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Harnack estimates for a nonlinear parabolic equation under Ricci flow ☆

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ABSTRACT

In this paper, we consider the Harnack estimates for a nonlinear parabolic equation under the Ricci flow. The gradient estimates for positive solutions as well as Li–Yau type inequalities are also given.

1. Introduction

The nonlinear parabolic equation is a classical subject that has been extensively studied, which leads to lots of important results especially in researches of differential geometry. One of the important technique in studying the heat equation is the differential Harnack inequality developed by Li and Yau [7]. This is also applied to Ricci flow by Hamilton [6], and plays an important role in solving the Poincaré conjecture [9].

Now we will study the case where M is a complete manifold without boundary. Consider positive solutions of a nonlinear parabolic equation on the manifold M, which evolves under the Ricci flow. A series of gradient estimates are obtained for such solutions, including several Li–Yau-type inequalities. Let $(M, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

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$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)}, \quad t \in [0,T]. \tag{1.1}$$

We assume that its Ricci curvature remains uniformly bounded for all $t \in [0,T]$. Consider a positive function $u = u(x,t)$ defined on $M \times [0,T]$ solving the equation

$$\left(\Delta_{g(t)} - q - \partial_t\right) u = au(\ln u)^{\alpha}, \quad t \in [0,T], \tag{1.2}$$

which has been first studied in [14] where $g(t) \equiv g$ is a fixed metric. Qian [10] and Wu [13] got a series of similar conclusions. Here $\Delta_{g(t)}$ stands for the Laplacian of $g(x,t)$ defined on $M \times [0,T]$ and $q(x,t)$ is a C^2 function defined on $M \times [0,T]$. Notice that the Laplacian $\Delta_{g(t)}$ depends on the parameter t, and we should study the nonlinear parabolic equation (1.2) coupled with the Ricci flow (1.1). The formula (1.1) provides us with additional information about the coefficients of the operator Δ appearing in (1.2) but is itself fully independent of (1.2).

2. Gradient estimates I: $\alpha = 1$

Firstly, we introduce a cut-off function (see [1,3,7,8,14]) on $B_{\rho,T} := \{(\chi,t) \in M \times [0,T] : \mathrm{dist}_{g(t)}(\chi,x_0) < \rho\}$, where $\mathrm{dist}_{g(t)}(\chi,x_0)$ stands for the distance between χ and x_0 with respect to the metric $g(t)$, which satisfies a basic analytical result stated in the following lemma.

Lemma 2.1. *Given $\tau \in (0,T]$, there exists a smooth function $\overline{\Psi} : [0,\infty) \times [0,T] \to R$ satisfying the following requirements:*

(1) *The support of $\overline{\Psi}(r,t)$ is a subset of $[0,\rho] \times [0,T]$, $0 \le \overline{\Psi}(r,t) \le 1$ in $[0,\rho] \times [0,T]$, and $\overline{\Psi}(r,t) = 1$ holds in $[0, \frac{\rho}{2}] \times [\tau,T]$.*
(2) *$\overline{\Psi}$ is decreasing as a radial function in the spatial variables.*
(3) *The estimate $|\partial_t \overline{\Psi}| \le \frac{\overline{C}}{\tau} \overline{\Psi}^{1/2}$ is satisfied on $[0,\infty) \times [0,T]$ for some $\overline{C} > 0$.*
(4) *The inequalities $-\frac{C_\alpha}{\rho} \overline{\Psi}^{\alpha} \le \partial_r \overline{\Psi} \le 0$ and $|\partial_r^2 \overline{\Psi}| \le \frac{C_\alpha}{\rho^2} \overline{\Psi}^{\alpha}$ hold on $[0,\infty) \times [0,T]$ for every $a \in (0,1)$ with some constant C_α dependent on a.*

Proof. See [1]. □

These properties are derived from Calabi's argument (see, e.g., [2,4,11]). Using this auxiliary function and applying the maximum principle, we are able to establish Li–Yau-type inequality for the system (1.1)–(1.2).

Theorem 2.2. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M with $\sup_{B_{\rho,T}} |\mathrm{Ric}_{g(t)}|_{g(t)} \le K$ for some $K > 0$, and u is a smooth positive function on $M \times [0,T]$ satisfying the nonlinear parabolic equation (1.2) where the function $q(x,t)$ is defined on $M \times [0,T]$ which is C^2 in the x-variable and C^1 in the t-variable. If $u(x,t) \le 1$ on $B_{\rho,T}$, $\alpha = 1, |\nabla_{g(t)} q|_{g(t)} \le \gamma$, and*

$$b := \frac{1}{8} + \min_{M\times[0,T]} q - \max\{a, 0\} > 0,$$

then there exists a constant C that depends only on n such that

$$\frac{|\nabla_{g(t)} u|_{g(t)}}{u} \le \frac{C}{b} \left(\frac{1}{\rho} + \frac{1}{\sqrt{t}} + \sqrt{K} + 1 + \sqrt{\gamma} + \sqrt{|a|}\right) \left(1 + \ln \frac{1}{u}\right) \tag{2.1}$$

on $B_{\rho/2,T}$ with $t \ne 0$.

