Energy Minimizing Multi-Crack Growth in Linear Elastic Fracture Using The Extended Finite Element Method

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Problem statement

- Consider a cracked linear-elastic isotropic solid subject to an external load whose quasistatic behavior can be described by the following total Lagrangian form:

$$\mathcal{L}(\mathbf{u}, a) = \Pi(\mathbf{u}, a) + \sum_{i=1}^{n_{\text{tip}}} \int_{a_i} G_c^i \, \mathrm{d}a_i$$

- The solution for $\boldsymbol{u}(a)$ and $a(t)$ are obtained by satisfying the stationarity of $\boldsymbol{L}(\boldsymbol{u},a)$ during the evolution of t, subject to $\Delta a_i \geq 0$:

$$\delta\mathcal{L}(\mathbf{u}, a) = \delta_{\mathbf{u}}\Pi(\mathbf{u}, a) + \sum_{i=1}^{n_{\text{tip}}} \left[\frac{\partial \Pi(\mathbf{u}, a)}{\partial a_i} + G_c^i \right] \delta a_i = 0$$

Problem statement

- The solution procedure at time t^k consists of
  1. solving the variational form for $\boldsymbol{u}(a^k)$:

$$\delta_{\mathbf{u}}\Pi(\mathbf{u}, a) = \delta_{\mathbf{u}}\mathcal{W}^{\text{int}}(\mathbf{u}, a) - \delta_{\mathbf{u}}\mathcal{W}^{\text{ext}}(\mathbf{u}) = 0$$

  2. advancing the fracture fronts, such that $\Pi(\boldsymbol{u},a^k) \rightarrow \Pi(\boldsymbol{u},a^{k+1})$ follows the path of steepest descent while satisfying Griffith's energy balance

$$\max_{\delta a_i \to 0}\left(-\frac{\partial \Pi(\mathbf{u}, a)}{\partial a_i}\right) = G_c^i$$

Crack growth
maximum hoop stress

- Post processing of solution to evaluate SIF [Yau, 1980]

$$I^{(1+2)} = \int_{\Omega} \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1+2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} \mathrm{d}\Omega = \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)})$$

- Crack growth direction [Erdogan & Shi, 1963]

$$\theta_c(K_I, K_{II}) = 2 \tan^{-1} \left[\frac{1}{4} \left(\frac{K_I}{K_{II}} - \mathrm{sign}(K_{II}) \sqrt{ \left(\frac{K_I}{K_{II}} \right)^2 + 8 } \right) \right]$$

- Growth criterion [Irwin, 1957; Hayashi & Nemat-Nasser, 1981]

$$\frac{k_I(K_I, K_{II}, \theta_c)^2 + k_{II}(K_I, K_{II}, \theta_c)^2}{E'} = G_c$$

Crack growth
energy minimization

- Energy release rate w.r.t. crack increment direction, θ_i:

$$G_i = -\frac{\partial \Pi(\mathbf{u}, \mathbf{a} + \Delta \mathbf{a})}{\partial \theta_i}$$

- The rates of energy release rates:

$$H_{ij} = \frac{\partial G_i}{\partial \theta_j}$$

- Updated directions (using Newton):

$$\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^{i} - \mathbf{H}^{-1}\mathbf{G}$$

Crack growth
energy minimization

- The discrete potential energy is given by:

$$\Pi = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{f}$$

- Energy release rate w.r.t. crack increment direction θ_i:

$$G_i = -\frac{1}{2}\mathbf{u}^{\mathrm{T}}\delta_i\mathbf{K}\mathbf{u} + \mathbf{u}^{\mathrm{T}}\delta_i\mathbf{f}$$, where: $\delta_i = \frac{\partial}{\partial \theta_i}$

- The rates of the energy release rate:

$$\mathrm{H}_{ij} = -\tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\delta_{ij}^2\mathbf{K}\mathbf{u} + \mathbf{u}^{\mathrm{T}}\delta_{ij}^2\mathbf{f} + (\delta_j\mathbf{K}\mathbf{u} - \delta_j\mathbf{f})^{\mathrm{T}}\mathbf{K}^{-1}(\delta_i\mathbf{K}\mathbf{u} - \delta_i\mathbf{f})$$

, where: $\delta_{ij} = \frac{\partial^2}{\partial \theta_i \theta_j}$

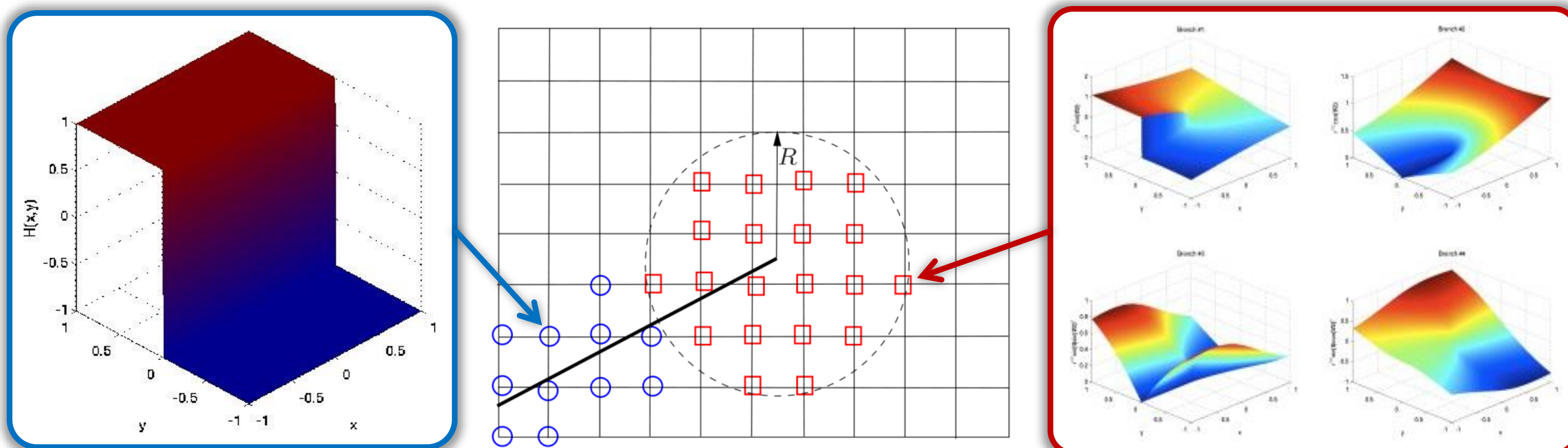
- Approximation function [Belytschko et al., 2001]

$$\mathbf{u}^h(\mathbf{x}) = \underbrace{\sum_{I\in\mathcal{N}_I} N_I(\mathbf{x})\mathbf{u}^I}_{\text{standard part}} + \underbrace{\sum_{J\in\mathcal{N}_J} N_J(\mathbf{x})H(\mathbf{x})\mathbf{a}^J}_{\text{discontinuous enrichment}} + \underbrace{\sum_{K\in\mathcal{N}_K} N_K(\mathbf{x})\sum_{\alpha=1}^{4} f_\alpha(\mathbf{x})\mathbf{b}^{K\alpha}}_{\text{singular tip enrichment}}$$

standard part — discontinuous enrichment — singular tip enrichment

$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } \mathbf{x} \text{ above crack} \\ -1 & \text{if } \mathbf{x} \text{ below crack} \end{cases}$$

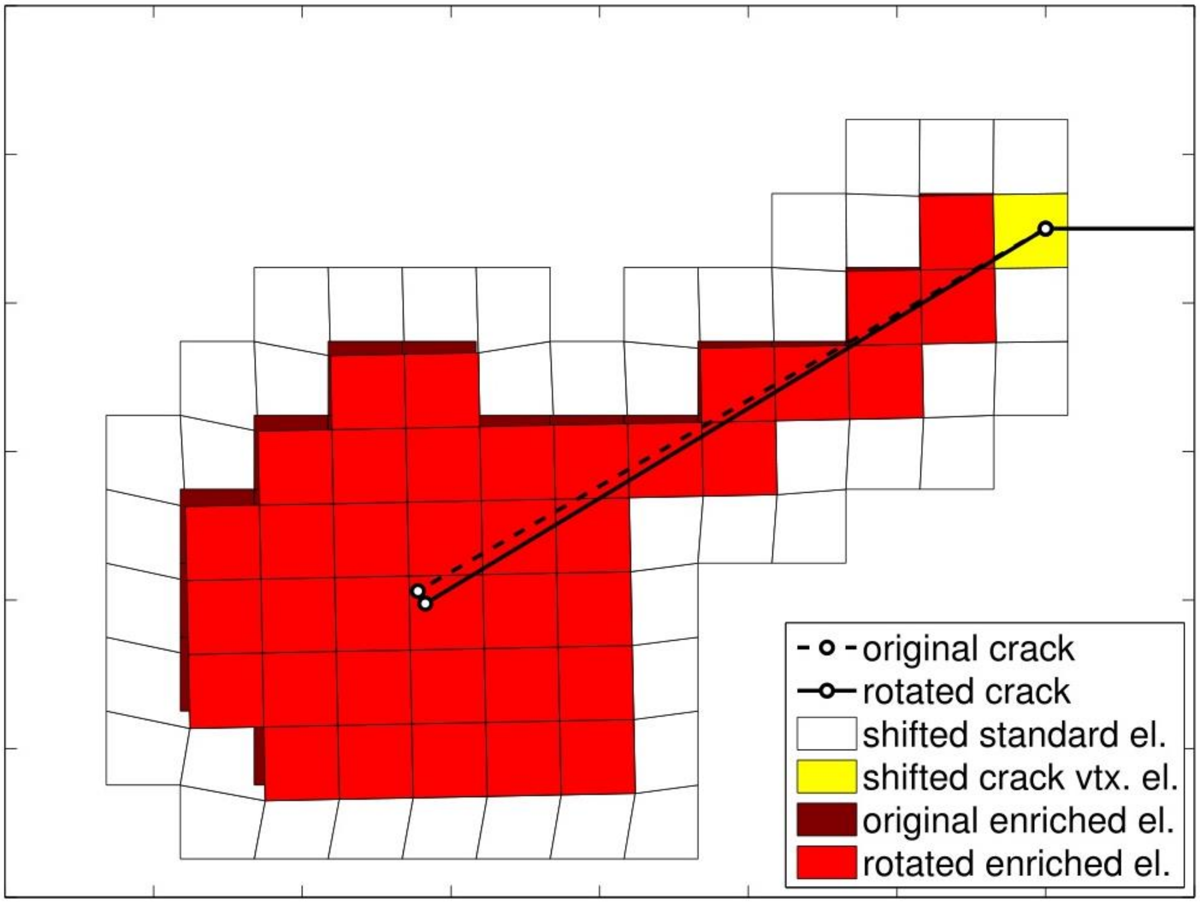
$$\{f_\alpha(r,\theta), \alpha = 1,4\} = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\}$$



Implementation

how to compute δK ?

Differentiation of the stiffness matrix
w.r.t. crack increment direction



Implementation

how to compute δK ?

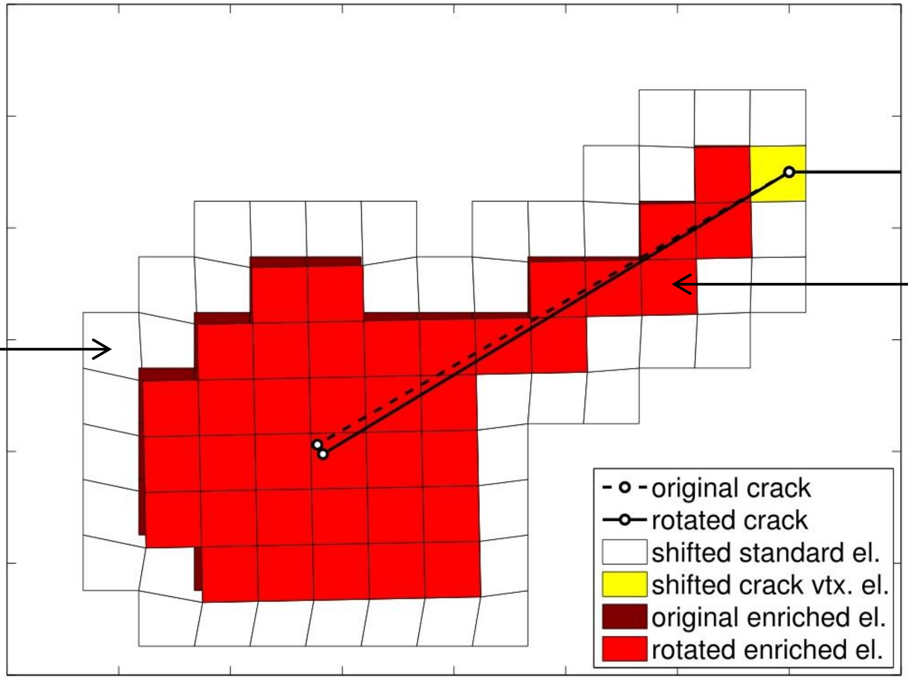
$$\delta \mathbf{K}_e = \int_{\Omega_e} (\delta \mathbf{B}^T \mathbf{D}\mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B}) \det(\mathbf{J})\, d\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D}\mathbf{B}\, \delta \det(\mathbf{J})\, d\bar{\Omega}$$

$$\delta^2 \mathbf{K}_e = \int_{\Omega_e} (\delta^2 \mathbf{B}^T \mathbf{D}\mathbf{B} + 2\delta \mathbf{B}^T \mathbf{D} \delta \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta^2 \mathbf{B}) \det(\mathbf{J})\, d\bar{\Omega} + \int_{\Omega_e} 2\,(\delta \mathbf{B}^T \mathbf{D}\mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B})\, \delta \det(\mathbf{J})\, d\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D}\mathbf{B}\, \delta^2 \det(\mathbf{J})\, d\bar{\Omega}$$

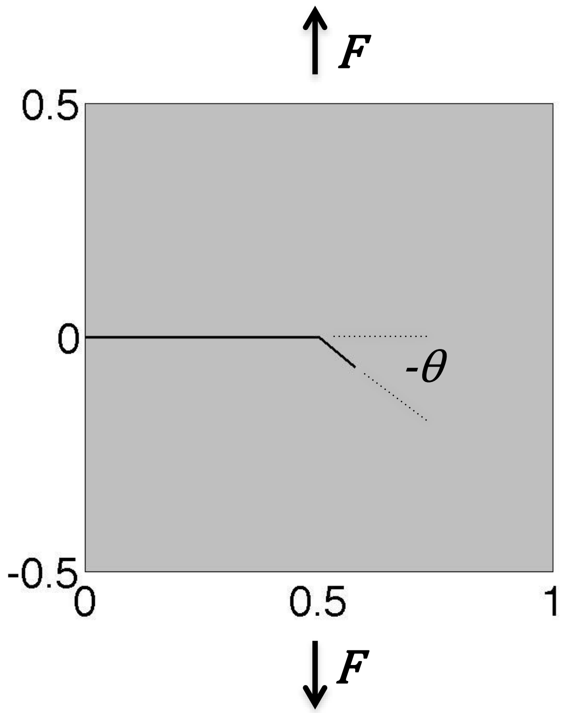
$$\delta \mathbf{K_e} = \mathbf{T}^T \mathbf{K_e} + \mathbf{K_e} \mathbf{T}$$

