Energy Minimizing Multi-Crack Growth in Linear Elastic Fracture Using The Extended Finite Element Method

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1. Motivation
2. Problem statement
3. Crack growth
4. Discretization by XFEM
5. Implementation
6. Verification
7. Results
8. Summary
Problem statement

• Consider a cracked linear-elastic isotropic solid subject to an external load whose quasistatic behavior can be described by the following total Lagrangian form:

\[ \mathcal{L}(u, a) = \Pi(u, a) + \sum_{i=1}^{n_{\text{tip}}} \int_{a_i} G^i_c \, da_i \]

• The solution for \( u(a) \) and \( a(t) \) are obtained by satisfying the stationarity of \( L(u,a) \) during the evolution of \( t \), subject to \( \Delta a_i \geq 0 \):

\[ \delta \mathcal{L}(u, a) = \delta_u \Pi(u, a) + \sum_{i=1}^{n_{\text{tip}}} \left[ \frac{\partial \Pi(u, a)}{\partial a_i} + G^i_c \right] \delta a_i = 0 \]

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The solution procedure at time $t^k$ consists of
1. solving the variational form for $u(a^k)$:

\[ \delta_u \Pi(u, a) = \delta_u \mathcal{W}^{\text{int}}(u, a) - \delta_u \mathcal{W}^{\text{ext}}(u) = 0 \]

2. advancing the fracture fronts, such that $\Pi(u, a^k) \rightarrow \Pi(u, a^{k+1})$ follows the path of steepest descent while satisfying Griffith’s energy balance:

\[ \max_{\delta a_i \rightarrow 0} \left( - \frac{\partial \Pi(u, a)}{\partial a_i} \right) = G^i_c \]
Crack growth
maximum hoop stress

- Post processing of solution to evaluate SIF [Yau, 1980]

\[ I^{(1+2)} = \int_{\Omega} \left( \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1+2)} \delta_{ij} \right) \frac{\partial q}{\partial x_j} d\Omega = \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)}) \]

- Crack growth direction [Erdogan & Shi, 1963]

\[ \theta_c(K_I, K_{II}) = 2 \tan^{-1} \left[ \frac{1}{4} \left( \frac{K_I}{K_{II}} - \text{sign}(K_{II}) \sqrt{\left( \frac{K_I}{K_{II}} \right)^2 + 8} \right) \right] \]


\[ \frac{k_I(K_I, K_{II}, \theta_c)^2 + k_{II}(K_I, K_{II}, \theta_c)^2}{E'} = G_c \]
Energy release rate w.r.t. crack increment direction, $\theta_i$:

$$G_i = -\frac{\partial \Pi(u, a + \Delta a)}{\partial \theta_i}$$

The rates of energy release rates:

$$H_{ij} = \frac{\partial G_i}{\partial \theta_j}$$

Updated directions (using Newton):

$$\theta^{i+1} = \theta^i - H^{-1}G$$
The discrete potential energy is given by:

\[ \Pi = \frac{1}{2} u^T K u - u^T f \]

Energy release rate w.r.t. crack increment direction \( \theta_i \):

\[ G_i = -\frac{1}{2} u^T \delta_i K u + u^T \delta_i f \]

The rates of the energy release rate:

\[ H_{ij} = -\frac{1}{2} u^T \delta_{ij}^2 K u + u^T \delta_{ij}^2 f + (\delta_j K u - \delta_j f)^T K^{-1} (\delta_i K u - \delta_i f) \]

where: \( \delta_i = \frac{\partial}{\partial \theta_i} \)
Discretization
XFEM

• Approximation function [Belytschko et al., 2001]

\[ u^h(x) = \sum_{I \in \mathcal{N}_I} N_I(x)u^I + \sum_{J \in \mathcal{N}_J} N_J(x)H(x)a^J + \sum_{K \in \mathcal{N}_K} N_K(x)\sum_{\alpha=1}^{4} f_\alpha(x)b^K_\alpha \]

- **Standard part**
- **Discontinuous enrichment**
- **Singular tip enrichment**

\[ H(x) = \begin{cases} 
+1 & \text{if } x \text{ above crack} \\
-1 & \text{if } x \text{ below crack}
\end{cases} \]

\[ \{f_\alpha(r,\theta), \alpha = 1, 4\} = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2} \sin \theta \right\} \]
Implementation
how to compute $\delta K$?