When q is nonnegative and $a \leq 0$, the constant C/b can be a universal constant which means a constant depending only on the dimension n. The number $1/8$ in b is not essential, because in the following proof we shall see that we can replace $1/8$ by 1. To prove the above theorem, we need the following crucial lemma.

Lemma 2.3. *Let $(M, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M with $\sup_{B_{\rho,T}} |\mathrm{Ric}_{g(t)}|_{g(t)} \leq K$ for some $K > 0$, and u is a smooth positive function on $M\times[0,T]$ satisfying the nonlinear parabolic equation (1.2) with $\alpha = 1$, $a < 0$. We assume that $u \leq 1$ on $B_{\rho,T}$. If $f := \ln u$ and $w := |\nabla_{g(t)} \ln(1-f)|^2_{g(t)} = |\nabla_{g(t)} f|^2_{g(t)}/(1-f)^2$, then the inequality*

$$\begin{aligned}(\Delta - \partial_t)\, w \geq{}& \frac{2f}{1-f}\langle\nabla f, \nabla w\rangle + 2(1-f)w^2 \\ &+\frac{2\langle\nabla f, \nabla q\rangle}{(1-f)^2} + 2a\frac{|\nabla f|^2}{(1-f)^2} + \frac{2|\nabla f|^2(q+af)}{(1-f)^3}\end{aligned} \tag{2.2}$$

holds on $B_{\rho,T}$.

Proof. Since u is a positive solution to the nonlinear parabolic equation (1.2) with $\alpha = 1$ and $a < 0$, direct calculation shows that

$$\Delta f + |\nabla f|^2 - f_t - q - af = 0, \quad f_t := \partial_t f.$$

The partial derivative of w with respect to t is given by

$$\begin{aligned}w_t ={}& \frac{2\langle\nabla f, \nabla f_t\rangle}{(1-f)^2} + \frac{2|\nabla f|^2 f_t}{(1-f)^3} + \frac{2\mathrm{Ric}(\nabla f, \nabla f)}{(1-f)^2} \\ ={}& \frac{2\langle\nabla f, \nabla(\Delta f + |\nabla f|^2 - q - af)\rangle}{(1-f)^2} + \frac{2\mathrm{Ric}(\nabla f, \nabla f)}{(1-f)^2} \\ &+ \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2 - q - af)}{(1-f)^3}.\end{aligned}$$

Using Bochner's identity $\langle\nabla f, \nabla\Delta f\rangle = \langle\nabla f, \Delta\nabla f\rangle - \mathrm{Ric}(\nabla f, \nabla f)$ we obtain

$$w_t = \frac{2\langle\nabla f, \Delta\nabla f\rangle + 2\langle\nabla f, \nabla(|\nabla f|^2 - q - af)\rangle}{(1-f)^2} + \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2 - q - af)}{(1-f)^3}.$$

The partial derivative of w with respect to x is given by

$$\Delta w = 2\frac{|\nabla^2 f|^2}{(1-f)^2} + 2\frac{\langle\nabla f, \Delta\nabla f\rangle}{(1-f)^2} + 4\frac{\langle\nabla f, \nabla|\nabla f|^2\rangle}{(1-f)^3} + 6\frac{|\nabla f|^4}{(1-f)^4} + 2\frac{|\nabla f|^2\Delta_{g(t)} f}{(1-f)^3}.$$

Combining those partial derivatives imply

$$\begin{aligned}(\Delta - \partial_t)\, w ={}& 2\frac{|\nabla^2 f|^2}{(1-f)^2} + 4\frac{\langle\nabla f, \nabla|\nabla f|^2\rangle}{(1-f)^3} + 6\frac{|\nabla f|^4}{(1-f)^4} - 2\frac{\langle\nabla f, \nabla|\nabla f|^2\rangle}{(1-f)^2} \\ &- 2\frac{|\nabla f|^4}{(1-f)^3} + 2a\frac{|\nabla f|^2}{(1-f)^2} + 2\frac{\langle\nabla f, \nabla q\rangle}{(1-f)^2} + 2\frac{q|\nabla f|^2}{(1-f)^3} + 2a\frac{f|\nabla f|^2}{(1-f)^3}.\end{aligned}$$

On the other hand, the gradient term $\langle\nabla f, \nabla w\rangle$ is determined by

$$\langle\nabla f, \nabla w\rangle = \frac{\langle\nabla f, \nabla|\nabla f|^2\rangle_{g(t)}}{(1-f)^2} + 2\frac{|\nabla f|^4}{(1-f)^3}.$$

Plugging it into the evolution of $(\Delta-\partial_t)w$ we conclude that

$$\begin{aligned}(\Delta-\partial_t)\,w &= 2\frac{|\nabla^2 f|^2}{(1-f)^2}+2\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{(1-f)^3}+\frac{2}{1-f}\langle\nabla f,\nabla w\rangle+2\frac{|\nabla f|^4}{(1-f)^4}\\ &\quad-2\langle\nabla f,\nabla w\rangle+2\frac{|\nabla f|^4}{(1-f)^3}+2a\frac{|\nabla f|^2}{(1-f)^2}\\ &\quad+2\frac{\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2\frac{q|\nabla f|^2}{(1-f)^3}+2a\frac{f|\nabla f|^2}{(1-f)^3}.\end{aligned}$$

