On conservative and associative operations on finite chains

55th ISFE

Jimmy Devillet

in collaboration with Miguel Couceiro and Jean-Luc Marichal

University of Luxembourg

Connectedness and Contour Plots

Let X be a nonempty set and let $F\colon X^2 \to X$

Definition

- The points $(x, y), (u, v) \in X^2$ are *connected for F* if

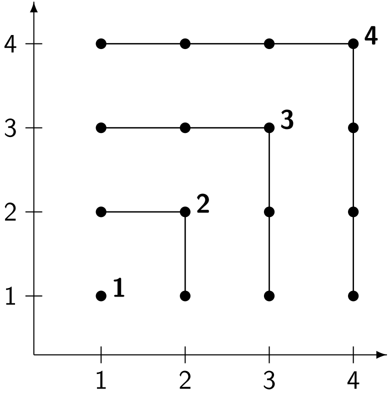
$$F(x, y) \;=\; F(u, v)$$

- The point $(x, y) \in X^2$ is *isolated for F* if it is not connected to another point in X^2

Connectedness and Contour Plots

For any integer $n \geq 1$, let $L_n = \{1, ..., n\}$ endowed with $\leq$

Example. $F(x, y) = \max\{x, y\}$ on L_4



Graphical interpretation of conservativeness

Definition

$F\colon X^2 \to X$ is said to be

- *conservative* if
$$F(x,y) \in \{x,y\}$$
- *reflexive* if
$$F(x,x) = x$$

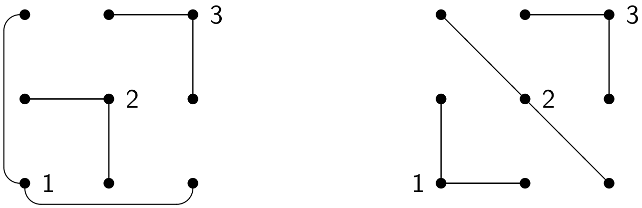
Graphical interpretation of conservativeness

Let $\Delta_X = \{(x, x) \mid x \in X\}$

Proposition

$F\colon X^2 \to X$ is conservative iff

- it is reflexive
- every point $(x, y) \notin \Delta_X$ is connected to either (x, x) or (y, y)



Graphical interpretation of the neutral element

Definition. An element $e \in X$ is said to be a *neutral element* of $F\colon X^2 \to X$ if

$$F(x,e) \;=\; F(e,x) \;=\; x$$

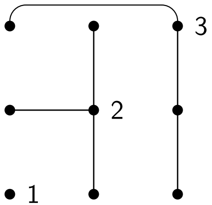
Proposition

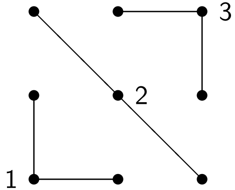
Assume $F\colon X^2 \to X$ is reflexive.
If $(x,y) \in X^2$ is isolated, then it lies on Δ_X, that is, $x = y$

Graphical interpretation of the neutral element

Proposition

Assume $F\colon X^2 \to X$ is conservative and let $e \in X$.
Then e is a neutral element iff (e, e) is isolated





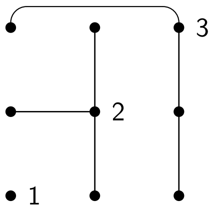
Graphical test for associativity under conservativeness

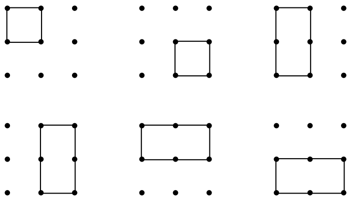
Proposition

Assume $F\colon X^2 \to X$ is conservative. The following assertions are equivalent.

