# Nitsche's method for patch coupling

## in isogeometric analysis

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# Why do we need patch coupling

- Model complex structures

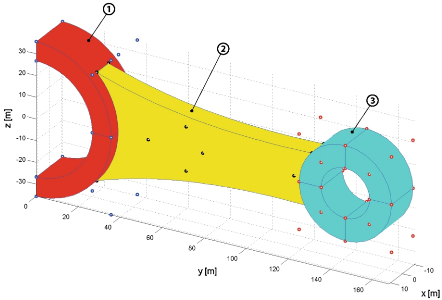



Figure: Connecting rod
[*V.P.Nguyen et.al.* 2013]

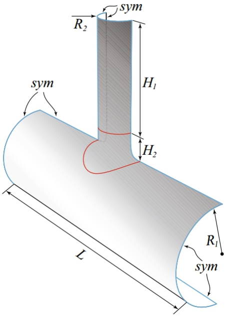



Figure: Intersecting tubular shell
[*Y.Guo et.al.* 2017]

# Why do we need patch coupling

- Model complex structures
- Assign various materials to sub-structures


Figure: iPHONE 6S [*www.visualcapitalist.com*]

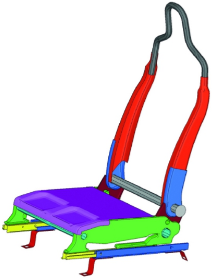


Figure: Seat frames [*aeplus.com*]

# Why do we need patch coupling

- Model complex structures
- Assign various materials to sub-structures
- Calculate using suitable elements (dimensions)

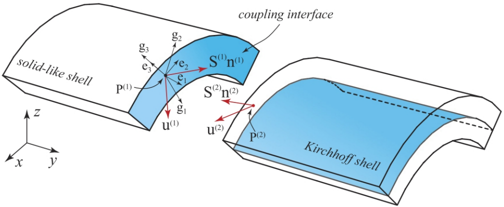



Figure: Mixed-dimensional coupling
[*Y.Guo and M.Ruess* 2015]

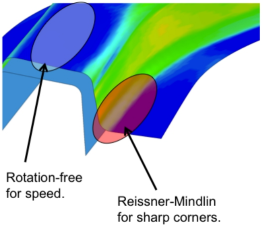



Figure: Simulation of metal forming
[*D.J.Benson et.al*. 2012]

# Why do we need patch coupling

- Model complex structures
- Assign various materials to sub-structures
- Choose reasonable element types (dimensions)
- Discrete model into elements of sufficient numbers

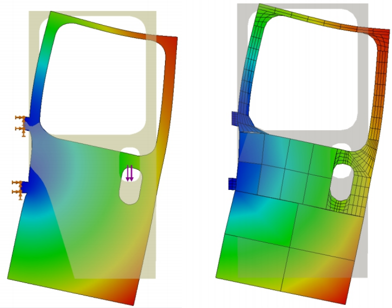


Figure: A truck door, left: commercial software results [*Marco Brino* 2015]

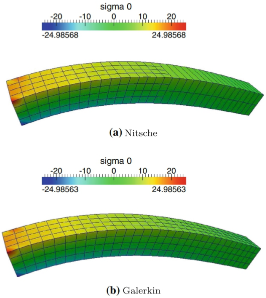



Figure: Saint Venant's Principle [*V.P.Nguyen et.al.* 2014]

# Why do we need patch coupling: from FEM to IGA

- IGA use NURBS (non-uniform rational B-spline) instead of polynomials

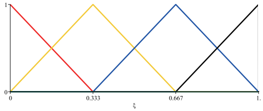



Figure: Lagrange basis functions in FEM

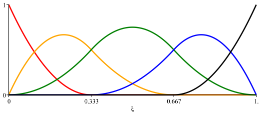



Figure: NURBS basis functions in IGA

- Non-interpolatory control points

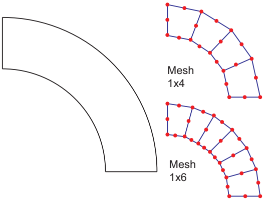



Figure: Meshes and nodes in FEM

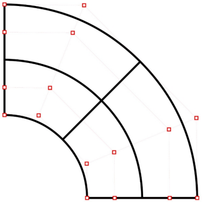


Figure: Control meshes and control points in IGA

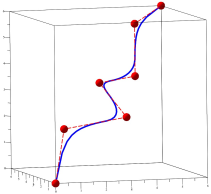


Figure: NURBS curve

# Stitching two fabrics together

Figure: Stitching fabrics [*bigbgsd.blogspot.com*]

Where there are displacement gaps,
there should be some kind of forces to prevent the two fabrics from separation,
and additional work to be done to stitching them together.

