Stable 3D XFEM with applications to non planar crack propagation and inverse problems

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Contact:

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Doc. ref.:

Outline

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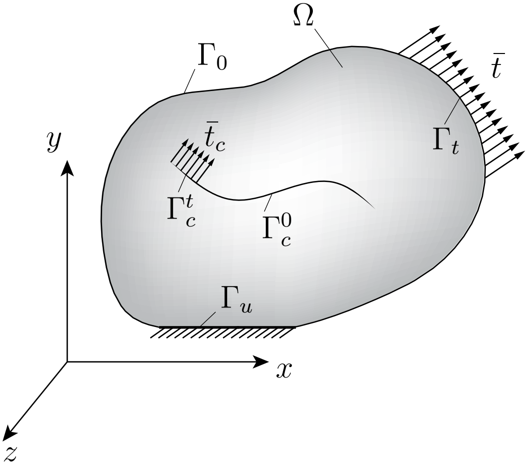


$$\Gamma = \Gamma_0 \cup \Gamma_u \cup \Gamma_t \cup \Gamma_c$$
$$\Gamma_c = \Gamma_c^t \cup \Gamma_c^0$$

Weak form of equilibrium equations

Find $\boldsymbol{u} \in \mathcal{U}$ such that $\forall \boldsymbol{v} \in \mathcal{V}$

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \epsilon(\boldsymbol{v}) \; d\Omega = \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} \; d\Omega + \int_{\Gamma_t} \bar{\boldsymbol{t}} \cdot \boldsymbol{v} \; d\Gamma + \int_{\Gamma_c^t} \bar{\boldsymbol{t}}_c \cdot \boldsymbol{v} \; d\Gamma_c^t$$

where :

$$\mathcal{U} = \left\{ \boldsymbol{u} | \boldsymbol{u} \in \left(H^1(\Omega) \right)^3, \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \Gamma_u \right\}$$

and

$$\mathcal{V} = \left\{ \boldsymbol{v} | \boldsymbol{v} \in \left(H^1(\Omega) \right)^3, \boldsymbol{v} = 0 \text{ on } \Gamma_u \right\}$$

Global enrichment XFEM

An XFEM variant is introduced which:

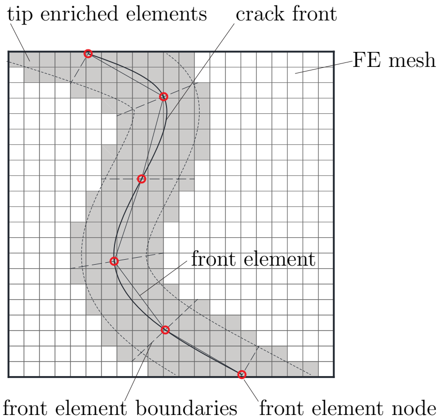
- Enables the application of geometrical enrichment to 3D.
- Extends dof gathering to 3D through global enrichment.
- Employs weight function blending.
- Employs enrichment function shifting.

A superimposed mesh is used to provide a p.u. basis.

Desired properties:

- Satisfaction of the partition of unity property.
- Spatial variation only along the direction of the crack front.

Front elements



- A set of nodes along the crack front is defined.
- Each element is defined by two nodes.
- A good starting point for front element thickness is h.

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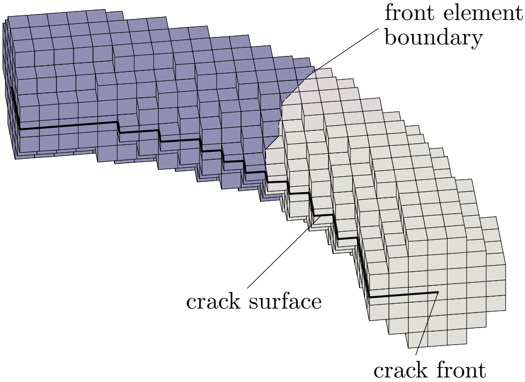
Volume corresponding to two consecutive front elements.



Different element colors correspond to different front elements.

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Front element shape functions

Linear 1D shape functions are used:

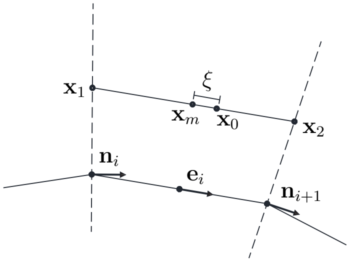
$$\mathbf{N}^{g}(\xi) = \left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2} \right]$$

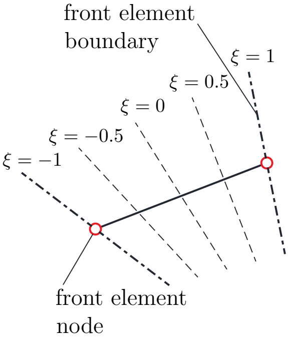
where ξ is the local coordinate of the superimposed element.

Those functions:

- form a partition of unity.
- are used to weight tip enrichment functions.

Definition of the front element parameter used for shape function evaluation.