(i) F is associative

(ii) For every rectangle in X^2 that has only one vertex on Δ_X, at least two of the remaining vertices are connected





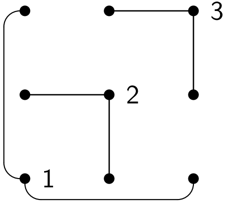
Graphical test for non associativity under conservativeness

Proposition

Assume $F\colon X^2 \to X$ is conservative. The following assertions are equivalent.

(i) F is not associative

(ii) There exists a rectangle in X^2 with only one vertex on Δ_X and whose three remaining vertices are pairwise disconnected



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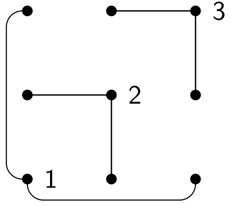
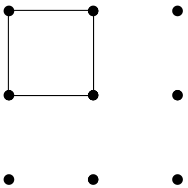




A class of associative operations

Recall that $L_n = \{1, \ldots, n\}$, with the usual ordering $\leq$

We are interested in the class of operations $F\colon L_n^2 \to L_n$ that

- have a neutral element $e \in L_n$

and are

- reflexive
- associative
- symmetric
- nondecreasing in each variable

A first characterization

Theorem (De Baets et al., 2009)

$F\colon L_n^2 \to L_n$ with neutral element $e \in L_n$ is reflexive, associative, symmetric, and nondecreasing iff there exists a nonincreasing map $g\colon [1,e] \to [e,n]$, with $g(e) = e$, such that

$$F(x,y) = \begin{cases} \min\{x,y\}, & \text{if } y \leq \overline{g}(x) \text{ and } x \leq \overline{g}(1), \\ \max\{x,y\}, & \text{otherwise,} \end{cases}$$

where $\overline{g}\colon L_n \to L_n$ is defined by

$$\overline{g}(x) = \begin{cases} g(x), & \text{if } x \leq e, \\ \max\{z \in [1,e] \mid g(z) \geq x\}, & \text{if } e \leq x \leq g(1), \\ 1, & \text{if } x > g(1) \end{cases}$$

A second characterization

Theorem

Assume $F\colon L_n^2 \to L_n$ is symmetric and nondecreasing.
Then F is conservative iff it is reflexive, associative and has a neutral element $e \in L_n$

Corollary

There are exactly 2^{n-1} conservative, symmetric, and non-decreasing operations on L_n

Single-peaked linear orderings

Definition. (Black, 1948) A linear ordering $\preceq$ on L_n is said to be *single-peaked* (w.r.t. the ordering $\leq$) if for any $a, b, c \in L_n$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$

Example. The ordering $\preceq$ on

$$L_4 = \{1 < 2 < 3 < 4\}$$

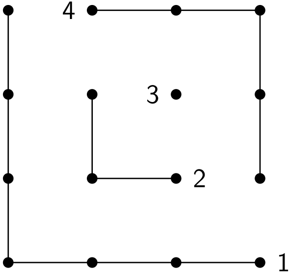
defined by

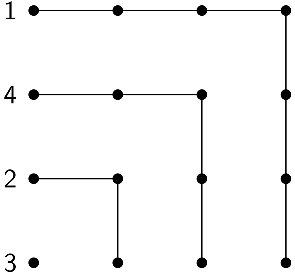
$$3 \prec 2 \prec 4 \prec 1$$

is single-peaked w.r.t. $\leq$

Note : There are exactly 2^{n-1} single-peaked linear orderings on L_n.

Single-peaked linear orderings





$\mathbf{1} < \mathbf{2} < \mathbf{3} < \mathbf{4}$ $\qquad$ $\mathbf{3} \prec \mathbf{2} \prec \mathbf{4} \prec \mathbf{1}$

A third characterization

Theorem

Let $F\colon L_n^2 \to L_n$. The following assertions are equivalent.

- F is reflexive, associative, symmetric, nondecreasing, and has a neutral element $e \in L_n$
- F is conservative, symmetric, and nondecreasing
- there exists a single-peaked linear ordering $\preceq$ on L_n such that

$$F \;=\; \max_{\preceq}$$

Selected references

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