# Analogy

Figure: Stitching fabrics

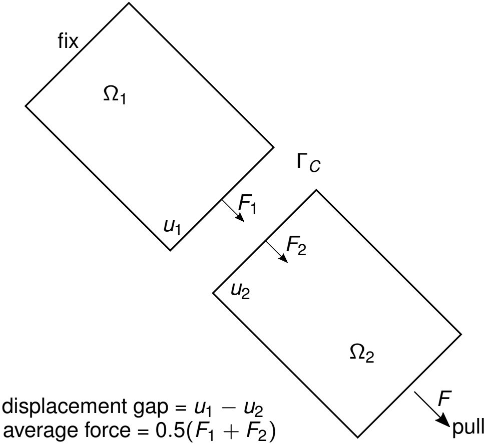



Figure: Patch coupling

## Constraints and additional work

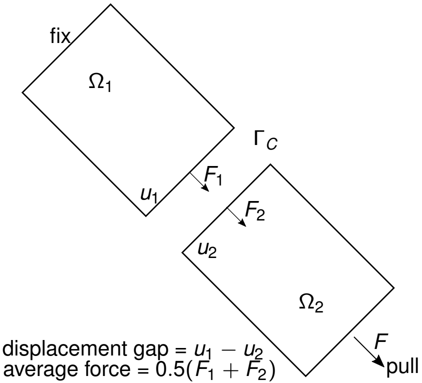



Figure: Patch coupling

Constraints on interface $\Gamma_C$

$$u_1 = u_2 \quad \text{on } \Gamma_C \tag{1a}$$
$$F_1 = F_2 \quad \text{on } \Gamma_C \tag{1b}$$

Define jump and average operators

$$\begin{aligned} [\![u]\!] &:= u_1 - u_2 \\ \langle F \rangle &:= \frac{1}{2}(F_1 + F_2) \end{aligned} \tag{2}$$

Additional work to be done

$$W_{add} = \langle F \rangle [\![u]\!] \tag{3}$$

# Problem setup

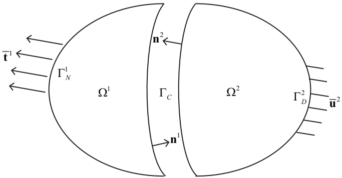



Figure: Couple two patches

$$\begin{aligned} \boldsymbol{u}_1 &= \boldsymbol{u}_2 && \text{on } \Gamma_C && \text{(4a)} \\ \boldsymbol{\sigma}_1 \cdot \boldsymbol{N}_1 &= -\boldsymbol{\sigma}_2 \cdot \boldsymbol{N}_2 && \text{on } \Gamma_C && \text{(4b)} \end{aligned}$$

The jump and average operators are defined as

$$\begin{aligned} [\![\boldsymbol{u}]\!] &:= \boldsymbol{u}_1 - \boldsymbol{u}_2 \\ \langle \boldsymbol{\sigma} \boldsymbol{N} \rangle &:= \frac{1}{2}(\boldsymbol{\sigma}_1 \boldsymbol{N} + \boldsymbol{\sigma}_2 \boldsymbol{N}) \end{aligned} \qquad (5)$$

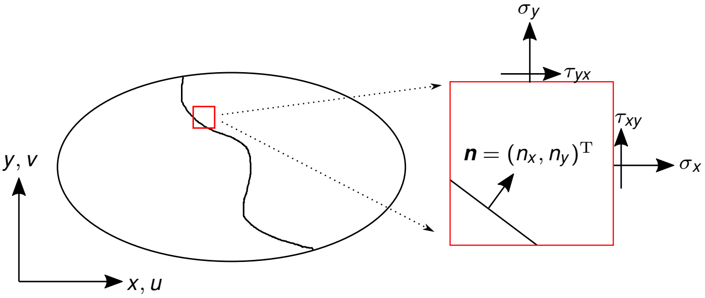
here $\boldsymbol{N}$ is chosen to be $\boldsymbol{N}_1$.

