Reduced basis Nitsche-based domain decomposition: a biomedical application

2nd exploratory workshop on applications of model order reduction methods in industrial research and development

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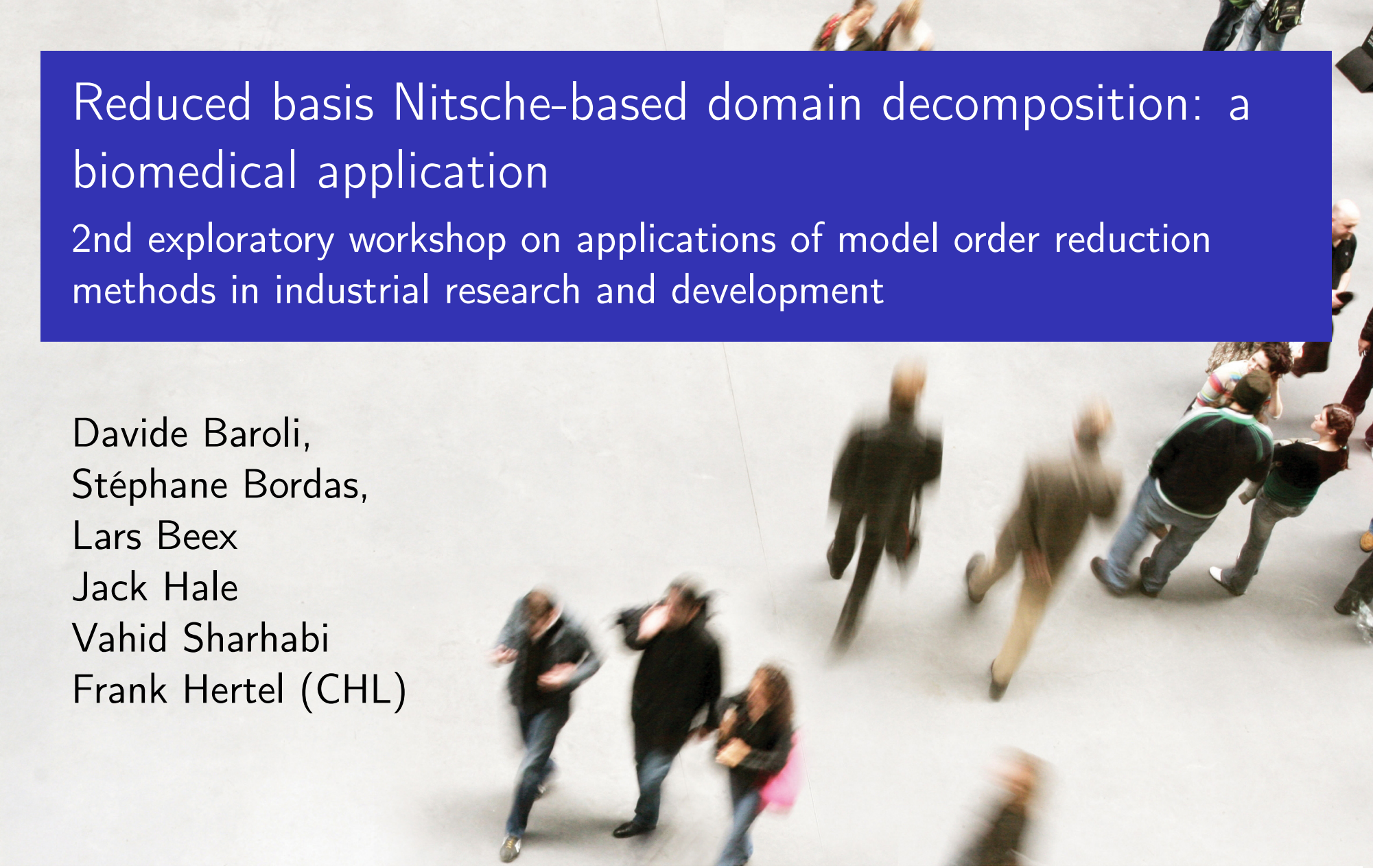

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Outline

Case of study: real-time simulation for Kyphoplasty

In personalized healtcare application, a current challenge is to provide fast and reliable simulation to support decisions in surgery planning.