Tip enrichment functions

Tip enrichment functions used:

$$F_j(\mathbf{x}) \equiv F_j(r,\theta) = \left[\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right]$$

Tip enriched part of the displacements:

$$\mathbf{u_t}(\mathbf{x}) = \sum_{K\in\mathcal{N}^s} N_K^g(\mathbf{x}) \sum_j F_j(\mathbf{x})\,\mathbf{c}_{Kj}$$

where

- N_K^g are the global shape functions
- $\mathcal{N}^s$ is the set of superimposed nodes

The weight function assumes the form:

$$\varphi(\mathbf{x}) = \sum_{T \in \mathcal{N}^{t1}} N_T(\mathbf{x})$$

where

- N_T are the FE shape functions.
- $\mathcal{N}^{t1}$ is a set including all nodes belonging to elements that contain the crack front.

This definition is identical to the one of Fries (?, ?).

The weight function is defined as in (?, ?). Nodal values:

$$\varphi_I = \begin{cases} 1, & g_I < 0 \\ (1 - g_I)^n, & 0 \leq g_I \leq 1 \\ 0, & g_I > 1 \end{cases}$$

where

- r_e is the enrichment radius.
- r_i is an additional distance such that $r_i < r_e$.
- r_I are the nodal values of parameter r.
- $g_I = \frac{r_I - r_i}{r_e - r_i}$

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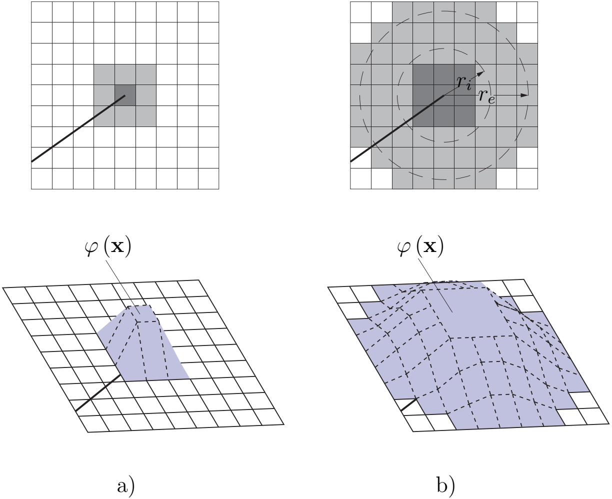
Weight function values are obtained through FE interpolation:

$$\varphi(\mathbf{x}) = \sum_{T \in \mathcal{N}^t} N_T(\mathbf{x}) \, \varphi_T$$

where $\mathcal{N}^t$ is the set of tip enriched nodes.

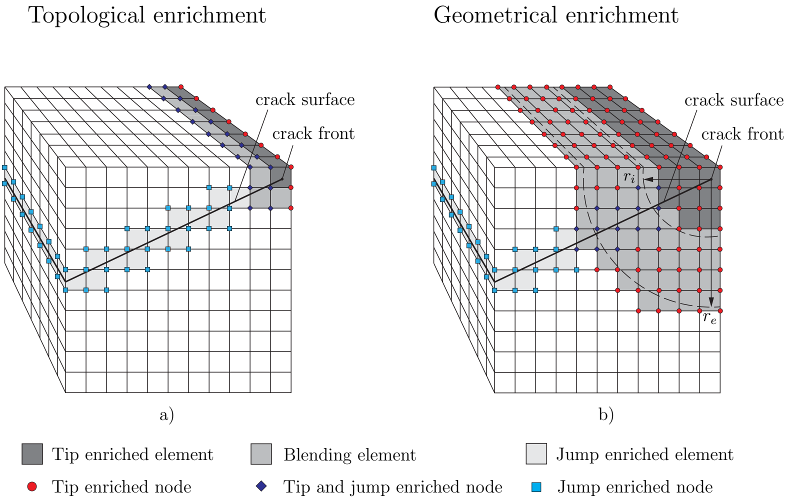
Weight functions for a) topological and b) geometrical enrichment.



Weight function for jump enrichment:

$$\bar{\varphi}(\mathbf{x}) = 1 - \varphi(\mathbf{x})$$

Enrichment strategies used for tip and jump enrichment.



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Displacement approximation

$$\mathbf{u}(\mathbf{x}) = \sum_{I \in \mathcal{N}} N_I(\mathbf{x})\,\mathbf{u}_I + \bar{\varphi}(\mathbf{x}) \sum_{J \in \mathcal{N}^j} N_J(\mathbf{x})\,(H(\mathbf{x}) - H_J)\mathbf{b}_J +$$

$$+\,\varphi(\mathbf{x}) \left(\sum_{K \in \mathcal{N}^s} N_K^g(\mathbf{x}) \sum_j F_j(\mathbf{x}) - \right.$$

$$\left. - \sum_{T \in \mathcal{N}^t} N_T(\mathbf{x}) \sum_{K \in \mathcal{N}^s} N_K^g(\mathbf{x}_T) \sum_j F_j(\mathbf{x}_T) \right) \mathbf{c}_{Kj}$$

where:

$\mathcal{N}$ is the set of all nodes in the FE mesh.

$\mathcal{N}^j$ is the set of jump enriched nodes.

$\mathcal{N}^t$ is the set of tip enriched nodes.

$\mathcal{N}^s$ is the set of nodes in the superimposed mesh.

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A method for the representation of 3D cracks is introduced which:

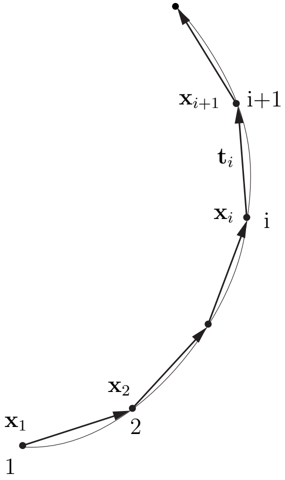
- Produces level set functions using geometric operations.
- Does not require integration of evolution equations.

