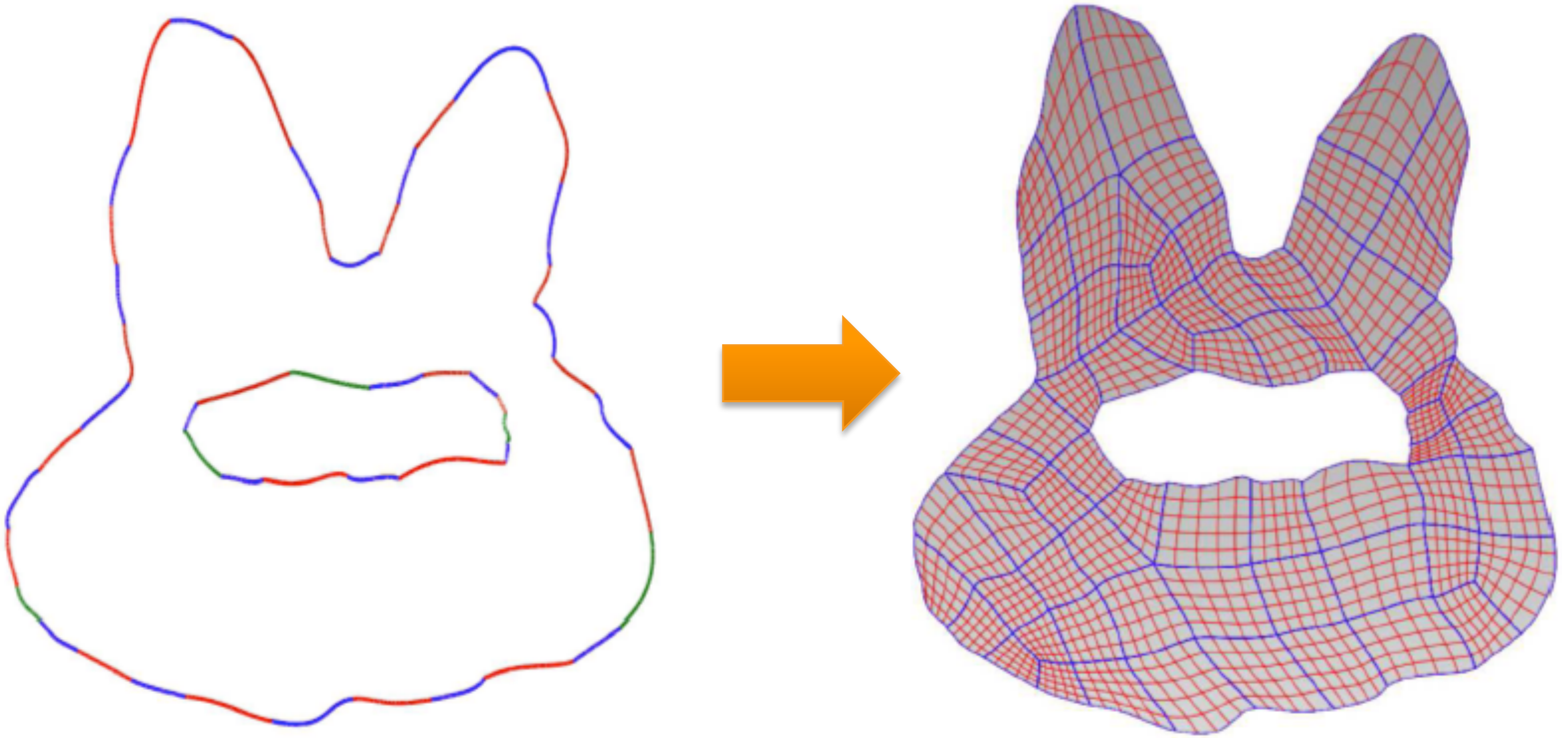
(d) Final isoparametric structure

Outline

- Parameterization in isogeometric analysis
- Analysis-suitable G^1 planar parameterization from complex boundary

Planar domain with arbitrary topology

- **Given the boundary spline curves of a planar domain with arbitrary topology, construct the patch structure and control points to obtain IGA-suitable parameterization**



(a) boundary Bézier curves

(e) parameterization result

Desired parameterization method

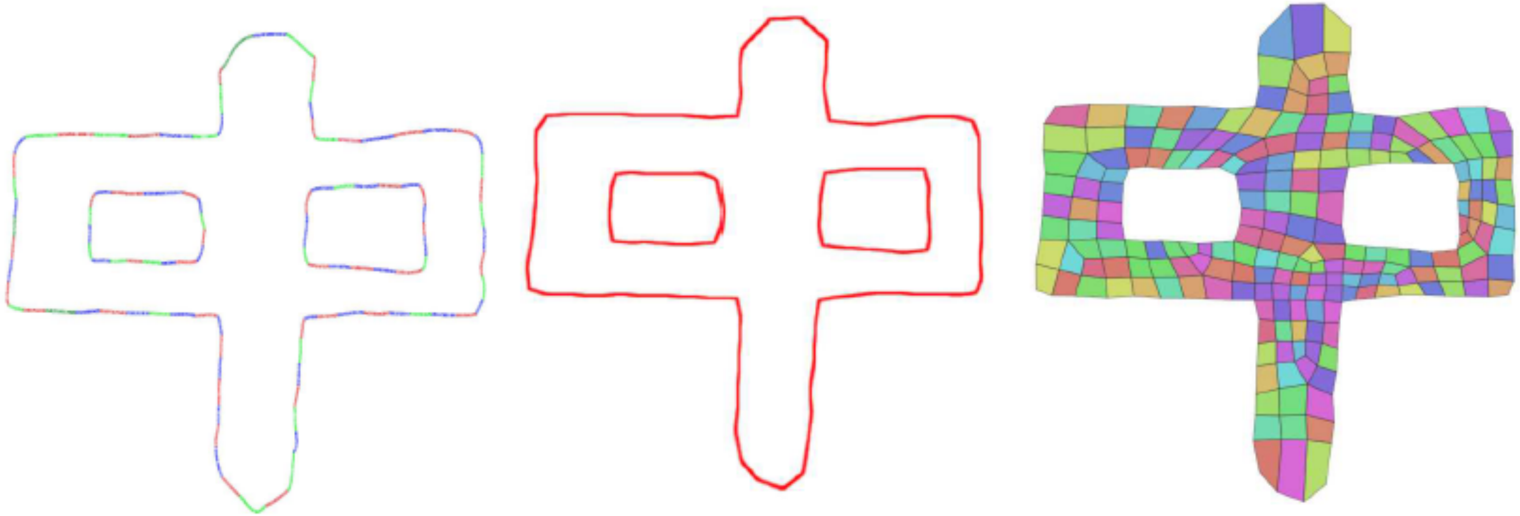
- Boundary-preserving
- Automatic continuity imposition
  ($\neq$ high-order meshing with C^0)
- Automatic construction of segmentation curve
- Injective
- Uniform patch size
- Orthogonal iso-parametric structure

Proposed framework

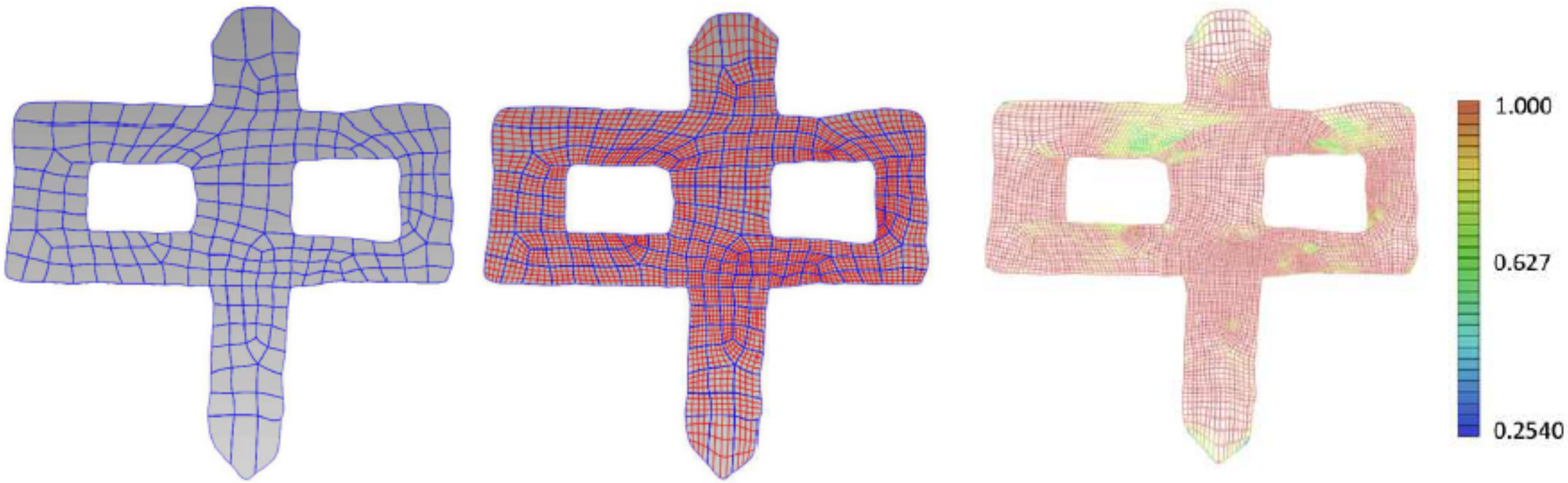
1. Pre-processing for high-quality parameterization
2. Topology information generation of quadrilateral decomposition
3. Quadrilateral patch partition by global optimization
4. High-quality patch parameterization by local optimization