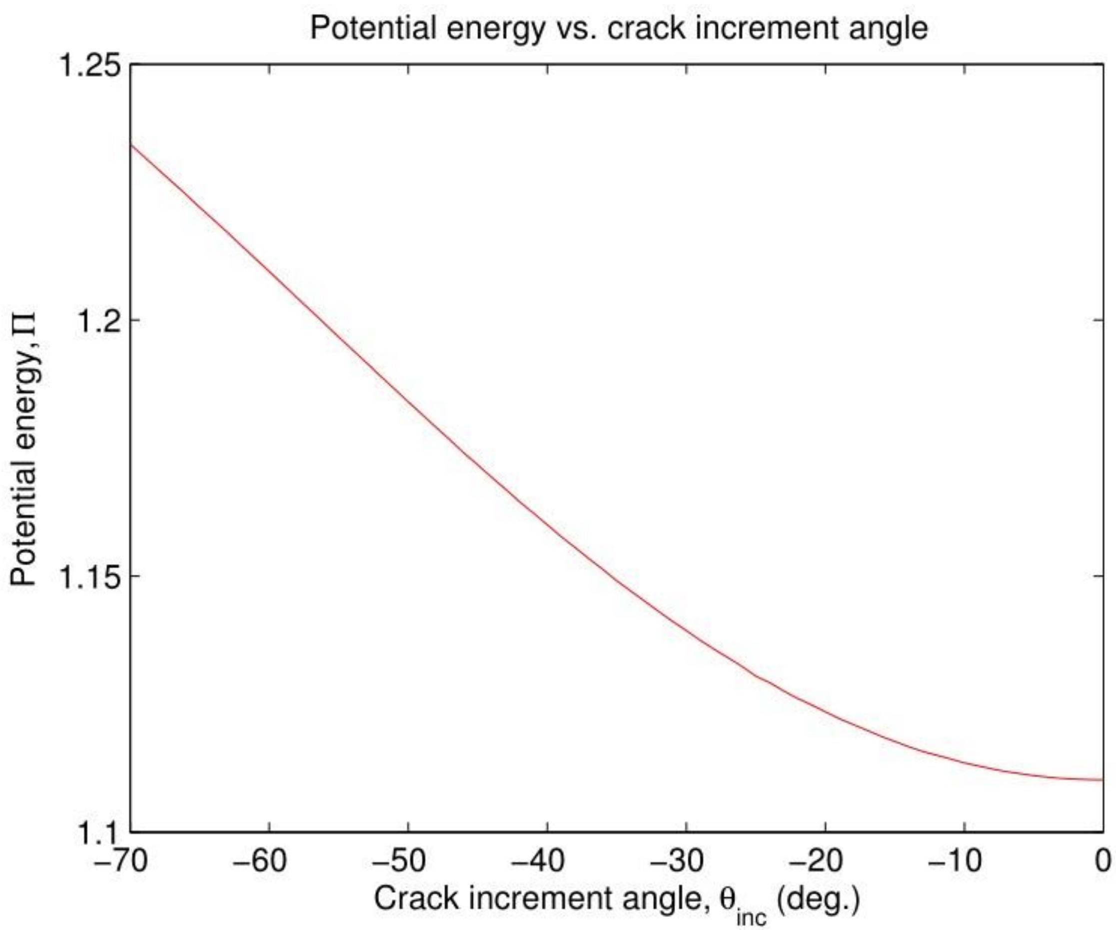
$$\delta^2 \mathbf{K_e} = 2(\mathbf{T}^T \mathbf{K_e} \mathbf{T} - \mathbf{K_e})$$

Differentiation of the stiffness matrix
w.r.t. crack increment direction



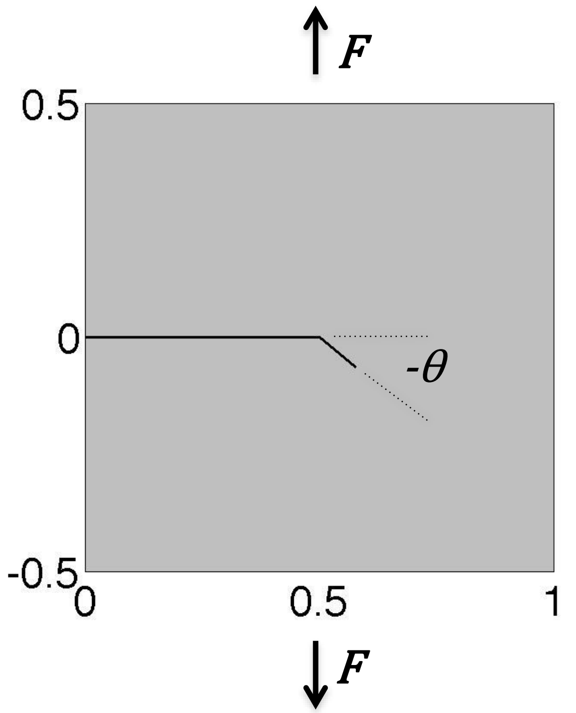
Test case: square plate with an edge crack with a small kink loaded in vertical tension

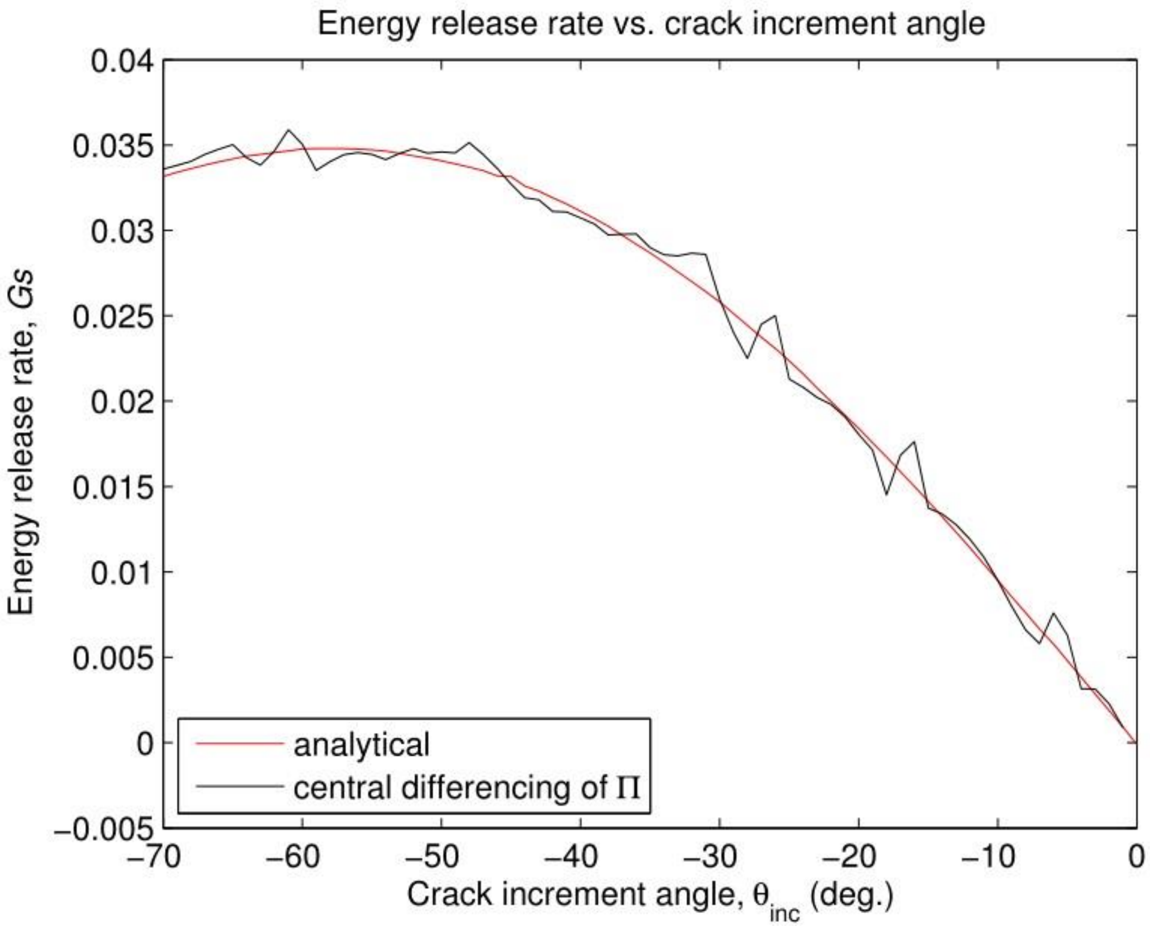




Π vs. θ

Test case: square plate with an edge crack with a small kink loaded in vertical tension



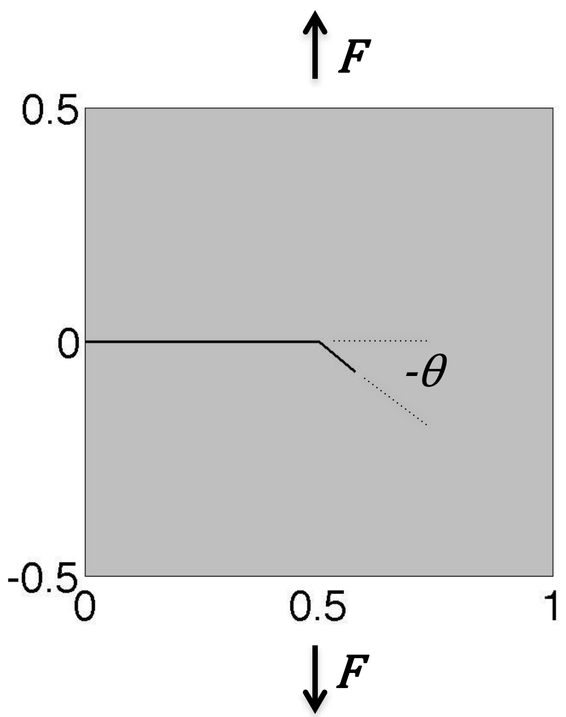


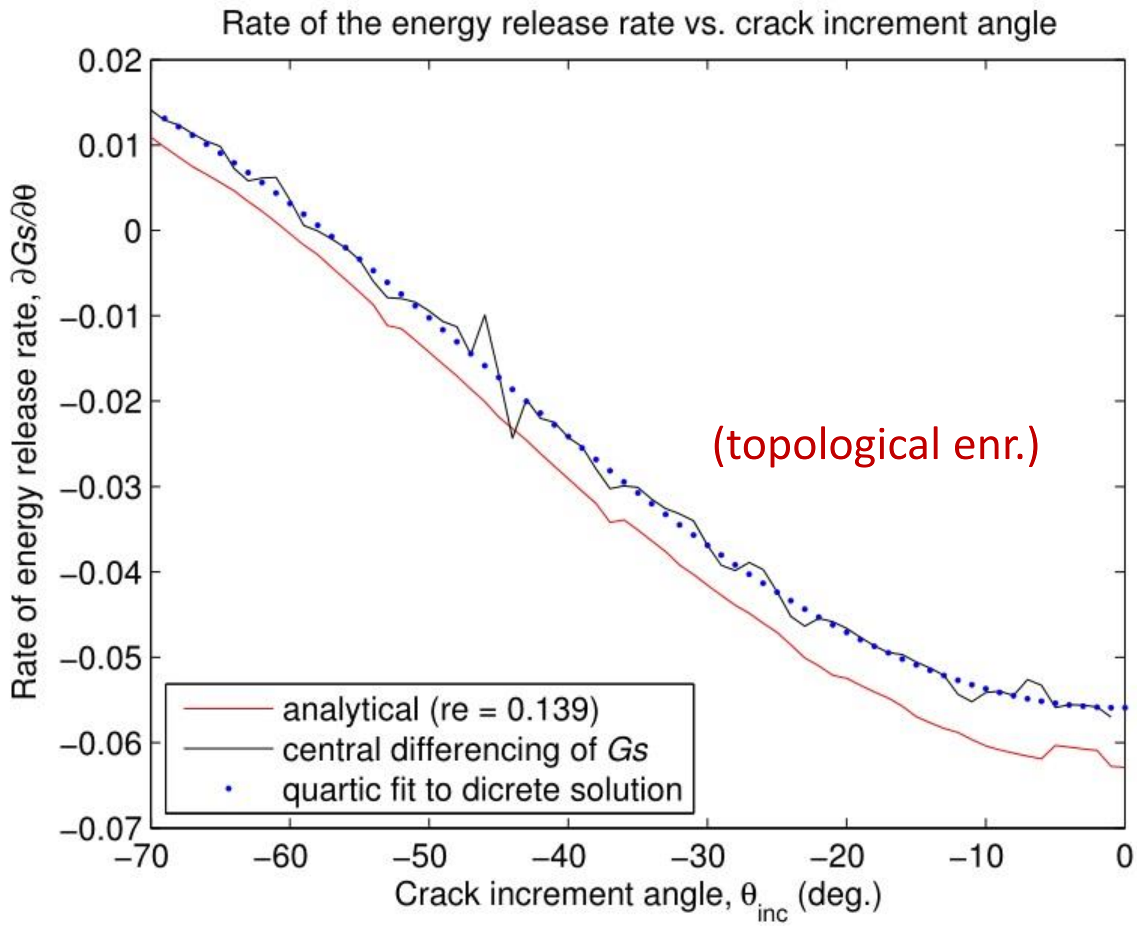
G vs. θ

Verification

rotational energy release rates

Test case: square plate with an edge crack with a small kink loaded in vertical tension