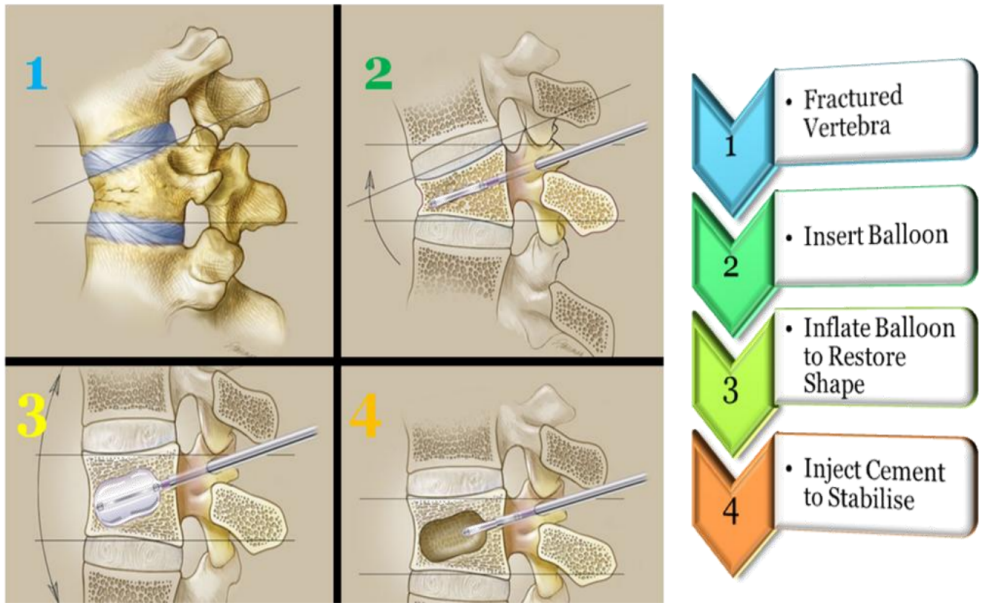

Image Segmentation

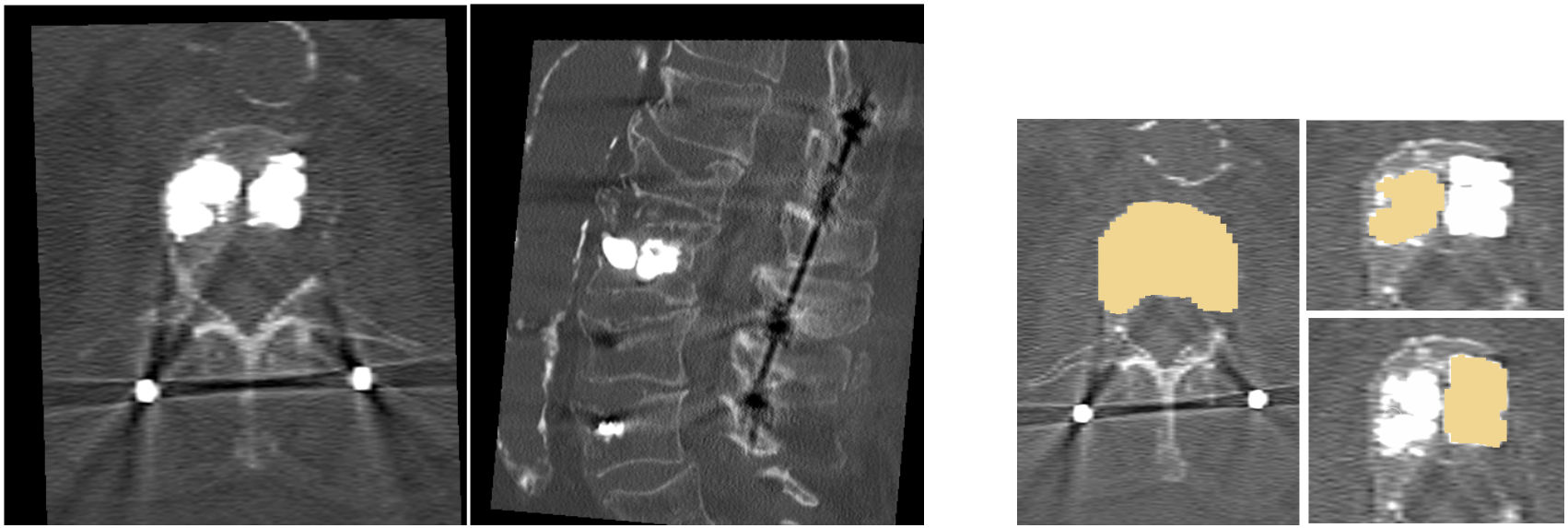

Figure: CT-Scan image provided by Dr. Hertel (CHL);Slicer segmentation of a layer

Computation model

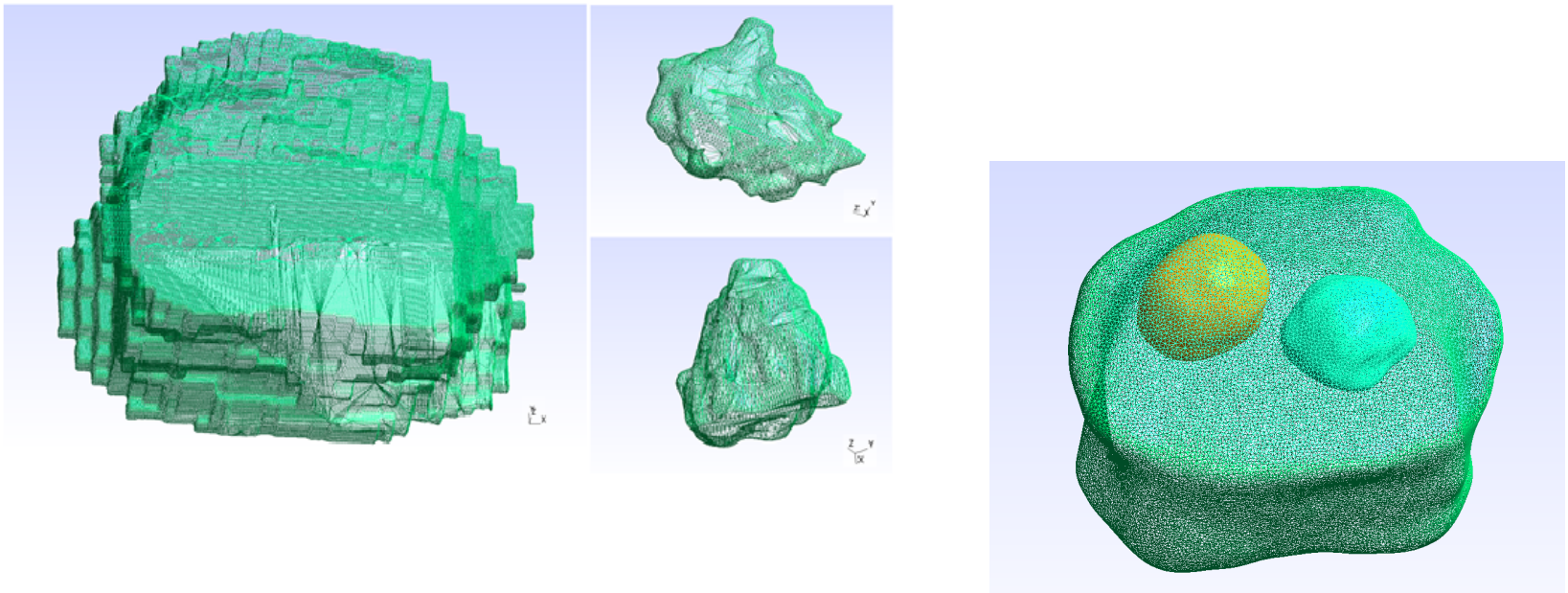

Figure: 3D surface reconstruction; geometrical processing: smoothing

Model: compressible linear eleasticity

The vertebra are considered to undergo small-deformations. The linear compressible elasticity problem reads: find the displacement u such that

$$\begin{aligned} -div(\sigma(\mathbf{u}, \mu, \lambda)) &= f(\beta, \rho, c_i) \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \partial\Gamma_D \\ \sigma(\mathbf{u}, \mu, \lambda) \cdot \mathbf{n} &= \mathbf{g} \quad \text{on } \partial\Gamma_N \end{aligned}$$

