Weakening the tight coupling between geometry and simulation in isogeometric analysis: from sub- and super- geometric analysis to Geometry Independent Field approximaTion (GIFT)

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Outline of the presentation

- Motivation
- Theoretical background
- Patch tests
- Numerical results
- Conclusions

Motivation: IsoGeometric Analysis (IGA)

- Tight link between CAD and analysis
- The same basis functions, which are used in CAD to represent the geometry, are used in the IGA as shape functions to approximation the unknown solution
- Geometry is exact at any stage of the solution refinement process
- Better accuracy per DOF in comparison with standard FEM
- Additional advantages, such as higher continuity of splines, makes IGA applicable for PDEs of higher order

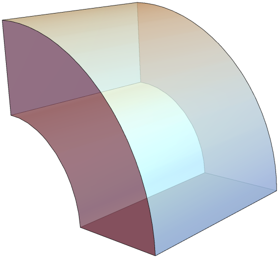
Disadvantages of IsoGeometric Analysis (IGA)

- Gaps can occur when different geometrical pieces are joined
- Additional coupling mechanisms are required for multi-patch geometries
- Tensor-product structure of NURBS does not allow local refinement

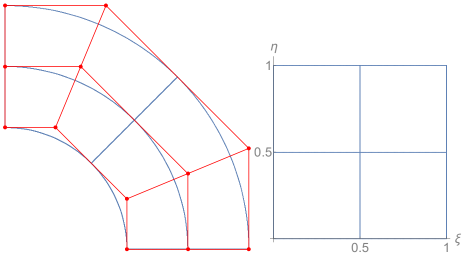
What is “I” in the IGA?

CAD model

(surface for 2d or volume for 3d)



Knot vectors and control points:



What is “I” in the IGA?

on each element: global assembly:

$$\boldsymbol{x} = \sum_{n=1}^{N_e} \boldsymbol{C}_i^e N_i(\xi, \eta)$$

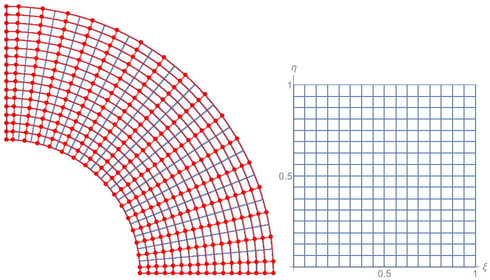
$$\boldsymbol{u} = \sum_{n=1}^{N_e} U_i^e N_i(\xi, \eta)$$

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{f}$$

$$\boldsymbol{K}^e\boldsymbol{U}^e = \boldsymbol{f}^e$$

What is “I” in the IGA?

refinement:



on each element:

$$\boldsymbol{x} = \sum_{n=1}^{N_e} \boldsymbol{C}_i^e N_i(\xi, \eta)$$

$$\boldsymbol{u} = \sum_{n=1}^{N_e} U_i^e N_i(\xi, \eta)$$

$$\boldsymbol{K}^e \boldsymbol{U}^e = \boldsymbol{f}^e$$

What is “I” in the IGA?

Isogeometric means that the same shape functions are used on each element to represent the geometrical variables $\boldsymbol{x}$ and the field $\boldsymbol{u}$:

$$\boldsymbol{x} = \sum_{n=1}^{N_e} \boldsymbol{C}_i^e N_i(\xi, \eta)$$

$$\boldsymbol{u} = \sum_{n=1}^{N_e} U_i^e N_i(\xi, \eta)$$

What is the difference between "iso-geometric" and "iso-parametric"?

In the standard FEM iso-parametric elements are used to **approximate** both, the field and the computational domain, while in the IGA, refinement does not improve geometry parameterization (it remains exact and equivalent to the original model at each refinement stage), therefore geometry refinement is redundant.

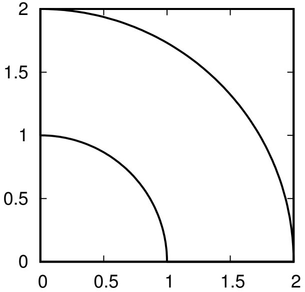
Are NURBS always the best choice to approximate the solution?

- Rational functions
- No local refinement

What can we improve?