# Different from fabrics stretching

- Instead of scalar $u$, use vector $\boldsymbol{u}$ for generalized cases, e.g. $\boldsymbol{u} = (u, v)^{\mathrm{T}}$ in 2D
- Instead of forces F, use traction $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{N}$,
  where the stress comes from displacement field

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{D}\varepsilon(\boldsymbol{u}) = \boldsymbol{D}\nabla\boldsymbol{u} \tag{6}$$

and $\boldsymbol{N}$ is the transformation matrix to collect area contribution.




$$\boldsymbol{N} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}$$

Figure: $\boldsymbol{u}$, $\boldsymbol{n}$ and $\boldsymbol{\sigma}$

## Nitsche formulation

Start from the classical weak form

$$a(\boldsymbol{u}, \boldsymbol{w}) = L(\boldsymbol{w}) \tag{7}$$

and introduce Nitsche contribution into the weak form

$$a(\boldsymbol{u}, \boldsymbol{w}) - \int_{\Gamma_C} \langle \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{N} \rangle [\![ \boldsymbol{w} ]\!] \mathrm{d}\Gamma - \int_{\Gamma_C} [\![ \boldsymbol{u} ]\!] \langle \boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{N} \rangle \mathrm{d}\Gamma + \alpha \int_{\Gamma_C} [\![ \boldsymbol{u} ]\!] [\![ \boldsymbol{w} ]\!] \mathrm{d}\Gamma = L(\boldsymbol{w}) \tag{8}$$

Note

- Two Nitsche terms are introduced to keep the stiffness matrix symmetric
- Additional stabilisation parameter $\alpha$ to guarantee coercive (positive definite)
- Boundary integrations are performed along slave boundary
- The Nitsche contributions are made by work–conjugate pairs:
  for membrane element they are displacement and traction force,
  for thin bending plate they are rotation and bending moment

# Penalty and Lagrange multiplier

Penalty method:

$$a(\boldsymbol{u}, \boldsymbol{w}) + \frac{\alpha}{2} \int_{\Gamma_C} [\![\boldsymbol{u}]\!][\![\boldsymbol{w}]\!] \mathrm{d}\Gamma = L(\boldsymbol{w}) \tag{9}$$

where $\alpha$ is the penalty parameter.

Lagrange multiplier method:

$$a(\boldsymbol{u}, \boldsymbol{w}) + \int_{\Gamma_C} \boldsymbol{\lambda} [\![\boldsymbol{w}]\!] \mathrm{d}\Gamma + \int_{\Gamma_C} \delta\boldsymbol{\lambda} [\![\boldsymbol{u}]\!] \mathrm{d}\Gamma = L(\boldsymbol{w}) \tag{10}$$

where $\boldsymbol{\lambda}$ is the vector of Lagrange multiplier.

| Methods | Pros | Cons |
|---|---|---|
| Penalty | No increased DOFs<br>Easy and straightforward | Depends on penalty parameter<br>sometimes ill-conditioned |
| Lagrange multiplier | $\boldsymbol{\lambda}$ means traction<br>Stable when satisfies LBB | Increase DOFs<br>Not positive define |
| Nitsche | No increased DOFs<br>Positive define, robust | Not parameter-free<br>Involve constitutive equation |

# How do they work

Penalty method:

$$a(\boldsymbol{u},\boldsymbol{w}) + \frac{\alpha}{2}\int_{\Gamma_C} [\![\boldsymbol{u}]\!][\![\boldsymbol{w}]\!]\mathrm{d}\Gamma = L(\boldsymbol{w}) \tag{11}$$

Lagrange multiplier method:

$$a(\boldsymbol{u},\boldsymbol{w}) + \int_{\Gamma_C} \boldsymbol{\lambda}[\![\boldsymbol{w}]\!]\mathrm{d}\Gamma + \int_{\Gamma_C} \delta\boldsymbol{\lambda}[\![\boldsymbol{u}]\!]\mathrm{d}\Gamma = L(\boldsymbol{w}) \tag{12}$$