Similar methods:

- 2D Vector level sets (?, ?).
- Hybrid implicit-explicit crack representation (?, ?).

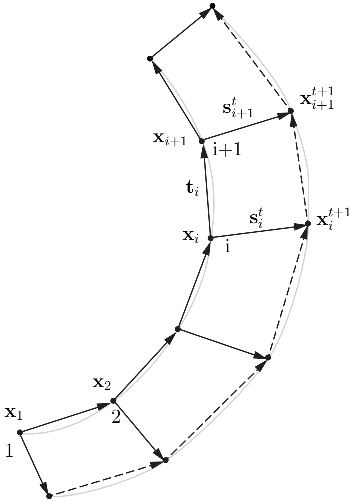
Crack front at time t:

- Ordered series of line segments $\mathbf{t}_i$
- Set of points $\mathbf{x}_i$



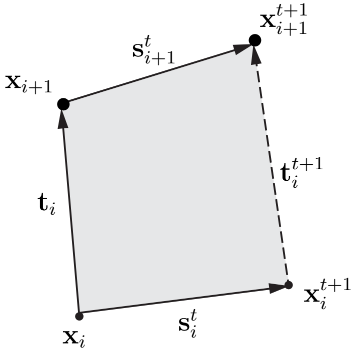
Crack front at time $t + 1$:

- Crack advance vectors $\mathbf{s}_i^t$ at points $\mathbf{x}_i$
- New set of points $\mathbf{x}_i^{t+1} = \mathbf{x}_i^t + \mathbf{s}_i^t$



Crack surface advance:

- Sequence of four sided bilinear segments.
- Vertexes: $\mathbf{x}_i^t$, $\mathbf{x}_{i+1}^t$, $\mathbf{x}_{i+1}^{t+1}$, $\mathbf{x}_i^{t+1}$



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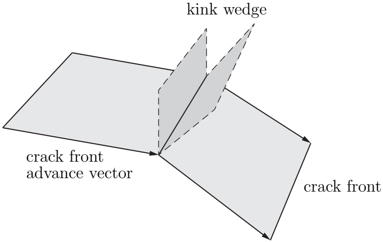
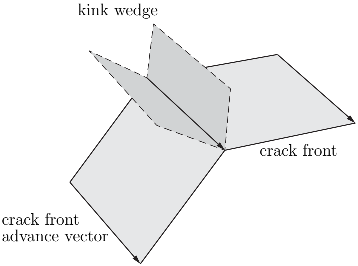
Discontinuities (*kink wedges*) are present:

- Along the crack front (a).
- Along the advance vectors (b).



a)

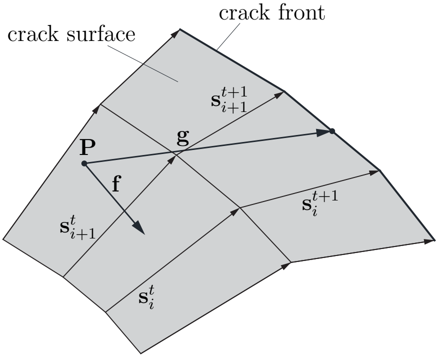


b)

Definition of the level set functions at a point **P**:

f distance from the crack surface.

g distance from the crack front.



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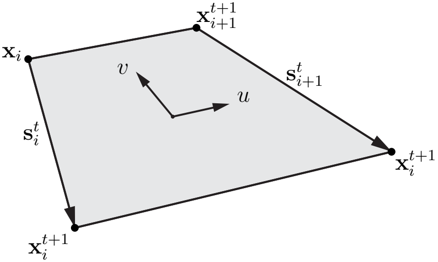
Point projection



Element parametric equations $\phi(u,v),\ u,v \in [-1,1]$:

$$\begin{cases} \phi_x &= g_1(u,v)\,x_i^t + g_2(u,v)\,x_{i+1}^t + g_3(u,v)\,x_{i+1}^{t+1} + g_4(u,v)\,x_i^{t+1} \\ \phi_y &= g_1(u,v)\,y_i^t + g_2(u,v)\,y_{i+1}^t + g_3(u,v)\,y_{i+1}^{t+1} + g_4(u,v)\,y_i^{t+1} \\ \phi_z &= g_1(u,v)\,z_i^t + g_2(u,v)\,z_{i+1}^t + g_3(u,v)\,z_{i+1}^{t+1} + g_4(u,v)\,z_i^{t+1} \end{cases}$$

where $g_i(u,v),\ u,v \in [-1,1]$ are linear shape functions.

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Equation of the tangent plane Π_0 at (u_0, v_0):

$$\det \begin{bmatrix} x - \phi_x(u_0, v_0) & y - \phi_y(u_0, v_0) & z - \phi_z(u_0, v_0) \\ \phi_{x,u}(u_0, v_0) & \phi_{y,u}(u_0, v_0) & \phi_{z,u}(u_0, v_0) \\ \phi_{x,v}(u_0, v_0) & \phi_{y,v}(u_0, v_0) & \phi_{z,v}(u_0, v_0) \end{bmatrix} = 0$$

Normal vector to the parametric surface at (u_0, v_0):

$$\mathbf{n}(u_0, v_0) = (A, B, C)$$

where A, B, C are the minors of the previous matrix at (u_0, v_0).