- Keep the exact representation of the geometry
- Choose more suitable approximation for the field

Patch test: will it pass or fail?



Laplace eqn:

$\Delta u = 0, \quad \text{in} \;\; \Omega$

$u|_{\partial\Omega}(x, y) = 1 + x + y.$

Elasticity eqn:

$\sigma_{ij,j} = 0, \quad \text{in} \;\; \Omega$

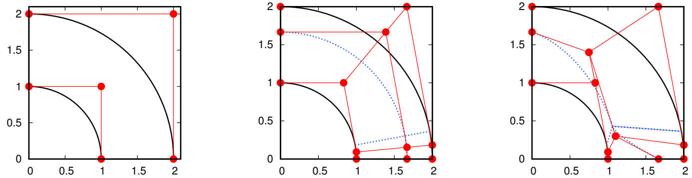
$t_i = \sigma_0 n_i \;\; \text{at} \;\; r = 1, 2,$

$u_2 = 0, t_1 = 0 \;\; \text{at} \;\; \theta = 0,$

$u_1 = 0, t_2 = 0 \;\; \text{at} \;\; \theta = \pi/2,$

Patch test: will it pass or fail?

Quarter annulus parameterizations:



Q_0	A_1	C_1
$\Sigma = [0, 0, 1, 1]$ $\Pi = [0, 0, 0, 1, 1, 1]$	$\Sigma = [0, 0, 2/3, 1, 1]$ $\Pi = [0, 0, 0, 1/8, 1, 1, 1]$	$\Sigma = [0, 0, 2/3, 1]$ $\Pi = [0, 0, 0, 1/8, 1, 1, 1]$
	B_1	
	$\Sigma = [0, 0, 0.17, 1, 1]$ $\Pi = [0, 0, 0, 0.81, 1, 1, 1]$	
	D_1 (B-Splines)	
	$\Sigma = [0, 0, 2/3, 1, 1]$ $\Pi = [0, 0, 0, 1/8, 1, 1, 1]$	

Five types of geometry parametrization

- $\mathbf{Q_0}$: the coarsest parameterization (single element) necessary to represent the geometry,
- $\mathbf{A_1}$: uniform parametrization (four elements), obtained by the refinement of Q_0 with knot insertion at 2/3 and 1/8
- $\mathbf{B_1}$: uniform parametrization (four elements), obtained by the refinement of Q_0 with knot insertion at 0.17 and 0.81
- $\mathbf{C_1}$: non-uniform parametrization (four elements), obtained from A_1 by moving the internal points randomly
  - Bases A_2, B_2, C_2, are obtained by elevating degree in both directions by 1 of A_1, B_1, and C_1, respectively.
- $\mathbf{D_1}$ **(B-Splines):** uniform parametrization (four elements), obtained from A_1 by setting all the weights to 1
  - Bases D_0 and D_2 are obtained from D_1 by reducing and elevating (respectively) degrees in both directions.

$T^{\ell}_{G_i,S_j}$	Laplace Eq.	Elasticity Eq.
$T^0_{Q_0,A_1}$	1.3815e-15	3.0871e-14
$T^0_{Q_0,A_2}$	5.2147e-15	1.7986e-14
$T^0_{Q_0,C_1}$	0.0182	0.0050
$T^0_{Q_0,C_2}$	0.0023	0.0012
$T^1_{A_1,A_1}$	1.0023e-15	1.1675e-14
$T^1_{A_1,A_2}$	4.3958e-14	1.2547e-14
$T^1_{A_2,A_1}$	1.4059e-15	1.5525e-15
$T^2_{B_1,A_1}$	1.4755e-15	2.9941e-15
$T^2_{B_1,A_2}$	2.1639e-15	1.2118e-14
$T^2_{B_2,A_1}$	1.0144e-15	5.4590e-15
$T^3_{C_1,C_1}$	1.1061e-15	1.6439e-14
$T^3_{C_1,C_2}$	1.8263e-15	2.8737e-15
$T^3_{C_2,C_1}$	1.2062e-15	5.6517e-14
$T^4_{C_1,A_1}$	0.0203	0.0085
$T^4_{C_1,A_2}$	0.0016	0.0009
$T^4_{C_2,A_1}$	0.0203	0.0085
$T^5_{A_1,D_1}$	0.0188	0.0214
$T^5_{A_1,D_2}$	0.0121	0.0039
$T^5_{A_1,D_0}$	0.5418	0.1411