Reference:
Gang Xu, Ming Li, Bernard Mourrain, Timon Rabczuk, Jinlan Xu, Stephane P.A. Bordas. Constructing IGA-suitable planar parameterization from complex CAD boundary by domain partition and global/local optimization,CMAME, in revision

Framework Overview



(a) boundary Bézier curves (b) discrete boundary (c) quad meshing result



(d) segmentation curves (e) parameterization result (f) Jacobian colormap

Pre-processing of input boundary curves

- Bézier extraction

$$\boxed{\mathbf{N}(\mathbf{t}) = \mathbf{C}\mathbf{B}(\mathbf{t})}$$

$$\boxed{\mathbf{P} = \mathbf{C}\mathbf{Q}}$$

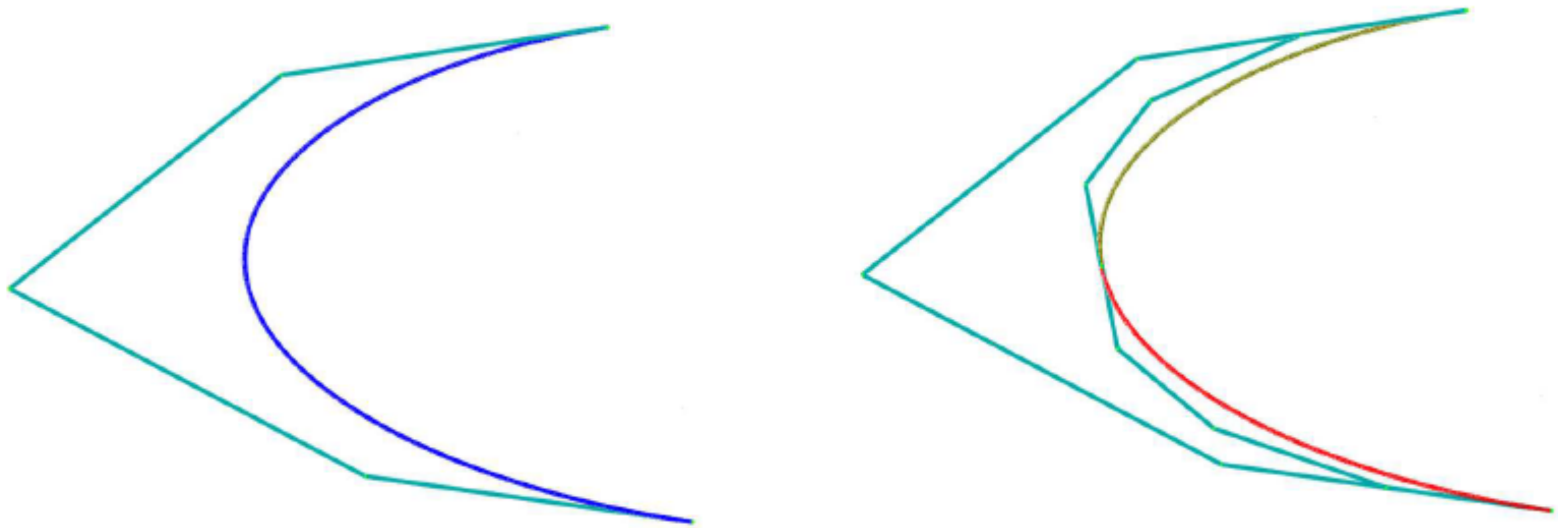
- Bézier subdivision

$$\boxed{\Gamma \geq \log_4 \frac{\sqrt{3}n(n-1)\eta}{8L_{ave}}}$$

$$\boxed{\eta = \max_{0 \leq i \leq n-2} \{|s^x_{i,k} - 2s^x_{i+1,k} + s^x_{i+2,k}|, |s^y_{i,k} - 2s^y_{i+1,k} + s^y_{i+2,k}|\}}$$

Pre-processing of input boundary curves

- Subdivision of a Bézier curve with concave shape



(a) original Bézier curve (b) Bézier subdivision

Proposed framework

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2. Topology information generation of quadrilateral decomposition
3. Quadrilateral patch partition by global optimization
4. High-quality patch parameterization by local optimization

Ref:

Gang Xu, Ming Li, Bernard Mourrain, Timon Rabczuk, Jinlan Xu, Stephane P.A. Bordas. Constructing IGA-suitable planar parameterization from complex CAD boundary by domain partition and global/local optimization,CMAME, in revision

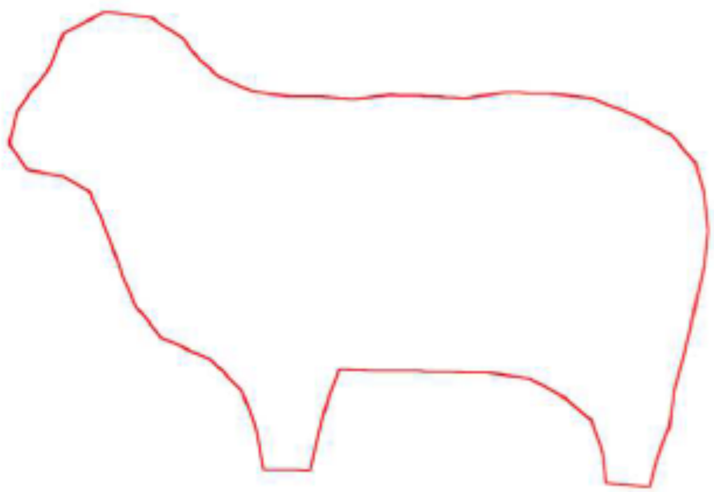
Global optimization method

- Propose a global optimization method to construct the four-sided curved partition of the computational domain

Topology generation of quadrilateral decomposition → Laplacian smoothing method → Construction of segmentation curves

Topology generation of quadrilateral decomposition

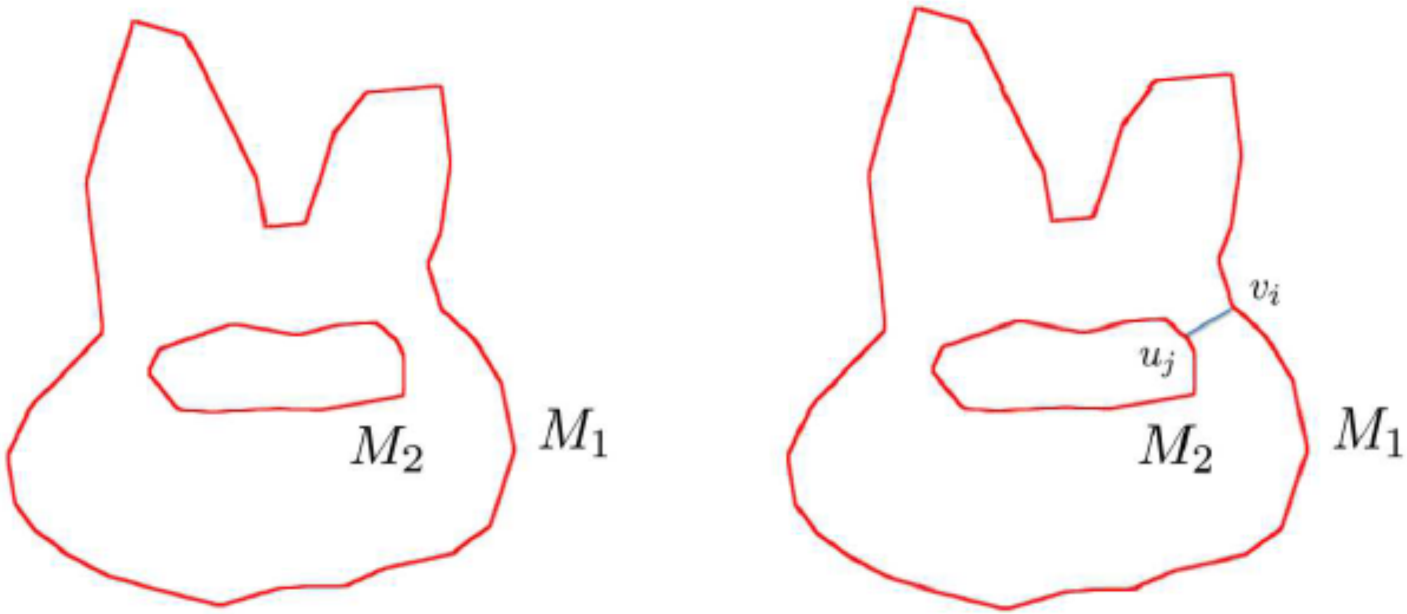
- Step. 1: Construct the discrete boundary by connecting the endpoints of the extracted Bézier curves.