Here the stress tensor σ is related to the displacement by Hooke's law:

$$\sigma(\mathbf{u}, \mu, \lambda) = 2\mu\epsilon(\mathbf{u}) + \lambda tr(\epsilon(\mathbf{u}))\mathbf{I} \quad \text{in } \Omega$$

And the source is defined as : $f(\beta, \rho, c_i) = (0.0,\ -\rho \cdot 9.8 \cdot e^{-\sum_{i=1}^{3} c_i \cdot (x_i - \beta_i)^2})$

Finite element simulation: FEniCS

On simulation performance perspective: We are using FEniCS:

- Flexible code style with respect different code languages
- DSEL based-high level programming (easy to simulate complex physical-mathematical model)
- High scalability performance in assembly and solver based on parallel distributed linear algebra
- Universal installation on different operating system using light containers (Docker)
- Modifying the constitutive material law is done using one line of code and differentiation is automatic

Reliable and fast method: model reduction method

Given a solution set $\mathcal{M}_h = \{u_h(\mu) : \mu \in \mathcal{P}\}$ of high-fidelity problem, the RB method finds a few functions $\{\xi_i\}_{i=1}^N$ such that:

$$u_h(\mu) = \sum_{i=1}^{N} u_N(\mu)^i \xi_i$$

where the coefficients $u_N(\mu)$ could be obtained via Galerkin reduced basis approximation:

$$[\xi_1|\ldots|\xi_N]^T \mathbb{A}_h(\mu)[\xi_1|\ldots|\xi_N] u_N = [\xi_1|\ldots|\xi_N]^T \mathbf{f}_h(\mu)$$

with $\mathbb{A}_h$ and $\mathbf{f}_h$ are the discrete counterpart of the differential operator.

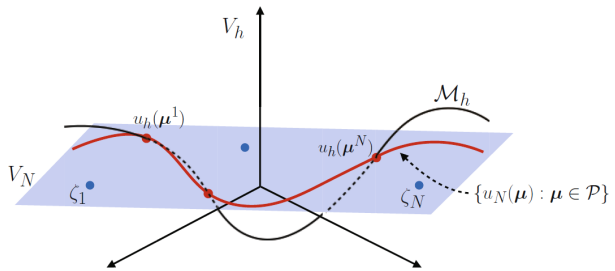

Figure: From "Reduced Basis Method for PDEs"(A.Quarteroni et al.)

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Reduced Basis technique: Greedy algorithm using SLEPc

Given tol and set of parameters $\{\mu_1, \ldots, \mu_n\}$.
Initialize the Reduced Basis space:

```
reducedbasis = SLEPc.BV().create(Comm)
```

Reduced Basis technique: Greedy algorithm using SLEPc

Given tol and set of parameters $\{\mu_1, \ldots, \mu_n\}$.

1. Compute high-fidelity solution $u_h(\mu_1)$ for μ_1

Reduced Basis technique: Greedy algorithm using SLEPc

Given tol and set of parameters $\{\mu_1, \ldots, \mu_n\}$.

1. Compute high-fidelity solution $u_h(\mu_1)$ for μ_1
2. Set $\mathbf{V}_{rb} = span\{u_h(\mu_1)\}$.
   `reducedbasis.insertVec(index, function)`

Reduced Basis technique: Greedy algorithm using SLEPc

Given tol and set of parameters $\{\mu_1, \ldots, \mu_n\}$.

1. Compute high-fidelity solution $u_h(\mu_1)$ for μ_1
2. Set $\mathbf{V}_{rb} = span\{u_h(\mu_1)\}$.
3. For each $\mu \in \mathcal{P}$
   Compute $u_{RB} = \sum_{i=1}^{N} u_N^i \xi_i(\mu)$ where ξ_i are the basis of $\mathbf{V}_{rb}$.
   Evaluate the error estimator $\eta(\mu) = \frac{(res_{RB}, M^{-1} res_{RB})}{\alpha_{LB}}$

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4. Choose $\tilde{\mu} = arg\ max_{\mu \in \mathcal{P}} \eta(\mu)$

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   Evaluate the error estimator $\eta(\mu) = \frac{(res_{RB}, M^{-1} res_{RB})}{\alpha_{LB}}$
4. Choose $\tilde{\mu} = arg\ max_{\mu \in \mathcal{P}} \eta(\mu)$
5. If $\eta(\tilde{\mu}) > tol$ and $V_{RB} = V_{RB} \cup \{u(\tilde{\mu})\}$
   `reducedbasis.orthogonalizeVec(function)`
   `reducedbasis.scaleColumn(norm)`

Online Galerkin Project with SLEPc

$$[\xi_1|\dots|\xi_N]^T \mathbb{A}_h(\mu)[\xi_1|\dots|\xi_N]u_N = [\xi_1|\dots|\xi_N]^T \mathbf{f}_h(\mu)$$

with $\mathbb{A}_h$ and $\mathbf{f}_h$ are the discrete counterpart of the differential operator.

```
reducedbasis.matProject(A, reducedbasis)
reducedbasis.dotVec(f)
```