Point projection

Point $\mathbf{P}$ can be expressed as:

$$\mathbf{P} = \mathbf{P}'(u, v) + \lambda \mathbf{n}(u, v)$$

where:

$\mathbf{P}'$ the projection of the point to the surface.

λ unknown parameter.

The above is solved for u, v and λ to obtain the projection.

Evaluation of the level set functions

At each step t:

- For each point all crack advance segments are tested.

- If for a certain element $u, v \in [-1, 1]$ then the point is projected on that element.

- If $u \notin [-1, 1]$ for all elements then the projection lies on the advance vector.

- If $v \notin [-1, 1]$ for all elements then the projection lies either:
  $\rightarrow$ at a previous crack advance segment
  $\rightarrow$ at the crack front at time $t - 1$ or t

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Evaluation of the level set functions

Level set function **f**:

$$\mathbf{f} = \mathbf{P} - \mathbf{P}'$$

where $\mathbf{P}'$ is either:

- Projection to an element of the crack surface
- Closest point projection to a kink wedge

Level set function **g**:

$$\mathbf{g} = \mathbf{P} - \mathbf{P}'$$

where $\mathbf{P}'$ is a closest point projection to the crack front

Inverse problem

→ Detection of cracks in existing structures

→ Measurements are available

→ A computational model is employed

→ The difference between the two is minimized

→ Information regarding the cracks is obtained

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Mathematical formulation:

$$\textit{Find } \beta_i \textit{ such that}$$
$$\mathcal{F}\left(r\left(\beta_i\right)\right) \rightarrow \min$$

where

- β_i Parameters describing the crack geometry
- $r(\cdot)$ Norm of the difference between measurements and computed values
- $\mathcal{F}$ Some function of the residual

The CMA-ES algorithm is employed to solve the problem.

Solution process:

→ Generation of initial population (β_i) with CMA-ES

→ Fitness function ($\mathcal{F}(r(\beta_i))$) evaluation using XFEM and measurements

→ Population is updated with CMA-ES

→ The procedure is repeated until convergence

During the optimization proccess:

- A large number of crack geometries is tested
- The computational model is solved several times
- An efficient and robust method is required

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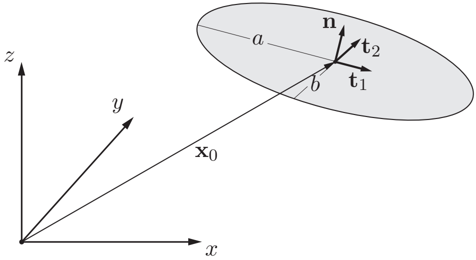
Elliptical cracks are considered:



Parameters:

- Coordinates of center point $\mathbf{x}_0$ ($\{x_0, y_0, z_0\}$)
- Rotation about the three axes θ_x, θ_y and θ_z
- Lengths a and b

Scaling of parameters:

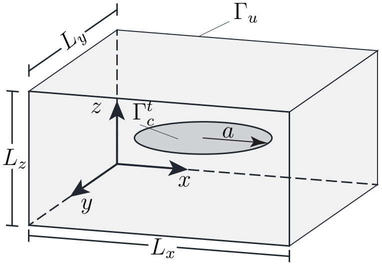
$$p_i = \frac{p_{i_1} + p_{i_2}}{2} + \frac{p_{i_2} - p_{i_1}}{2} \sin\left(\frac{\beta_i}{10} \cdot \frac{\pi}{2}\right)$$

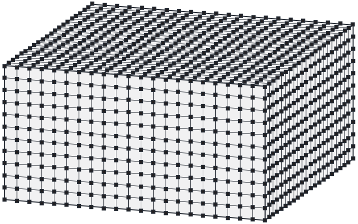
where:

β_i are design variables

p_i are geometrical parameters of the crack

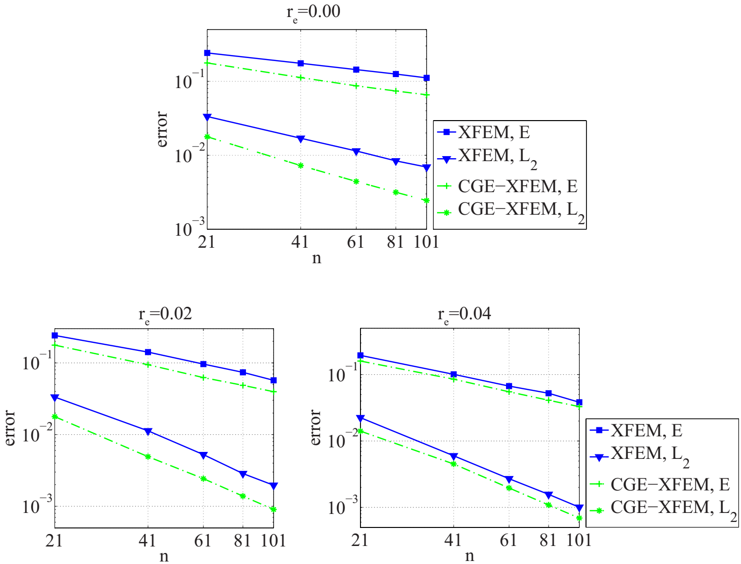
p_{i_1}, p_{i_2} are lower and upper values for the parameters