(a)input discrete boundary

Step. 2

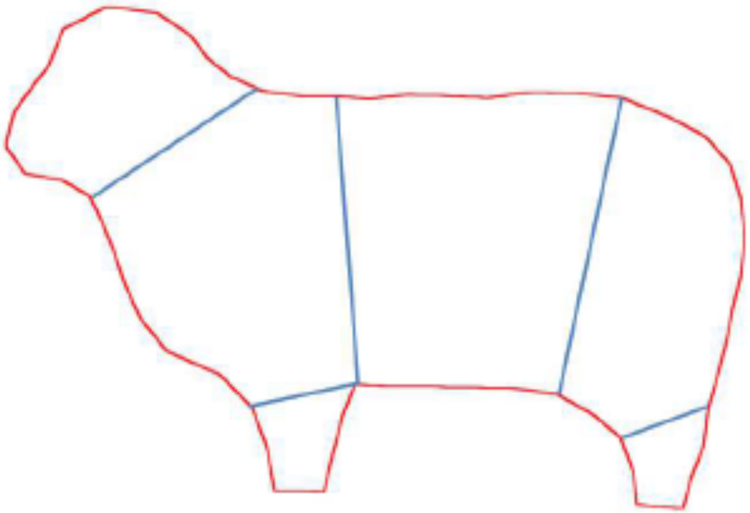
- Multiply-connected region → simply-connected region.



(a)multiply-connected domain (b) simply-connected domain

Step. 3

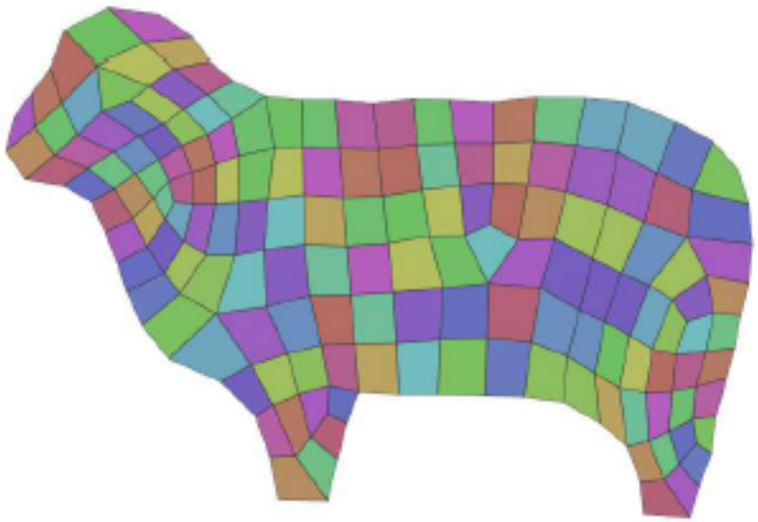
- Approximate convex decomposition of the simply-connected regions



(b) quasi-convex polygon decomposition

Step. 4

- For each quasi-convex polygon obtained in Step.3, generate the quadrangulation topology information
- Only introduce irregular vertices with valence 3 or 5, which guarantee the solution existence for G1 planar parameterization around the irregular vertex



(c) quad-meshing result by our method with 147 elements and 14 irregular vertices

Reference:
K.Takayama,D.Panozzo,O. Sorkine-Hornung Pattern-based quadrangulation for N-sided patches. CGF, 2015

Laplacian smoothing

- We adapt an iterative Laplacian smoothing method to improve the quality of the quad mesh.

$$x_i^k = \frac{\sum_{j=1}^{N_i} x_j^{k-1}}{N_i}, \qquad y_i^k = \frac{\sum_{j=1}^{N_i} y_j^{k-1}}{N_i}$$

Termination rules:

$$\frac{[\sum_{i=1}^{m}[(x_i^k - x_i^{k-1})^2 + (y_i^k - y_i^{k-1})^2]]^{1/2}}{[\sum_{i=1}^{m}[(x_i^{k-1})^2 + (y_i^{k-1})^2]]^{1/2}} < \delta$$

Construction of segmentation curves

- The segmentation curves should interpolate two vertices on the quad mesh Q(V,E).
- Global optimization method to construct the optimal shape of segmentation curves.

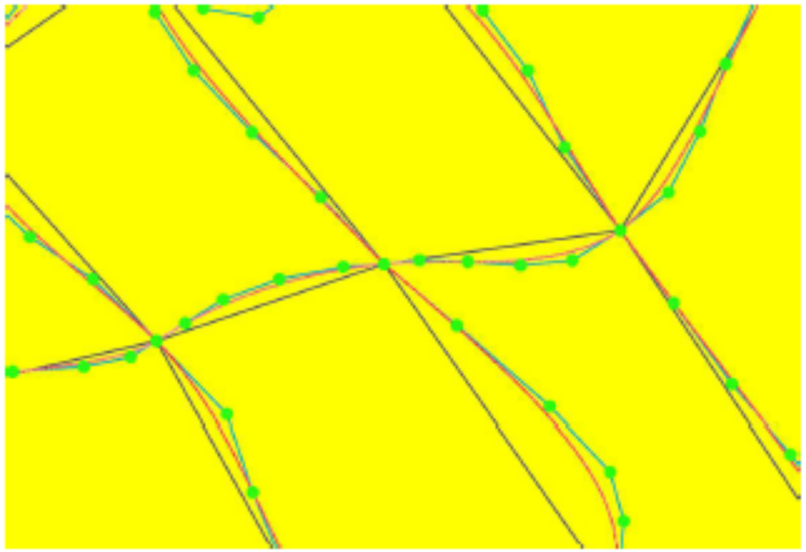


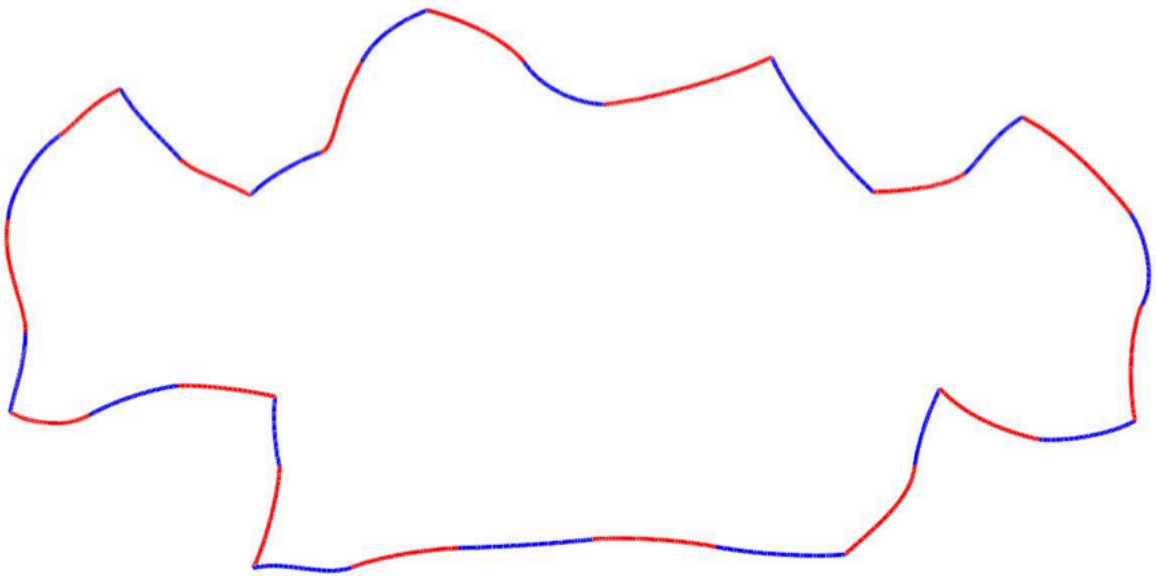
Fig.5(a) segmentation curves(red) and quad edges(black)

Desired parameterization method

- Boundary-preserving
- Automatic continuity imposition
  ($\neq$ high-order meshing with C^0)
- Automatic construction of segmentation curve
- Injective
- Uniform patch size
- Orthogonal iso-parametric structure

Uniform patch size

Computing the area of planar region bounded by B-spline curves?