Reduced Integration

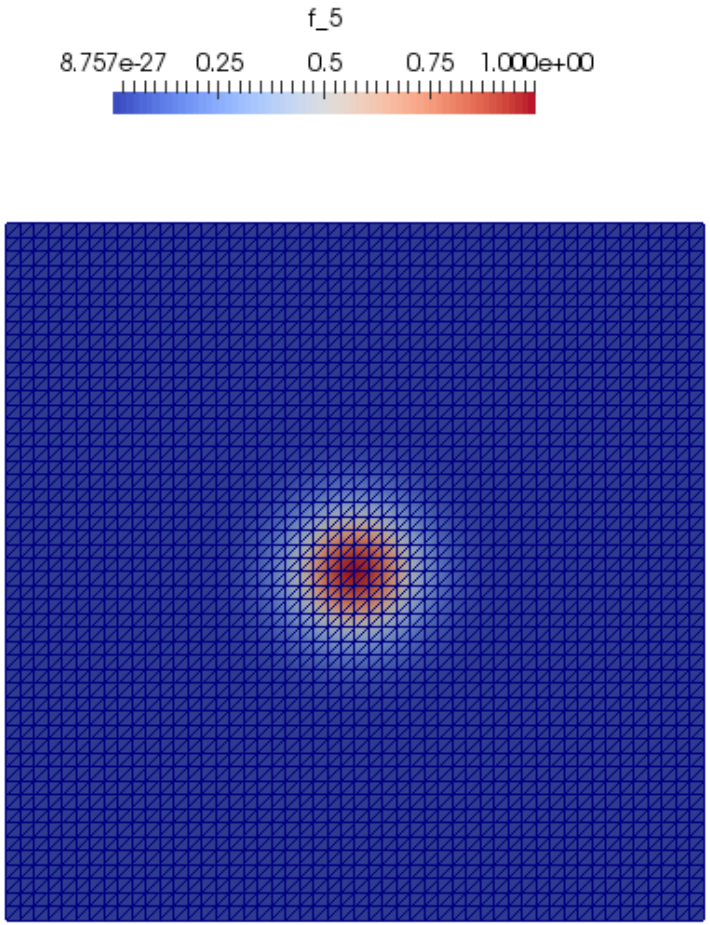
One of the major computational efforts in application of standard model reduction to linear (non-linear) PDEs is the assemble the finite element space operators and project onto the ROM basis.
We employ the novel technique "reduced integration“ which consists in: Given a subspace of reduced basis $V_{RB} = \{\xi_1, \ldots, \xi_N\}$, quadrature rule order m and a coarse mesh $\Omega_H := \mathcal{T}_H(\Omega)$: For each basis ξ_i and a tolerance *tol*.

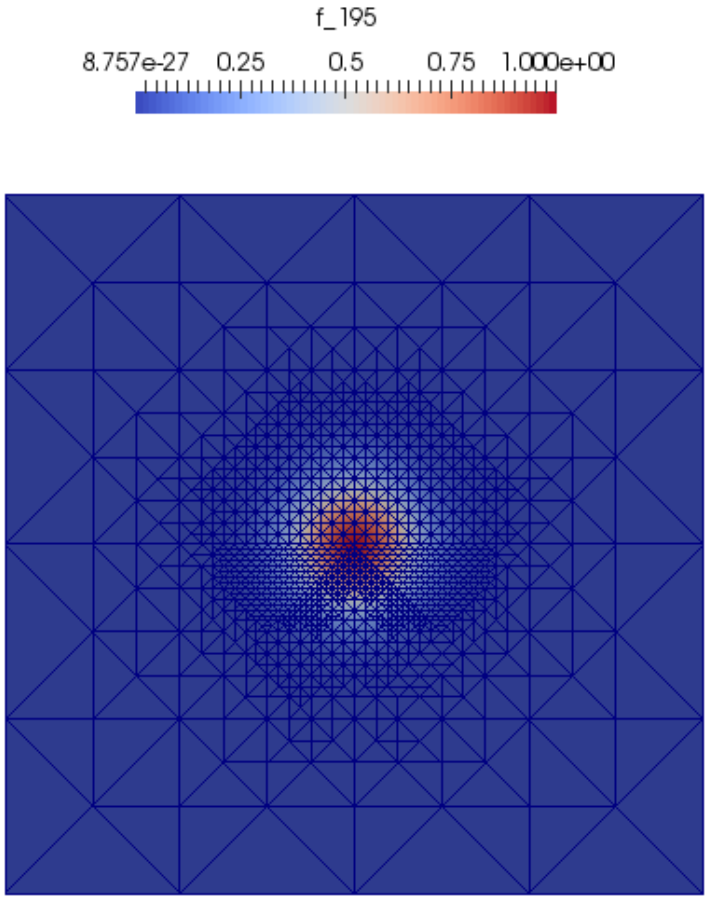
1. Compute the estimator

$$\eta(\xi_i, \Omega_H) = \int_{\Omega_H, m} |\xi_i| - \int_{\Omega_H, m+1} |\xi_i| \tag{1}$$

2. If $\eta(\xi_i, \Omega_H) > tol$ then $\Omega_H = \mathrm{REFINE}(\Omega_H)$ and go to **1**..

Reduced Integration test



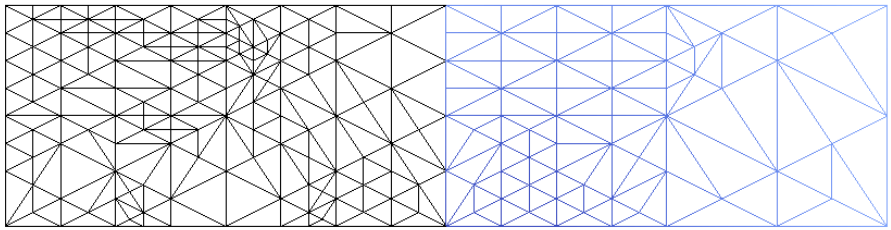


Locally reduced integration

Due to local behaviour of modes/basis which is sensitive to the zone of application of the external force/moments, we combine the domain-decomposition paradigm "divide-et-impera" with reduced integration technique.
The algorithm consist in

- Decompose the physical domain in part (regular - ad hoc)
- Restrict each reduced basis on each subdomain, constructing the space $V_{RB,i}$ for $i = 1, \ldots, \mathrm{NbSubDomains}$
- Perform in each subdomain the reduced-integration
- Re-orthonormalization of each local space.

Non-conforming impera algorithm



Some domain decomposition technique: FETI-DP, BDDC, Mortar, Discontinuous Galerkin, InterNODES, Nitsche, ...
ROM + domain decomposition: Reduced Basis Element [RBE], DG-RBE, static condensation reduced basis element, Latin Multiscale

Why we choose Nitsche technique?

Pros.

- do not require any additional unknown
- has the same accurancy property of the underling discretization
- avoid of mesh burden near interface
- could treat also overlapping subdomains

Cons.
The stability of Nitsche method depends on the choose of a penalty scalar value. However, it is possible to estimate this constant using trace inequality (S. Claus et al.).