Differentiation of the stiffness matrix
w.r.t. crack increment direction

- ○ - original crack
- · - rotated crack
- ◯ - shifted standard el.
- □ - shifted crack vtx. el.
- ■ - original enriched el.
- □ - rotated enriched el.
Implementation

how to compute $\delta K$?

\[
\delta K_e = \int_{\Omega_e} (\delta B^TDB + B^T\delta D B) \det(J) \, d\Omega + \int_{\Omega_e} B^TDB \delta \det(J) \, d\Omega
\]

\[
\delta^2 K_e = \int_{\Omega_e} (\delta^2 B^TDB + 2\delta B^T\delta DB + B^T\delta^2 B) \det(J) \, d\Omega + \int_{\Omega_e} 2 (\delta B^TDB + B^T\delta DB) \delta \det(J) \, d\Omega + \int_{\Omega_e} B^TDB \delta^2 \det(J) \, d\Omega
\]

Differentiation of the stiffness matrix w.r.t. crack increment direction

\[
\delta K_e = T^T K_e + K_e T
\]

\[
\delta^2 K_e = 2(T^T K_e T - K_e)
\]
Verification
rotational energy release rates

Test case: square plate with an edge crack with a small kink loaded in vertical tension

\[ \Pi \text{ vs. } \theta \]

Potential energy vs. crack increment angle

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Verification
rotational energy release rates

**Test case:** square plate with an edge crack with a small kink loaded in vertical tension.

![Diagram of a square plate with an edge crack and a small kink](image)

Energy release rate vs. crack increment angle

- **$G$ vs. $\theta$**
- Analytical
- Central differencing of $\Pi$

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Rate of the energy release rate vs. crack increment angle

(dG/dθ vs. θ)

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Verification
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Test case: square plate with an edge crack with a small kink loaded in vertical tension

Rate of the energy release rate vs. crack increment angle

\( \frac{dG}{d\theta} \) vs. \( \theta \)

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Verification
energy min. VS. max-hoop

**Test case:** square plate with an inclined center crack in vertical tension

![Diagram of square plate with inclined center crack](image)

**Comparison of crack propagations (n_{elm} = 200 \times 200)**

simply supported square plate with an inclined center crack subjected to a uniform vertical tension

- **Initial crack**
- **Max–hoop criterion**
- **min(II) (just 1st. inc.)**
- **Max–hoop (predictor)**
Verification
energy min. VS. max-hoop

**Test case:** square plate with an inclined center crack in vertical tension

Comparison of the energy release rate ($\Delta a_{\text{inc}} \propto h_e$)
simply supported square plate with an inclined center crack subjected to a uniform vertical tension

$G_{\min(II)}/G_{\text{hoop}}$ vs. $\theta$

$\Delta \theta_{200 \times 200}^{\text{min(II)}} < -70.53^\circ$

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Energy minimizing multi-crack growth in linear elastic fracture using XFEM
Results

10 crack problem

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

Max hoop stress
Global energy min.

\[ n_{\text{nod}} = 80 \times 160, \Delta a \propto h_e \]
Results
10 crack problem

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

- Max hoop stress
- Global energy min.

$n_{nod} = 160 \times 320, \Delta a \propto h_e$
Results
10 crack problem

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

Max hoop stress
Global energy min.

$n_{nod} = 160 \times 320, \Delta a \propto h_e$
Results

10 crack problem

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

$\eta_{nod} = 160 \times 320$, $\Delta a \propto h_e$
Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

Max hoop stress
Global energy min.