Computing the area of planar region with Bézier boundary

- For the planar region bounded by N pieces of Bézier curves

$$\boldsymbol{S}_k(t) = (S_k^x(t), S_k^y(t)) = \sum_{i=1}^{n} (s_{i,k}^x, s_{i,k}^y) B_i^n(t)$$

Then the area A(Ω) of the planar region is

$$A(\Omega) = \frac{1}{4n} \sum_{k=1}^{N} \sum_{j=0}^{2n-1} (c_j^k - d_j^k)$$

$$c_j^k = \sum_{r=max(0,j-n)}^{min(j,n-1)} \frac{\binom{n}{r}\binom{n-1}{j-r}}{\binom{2n-1}{j}} s_{r,k}^x (s_{j-r+1,k}^y - s_{j-r,k}^y)$$

$$d_j^k = \sum_{r=max(0,j-n)}^{min(j,n-1)} \frac{\binom{n}{r}\binom{n-1}{j-r}}{\binom{2n-1}{j}} s_{r,k}^y (s_{j-r+1,k}^x - s_{j-r,k}^x)$$

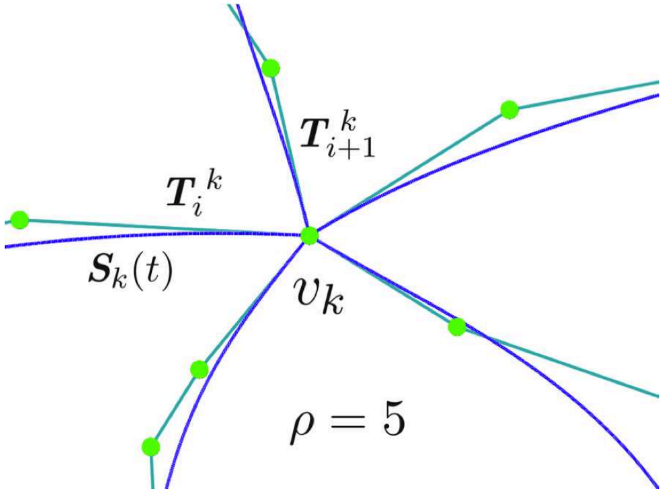
Global optimization method

- Objective functions:

$$F_{\text{uniform}} = \frac{1}{L}\sum_{i=0}^{L}(A_i - A_{ave})^2$$

$$F_{\text{shape}} = \sum_{k=0}^{N}\int_0^1 \sigma_1\|\boldsymbol{S}_k^{'}(t)\|^2 + \sigma_2\|\boldsymbol{S}_k^{''}(t)\|^2 dt$$

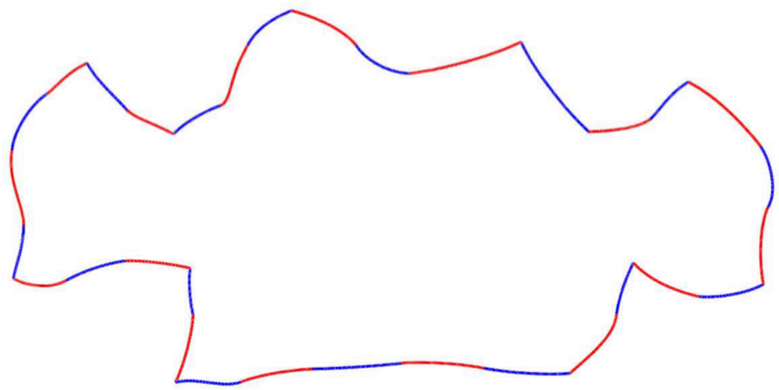
$$F_{\text{tangent}} = \sum_{k=0}^{N}\sum_{i=1}^{\rho}\left(\frac{\boldsymbol{T}_i^{k}\cdot\boldsymbol{T}_{i+1}^{k}}{\|\boldsymbol{T}_i^{k}\|\|\boldsymbol{T}_{i+1}^{k}\|} - \cos\frac{2\pi}{\rho}\right)^2$$



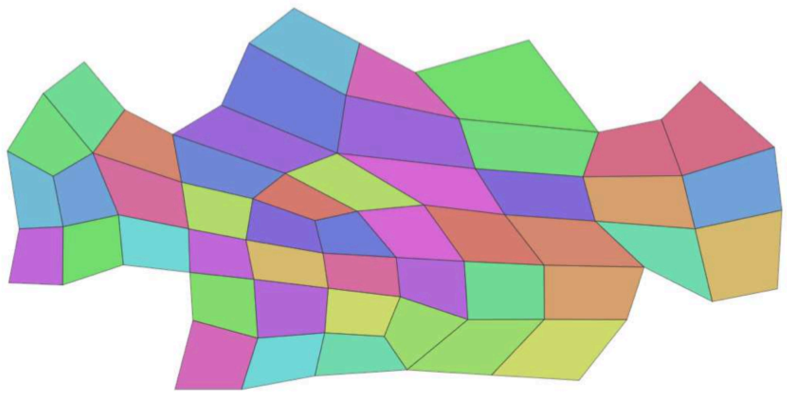
$$F = \omega_1 F_{\text{uniform}} + \omega_2 F_{\text{shape}} + \omega_3 F_{\text{tangent}}$$

$$\underset{s_{i,k}}{\arg\min}\quad F$$

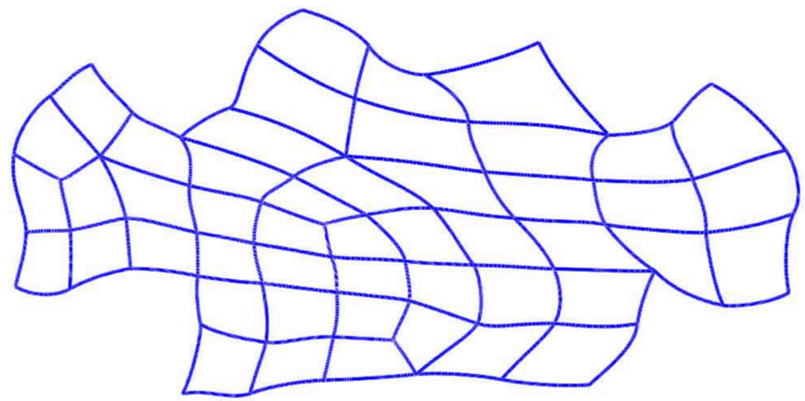
An example



(a) boundary Bézier curves



(e) quad meshing result II



(f) segmentation curves II

Proposed framework

1. Pre-processing for high-quality parameterization
2. Topology information generation of quadrilateral decomposition
3. Quadrilateral patch partition by global optimization
4. High-quality patch parameterization by local optimization

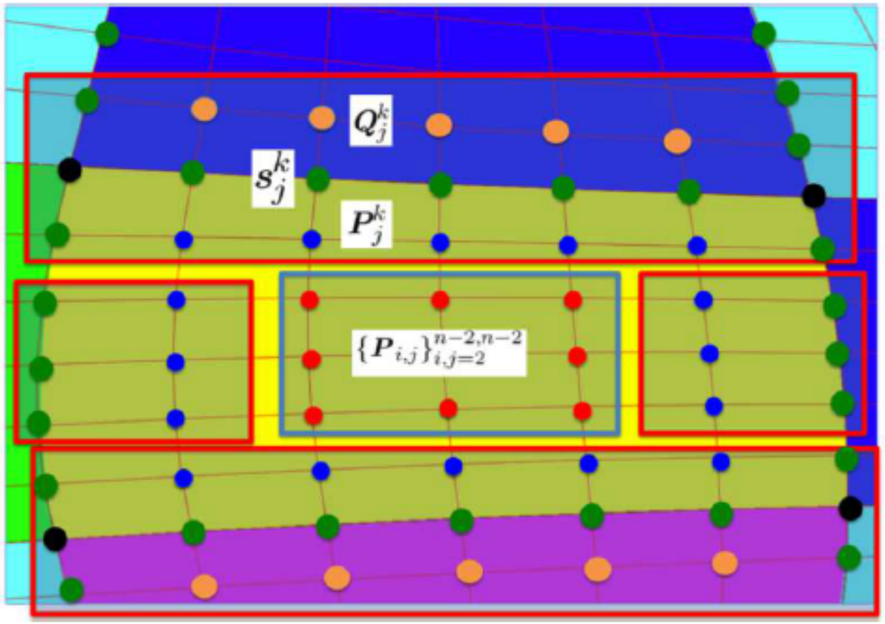
Ref:
Gang Xu, Ming Li, Bernard Mourrain, Timon Rabczuk, Jinlan Xu, Stephane P.A. Bordas. Constructing IGA-suitable planar parameterization from complex CAD boundary by domain partition and global/local optimization,CMAME, in revision

High-quality patch parameterization

Step 1: Construction of boundary second-layer control points with orthogonality optimization and continuity constraints.