Nitsche's method:

$$a(\boldsymbol{u},\boldsymbol{w}) - \int_{\Gamma_C} \langle\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{N}\rangle[\![\boldsymbol{w}]\!]\mathrm{d}\Gamma - \int_{\Gamma_C} [\![\boldsymbol{u}]\!]\langle\boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{N}\rangle\mathrm{d}\Gamma + \alpha\int_{\Gamma_C} [\![\boldsymbol{u}]\!][\![\boldsymbol{w}]\!]\mathrm{d}\Gamma = L(\boldsymbol{w}) \tag{13}$$

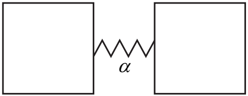



Figure: $\alpha$ is spring stiffness

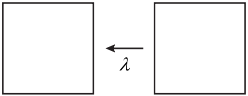



Figure: $\lambda$ is external traction

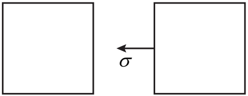
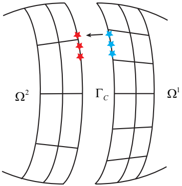



Figure: $\sigma$ is from slave boundary

# Slave boundary to perform boundary integration

- Choose slave boundary that has more elements




- Choose slave boundary that has shorter edge

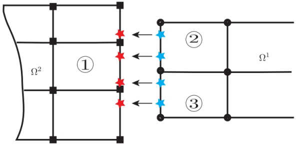



Figure: 2D
[*V.P.Nguyen et.al*. 2013]

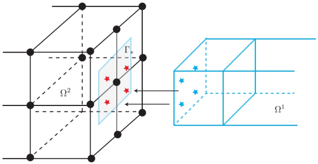
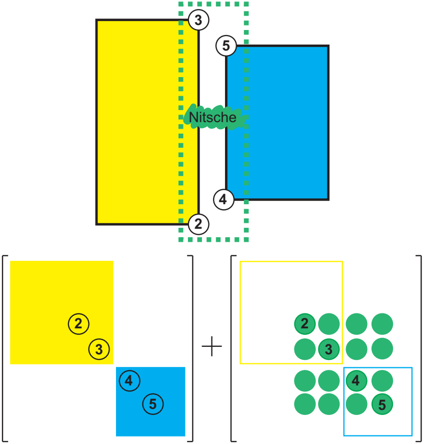


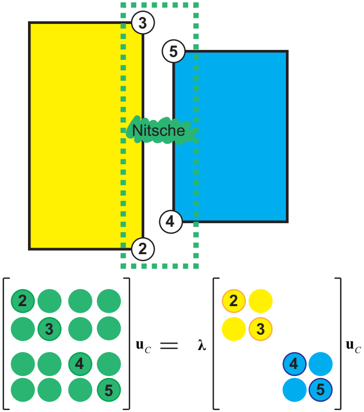

Figure: 3D
[*V.P.Nguyen et.al*. 2013]

# Stiffness matrix illustration




1. Calculate stiffness matrix for slave and master
   $\boldsymbol{K} = \sum \boldsymbol{K}_s + \sum \boldsymbol{K}_m$
2. Calculate Nitsche contribution along coupled boundary
   $\boldsymbol{K} += \sum \boldsymbol{K}_N$

# Stabilisation parameter $\alpha$




1. Solve generalized eigenvalues $\boldsymbol{\lambda}$ along coupled boundary $\boldsymbol{K}_N \boldsymbol{u}_c = \lambda \boldsymbol{K} \boldsymbol{u}_c$
2. $\alpha = 2max(\lambda)$ ref A.Apostolatos et.al. IJNME. 2013

# Bending plate

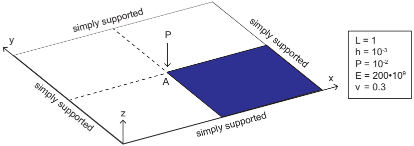
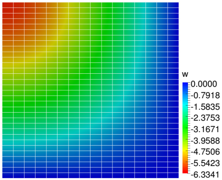
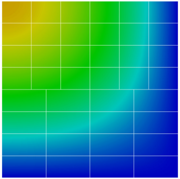
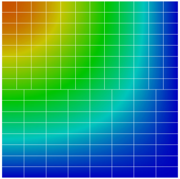