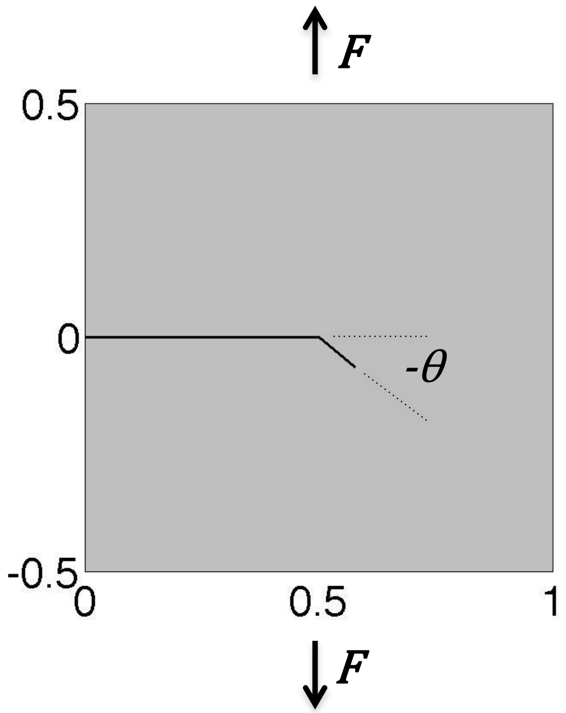


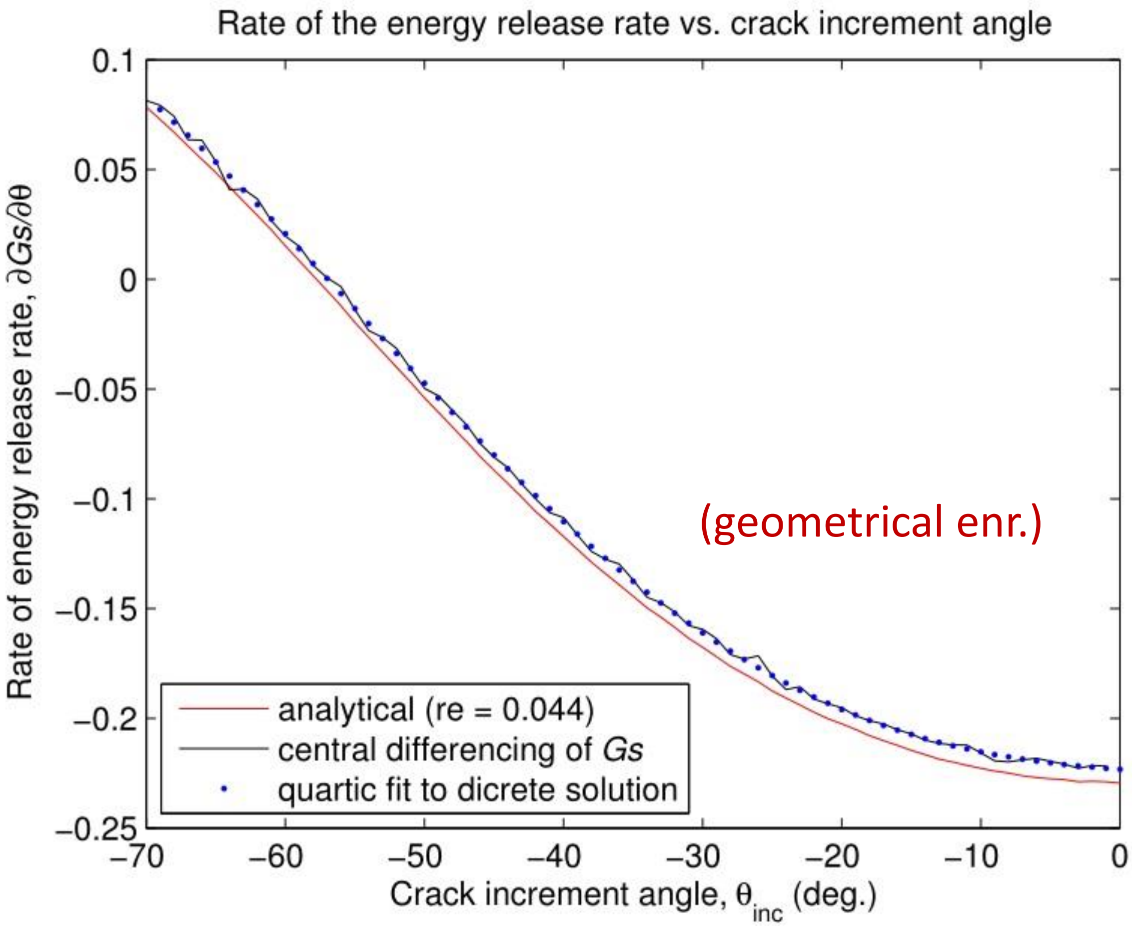


$\mathrm{d}G/\mathrm{d}\theta$ vs. θ

Verification
rotational energy release rates

Test case: square plate with an edge crack with a small kink loaded in vertical tension