Step 2: Local C1 linear-energy-minimizing method for constructing inner control points.

Step 3: Find out the invalid patches on the parameterization, then recover patch validity.



Step. 1: Construction of boundary control points

Initial construction

Imposition of C^1-continuity at regular region

Imposition of G^1-continuity around the irregular vertices

Initial construction

- Firstly, we will describe the initial construction by orthogonality optimization.

$$\boxed{\boldsymbol{P}^0_{n-1,j} = \boldsymbol{P}_{n,j} + \frac{(\boldsymbol{P}_{0,j} - \boldsymbol{P}_{n,j})}{n}}$$

$$\boxed{\arg\min_{\boldsymbol{P}_{n-1,j}} \int_0^1 (<\boldsymbol{r}_{1,u}(1,v), \boldsymbol{r}_{1,v}(1,v)>)^2 dv}$$

$$\boldsymbol{r}_{1,u}(1,v) = n\sum_{j=0}^{n} B_l^n(v)\Delta^{1,0}\boldsymbol{P}_{n-1,l},$$

$$\boldsymbol{r}_{1,v}(1,v) = n\sum_{j=0}^{n-1} B_l^{n-1}(v)\Delta^{0,1}\boldsymbol{P}_{n,l},$$

Imposition of C¹-continuity by Lagrange Multiplier method

Minimize the change of related control points along the segmentation curves such that they satisfy the C^1-constraints

$$\boldsymbol{s}_j^k - \boldsymbol{P}_j^k = \boldsymbol{Q}_j^k - \boldsymbol{s}_j^k,$$

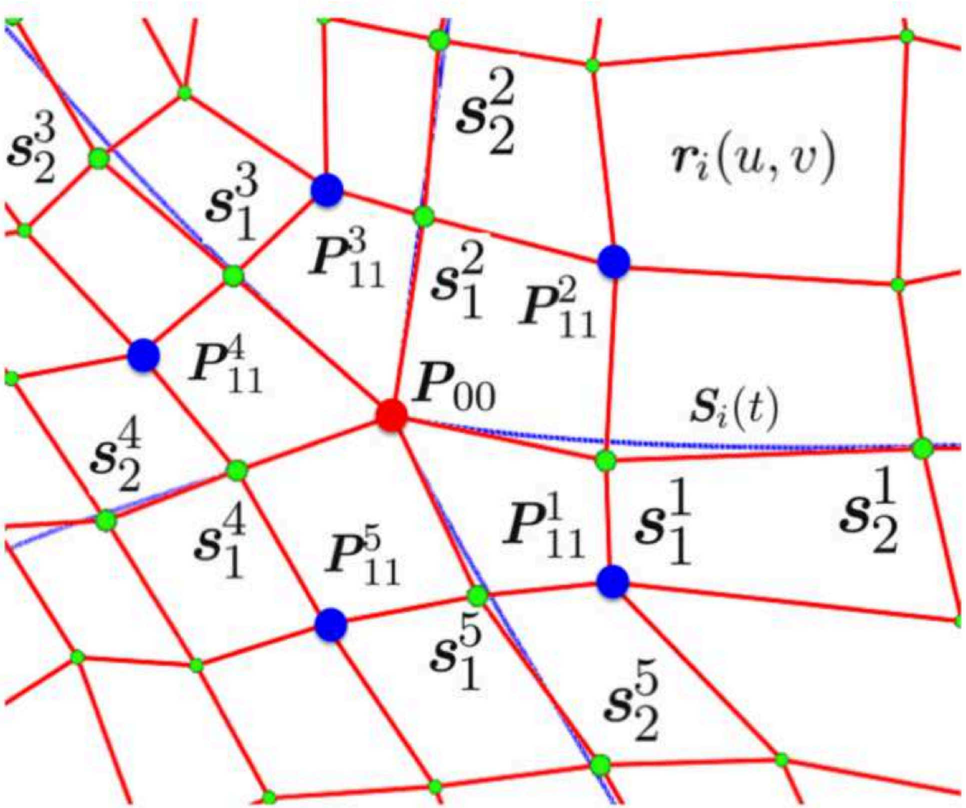
$$\mathrm{Min} \sum_{k=1}^{N} \sum_{j=0}^{n} \left(\| \boldsymbol{P}_j^k - \bar{\boldsymbol{P}}_j^k \|^2 + \| \boldsymbol{Q}_j^k - \bar{\boldsymbol{Q}}_j^k \|^2 \right)$$

the Lagrange function:

$$L = \sum_{i=0}^{N} \sum_{j=0}^{n} \left(\| \boldsymbol{P}_j^k - \bar{\boldsymbol{P}}_j^k \|^2 + \| \boldsymbol{Q}_j^k - \bar{\boldsymbol{Q}}_j^k \|^2 \right) + \sum_{i=0}^{N} \sum_{j=0}^{n} \lambda_{k,j} \left(2\boldsymbol{s}_j^k - \boldsymbol{P}_j^k - \boldsymbol{Q}_j^k \right)$$

Imposition of G^1-continuity around irregular vertex

- Some special treatments should be done achieve G^1-continuity at the irregular vertices.



G^1-continuity

Imposition of G1-continuity (Mourrain et al， CAGD 2016)

- The G1-continuity constraints around the irregular vertex can be described as follows:

$$(\boldsymbol{s}_1^i - \boldsymbol{P}_{00}) = \alpha_i(\boldsymbol{s}_1^{i+1} - \boldsymbol{P}_{00}) + \beta_i(\boldsymbol{s}_1^{i-1} - \boldsymbol{P}_{00}),$$

$$\boldsymbol{0} = n\alpha_i(\boldsymbol{P}_{11}^i - \boldsymbol{s}_1^i) + n\beta_i(\boldsymbol{P}_{11}^{i-1} - \boldsymbol{s}_1^i) - (n-1)(\boldsymbol{s}_2^i - \boldsymbol{s}_1^i) + (\boldsymbol{s}_1^i - \boldsymbol{P}_{00})$$

$$\begin{pmatrix} \alpha_1 & 0 & \dots & 0 & \beta_1 \\ \beta_2 & \alpha_2 & \dots & 0 & 0 \\ 0 & \beta_3 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \alpha_{M-1} & 0 \\ 0 & 0 & \dots & \beta_M & \alpha_M \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_{11}^1 \\ \boldsymbol{P}_{11}^2 \\ \boldsymbol{P}_{11}^3 \\ \vdots \\ \boldsymbol{P}_{11}^{M-1} \\ \boldsymbol{P}_{11}^M \end{pmatrix} = \begin{pmatrix} \boldsymbol{H}_1 \\ \boldsymbol{H}_2 \\ \boldsymbol{H}_3 \\ \vdots \\ \boldsymbol{H}_{M-1} \\ \boldsymbol{H}_M \end{pmatrix}$$

There exists unique solution for M=3 and M=5

High-quality patch parameterization

Step 1: Construction of boundary second-layer control points with orthogonality optimization and continuity constraints.