The Nitsche-type weak formulation associated to this problem is: Find a $u_h \in V_h^k$ such that

$$a_h(u_h, v_h) - b_h(u_h, v_h) - b_h(v_h, u_h) + j_h(u_h, v_h) = L_h(v_h) \quad \forall\, v_h \in V_h^k$$

The bilinear form a_h and b_h are defined as

$$a(u_h, v_h) = \sum_{i=1}^{2} \int_{\Omega_i} \sigma(u_h^i, mu_i, \lambda_i) : \epsilon(v_h^i) dX$$

$$b(u_h, v_h) = \int_{\Gamma} \{(\sigma(u_h, mu, \lambda))\}_{\kappa}, [\![(v \cdot n)]\!] d\Gamma$$

and the penalty-ghost form is

$$j(u_h, v_h) = \int_{\Gamma} \frac{\alpha}{\{h\}} [\![u^h]\!][\![v^h]\!] d\Gamma$$

Create decomposed mesh:

```
multimesh=dolfin.MultiMesh()
multimesh.add(mesh)
multimesh.build()
```

The MultiMeshForm for the first subdomain:

```
a = inner(sigma(u,mu0,lmbda0), grad(v))*dX
b_u= - inner(w_avg(sigma(u,mu0,lmbda0),kappa0,kappa1),
          tensor_jump(v,n))*dI
b_v=- inner(w_avg(sigma(v,mu0,lmbda0),kappa0,kappa1),
          tensor_jump(u,n))*dI
j= alpha/avg(h)*inner(jump(u), jump(v))*dI
```

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$$j(u_h, v_h) = \int_{\Gamma} \frac{\alpha}{\{h\}} [\![u^h]\!][\![v^h]\!] d\Gamma$$

where the penalty parameter is

$$\alpha = \frac{C_I(d, p) \cdot (E_1 \cdot E_2)}{E_1 + E_2}$$

with d is physical dimension and p the order of the polynomial Lagrange discretization space.

The bilinear form a_h and b_h are defined as

$$a(u_h, v_h) = \sum_{i=1}^{2} \int_{\Omega_i} \sigma(u_h^i, mu_i, \lambda_i) : \epsilon(v_h^i) dX$$

$$b(u_h, v_h) = \int_{\Gamma} \{(\sigma(u_h, mu, \lambda))\}_{\kappa}, [\![(v \cdot n)]\!] d\Gamma$$

and the penalty-ghost form is

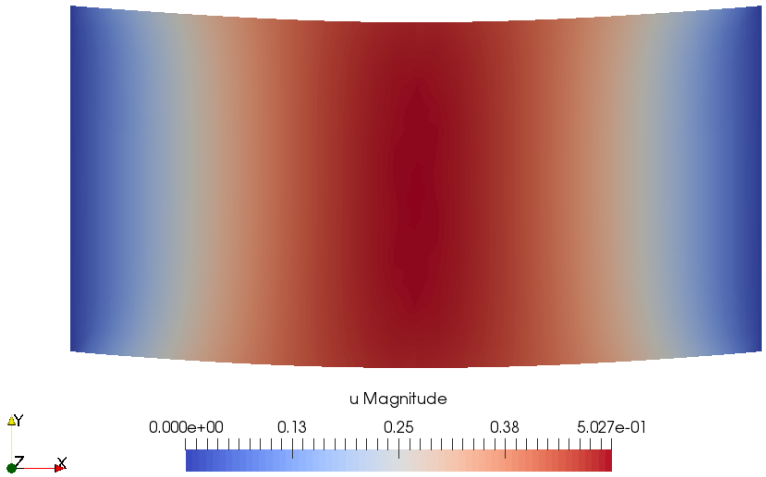
$$j(u_h, v_h) = \int_{\Gamma} \frac{\alpha}{\{h\}} [\![u^h]\!][\![v^h]\!] d\Gamma$$

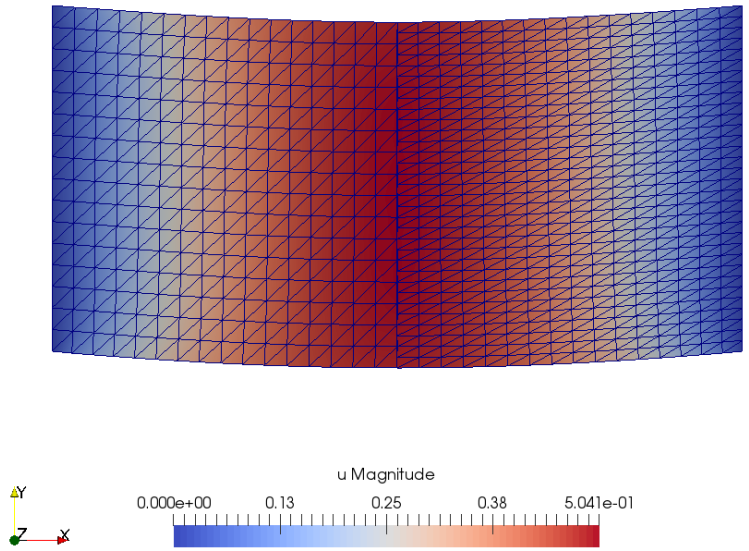
$$\{\mu u\}_{\kappa} = (\kappa_1 \mu_1 u^1 + \kappa_2 \mu_2 u^2)_{|\Gamma}$$

where the weights are

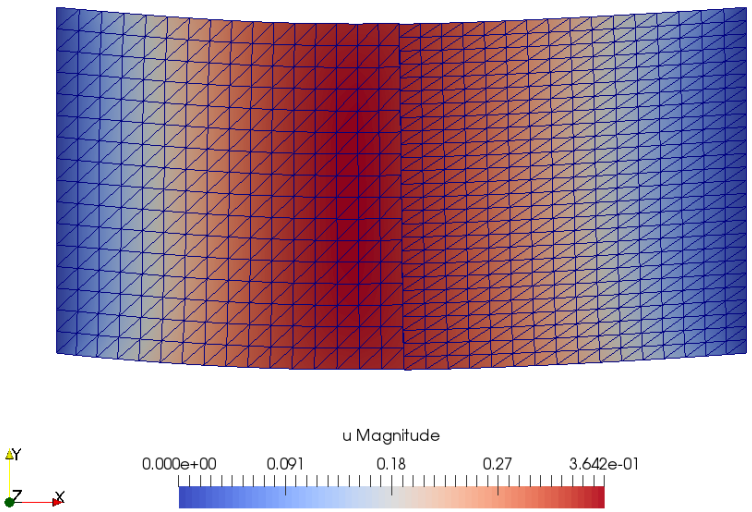
$$\kappa_1 = \frac{E_1}{E_1 + E_2}, \quad \kappa_2 = \frac{E_2}{E_1 + E_2}$$

