

Energy minimizing multi-crack growth in linear elastic fracture using the extended finite element method

*Danas Sutula¹, Pierre Kerfriden², Stephane P. A. Bordas³

^{1,2}Cardiff University, School of Engineering, The Parade, Cardiff, CF24 3AA, Wales, UK

³University of Luxembourg, Engineering, 6 rue Richard Coudenhove-Kalergi, L-1359 Luxembourg

*sutulad@cardiff.ac.uk

ABSTRACT

We investigate multiple fracture evolution under quasi-static conditions in an isotropic linear elastic solid based on the principle of minimum potential elastic energy in the framework of the extended finite element method. The technique enables a minimization of the potential energy with respect to all crack increment directions. Results show that the maximum hoop stress criterion and the energy minimization approach converge to the same fracture path. It is found that the converged solution lies in between the fracture paths obtained by each criterion for coarser meshes. This presents an opportunity to estimate an upper and lower bound of the true fracture path as well as an error on the crack path.

Key Words: Energy minimisation ; linear elastic fracture ; crack growth criterion ; XFEM.

1. Introduction

In computational fracture mechanics as applied, for example, to damage tolerance assessment, it has been common practice to determine the onset of fracture growth and the growth direction by post-processing the solution of the linear elastostatics problem, at a particular instance in time. For mixed mode loading the available analytically derived criteria that can be used for determining the onset of crack growth typically rely on the assumptions of an idealized geometry e.g. a single crack subjected to remote loading [9, 5] and that the kink angle of the infinitesimal crack increment is small [7]. Moreover, the growth direction given by a criterion that is based on an instantaneous local crack tip field can only be valid for infinitesimally small crack growth increments. Consequently, the maximum hoop stress criterion [4] and other similar criteria [2] disregard the changes in the solution that take place as fractures advance over a finite size propagation. Hence, due to the error committed in time-integration, fractures may no longer follow the most energetically favorable paths that theoretically could be achieved for a specific discrete problem.

2. Method

In our approach, we investigate multiple fracture evolution under quasi-static conditions in an isotropic linear elastic solid based on the principle of minimum potential elastic energy, which can help circumvent the aforementioned difficulties. The technique enables a minimization of the potential energy with respect to all crack increment directions taking into consideration their relative interactions. The directions are optimized (in the energy sense) by considering virtual crack rotations to find the energy release rates and its first derivatives in order to determine, via an iterative process, the directions that yield zero energy release rates with respect to all virtual rotations [6]. We use the extended finite element method (XFEM) [1, 8] for discretization of a 2D continuum in order to model an elaborate crack evolution over time, similar in principle to [3], although here we would like to consider hundreds of propagating cracks.

3. Governing equations

The energy release rate with respect to a fracture growth direction θ_i can be obtained by differentiation of the potential energy Π of the system:

$$Gs_i = -\frac{\partial \Pi}{\partial \theta_i} \quad (1)$$

Considering a general case of multiple fractures, the rate of the energy release rate can be obtained as:

$$Hs_{i,j} = \frac{\partial Gs_i}{\partial \theta_j} = -\frac{\partial^2 \Pi}{\partial \theta_i \partial \theta_j} \quad (2)$$

In a discrete setting, the potential energy of a static system can be written as:

$$\Pi = \frac{1}{2} u' K u - u' f \quad (3)$$

where u , K , and f are the displacement vector, the stiffness matrix, and the applied force vector. The energy release rate with respect to an arbitrary crack incitement angle θ_i is defined as the negative variation of the potential energy:

$$Gs_i = -\frac{1}{2} u' \delta_i K u + u' \delta_i f - \delta_i u' (K u - f) \quad (4)$$

in which case the last term in (4) disappears due to assumed equilibrium of the discrete system i.e. $K u = f$. Hence, the expression for the energy release rate becomes:

$$Gs_i = -\frac{1}{2} u' \delta_i K u + u' \delta_i f \quad (5)$$

where $\delta_i f$ only needs to be accounted for if the applied loads influence the virtual crack rotation, e.g. due to crack face tractions and body-type loads. The rates of the energy release rate, Hs_{ij} are obtained by differentiating Gs_i in (5) with respect to θ_j :

$$Hs_{ij} = -\left(\frac{1}{2} u' \delta_{ij}^2 K u - u' \delta_{ij}^2 f\right) - \delta_j u' (\delta_i K u - \delta_i f) \quad (6)$$

The variations of displacements $\delta_j u$ in (6) are global, and can be determined from the equilibrium condition and that the variation must vanish, i.e. $\delta_j (K u - f) = 0$ and thus:

$$\delta u = -K^{-1} (\delta K u - \delta f) \quad (7)$$

Substituting (7) in (6) gives:

$$Hs_{ij} = -\left(\frac{1}{2} u' \delta_{ij}^2 K u - u' \delta_{ij}^2 f\right) + (\delta_j K u - \delta_j f)' K^{-1} (\delta_i K u - \delta_i f) \quad (8)$$