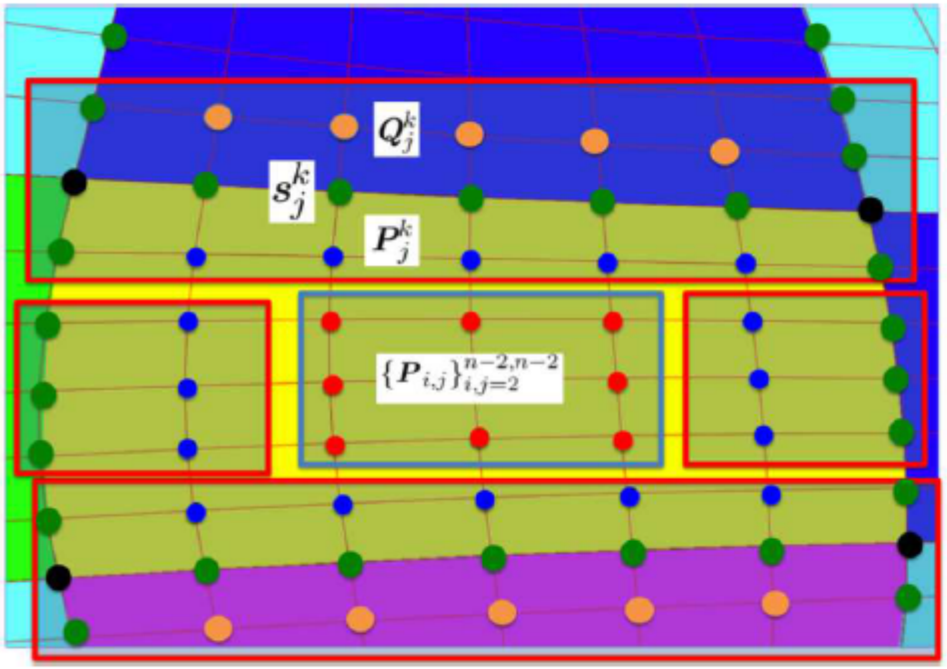
Step 2: Local C1 linear-energy-minimizing method for constructing inner control points.

Step 3: Find out the invalid patches on the parameterization, then recover patch validity.

Local C[1] linear-energy-minimizing m

- Constructing interior (n−3)×(m−3) control points of each patch

$$E(\boldsymbol{r}) = \int_{\Omega} \tau_1(\|\boldsymbol{r}_u\|^2 + \|\boldsymbol{r}_v\|^2) + \tau_2(\|\boldsymbol{r}_{uu}\|^2 + 2\|\boldsymbol{r}_{uv}\|^2 + \|\boldsymbol{r}_{vv}\|^2)dudv$$



Inner control points construction

- A tensor product Bézier surface r(u,v) has minimal energy E(r) if and only if remaining inner control points satisfy

$$
\begin{aligned}
\boldsymbol{0} \;=\; & \frac{\tau_1}{4(n-1)}\left(\sum_{k=0}^{n-1}\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{2n}{l+j}}C_{n,i}^{k}\Delta^{1,0}\boldsymbol{P}_{kl}+\sum_{k=0}^{n}\sum_{l=0}^{n-1}\frac{\binom{n}{k}}{\binom{2n}{i+k}}C_{n,j}^{l}\Delta^{0,1}\boldsymbol{P}_{kl}\right) \\
& +\frac{2\tau_2}{(2n-1)^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\frac{\binom{n-1}{k}\binom{n-1}{l}}{\binom{2n-2}{i+k-1}\binom{2n-2}{l+j-1}}B_{n,i}^{k}B_{n,j}^{l}\Delta^{1,1}\boldsymbol{P}_{kl} \qquad (29)\\
& +\frac{\tau_2}{(2n-3)(2n+1)}\left(\sum_{k=0}^{n-2}\sum_{l=0}^{n}\frac{\binom{n-2}{k}\binom{n}{l}}{\binom{2n-4}{i+k-2}\binom{2n}{l+j}}A_{n,i}^{k}\Delta^{2,0}\boldsymbol{P}_{kl}+\sum_{k=0}^{n}\sum_{l=0}^{n-2}\frac{\binom{n}{k}\binom{n-2}{l}}{\binom{2n}{i+k}\binom{2n-4}{l+j-2}}A_{n,j}^{l}\Delta^{0,2}\boldsymbol{P}_{kl}\right)
\end{aligned}
$$

- A linear system with (n−3)×(n−3) equations and (n−3)×(m−3) variables

$$\boxed{\mathbf{MP} = \mathbf{B}\ .}$$

High-quality patch parameterization

Step 1: Construction of boundary second-layer control points with orthogonality optimization and continuity constraints.

Step 2: Local C1 linear-energy-minimizing method for constructing inner control points.

Step 3: Find out the invalid patches on the parameterization, then recover patch validity.

Step 3: Injective parameterization

- Jacobian

$$J(u,v) = \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1} \alpha_{ij} B_i^{2n-1}(u) B_j^{2n-1}(v)$$

$$\min_{0\le i,j\le 2n-1} \alpha_{ij} \le J(u,v) \le \max_{0\le i,j\le 2n-1} \alpha_{ij}$$

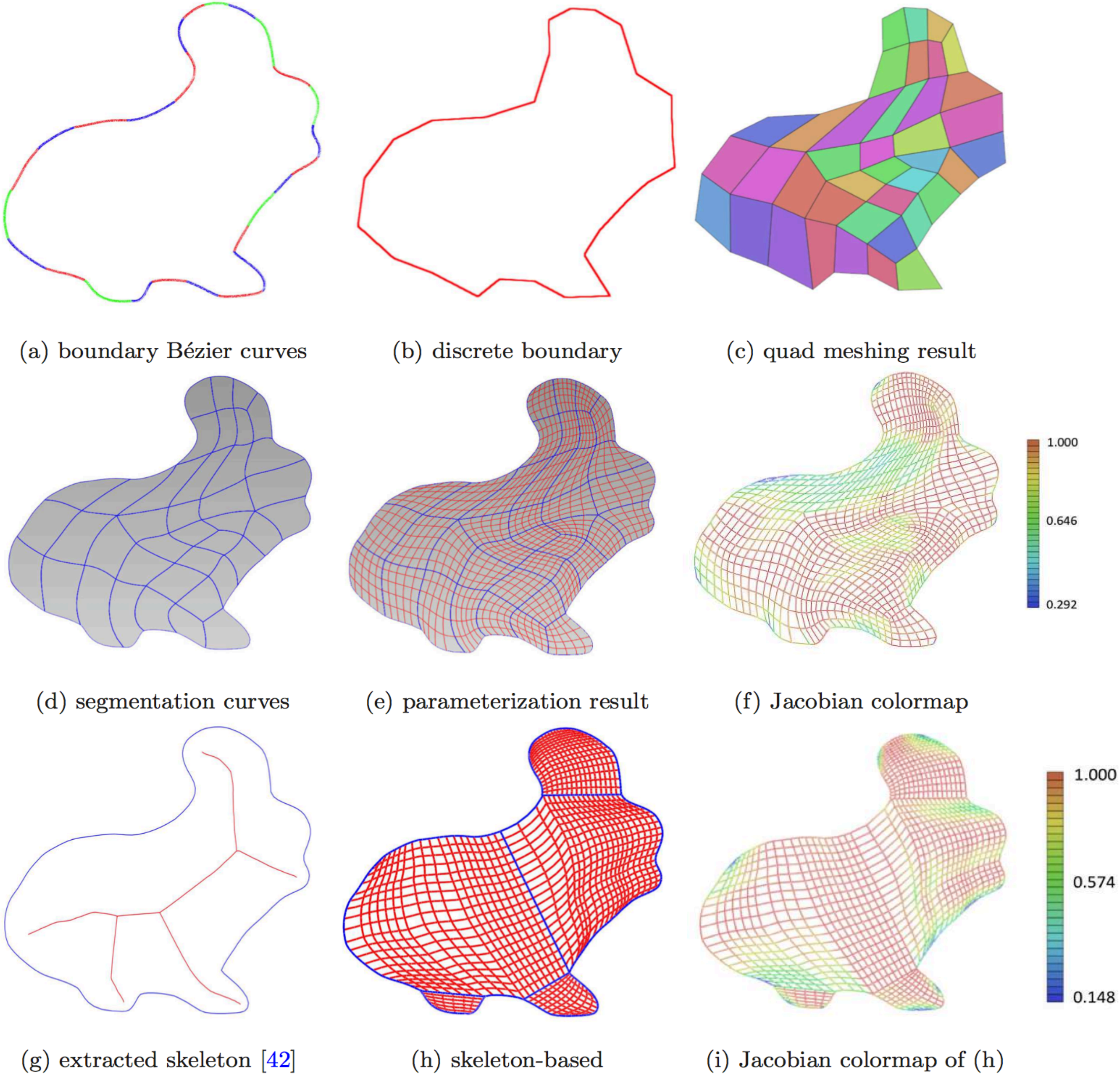
$$\min_{0\le i,j\le 2n-1} \alpha_{ij} > 0$$ → Valid patch