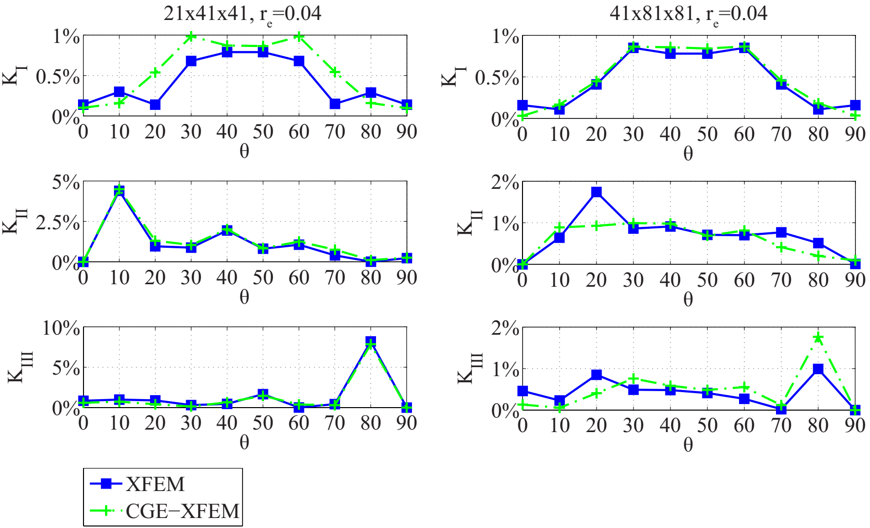
■ node where boundary conditions are applied

- Uniform normal and shear loads of magnitude 1 are applied at Γ_c^t.
- Problem dimensions: $L_x = L_y = 2L_z = 0.4$ units and $a = 0.1$ unit.
- Material parameters: $E = 100$ units and $\nu = 0.3$.
- Mesh consists of $n_x \times n_y \times n_z$ hexahedral elements, $n_x = n_y = 2n_z = n$ and $n \in \{21, 41, 61, 81, 101\}$.


r_e=0.00
error
n
21
41
61
81
101
10^-1
10^-2
10^-3
XFEM, E
XFEM, L_2
CGE−XFEM, E
CGE−XFEM, L_2
r_e=0.02
r_e=0.04

Convergence rates

	$r_e = 0.00$	$r_e = 0.02$	$r_e = 0.04$
XFEM E	0.492	0.911	1.015
XFEM L_2	1.009	1.824	1.976
CGE-XFEM E	**0.635**	**0.957**	**1.014**
CGE-XFEM L_2	**1.265**	**1.890**	**1.930**


21x41x41, r_e=0.04
41x81x81, r_e=0.04
K_I
K_II
K_III
θ
1%
0.5%
0%
5%
2.5%
10%
2%
0 10 20 30 40 50 60 70 80 90
XFEM
CGE−XFEM

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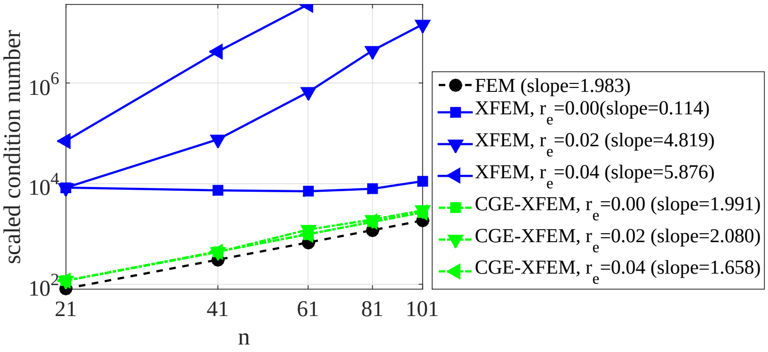
Condition numbers for three different enrichment radii.



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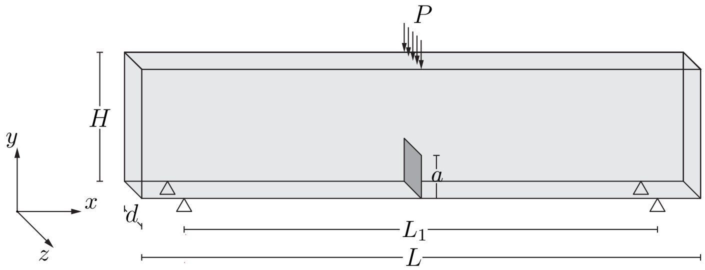
Edge crack in a beam under three point bending.



Geometrical parameters:
$L = 260$ mm, $L_1 = 240$ mm, $H = 60$ mm, $d = 10$ mm, $\alpha = 20$ mm
Material parameters:
$E = 2.1 \times 10^5$ N/mm^2, $\nu = 0.0$

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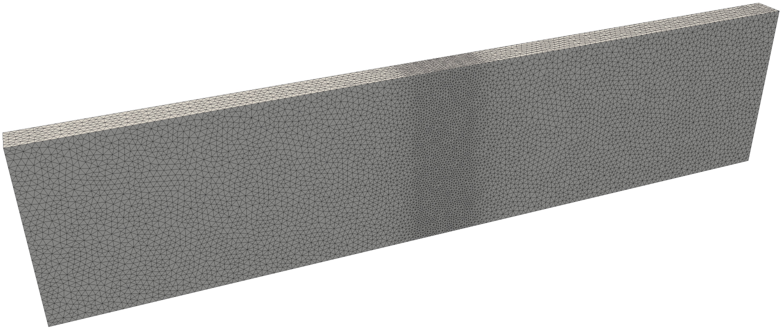
An unstructured tetrahedral mesh is used:



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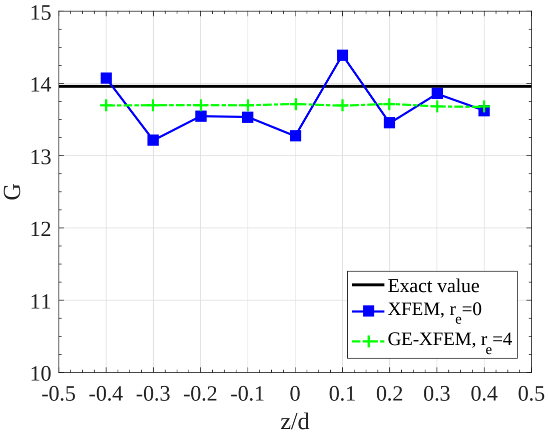
Energy release rates along the crack front:



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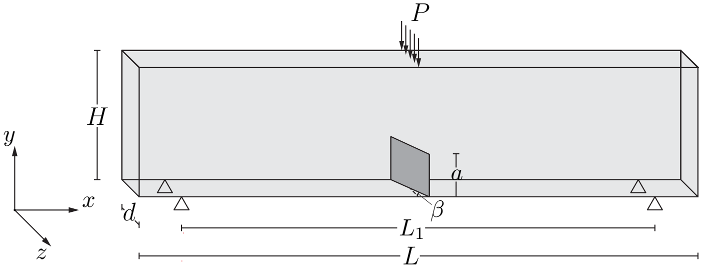
Inclined edge crack in a beam under three point bending.



Geometrical parameters:

$L = 260$ mm, $L_1 = 240$ mm, $H = 60$ mm, $d = 10$ mm, $\alpha = 20$ mm, $\beta = 45°$

Material parameters:

$E = 2.1 \times 10^5$ N/mm^2, $\nu = 0.3$

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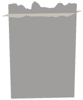
Inclined edge crack in a beam



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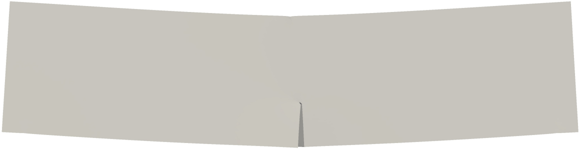
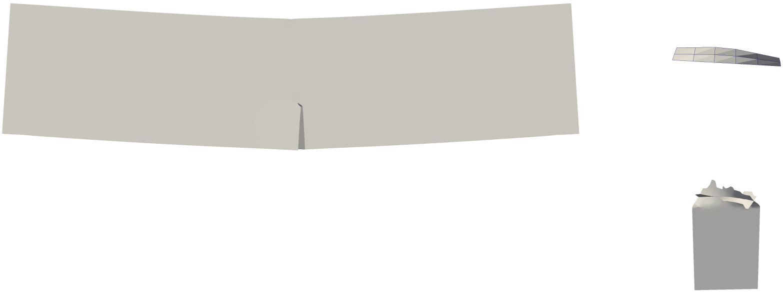
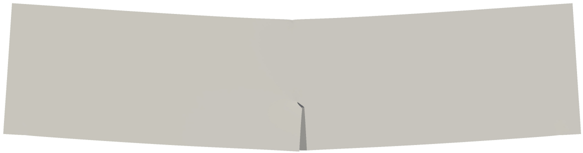
Inclined edge crack in a beam

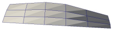


Inclined edge crack in a beam



Inclined edge crack in a beam





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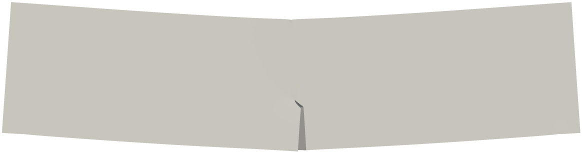

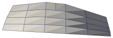

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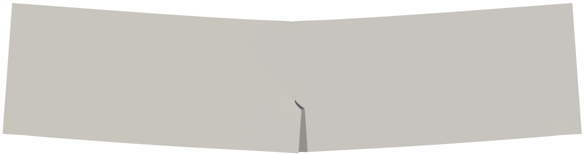

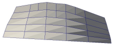

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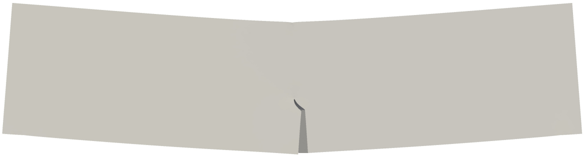

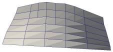

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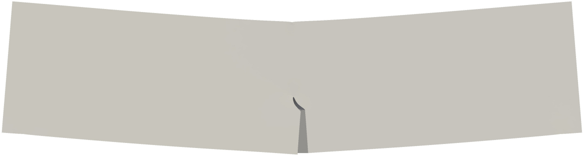

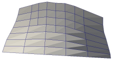

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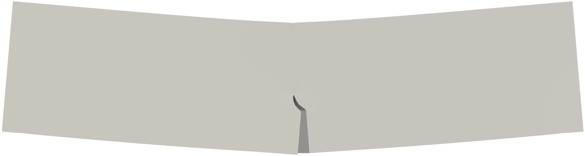