Numerical test with single material

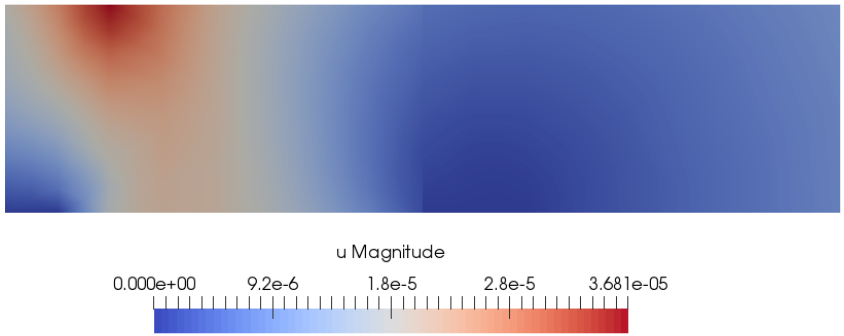


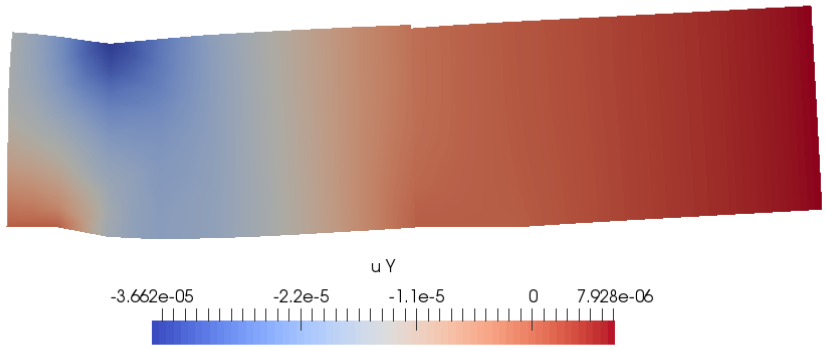


Numerical test with two materials



Numerical Validation Test:2D





Number of dof of high-fidelity discretization:7442
Number of dof of reduced integration discretization: 2796
Speed-up: x40 time with iRB+Nitsche
Speed-up: x10 time with reduced integration in assemble. Error in first subdomain L_2-norm:=3.92e-07, H_1-norm=2.03e-06
Error in first subdomain L_2-norm:=2.23e-08, H_1-norm=5.47e-08

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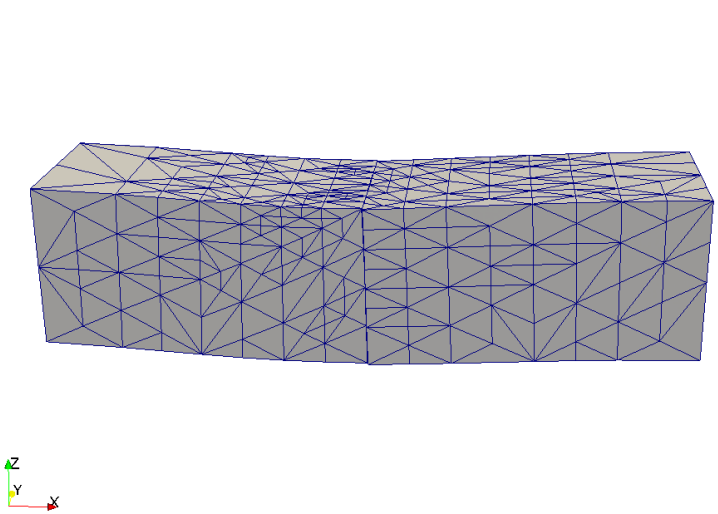
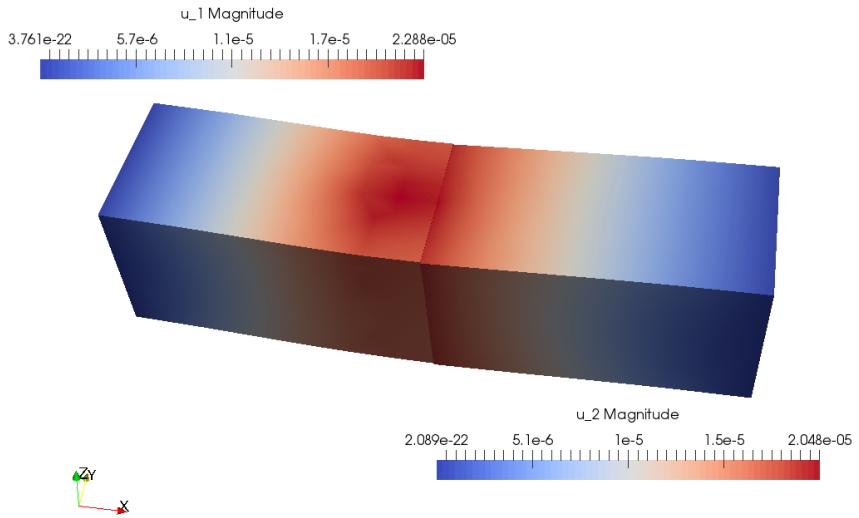
Numerical Validation Test:3D





Number of dof of high-fidelity discretization:9600
Number of dof of reduced integration discretization: 6953
Speed-up: x1000 time with iRB+Nitsche
Speed-up: x8 time with reduced integration in assemble.