- Logarithmic-barrier method for invalid patch

$$\min \quad E(\boldsymbol{r}(u,v)) \quad \text{s.t.} \quad \alpha_{ij} > 0$$

$$\arg\min_{\boldsymbol{P}_{i,j}} \quad E(\boldsymbol{r}(u,v)) - \mu \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1} \ln(\alpha_{ij})$$

Example I

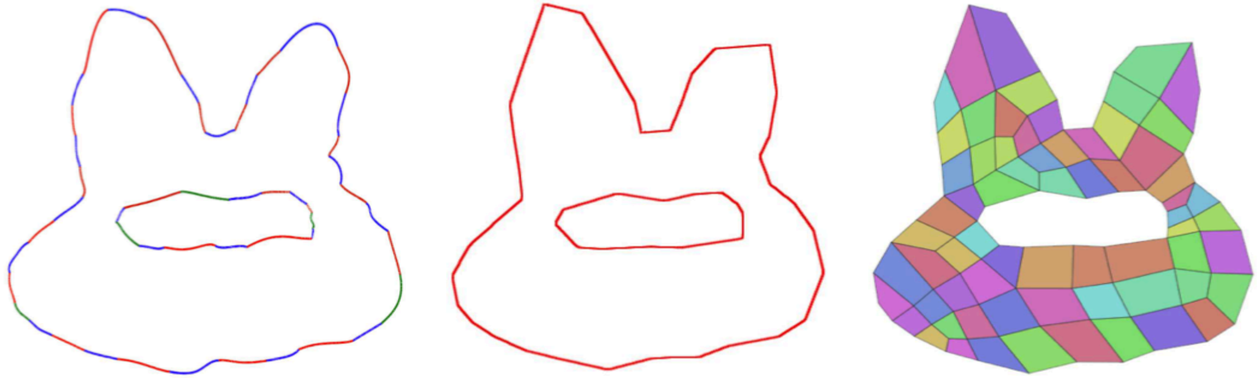


(a) boundary Bézier curves (b) discrete boundary (c) quad meshing result

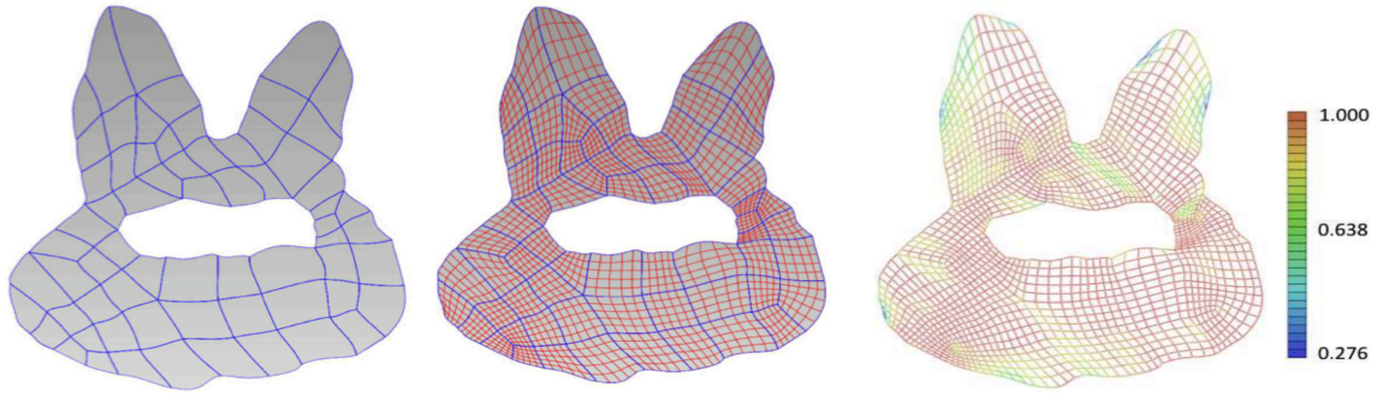
(d) segmentation curves (e) parameterization result (f) Jacobian colormap

(g) extracted skeleton [42] (h) skeleton-based (i) Jacobian colormap of (h)

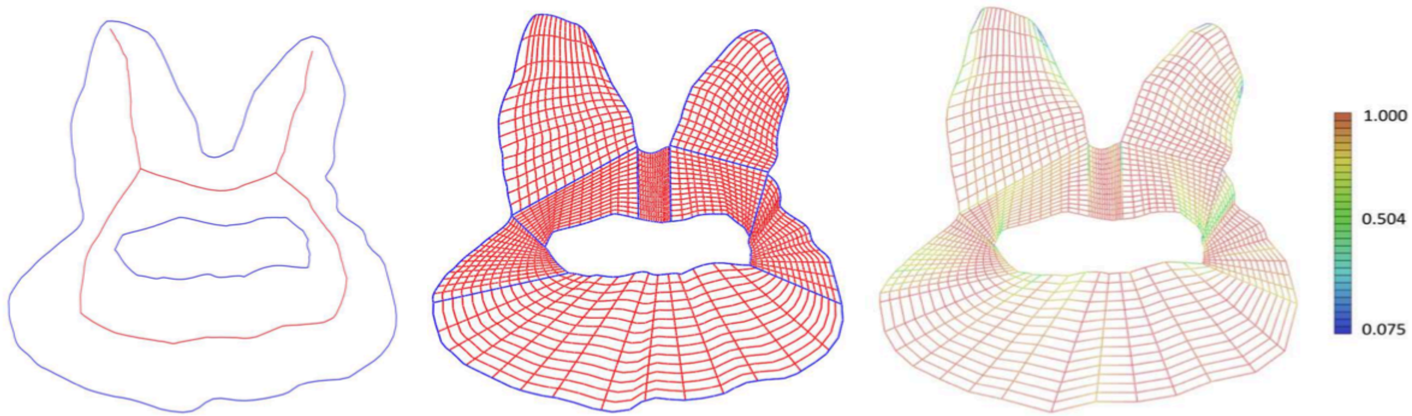
Example II



(a) boundary Bézier curves (b) discrete boundary (c) quad meshing result

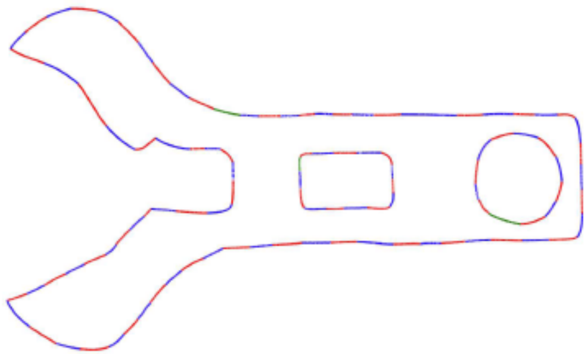


(d) segmentation curves (e) parameterization result (f) Jacobian colormap

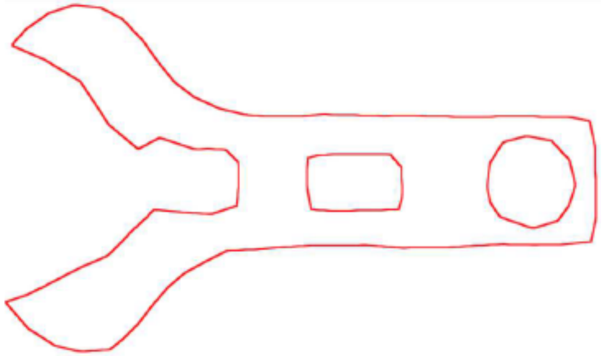


(g) extracted skeleton [42] (h) skeleton-based parameterization [42] (i) Jacobian colormap of (h)