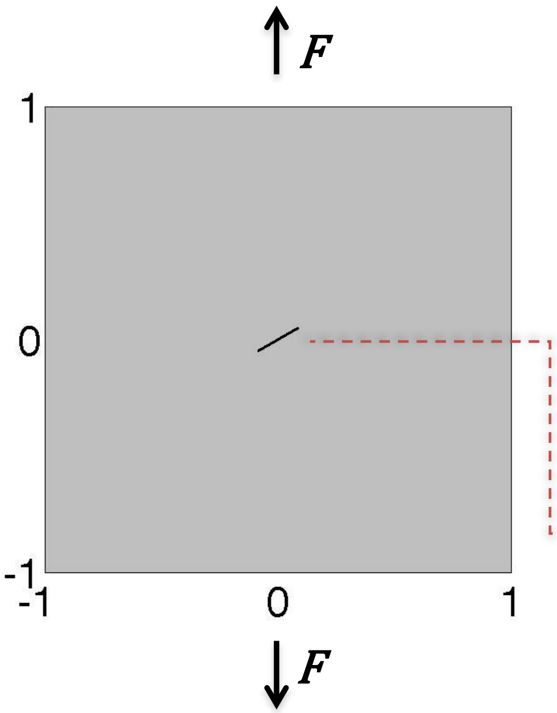


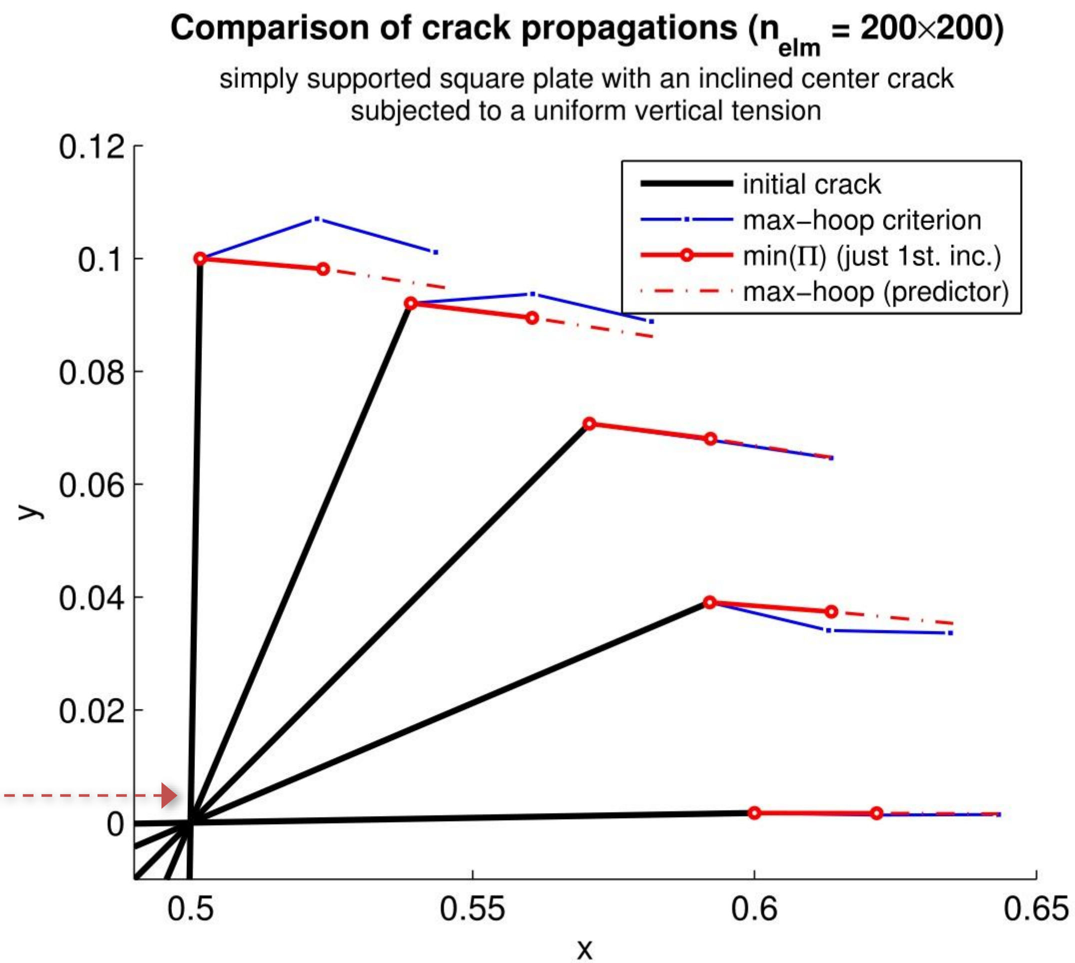


$\mathrm{d}G/\mathrm{d}\theta$ vs. θ

Verification
energy min. VS. max-hoop

Test case: square plate with an inclined center crack in vertical tension

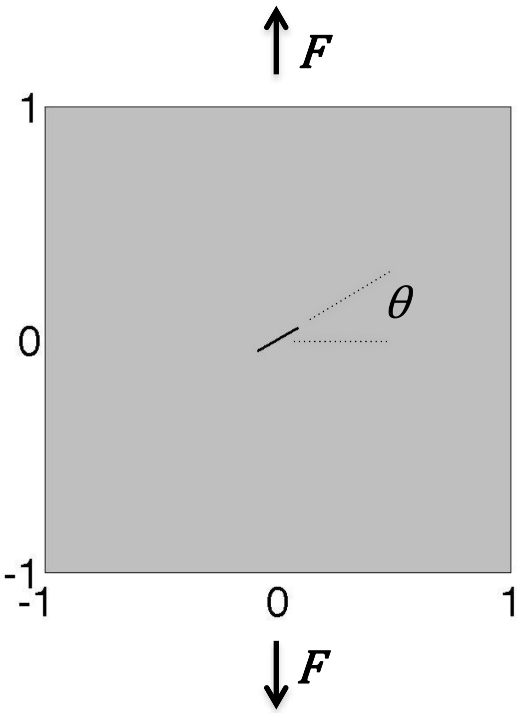


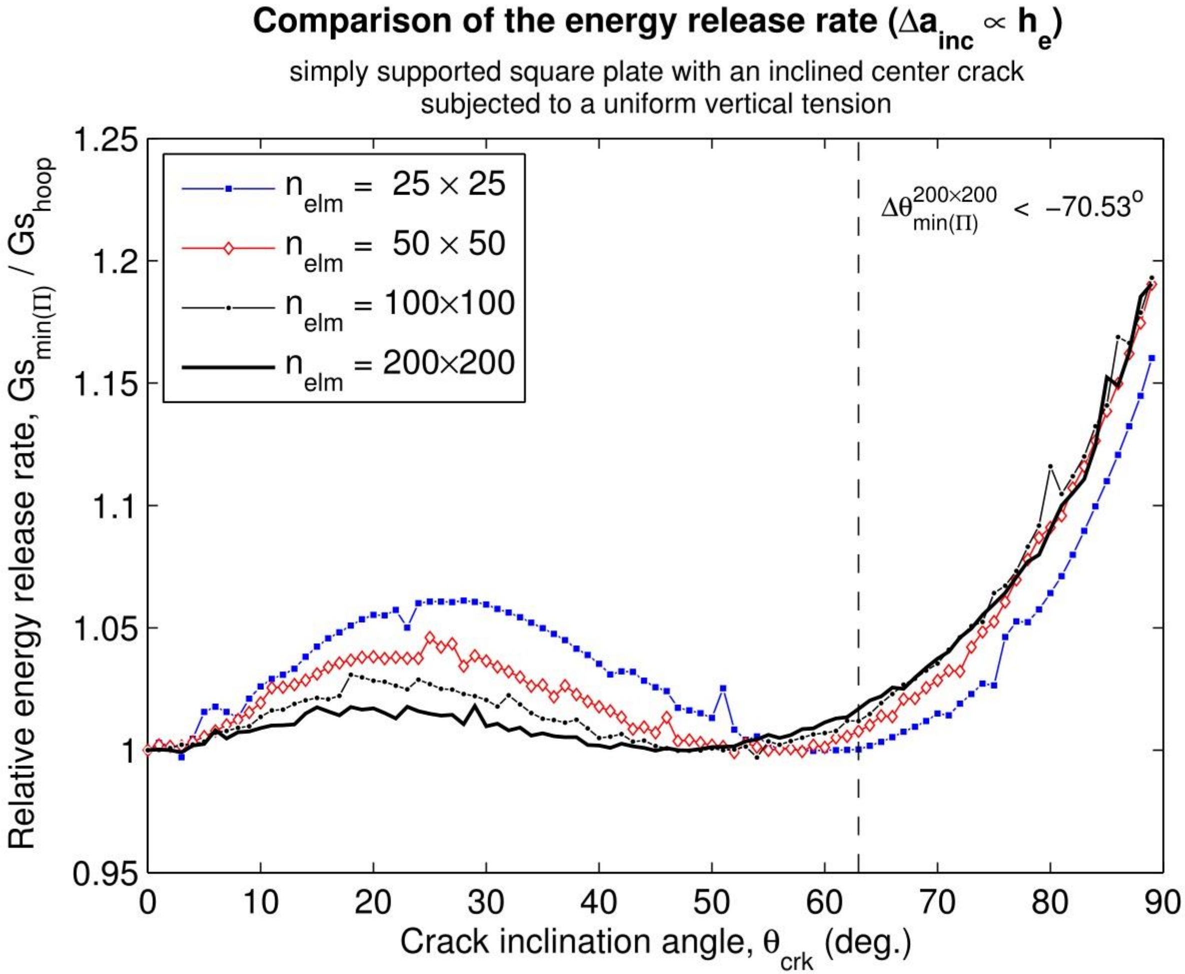


Verification

energy min. VS. max-hoop

Test case: square plate with an inclined center crack in vertical tension

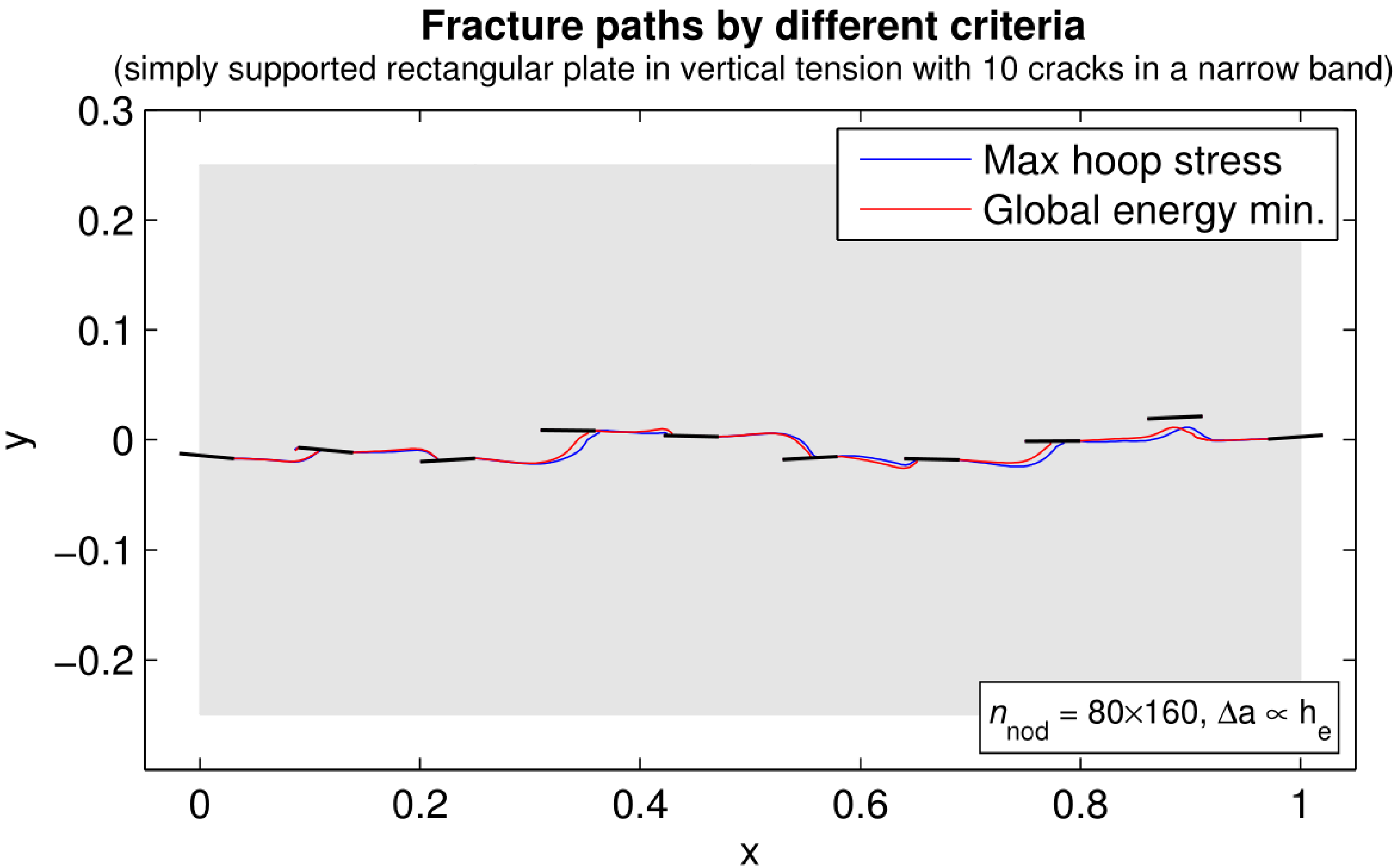




$G_{min(\Pi)}/G_{hoop}$ vs. θ

Results

10 crack problem



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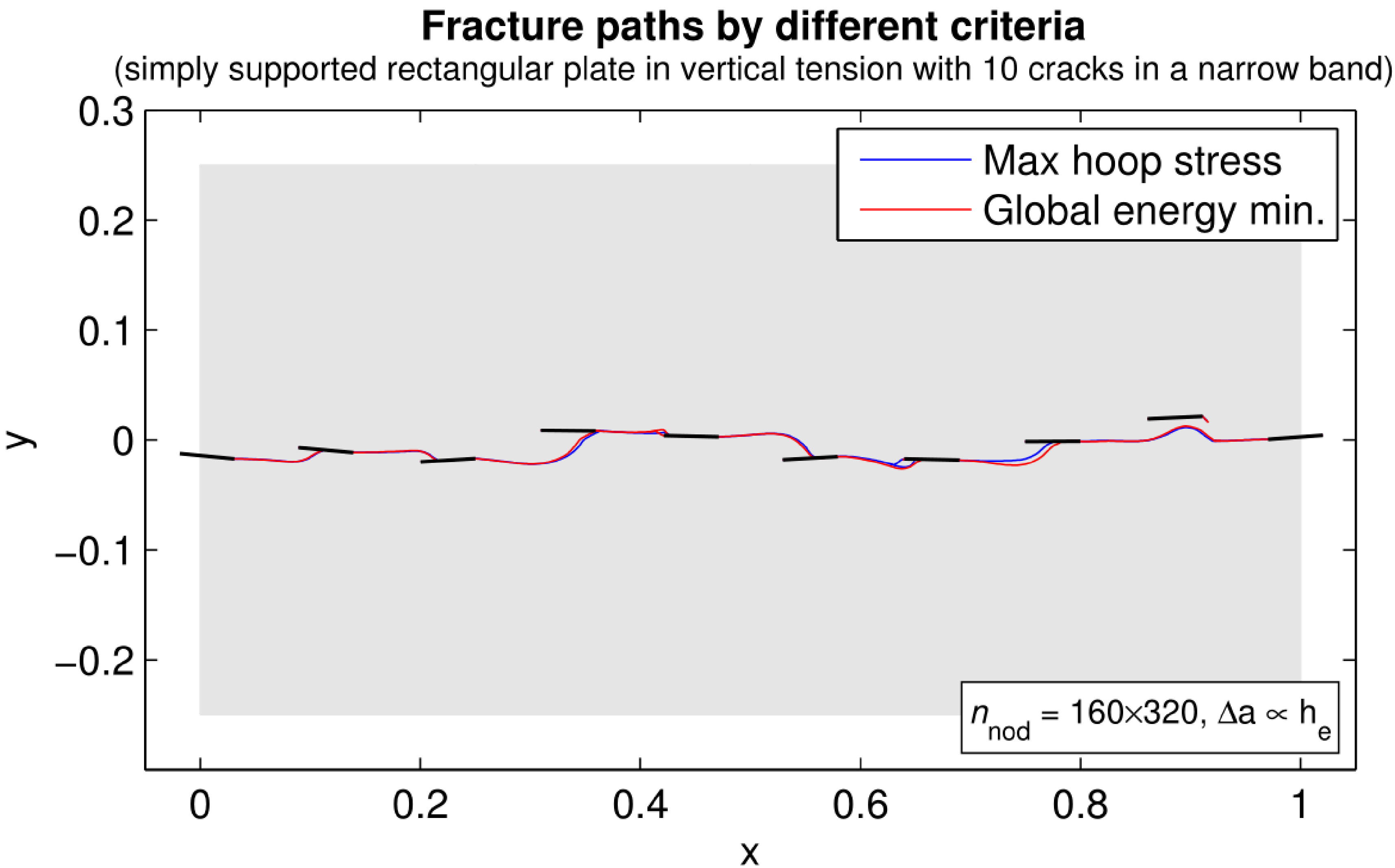
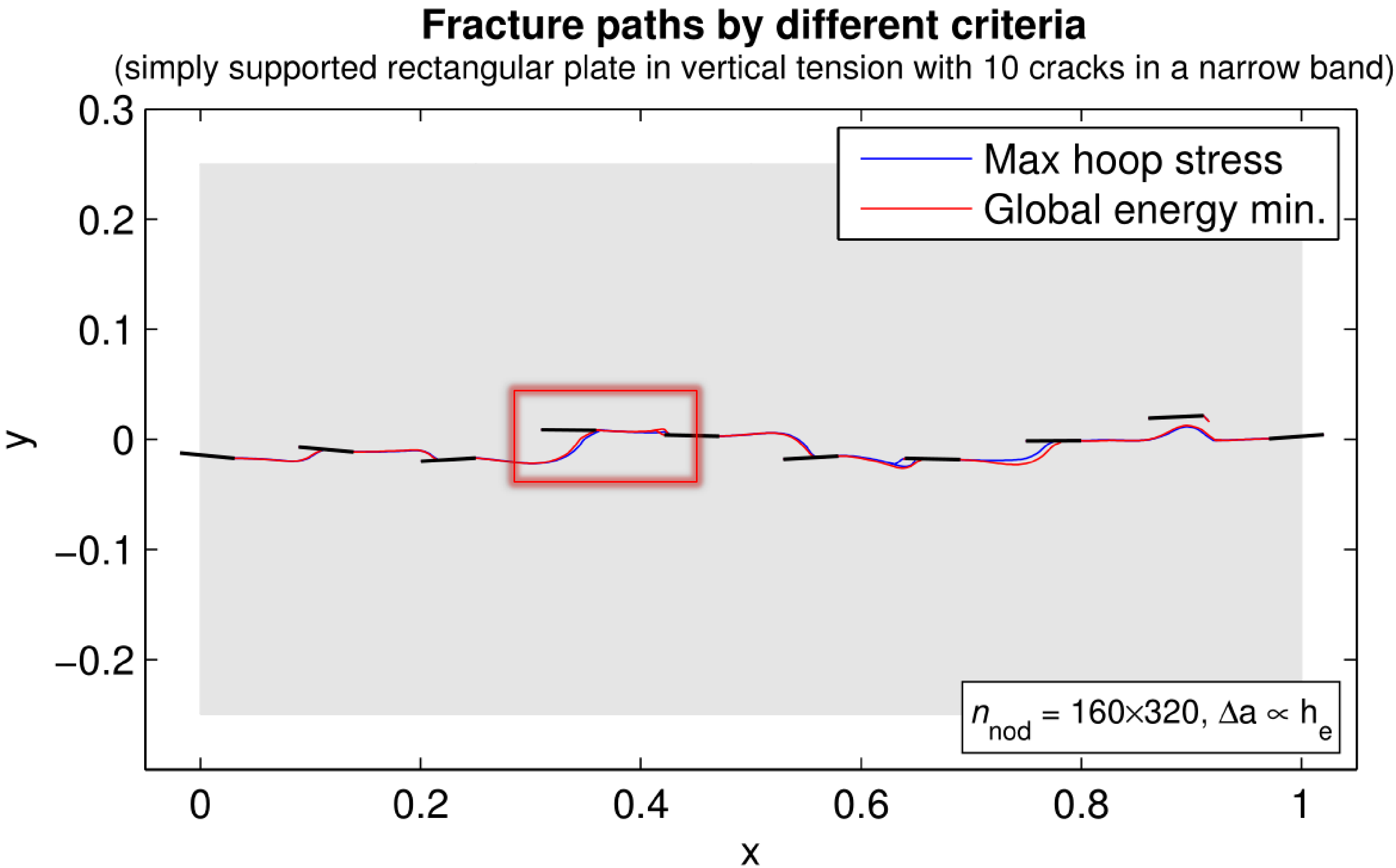
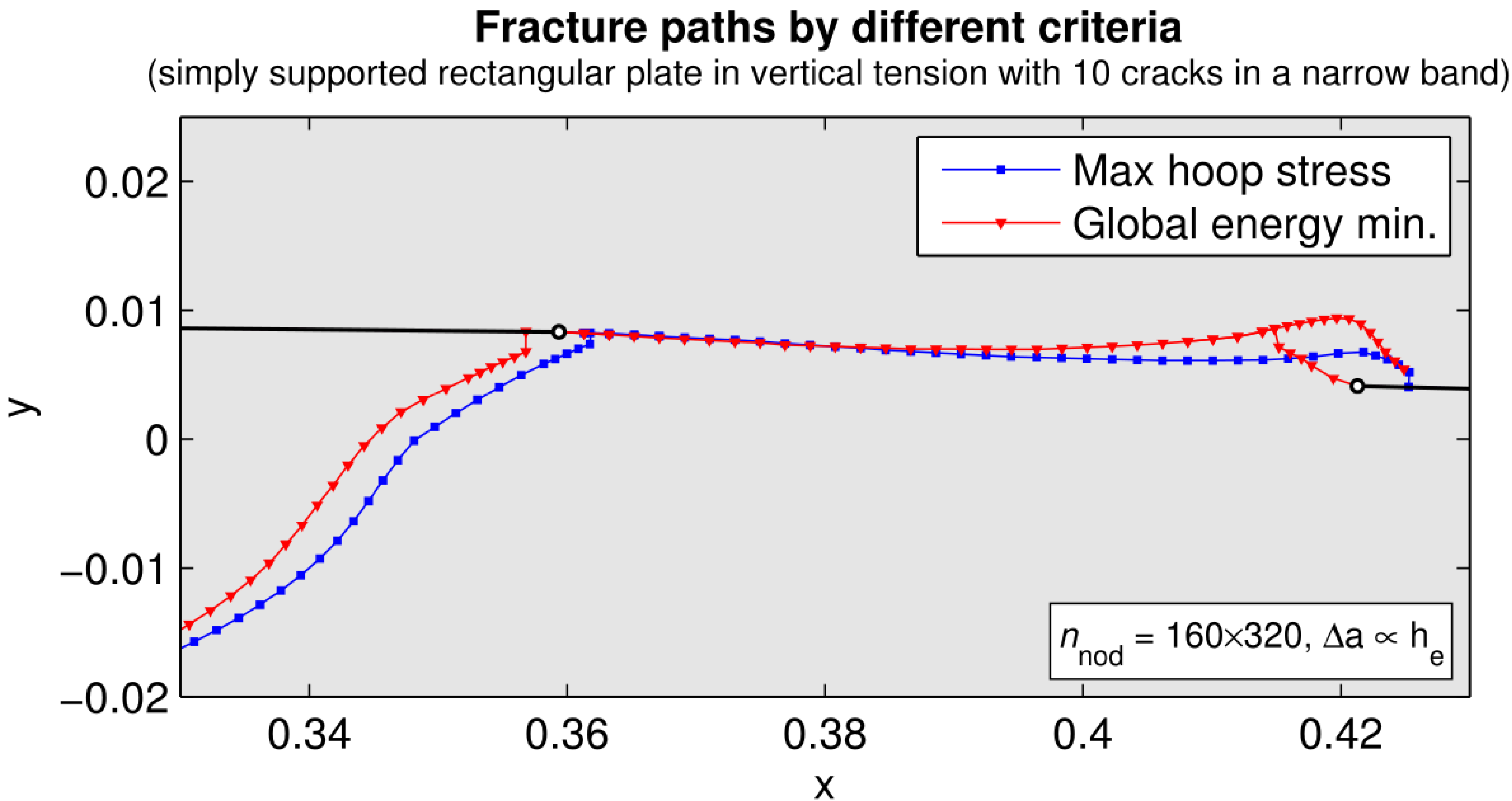


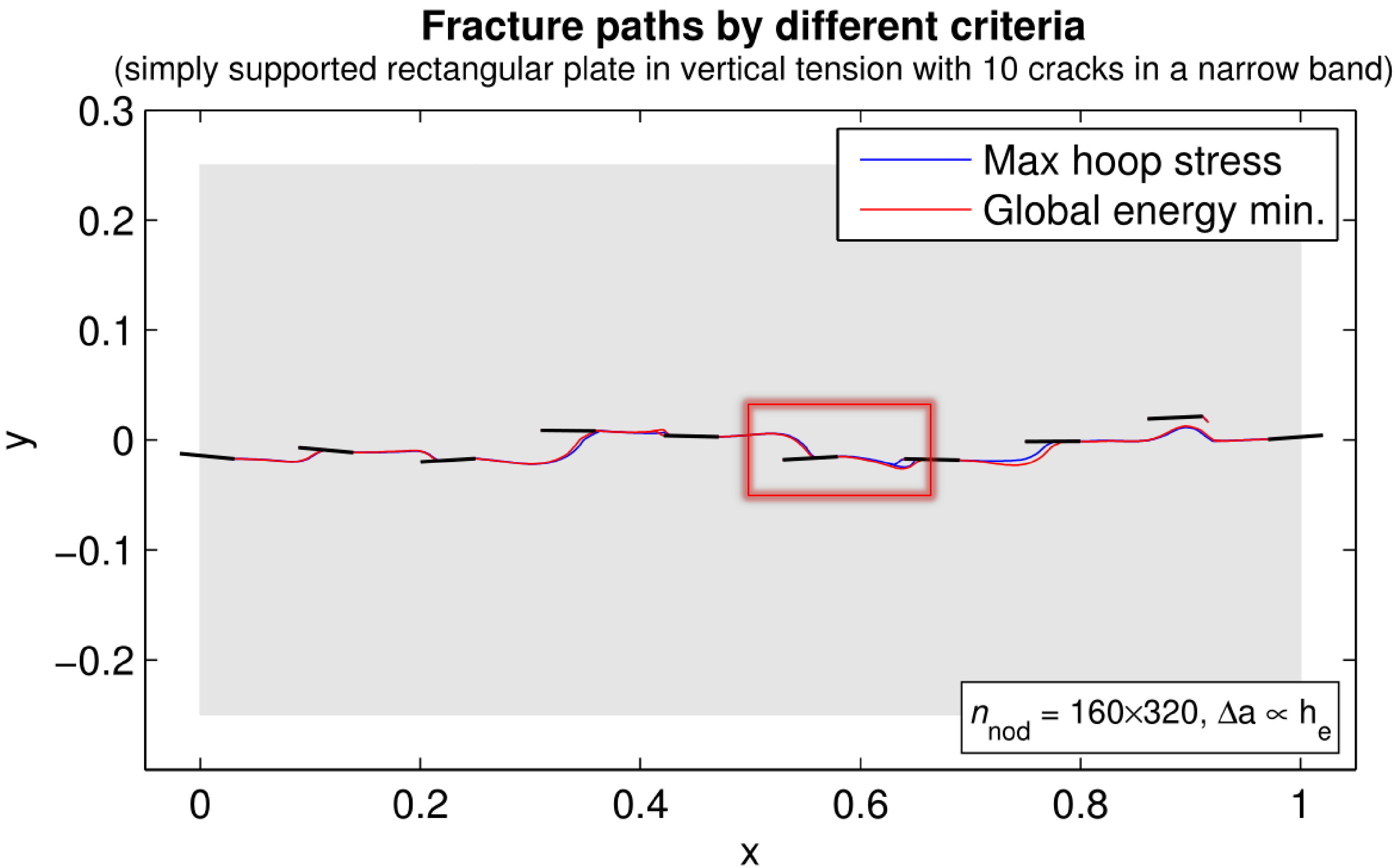


Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)
Max hoop stress
Global energy min.
n_nod = 160×320, Δa ∝ h_e
0.3
0.2
0.1
0
–0.1
–0.2
y
0
0.2
0.4
0.6
0.8
1
x