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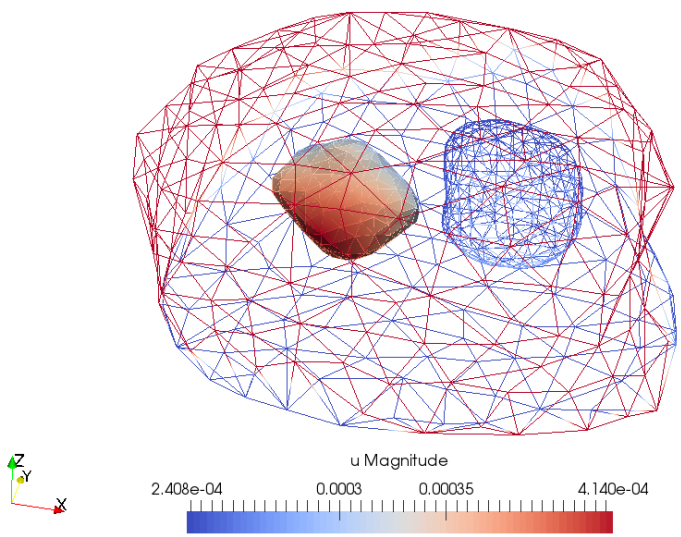
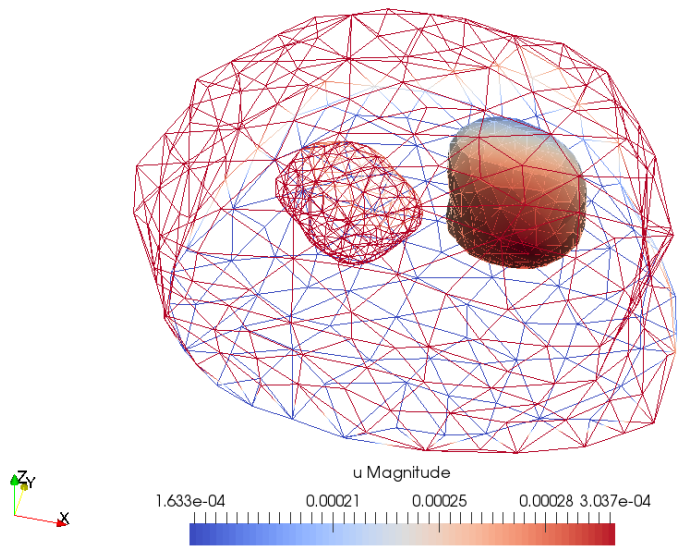
Patient-specific application: Vertebra augemented by kyphoplasty





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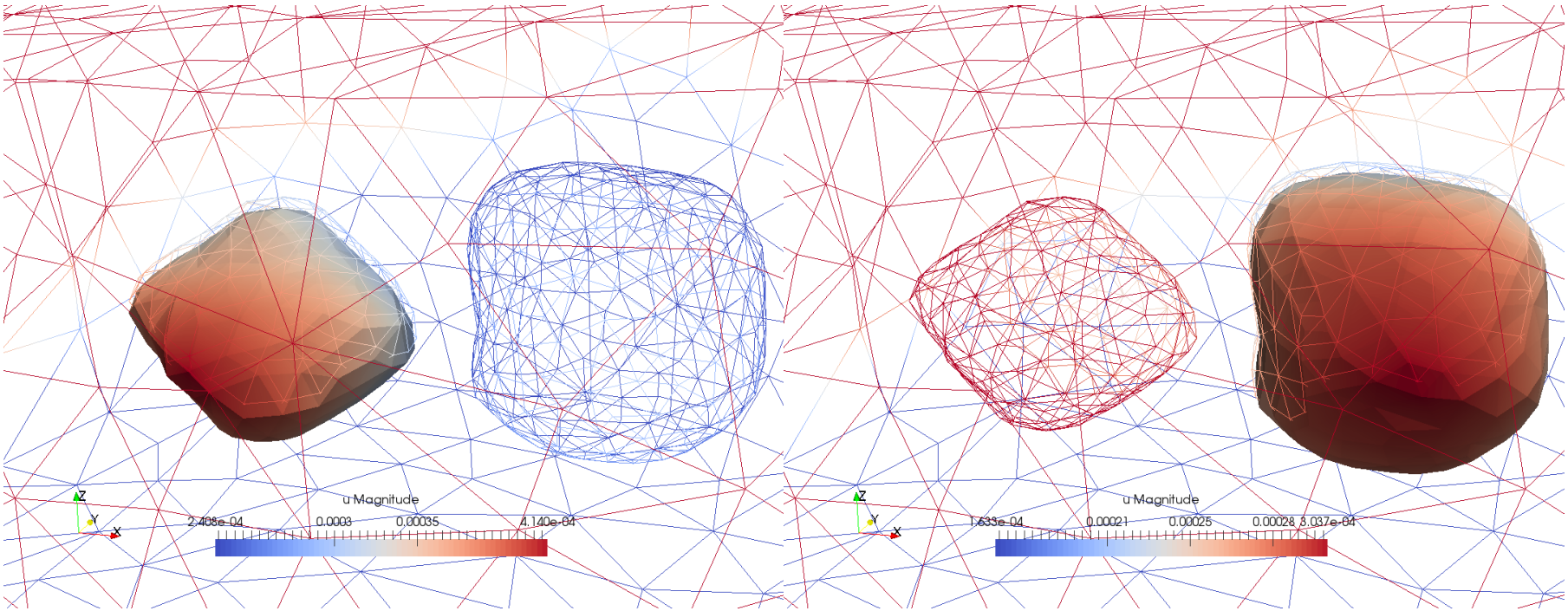
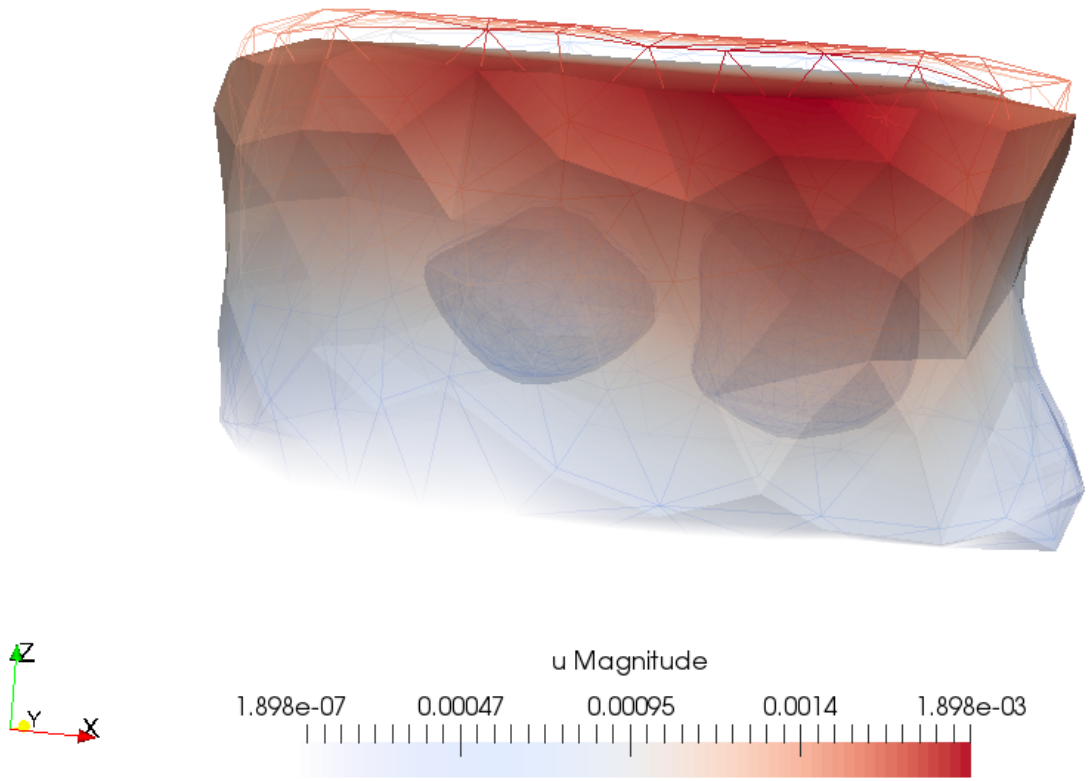
Reference vs Deformed Domain for each cemented domain