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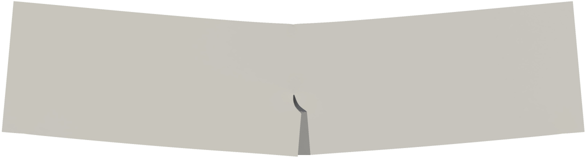
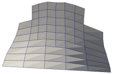
Inclined edge crack in a beam





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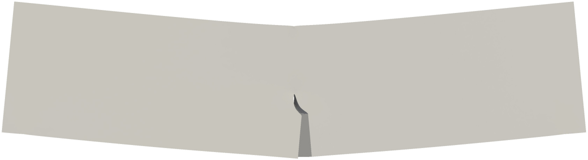
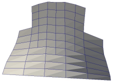
Inclined edge crack in a beam





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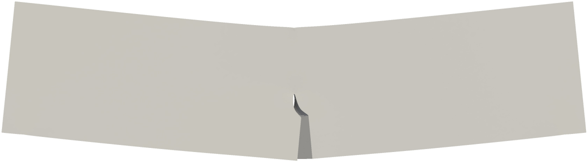
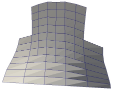
Inclined edge crack in a beam





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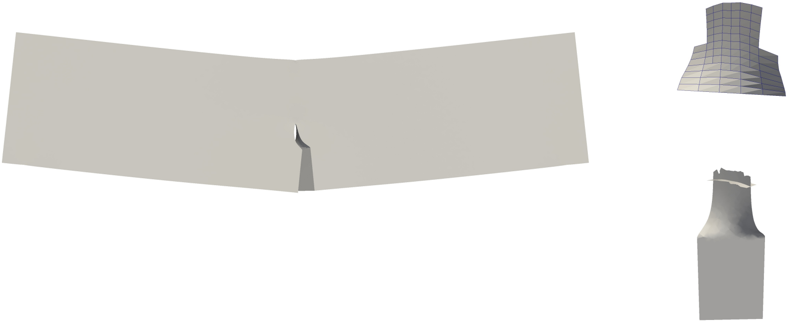

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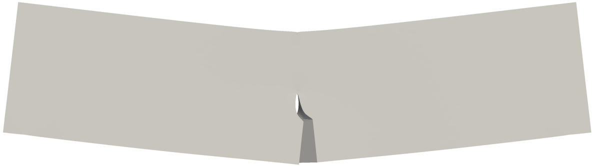
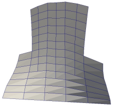
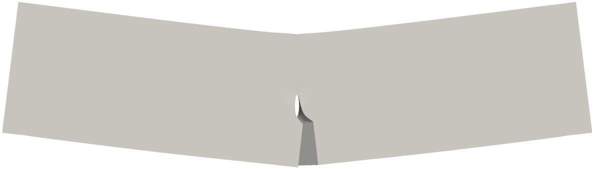
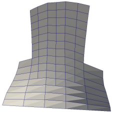
Inclined edge crack in a beam









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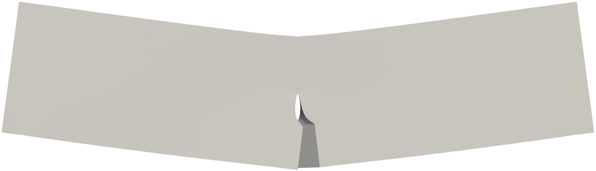

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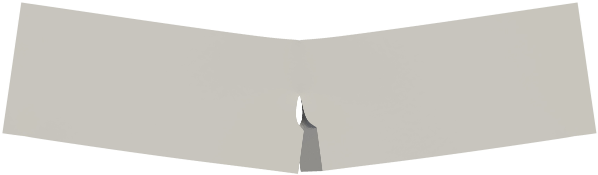

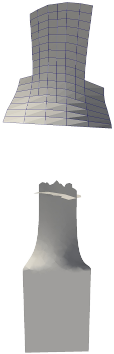

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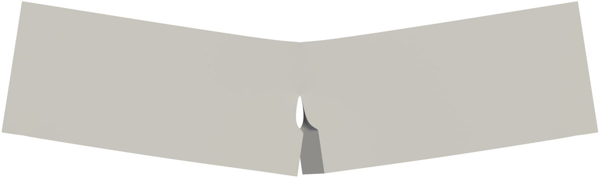
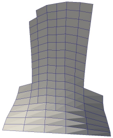
Inclined edge crack in a beam





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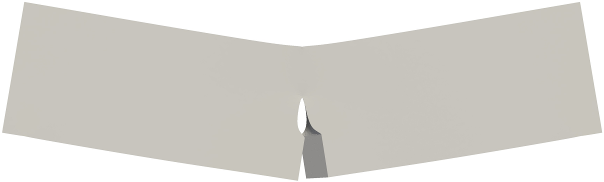

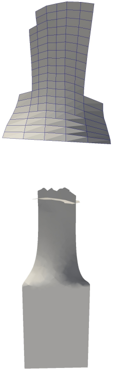

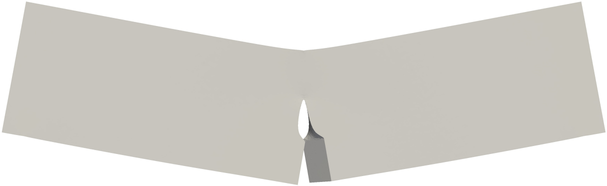

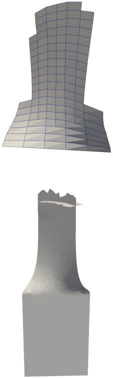

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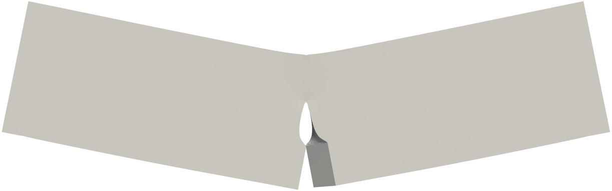