Figure: Bending plate





# Vibration square plate

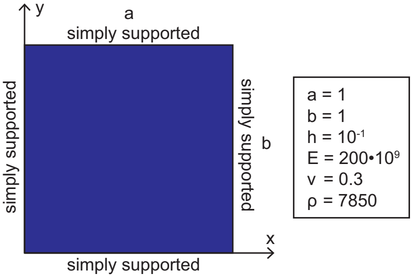



Figure: Square plate

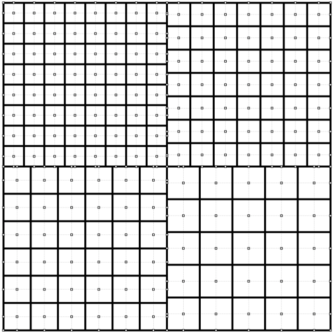
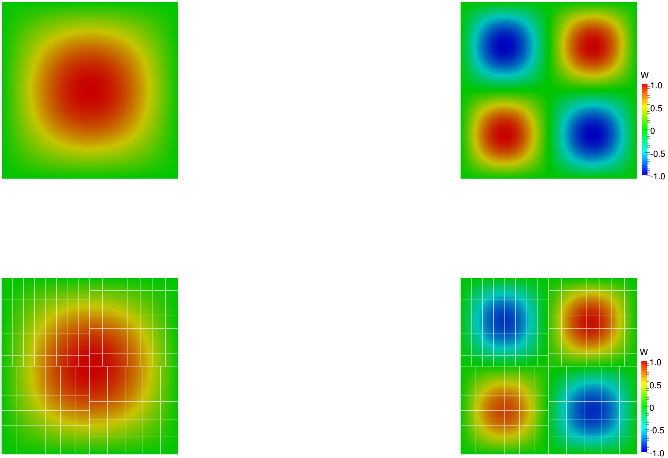
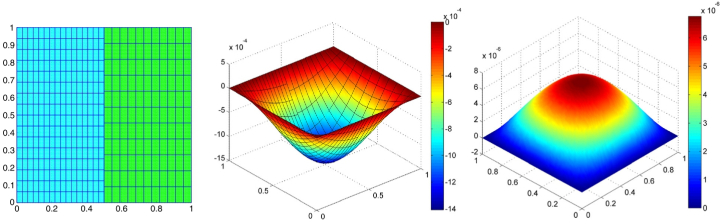


Figure: Meshes

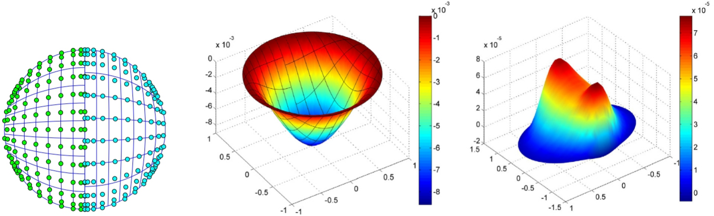
# Vibration square plate



# Clamped plate and errors [X.Du et.al. 2015]

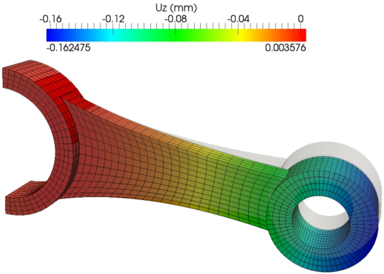



(a) Non-conforming mesh (b) Deflection *w* (c) Absolute error

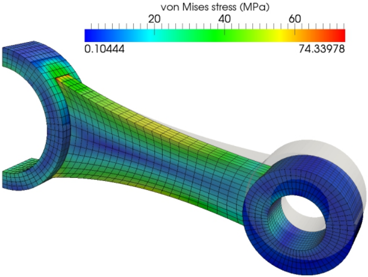



(a) Non-conforming mesh and control points (b) Deflection (c) Absolute error

# Connecting rod [V.P.Nguyen et.al. 2014]




(a) z-displacement field




(b) Stress field

# Intersecting tubular shell [Y.Guo et.al. 2017]