Deformation of the surface of Vertebra



Conclusion

Preliminary numerical comparison evidences:

- flexible approach in online gluing with Nitsche formulation;
- assemble speed-up by local reduced integration;
- iRB-Nitsche provides a fast and reliable simulation in patient-specific scenario;

Working in progress:

- integration of iRB-Nitsche in multi-query UQ algorithm.

Future perspective: Contact Bio-mechanics problem

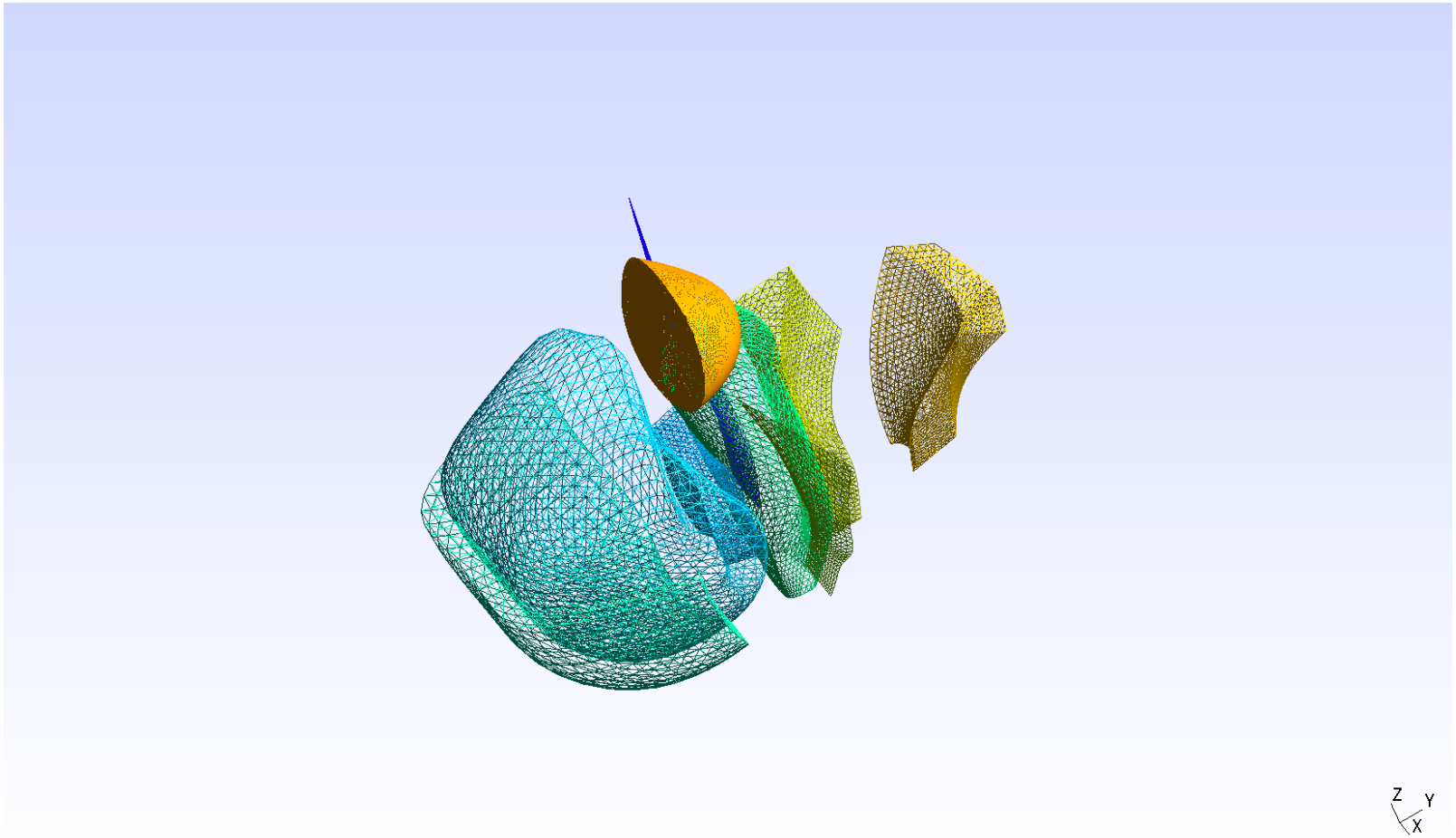


Figure: Computational mesh of pelvic organ surface provided by Prof. M.Brieu (Lille)

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Thanking: Dr.Hertel, S. Claus, V. Sharhabi.

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Thank you for your attention