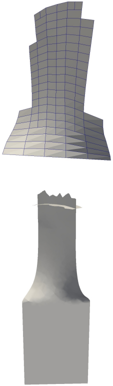

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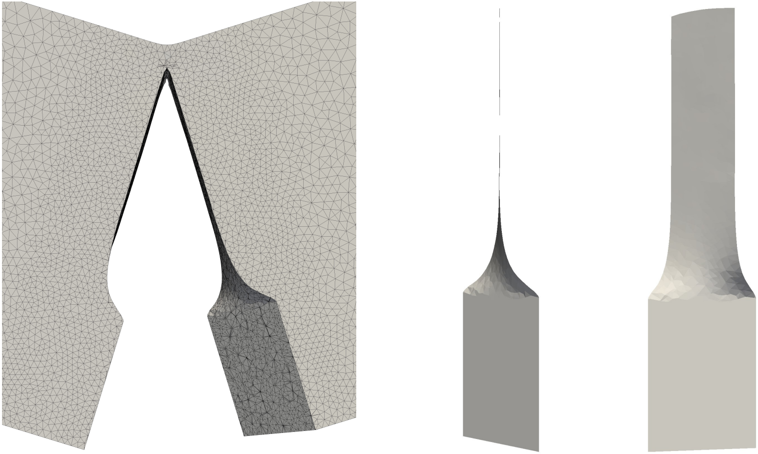
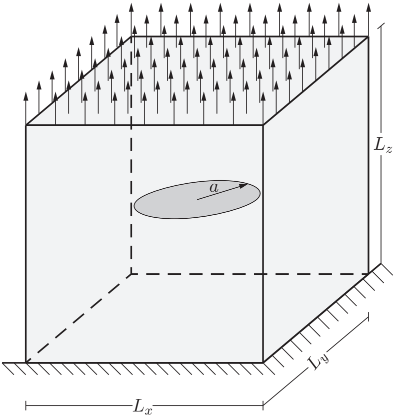
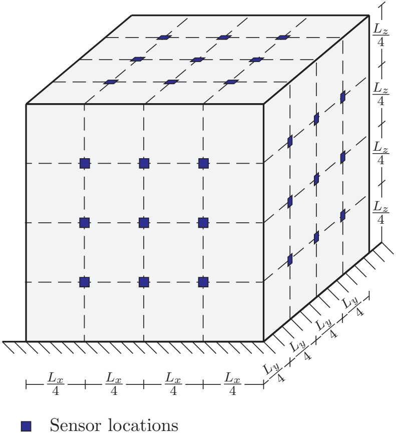


(a) (b) (c)

Geometry and sensors:





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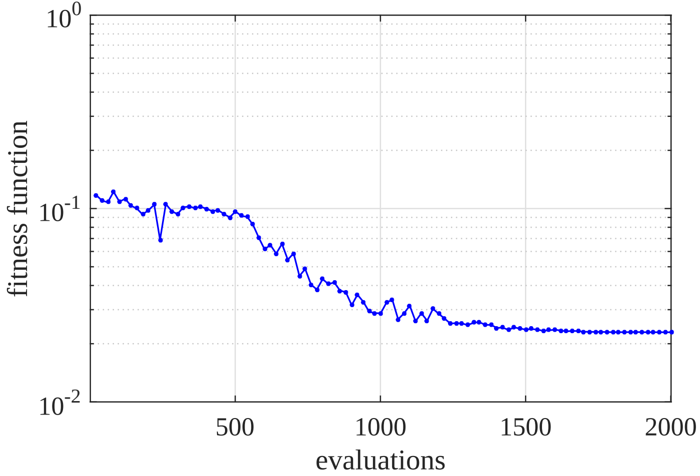
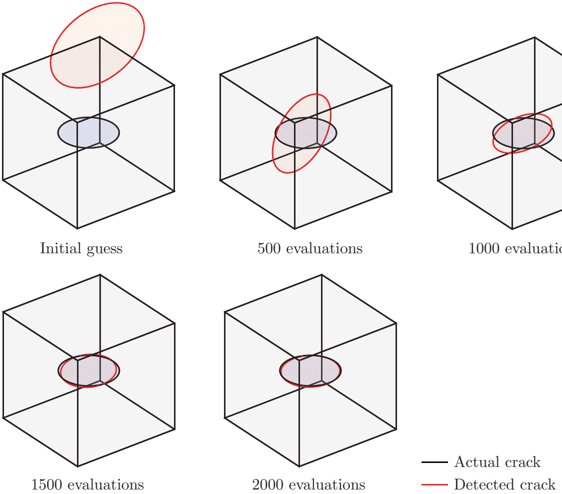
Optimization problem convergence:



Best solution after different numbers of iterations



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Conclusions

A method was presented which:

- Utilizes a novel form of enrichment.
- Provides improved conditioning.
- Enables the use of geometrical enrichment.
- Provides high accuracy and optimal convergence.
- Was combined to vector level sets to solve crack propagation problems
- Was applied to inverse problems.

Bibliography

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