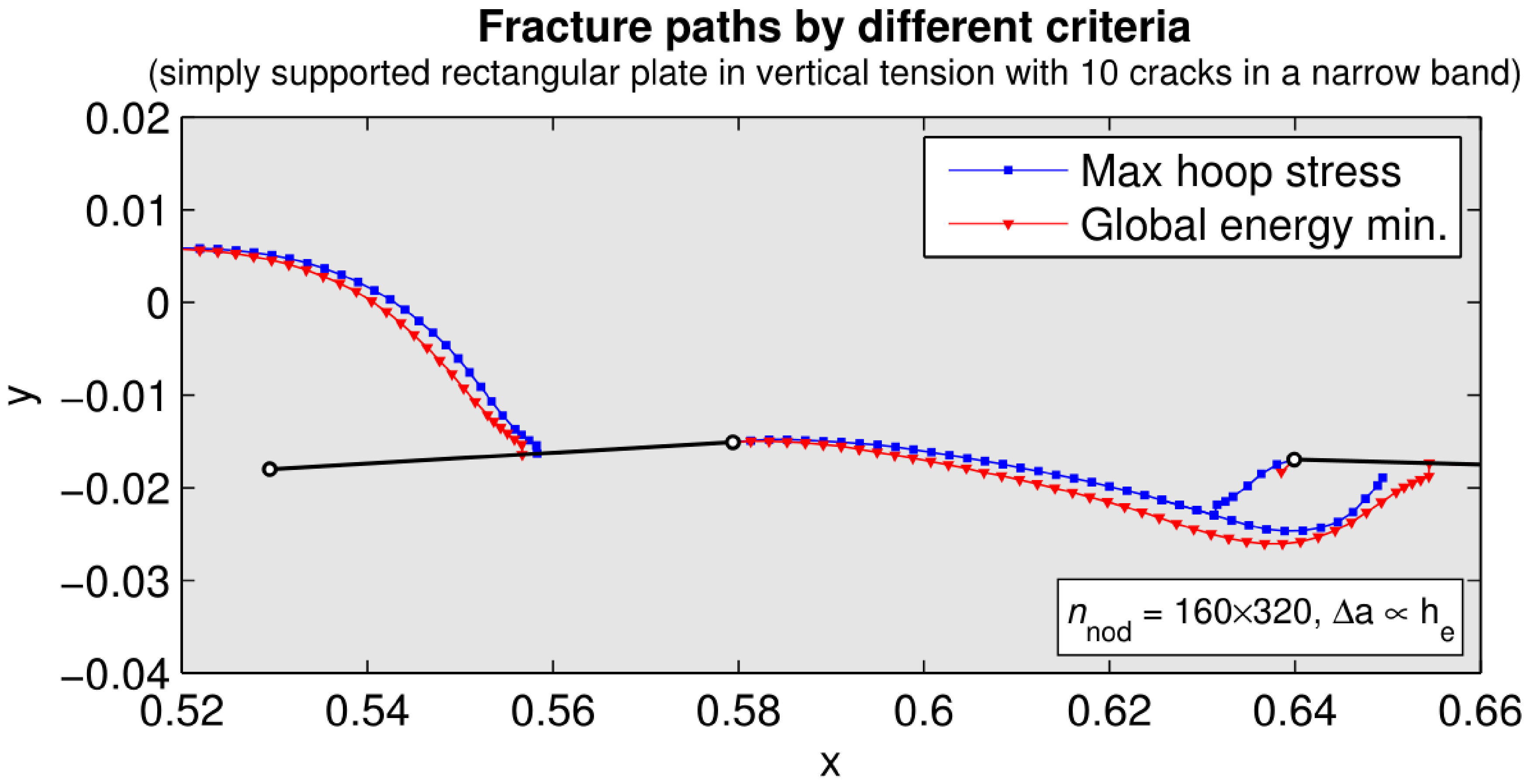
Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)
Max hoop stress
Global energy min.
n_nod = 160×320, Δa ∝ h_e
0.02
0.01
0
−0.01
−0.02
y
0.34
0.36
0.38
0.4
0.42
x