Table 1: Results of various patch tests, denoted by T. Superscript ℓ denotes the test case, G_i denotes the bases for the geometry, and S_j denotes the bases for the solution approximation

Conclusion from the patch test studies

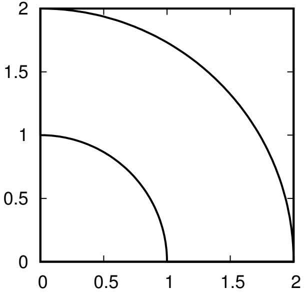
Any of the following combination of bases, which are equal up to operations of knot insertion or degree elevation, pass the patch test

- Geometry by Q_0, together with A_i or B_i for the solution
- Geometry by A_i and solution by A_j
- Geometry by B_i and solution by A_j
- Geometry by C_i and solution by C_j

Patch test for

A mixture of C_i with either of Q_0, A_i or B_i fails because C_i can not be obtained from these bases (only internal points randomly moved)

Convergence studies



Laplace eqn:

$\Delta u = 0, \quad \text{in } \ \Omega$

$u|_{\partial\Omega}(x, y) = r^{-3} \cos 3\theta.$

Elasticity eqn:

$\sigma_{ij,j} = 0, \quad \text{in } \ \Omega$

$t_i = \sigma_1 n_i \quad \text{at } \ r = 1,$

$t_i = \sigma_2 n_i \quad \text{at } \ r = 2,$

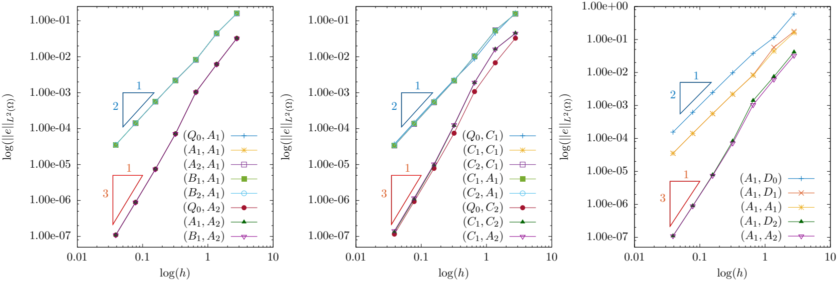
$u_2 = 0, t_1 = 0 \quad \text{at } \ \theta = 0,$

$u_1 = 0, t_2 = 0 \quad \text{at } \ \theta = \pi/2,$

Numerical examples: Example 1 - Laplace

Convergence studies for various combinations of bases

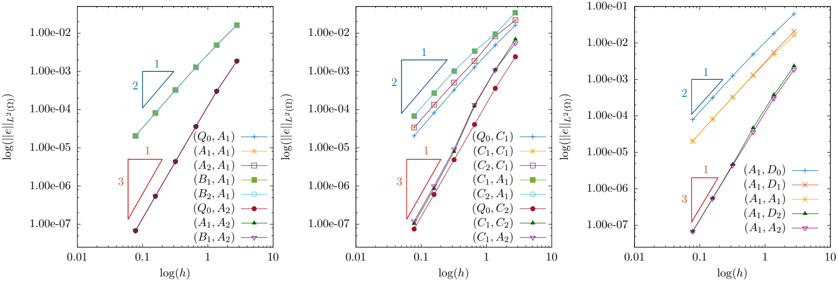
For exact representation of the geometry, all combinations of bases, including those which fail the patch test, deliver the optimal convergence



Numerical examples: Example 1 - Elasticity

Convergence studies for various combinations of bases

For exact representation of the geometry, all combinations of bases, including those which fail the patch test, deliver the optimal convergence



Conclusion:

Together with the given (exact) geometry parametrization at the coarsest level, the convergence rate is entirely defined by the solution basis, and does not depend on the further refinement of the geometry parametrization:

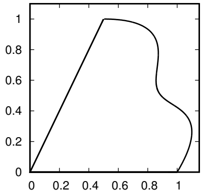
- For a given geometry parameterization, the degree of the solution basis can be increased or decreased without changing the degree of the geometry (from iso-geometric to super-geometric and sub-geometric elements)
- For solution approximation, using same degree B-Splines or NURBS yields almost identical results