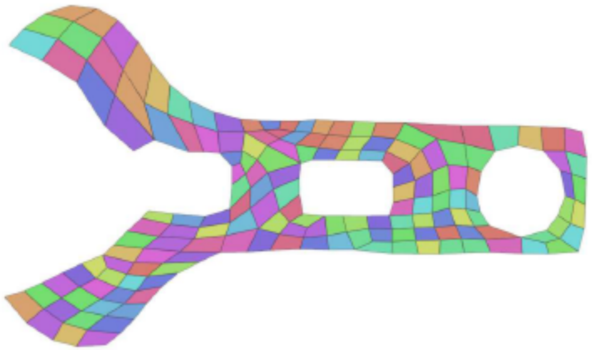
Example III



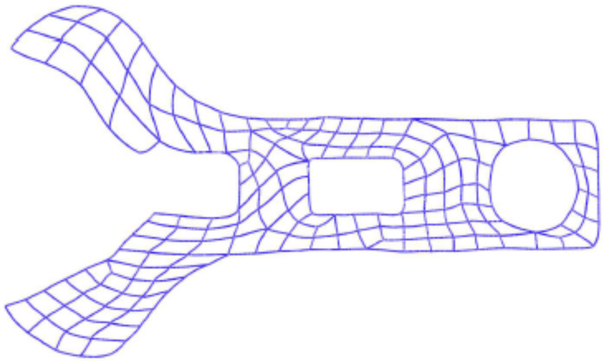
(a) boundary Bézier curves



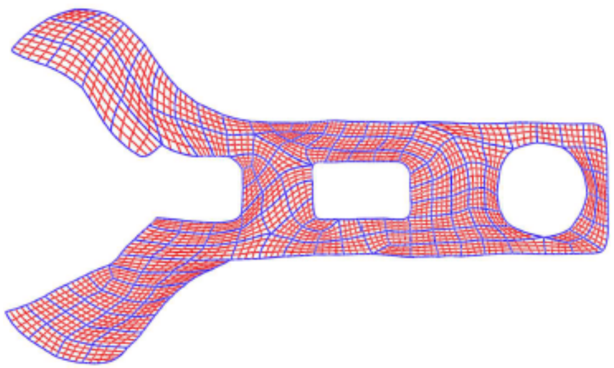
(b) discrete boundary



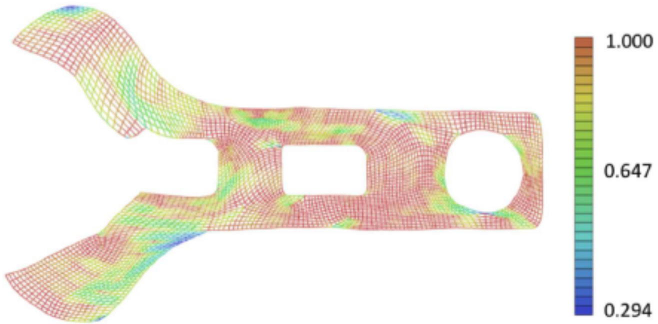
(c) quad meshing result



(d) segmentation curves

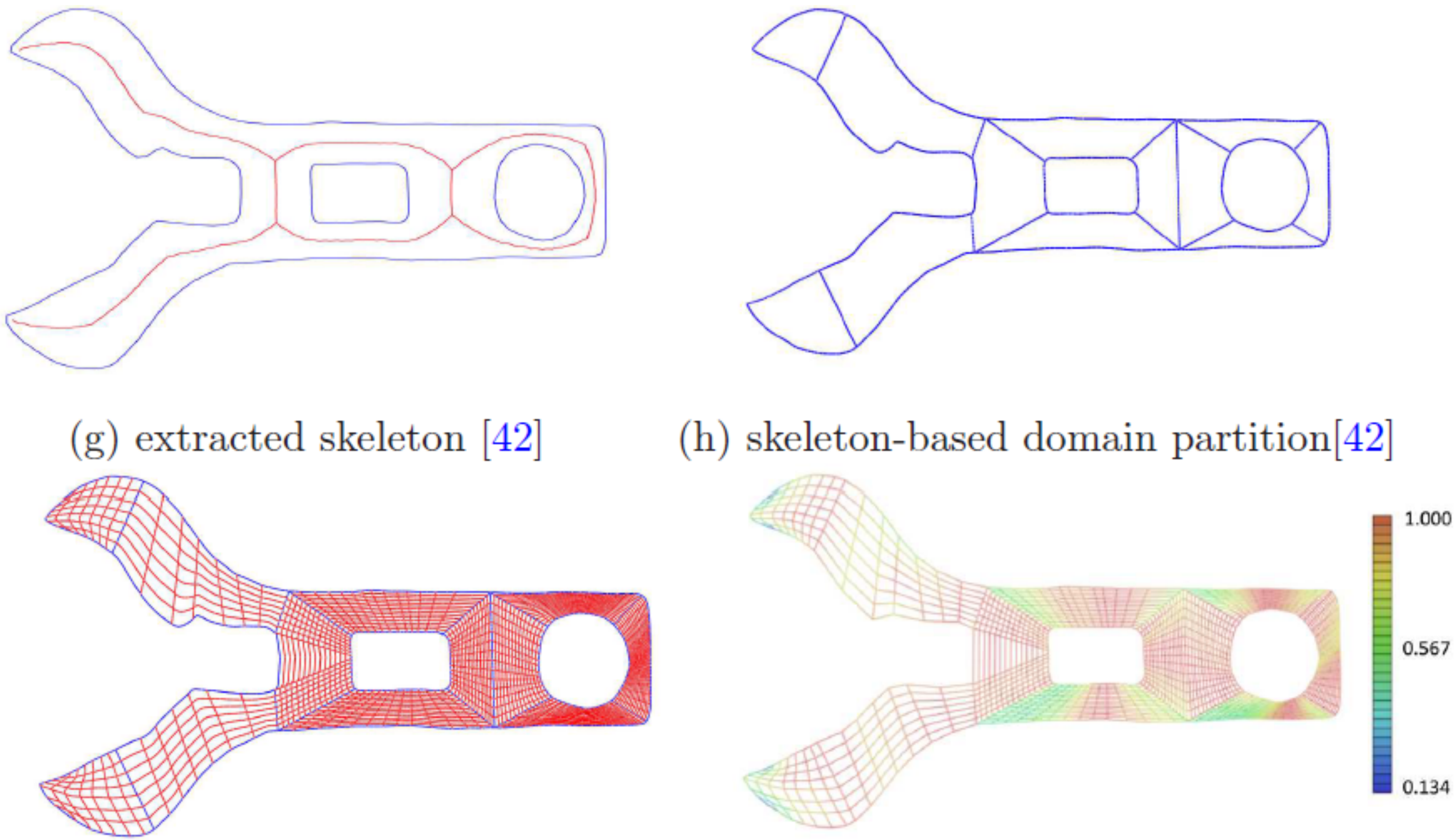


(e) parameterization result



(f) Jacobian colormap

Example III with the skeleton-based decomposition



(g) extracted skeleton [42] (h) skeleton-based domain partition[42]

(i) skeleton-based parameterization [42] (j) Jacobian colormap of (i)

Fig. 11. Example V.

Quality comparison

Table 2: Quantitative data for planar parameterization in Fig. 9, Fig. 10 and Fig. 11. p: degree of planar parameterization; # SD: number of subdomains by domain decomposition;# Patch: number of Bézier patches; # Con.: number of control points.

Example	Method	p	# SD	#Patch	# Con.	Scaled Jacobian		Conditional number	
						Average	Min	Average	Max
Fig. 9	Our method	6	39	39	1467	0.8843	0.292	2.76	8.06
	Xu et al.[42]	6	5	35	1309	0.5172	0.148	5.36	16.31
Fig. 10	Our method	5	66	66	1768	0.9194	0.276	2.42	10.18
	Xu et al.[42]	5	8	56	1507	0.7801	0.075	4.35	18.23
Fig. 11	Our method	5	155	155	3720	0.9017	0.294	2.57	7.86
	Xu et al.[42]	5	12	132	3282	0.7894	0.134	4.23	15.64

Parameters and computing time

Example	F_{shape} in (14)		F in (16)			$E(\boldsymbol{r})$ in (28)		$\# T_1$	$\# T_2$	$\# T$
	σ_1	σ_2	ω_1	ω_2	ω_3	τ_1	τ_2			
Fig. 1	2.0	1.0	2.0	1.0	50.0	2.0	1.5	73.22	177.86	251.08
Fig. 8(c)	1.0	1.0	1.0	1.0	50.0	1.0	1.5	22.68	26.64	49.32
Fig. 8(g)	1.0	1.0	1.0	1.0	50.0	1.0	1.5	22.96	27.18	50.14
Fig. 8(k)	1.0	1.0	1.0	1.0	50.0	1.0	1.5	27.68	36.32	63.90
Fig. 9	1.0	2.0	1.0	2.0	50.0	1.0	2.0	32.74	53.84	86.58
Fig. 10	2.0	1.0	1.0	2.0	50.0	2.0	1.0	41.02	61.36	102.38
Fig. 11	2.0	1.0	2.0	2.0	50.0	2.0	1.0	75.70	209.68	285.38