\( n_{\text{nod}} = 160 \times 320, \Delta a \propto h_e \)
Results

10 crack problem

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

$\eta_{\text{nod}} = 160 \times 320$, $\Delta a \propto h_e$
Results
10 crack problem

Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

- Max hoop stress
- Global energy min.

$x_nod = 160 \times 320, \Delta a \propto h_e$
Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

\[ n_{\text{nod}} = 160 \times 320, \Delta a \propto h_e \]

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10 crack problem

Convergence to same fracture path by hoop-stress and energy-min. criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)

\[ L_2 \text{-norm of distance between fracture surfaces} \]

\[ \text{Crack increment length, } \Delta \ell_{\text{inc}} \]

\[ 10^{-2} \]

\[ 10^{-3} \]

\[ 10^{-4} \]

\[ 10^{-3} \]

\[ 10^{-2} \]

\[ \text{best fit (slope = 1.04)} \]
Results
double cantilever problem

\[ n_{\text{elm}} = \{30 \times 60, 60 \times 120, 120 \times 240, 240 \times 480\} \]
Results
double cantilever problem

Fracture paths by different criteria
(double cantilever problem with an edge crack offset by 0.01 above the x-axis)

$\mathbf{n_{elm} = \{30\times60, 60\times120, 120\times240, 240\times480\}, \Delta a \propto h_e}$
Results
2 edge cracks; internal pressure loading

Fracture paths by different criteria
(simply supported square plate with two pressure loaded edge cracks: $\Delta x=0.6$, $\Delta y=0.04$)

- Max hoop stress
- Global energy min.
- Averaged direction

$n_{elm} = \{100\times100, 200\times200, 400\times400\}, \Delta a \propto h_e$
Results
3 cracks; center crack pressure loading

Fracture paths by different criteria
(simply supported cracked square plate with a pressure loaded center crack)

- Max hoop stress
- Global energy min.
- Averaged direction

\[ n_{\text{elm}} = \{50 \times 50, 100 \times 100, 200 \times 200\}, \Delta a \propto h_e \]
Results

Edge crack in a PMMA beam with 3 holes
Results

Edge crack in a PMMA beam with 3 holes

![Graph showing an edge crack in a PMMA beam with 3 holes.]
Results
Edge crack in a PMMA beam with 3 holes
Results

Edge crack in a PMMA beam with 3 holes

\[ (-5, -2.5) \]
Results

Edge crack in a PMMA beam with 3 holes
Results
2 edge cracks and 2 holes (Khoeil et al. 2008)

Fracture paths by different criteria
(rectangular plate with two holes and two edge cracks subjected to vertical extension)

- Max hoop stress
- Global energy min.
- Averaged direction

\[ n_{\text{nod}} = \{20k, 80k, 320k\}, \Delta a \propto h_e \]
Results

2 edge cracks and 2 holes (Khoeil et al. 2008)

Fracture paths by different criteria
(rectangular plate with two holes and two edge cracks subjected to vertical extension)

- Blue: Max hoop stress
- Red: Global energy min.
- Green: Averaged direction

$n_{nod} = \{20k, 80k, 320k\}, \Delta a \propto h_e$
Results
2 edge cracks and 2 holes (Khoeil et al. 2008)

Fracture paths by different criteria
(rectangular plate with two holes and two edge cracks subjected to vertical extension)

Max hoop stress
Global energy min.
Averaged direction

$n_{nod} = \{20k, 80k, 320k\}$, $\Delta a \propto h_e$
Summary

• A robust approach to determining multiple crack growth based on the principle of minimum energy within XFEM;

• Limitations undermining the max. hoop-stress criterion are overcome, e.g. assumptions about geometry and loading;

• The energy minimization approach is characterized by mode-I field dominance at the crack tip (post-increment);

• Both criteria lead to fracture paths solutions that are in close agreement (strong correlation with local symmetry, i.e. $K_{II}=0$);

• Better accuracy and faster convergence of fracture path solutions can be obtained by taking a bi-section of the interval that is bounded by the respective criteria.