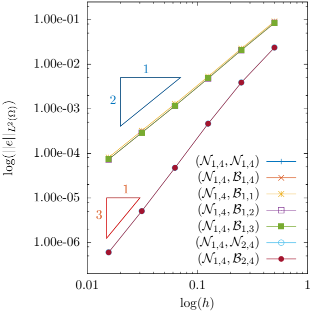
Numerical examples: Example 2 - Laplace

Convergence studies for various combinations of bases

Exact representation of the geometry by $\mathcal{N}_{1,4}$, various bases for the solution approximation, iso/super/sub-geometric, deliver optimal convergence (governed by the minimum degree), second order for first five choices, and third order for last two choices. $\mathcal{B}$ represents B-splines.

$$u(x,y) = \ln((x+0.1)^2+(y+0.1)^2)$$

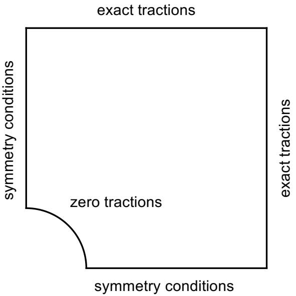


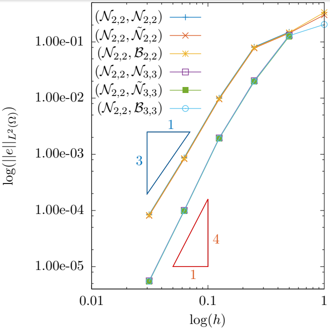


Numerical examples: Example 3 - Elasticity

Convergence studies for various combinations of bases

Exact representation of the geometry by $\mathcal{N}_{2,2}$, various bases for the solution approximation, iso/super-geometric, deliver optimal convergence (governed by the minimum degree), third order for first three choices, and fourth order for last three choices. $\tilde{\mathcal{N}}$ are NURBS with weights of two inner points changed.

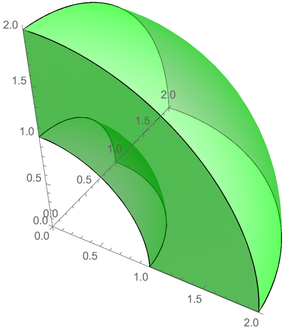


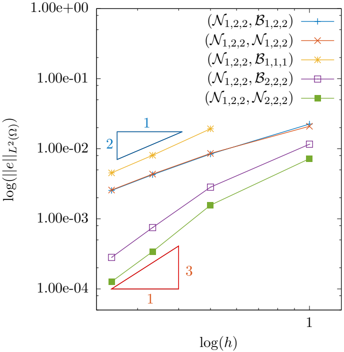


Numerical examples: Example 4 - Elasticity 3D

Convergence studies for various combinations of bases

Same observation as in Example 2 and 3. Optimal convergence governed by the minimum degree, second order for first three choices, and third order for last two choices

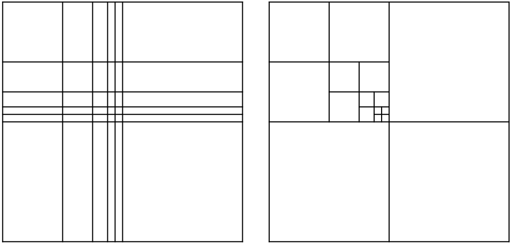




Numerical examples: NURBS + PHT splines

Main properties of PHT-splines:

- Polynomial splines of degree $p = 3$ defined over T-meshes:

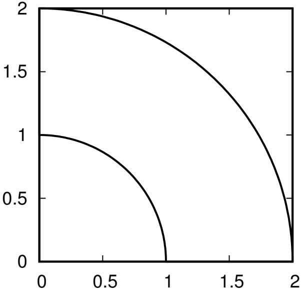


Global refinement (tensor-product mesh) vs local refinement (T-mesh)

- C^1 continuity across the elements

Numerical examples: NURBS + PHT splines

Laplace equation in the quarter annulus with the solution exhibiting a high peak:

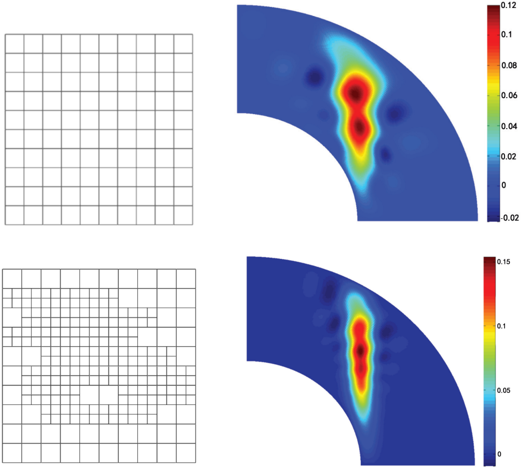


$$u(x, y) = (r-1)(r-2)\theta(\theta - \pi/2)\exp(-100(r\cos\theta - 1)^2)$$

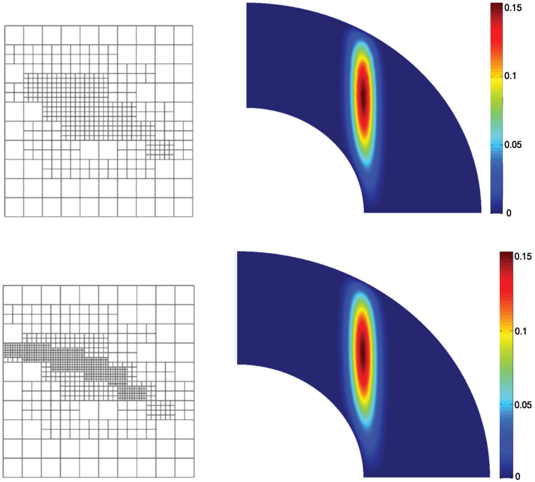
Numerical examples: NURBS + PHT splines

- IGA with cubic NURBS (for the geometry as well as numerical solution). Note that, a quadratic NURBS is sufficient for this geometry, however, to have a fair comparison with the remaining studies, we elevate the degree while maintaining the exact geometry representation.
- IGA with cubic PHT-splines (for the geometry as well as the numerical solution). Note that, in this case, the computational geometry is only approximate (not exact as in IGA with cubic NURBS).
- GIFT with cubic B-splines for the numerical solution, and quadratic NURBS for exact geometry representation.
- GIFT with cubic PHT-splines for the numerical solution, and quadratic NURBS for exact geometry representation.

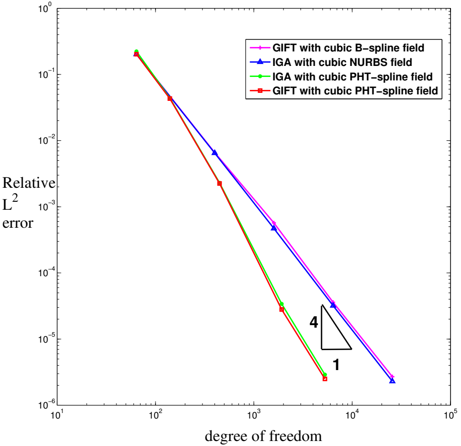
Adaptive refinement with PHT splines:



Adaptive refinement with PHT splines:



Numerical examples: NURBS + PHT splines



Comparison between IGA with cubic PHT-spline field and GIFT with cubic PHT-spline field:

- The advantage of the exact geometry representation in the latter case over an approximate geometry in the former case is very minor in the given example, but in realistic industrial problems with complex domains, this advantage will become more pronounced.
- Use of GIFT concept eliminates the need to communicate with the original CAD model at each step of the solution refinement process, and the approximation of the boundaries.
- Use of GIFT concept also eliminates the need to refine the original coarse geometry, as well as to store and process the refined data, which can lead to significant computational savings for big problems.

Conclusions

- It is possible to retain the advantages of IGA but decouple the geometry and the field approximation
- Standard patch tests may not always pass, yet the convergence rates are optimal as long as the geometry is exactly represented by the geometry basis
- With geometry exactly represented by NURBS, using same degree B-splines or NURBS for the approximation of the solution field yields almost identical results
- With geometry exactly represented by NURBS, using PHT splines for the approximation of the solution gives additional advantage of local adaptive refinement