In (8) the second order mixed derivatives $\delta_{ij}^2 K$ and $\delta_{ij}^2 f$ capture the local interaction between the rotations of different crack increments. However, if the crack tips are sufficiently far apart such that no geometrical interactions exist between different rotations, then for $i \neq j$ the interacting terms vanish, i.e. $\delta_{ij}^2 K = 0$ and $\delta_{ij}^2 f = 0$. As such, it only becomes necessary to retain the non-zero self-interactions i.e. $\delta_{ii}^2 K$ and $\delta_{ii}^2 f$. Consequently, by leaving out the cross-interactions, equation (8) reads as:

$$Hs_{ij} = -\left(\frac{1}{2} u' \delta_{ii}^2 K u - u' \delta_{ii}^2 f\right) + (\delta_j K u - \delta_j f)' K^{-1} (\delta_i K u - \delta_i f) \quad (9)$$

Equations (5) and (9) can be used to determine the energy release rates and the rates of the energy release rates associated with the rotation of different crack increments. The problem of finding the most energetically favorable growth directions for the candidate finite length crack increments, denoted by a set I_{inc} , is one requiring that the corresponding energy release rates must vanish i.e. $G_{S_i} = 0, \forall i \in I_{\text{inc}}$. The solution procedure at every time step, t^{n+1} can be cast as Newton-Raphson iterations:

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k - \mathbf{H}_k^{-1} \mathbf{G} \mathbf{s}_k \quad (10)$$

where k is the iteration count. The converged solution is attained when $|\boldsymbol{\theta}_{k+1} - \boldsymbol{\theta}_k| \leq \epsilon$, ϵ being the tolerance in the change in the angle of the finite crack increment, e.g. $\epsilon = 0.1^\circ$.

4. Implementation

Although XFEM facilitates mesh independent fracture propagation the enrichment must be updated at each time step. In the current implementation this is achieved by means of a systematic book-keeping of the element enrichment data, addition and removal of enrichment only where necessary, and a consistent updating of the global system of equations. Consequently, moderate computational times are obtained, even in our *Matlab* implementation. In the problems we solve, the greatest cost, by far, is in the solution of the linear system of equations rather than in the assembly/updating.

5. Results and discussion

We compare the fracture paths obtained by different criteria for problems consisting of multiple cracks and verify that, with mesh refinement, both criteria converge to the same fracture path provided the criterion for growth is the same. However, the convergence rate of the energy minimization technique to the converged crack path is found to be only marginally superior to that of the maximum hoop stress criterion. It is found that the converged fracture path lies in between the fracture paths obtained by each criterion for coarser meshes. This presents an opportunity to estimate an upper and lower bound of the true fracture path as well as an error on the crack path. It is found that a more accurate approximation of the fracture path for coarser meshes can be obtained by averaging the directions determined by each criteria individually at every time step. Some results are demonstrated in Appendix A. Although there is no limitation on the number of cracks in the implementation, the example cases presented consider only few cracks as it is sufficient to demonstrate the key idea clearly.

6. Conclusions

Convergence of the maximum hoop stress criterion and the energy minimization towards the true fracture path is found to be similar. However, from numerical experiments it is found that the converged fracture path lies in between the fracture paths obtained by each criterion for coarser meshes. Besides the opportunity to estimate the error on the fracture path for a given mesh, a more accurate approximation of the true fracture path can be obtained by taking the average of the propagation directions given by each criterion separately at every time step.

References

- [1] T. Belytschko and T. Black. Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical*, 620(July 1998):601–620, 1999.
- [2] P. O. Bouchard, F. Bay, and Y. Chastel. Numerical modelling of crack propagation: automatic remeshing and comparison of different criteria. *Computer Methods in Applied Mechanics and Engineering*, 192(35-36):3887–3908, August 2003.
- [3] E. Budyn, G. Zi, N. Moës, and T. Belytschko. A method for multiple crack growth in brittle materials without remeshing. *International Journal for Numerical Methods in Engineering*, 61(10):1741–1770, November 2004.

- [4] F. Erdogan and G. C. Sih. On the crack extension in plates under plane loading and transverse shear. *Journal of basic engineering*, 85:519, 1963.
- [5] K. Hayashi and S. Nemat-Nasser. Energy-release rate and crack kinking under combined loading. *Journal of Applied Mechanics*, (September 1981):520–524, 1981.
- [6] M. A. Hussain, S. L. Pu, and J. Underwood. Strain Energy Release Rate for a Crack Under Combined Mode I and Mode. *Fracture analysis*, 560:1, 1974.
- [7] B. L. Karihaloo, L. M. Keer, and S. Nemat-Nasser. Crack kinking under nonsymmetric loading. *Engineering Fracture Mechanics*, 13(4):879–888, 1980.
- [8] N. Moës, J. Dolbow, and T. Belytschko. A finite element method for crack growth without remeshing. 150(February):131–150, 1999.
- [9] R. J. Nuismer. An energy release rate criterion for mixed mode fracture. *International journal of fracture*, 11(2):245–250, 1975.