Results
10 crack problem

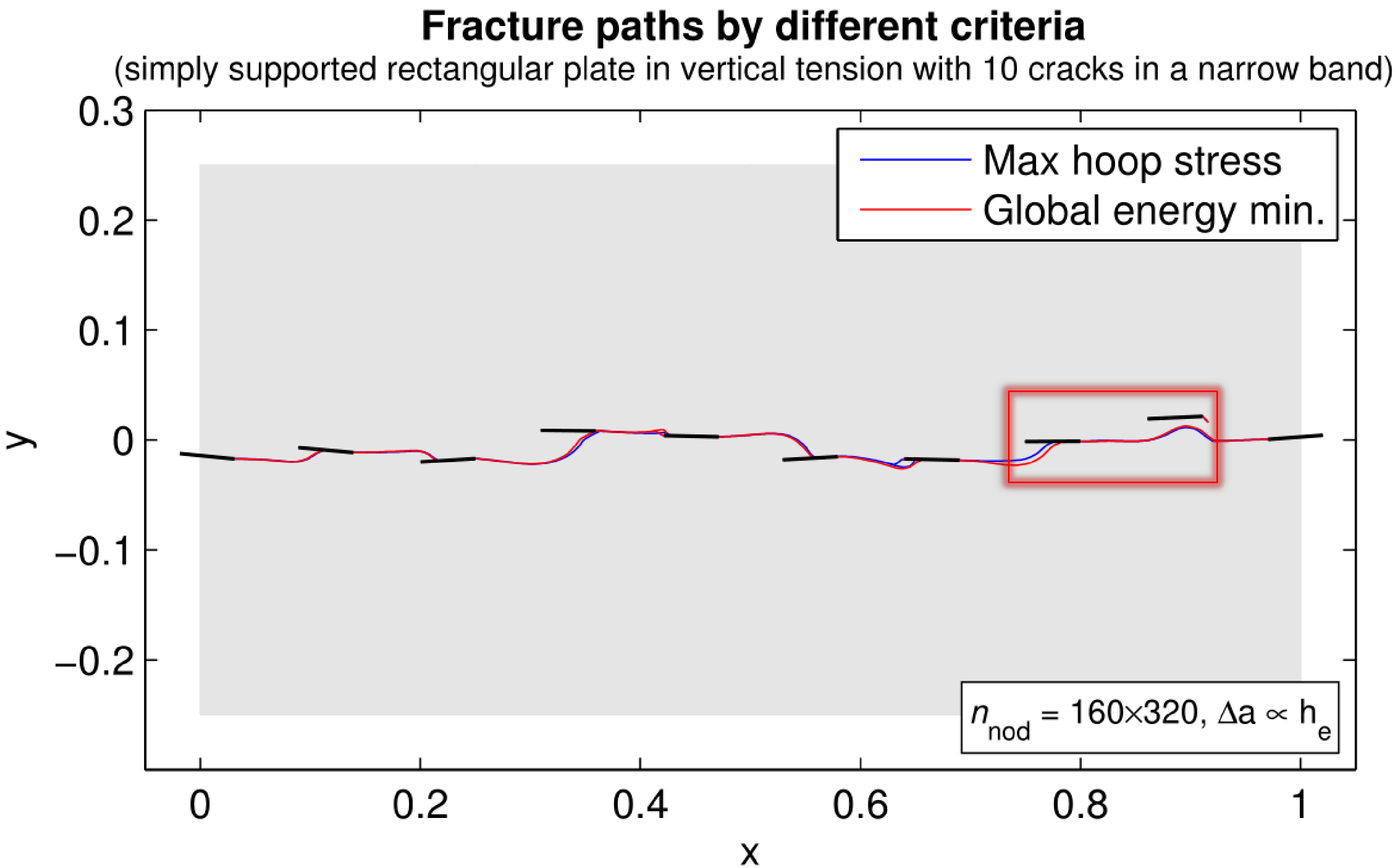


Results

10 crack problem

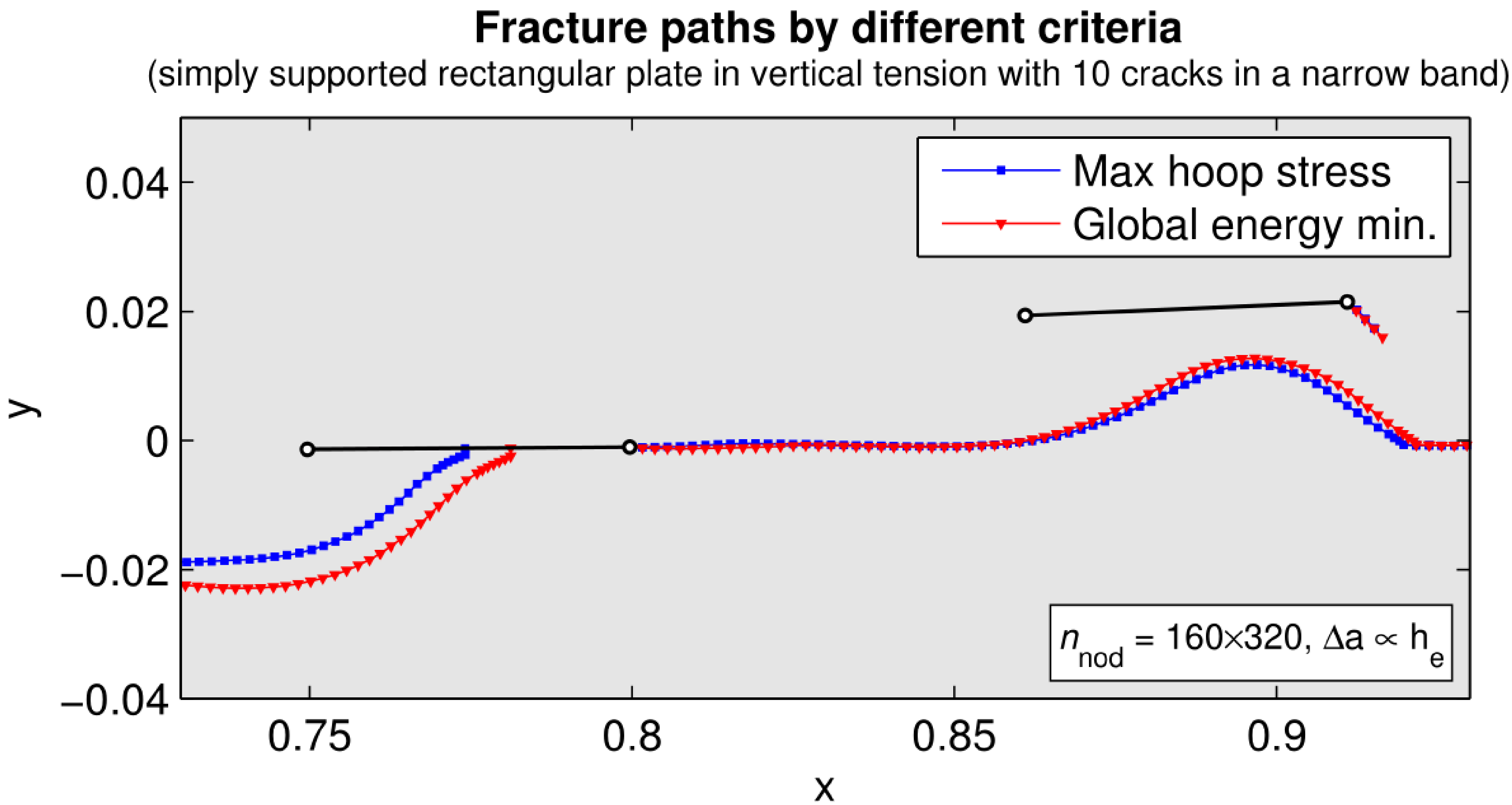


Results
10 crack problem



Results

10 crack problem



Results

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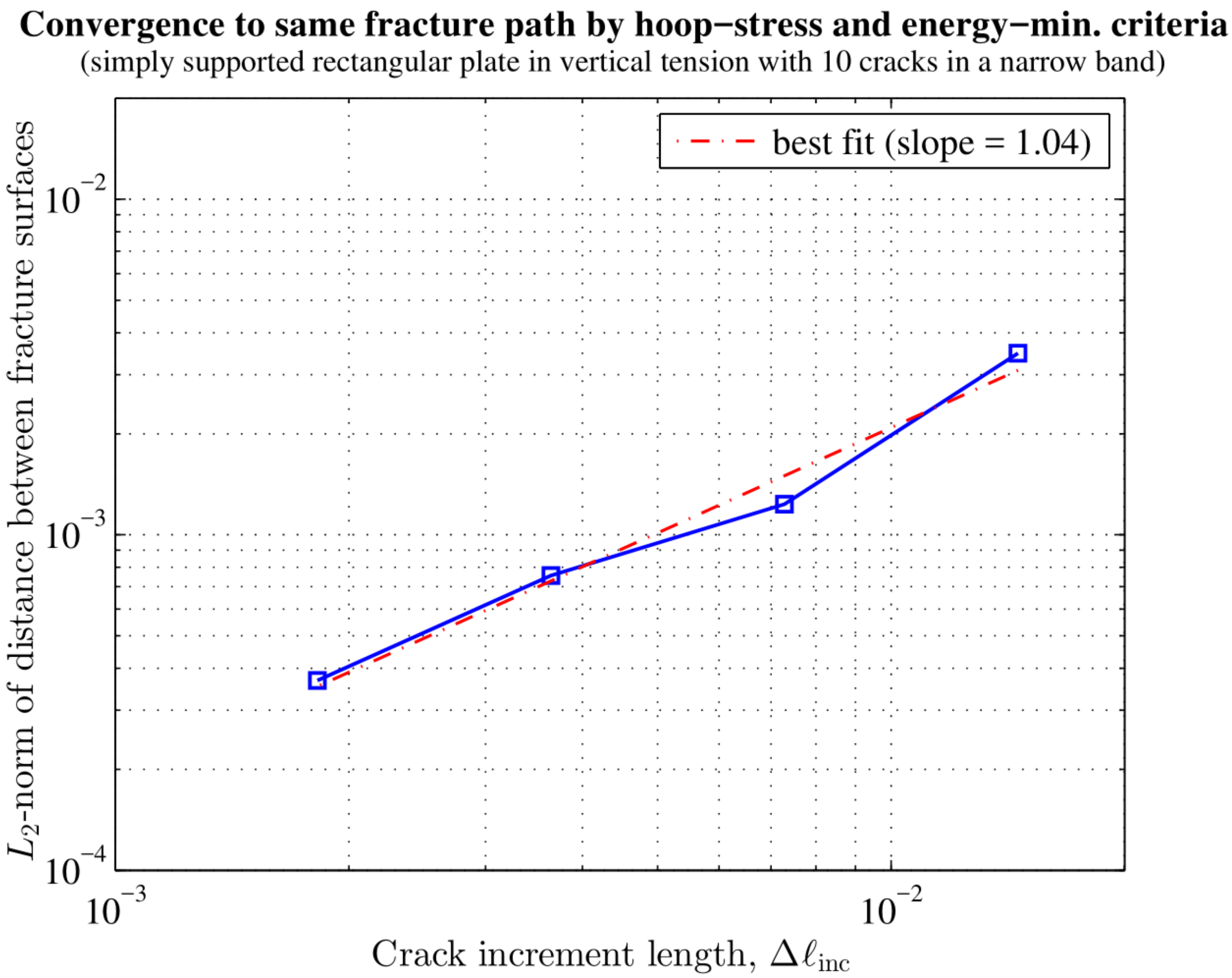
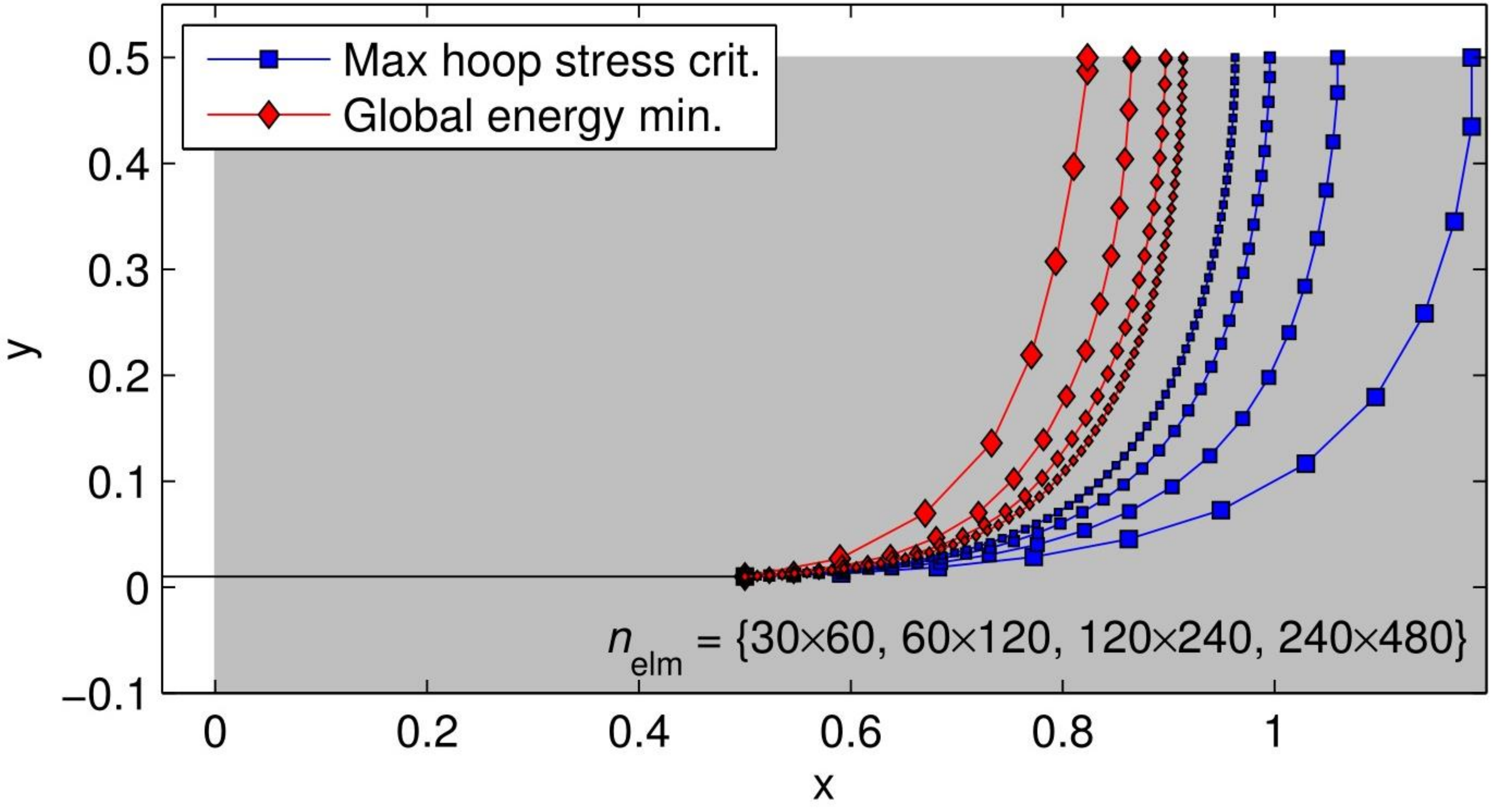
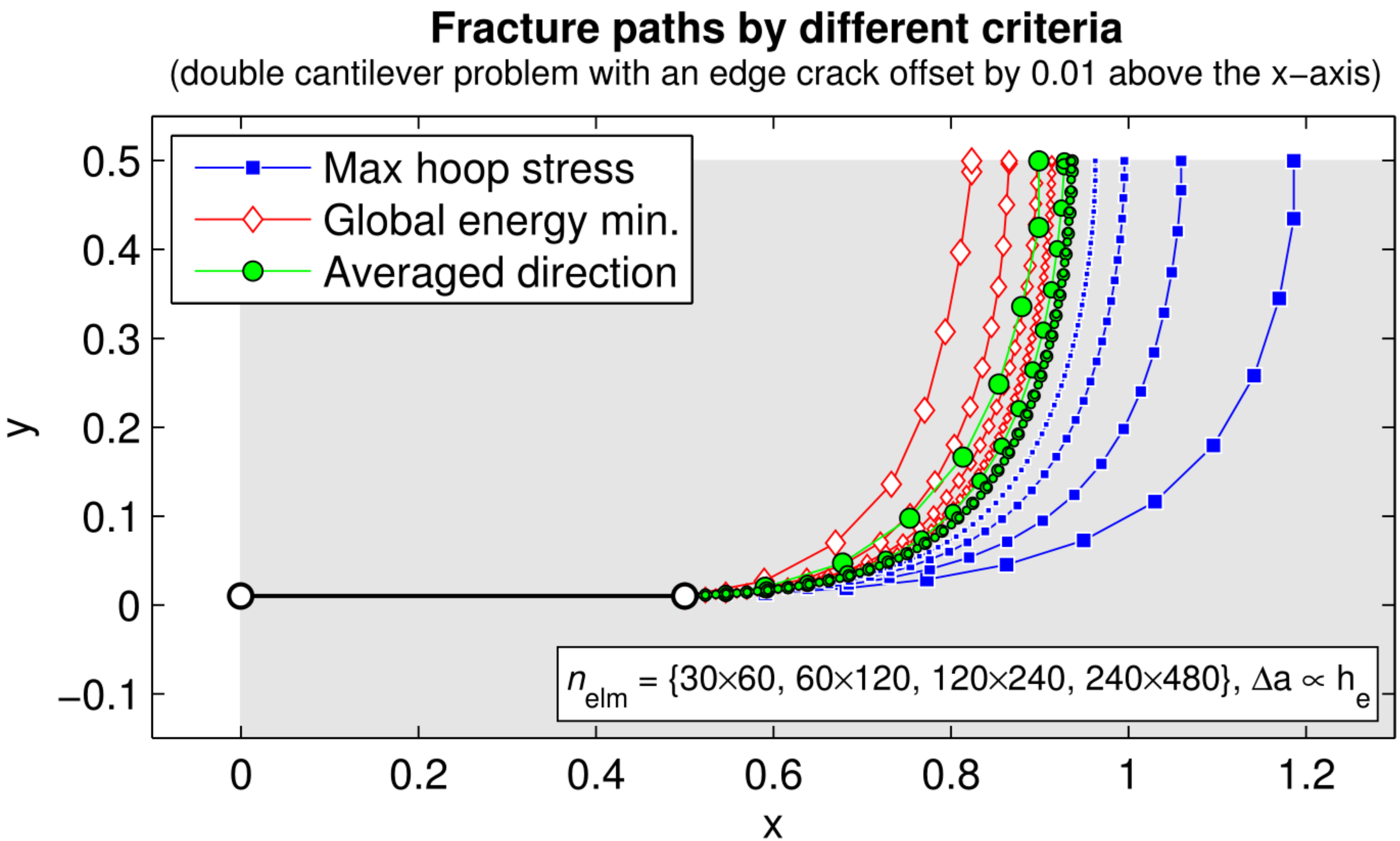