Appendix A. Figures

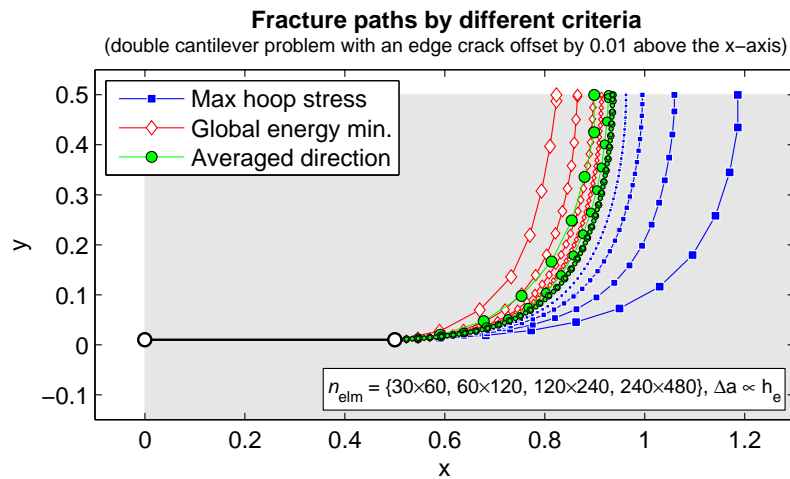


Figure A.1: Fracture paths considering different growth criteria for the double cantilever problem with the initial crack positioned 0.01 above the x-axis. The prying action is exerted by prescribed displacements on the left edge.

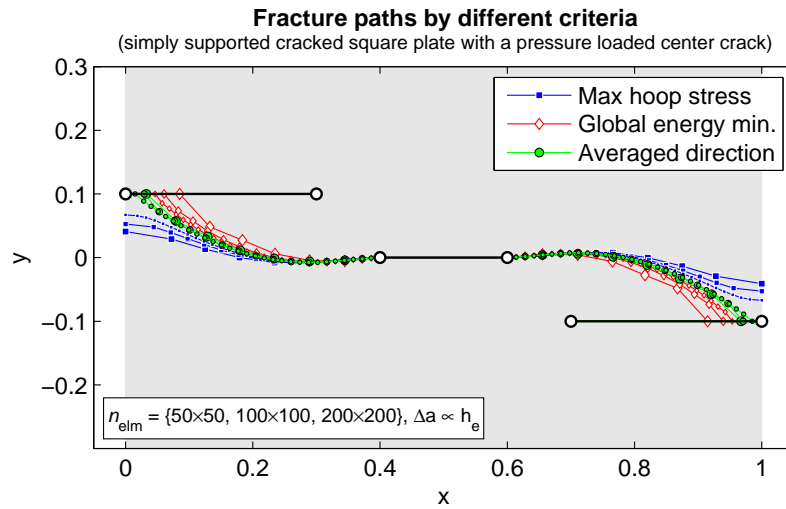


Figure A.2: Fracture paths considering different growth criteria for a simply supported square plate with three pre-existing cracks, where the center crack is subjected to a pressure load acting normal to the crack surface.

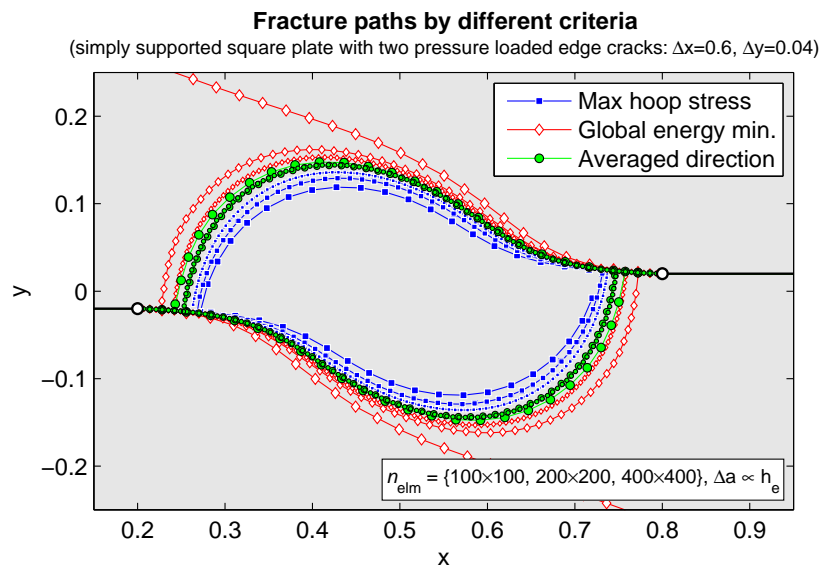


Figure A.3: Fracture paths considering different growth criteria for a simply supported square plate with two initial edge cracks that are loaded by pressure acting normal to the crack surface.