Results
double cantilever problem

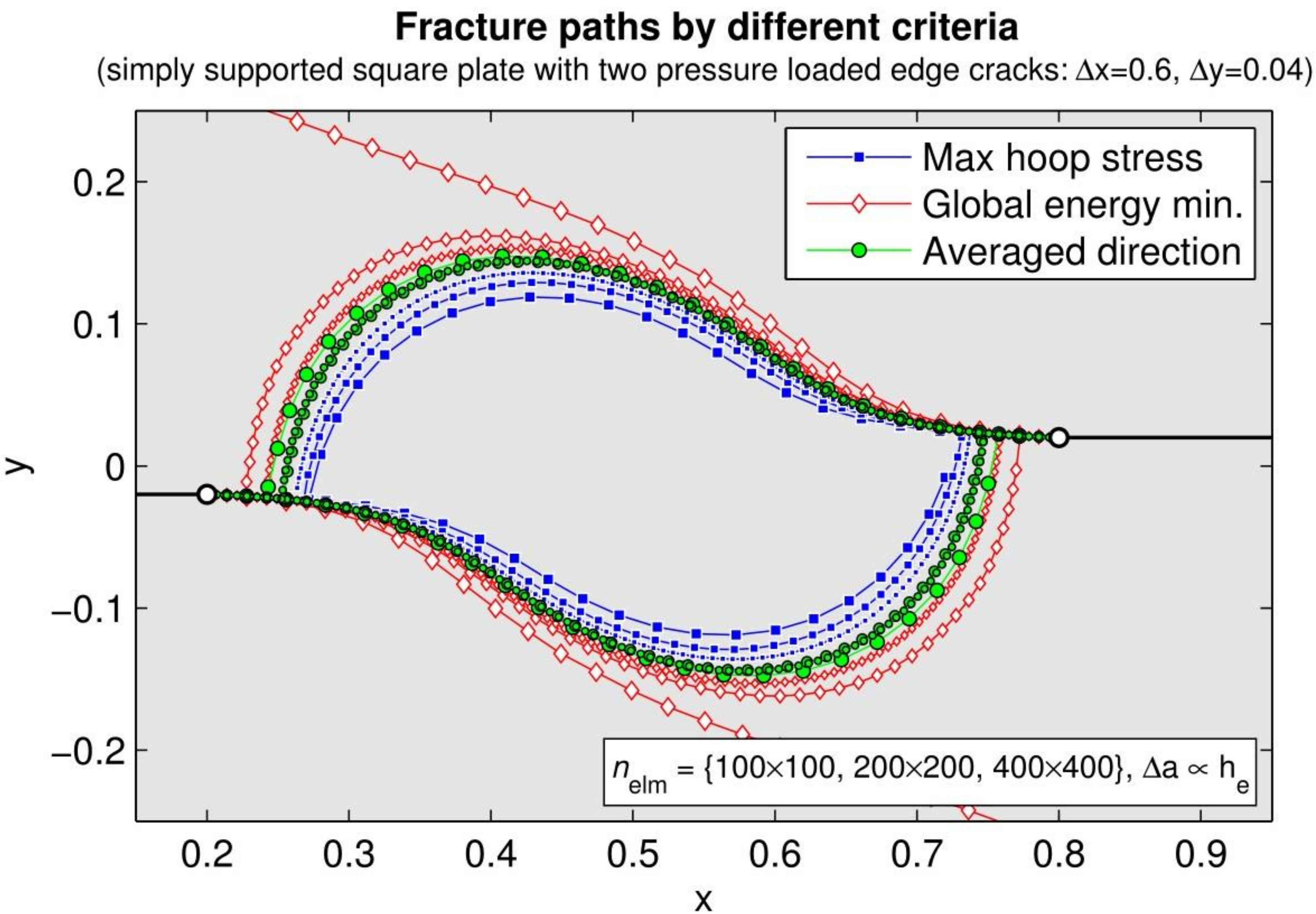


Results
double cantilever problem



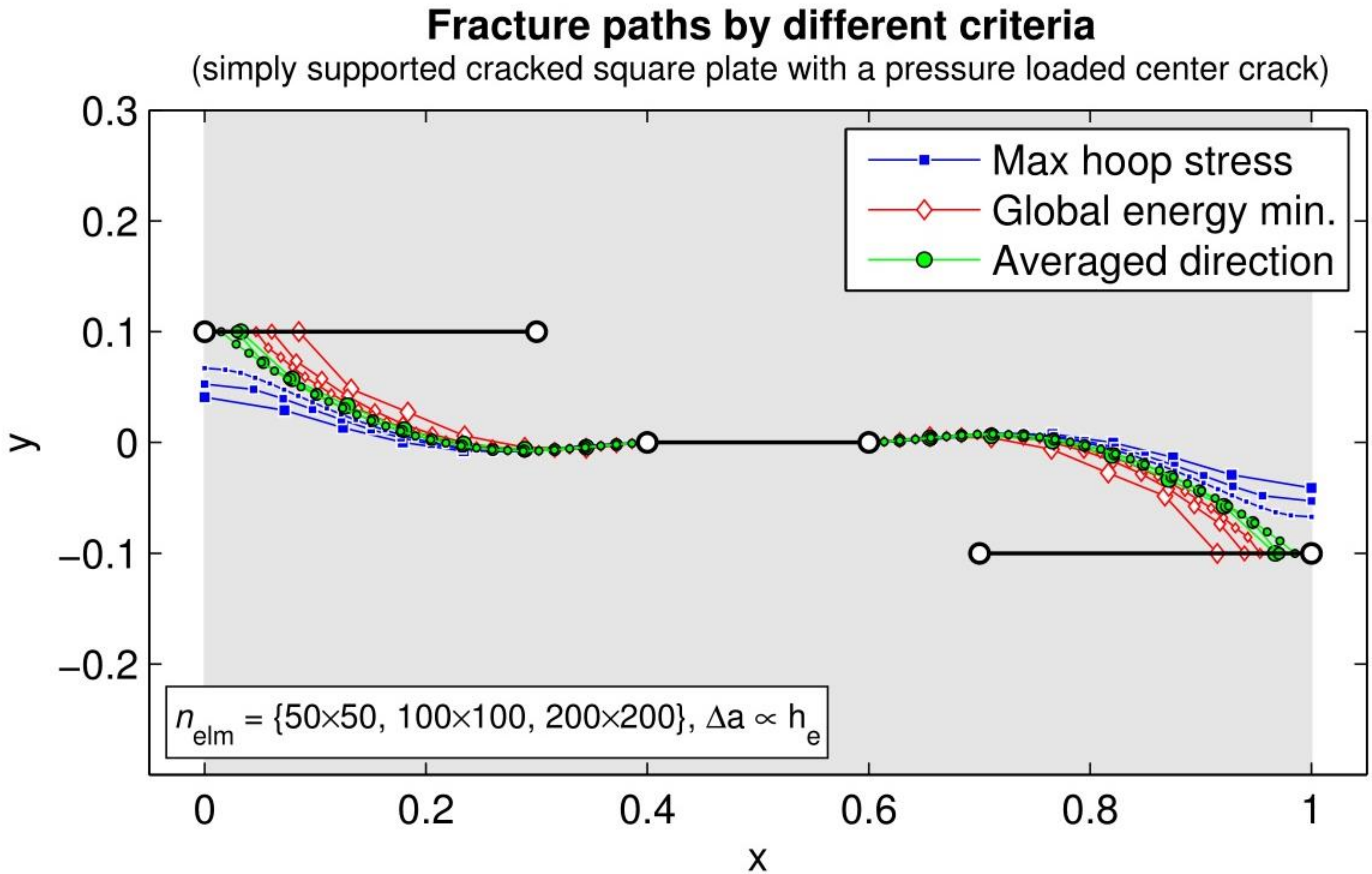
Results

2 edge cracks; internal pressure loading



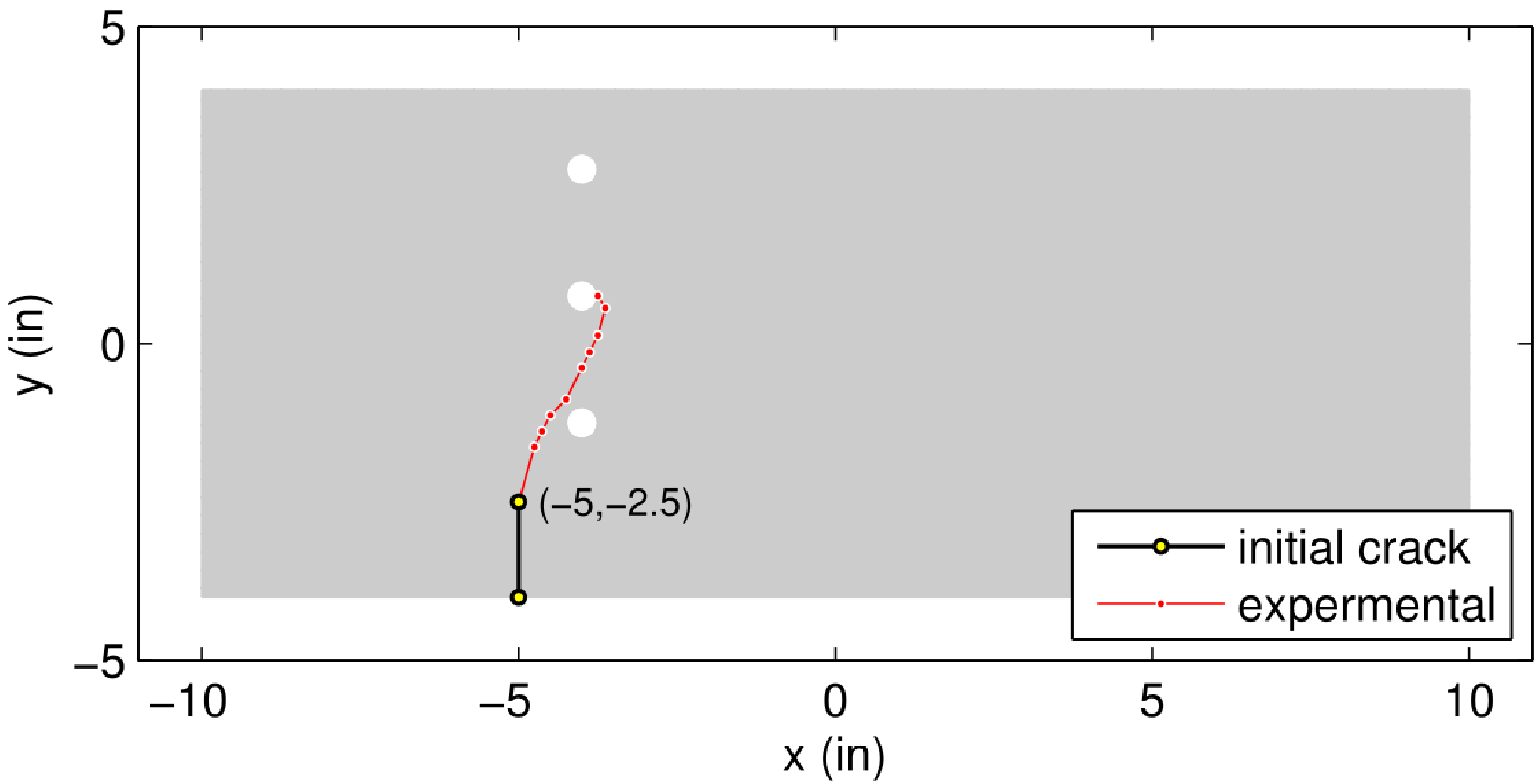
Results

3 cracks; center crack pressure loading



Results

Edge crack in a PMMA beam with 3 holes



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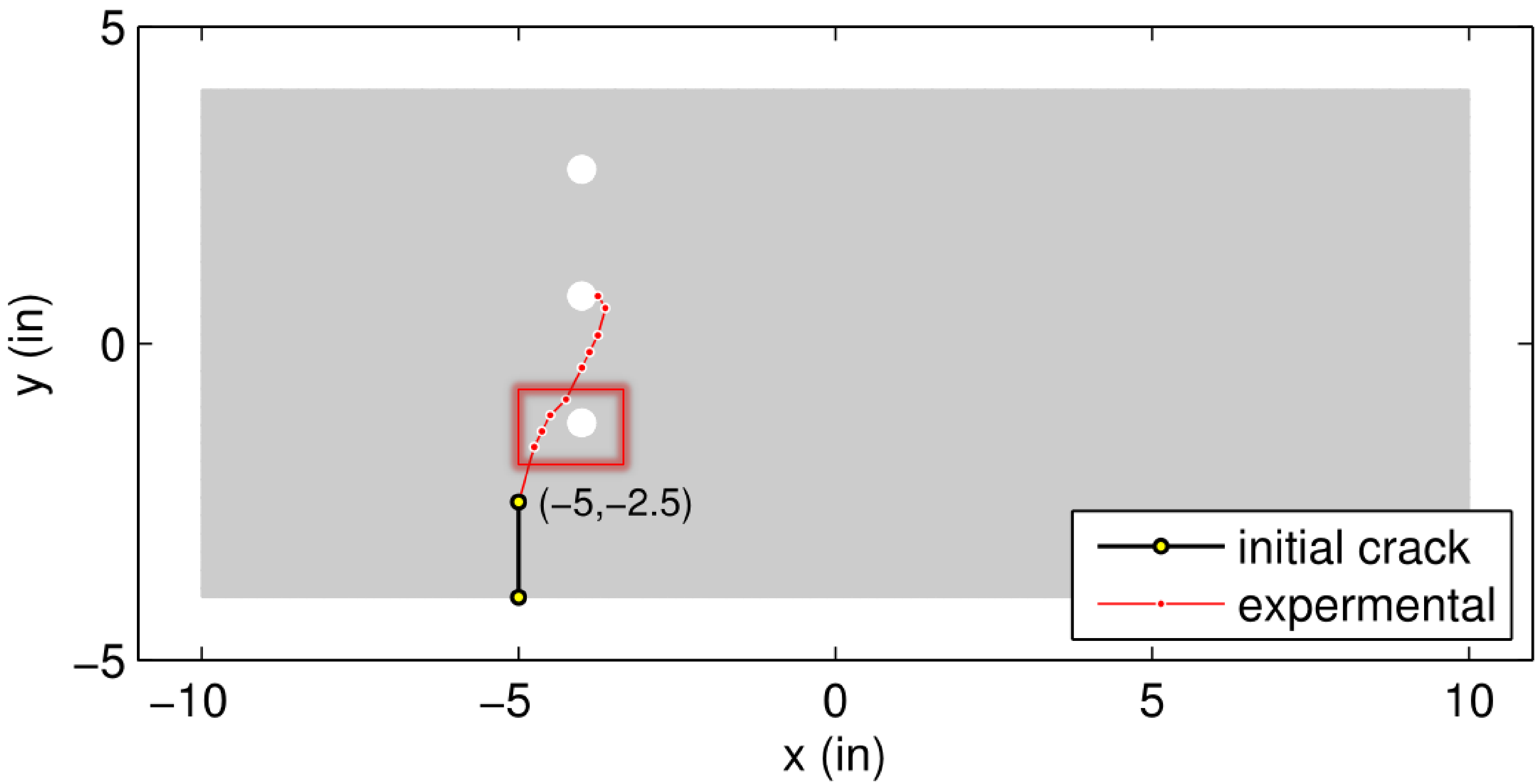

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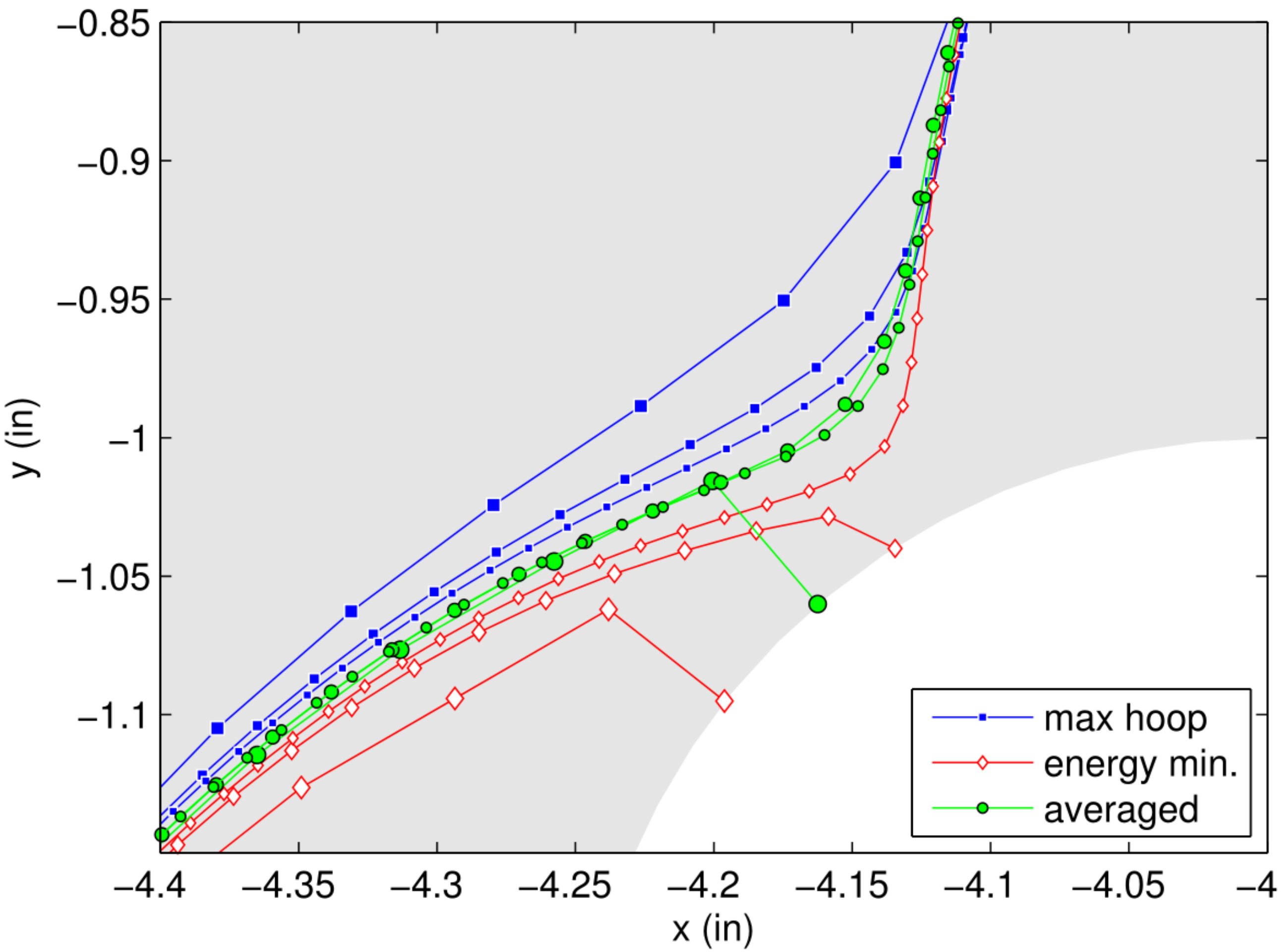

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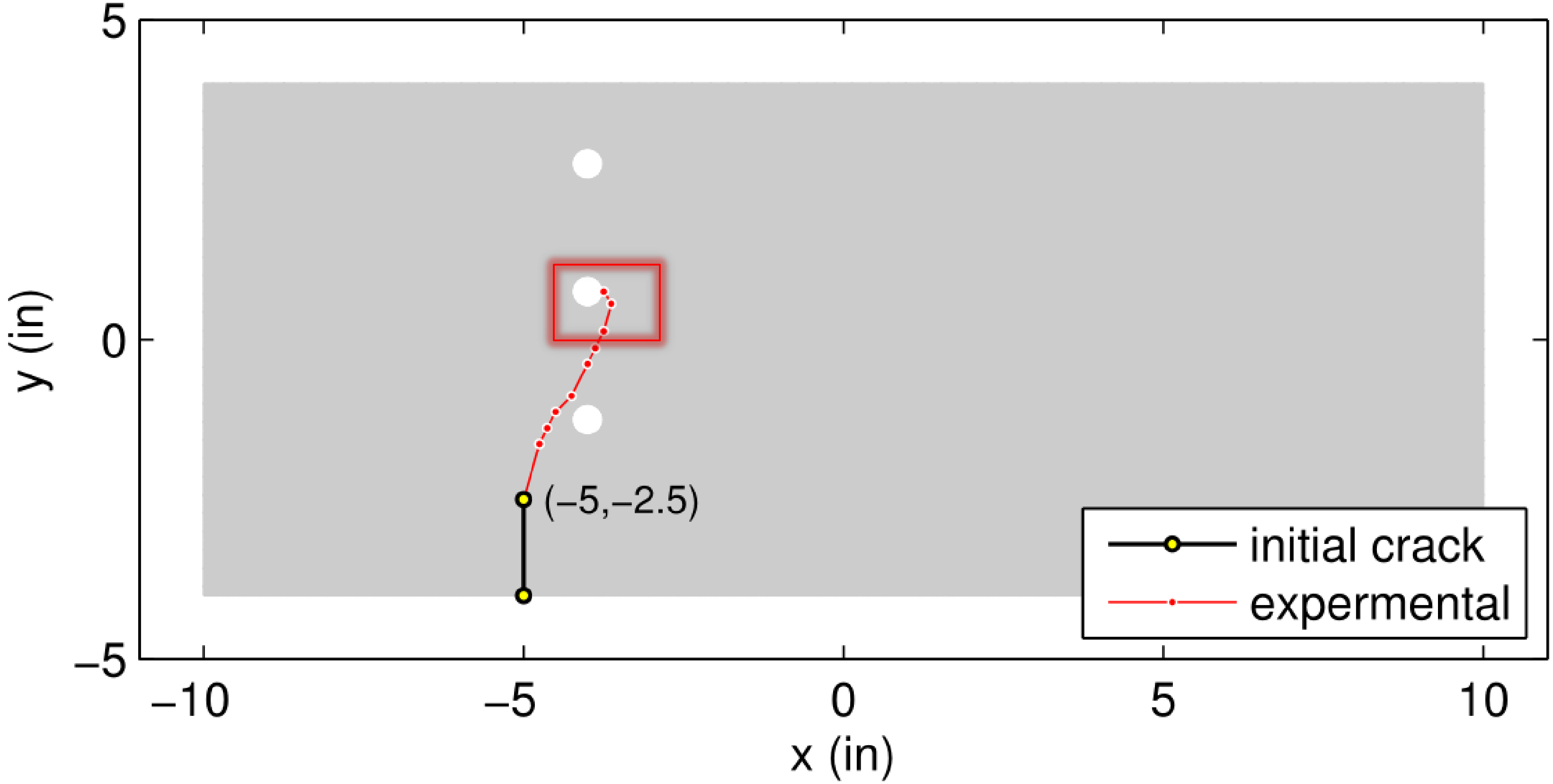

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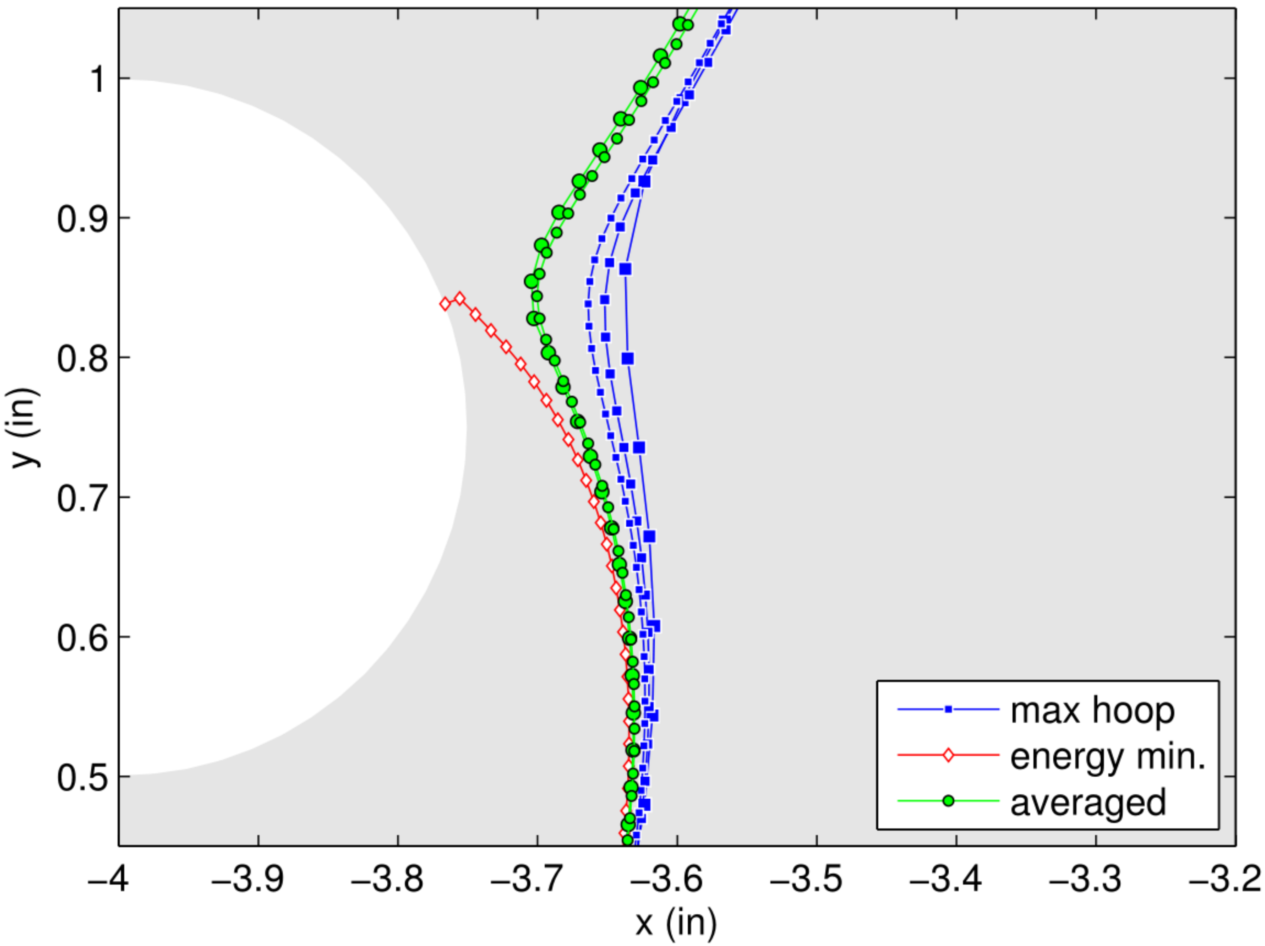
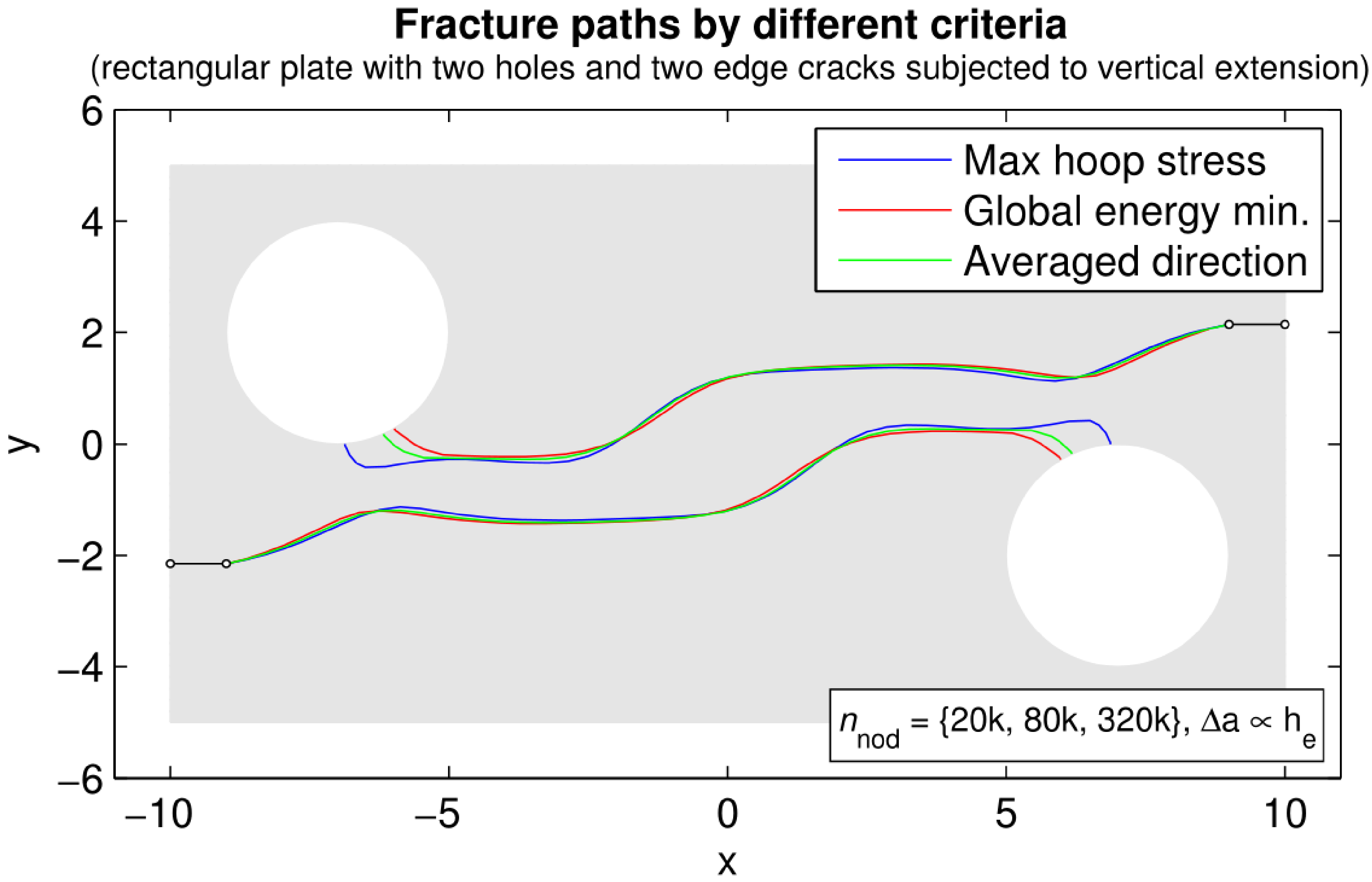


Results

2 edge cracks and 2 holes (Khoeil et al. 2008)



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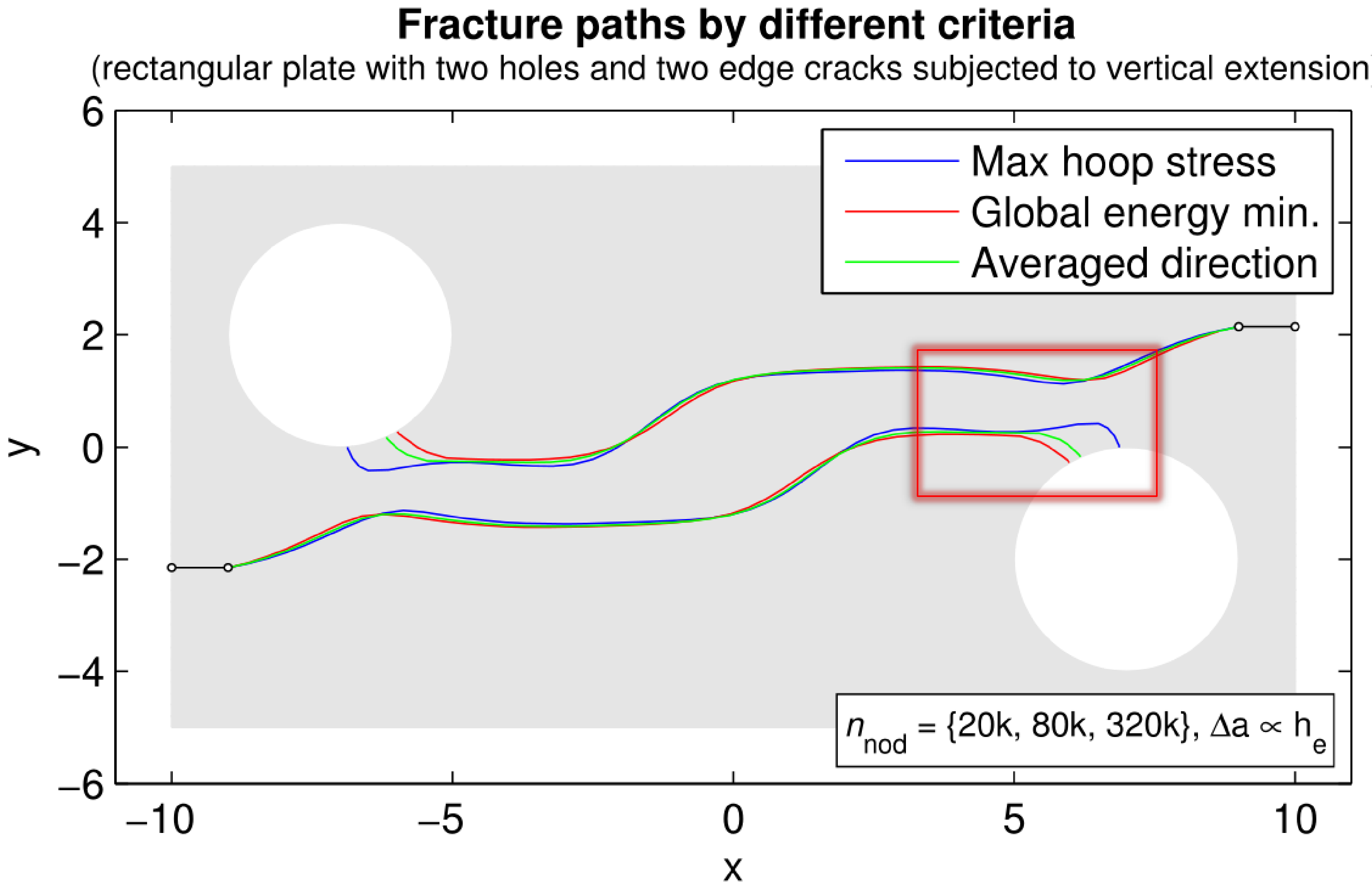

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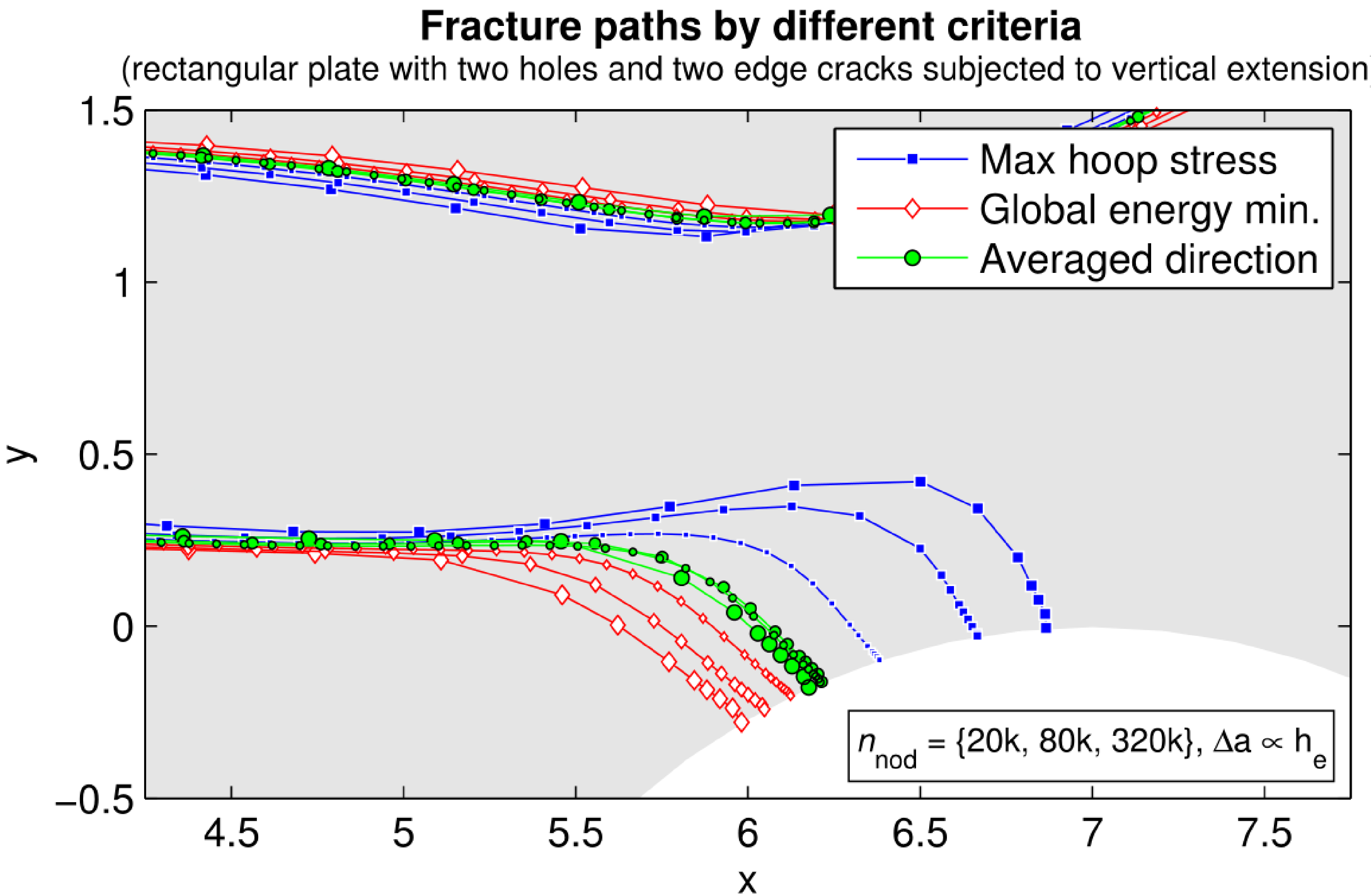

Summary

- A robust approach to determining multiple crack growth based on the principle of minimum energy within XFEM;
- Limitations undermining the max. hoop-stress criterion are overcome, e.g. assumptions about geometry and loading;
- The energy minimization approach is characterized by mode-I field dominance at the crack tip (post-increment);
- Both criteria lead to fracture paths solutions that are in close agreement (strong correlation with local symmetry, i.e. K_{II}=0);
- Better accuracy and faster convergence of fracture path solutions can be obtained by taking a bi-section of the interval that is bounded by the respective criteria.