Because of the identity

$$\frac{|\nabla^2 f|^2}{(1-f)^2}+\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{(1-f)^3}+\frac{|\nabla f|^4}{(1-f)^4}=\left|\frac{\nabla^2 f}{1-f}+\frac{\nabla f\otimes\nabla f}{(1-f)^2}\right|^2$$

we therefore arrive at

$$\begin{aligned}(\Delta-\partial_t)\,w &= 2\left|\frac{\nabla^2 f}{1-f}+\frac{\nabla f\otimes\nabla f}{(1-f)^2}\right|^2+\frac{2}{1-f}\langle\nabla f,\nabla w\rangle+2\frac{|\nabla f|^4}{(1-f)^3}-2\langle\nabla f,\nabla w\rangle\\ &\quad+2\frac{\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2a\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^2(q+af)}{(1-f)^3}\end{aligned}$$

which immediately implies

$$(\Delta-\partial_t)\,w\geq\frac{2f}{1-f}\langle\nabla f,\nabla w\rangle+\frac{2|\nabla f|^4}{(1-f)^3}+\frac{2\langle\nabla f,\nabla q\rangle}{(1-f)^2}+\frac{2a|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^2(q+af)}{(1-f)^3}.$$

This complete the proof. $\square$

Proof of Theorem 2.2. Pick a number $\tau\in(0,T]$ and fix a function $\overline{\Psi}(x,t)$ satisfying the conditions of Lemma 2.1. We will establish (2.1) at (x,τ) for all x such that $\mathrm{dist}_{g(\tau)}(x,x_0)<\frac{\rho}{2}$. Define $\Psi: M\times[0,T]\to\mathbf{R}$ by setting

$$\Psi(x,t):=\overline{\Psi}\left(\mathrm{dist}_{g(t)}(x,x_0),t\right).$$

Then, using (2.2) and a straightforward calculation, one has

$$\begin{aligned}(\Delta-\partial_t)\,(\Psi w) &\geq \Psi\langle-\Lambda,\nabla w\rangle+\left[2(1-f)w^2+\frac{2\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2a\frac{|\nabla f|^2}{(1-f)^2}\right.\\ &\quad\left.+2\frac{|\nabla f|^2(q+af)}{(1-f)^3}\right]\Psi+2\langle\nabla w,\nabla\Psi\rangle+w\Delta\Psi-w\partial_t\Psi\\ &\geq\langle-\Lambda,\nabla(\Psi w)\rangle+2\Psi(1-f)w^2+w\langle\Lambda,\nabla\Psi\rangle\\ &\quad+w\Delta\Psi-w\Psi_t+\frac{2}{\Psi}\langle\nabla\Psi,\nabla(\Psi w)\rangle-2\frac{|\nabla\Psi|^2}{\Psi}w\\ &\quad+\left[\frac{2\,\langle\nabla f,\nabla q\rangle}{(1-f)^2}+2a\frac{|\nabla f|^2}{(1-f)^2}+\frac{2|\nabla f|^2(q+af)}{(1-f)^3}\right]\Psi\end{aligned}$$

where $\Lambda=-\frac{2f}{1-f}\nabla f$. By our assumption that $|\mathrm{Ric}|\leq K$ on $B_{\rho,T}$ and Lemma 2.1 that $-\frac{C_1}{\rho}\overline{\Psi}^{1/2}\leq\overline{\Psi}_r\leq 0$, and the identity

$$-w\Psi_t = -\left[\overline{\Psi}_t + \overline{\Psi}_r \partial_t \mathrm{dist}_{g(t)}(\cdot, x_0)\right] w,$$

we have (because $-\partial_t \mathrm{dist}_{g(t)}(\cdot, x_0) \leq 4\sqrt{(m-1)K}$, cf. Lemma 8.33 in [5])

$$-w\Psi_t \geq -\overline{\Psi}_t w - \frac{4C_1\sqrt{(m-1)K}}{\rho} w \overline{\Psi}^{1/2}.$$

Suppose that Ψw achieves its maximum at (x_0, t_0). By [7], without loss of generality, we may assume that x_0 is not in the cut-locus of M. At the point (x_0, t_0), one has $\Delta(\Psi w) \leq 0$, $\nabla(\Psi w) = 0$, $(\Psi w)_t \geq 0$. Therefore

$$\begin{aligned} 2\Psi(1-f)w^2 \leq & -w\langle \Lambda, \nabla\Psi\rangle + 2\frac{|\nabla\Psi|^2}{\Psi} w - w\Delta\Psi + w\Psi_t \\ & - \left[\frac{2\langle \nabla f, \nabla q\rangle}{(1-f)^2} + 2a\frac{|\nabla f|^2}{(1-f)^2} + \frac{2|\nabla f|^2(q+af)}{(1-f)^3}\right]\Psi \end{aligned} \tag{2.3}$$

at (x_0, t_0). We need to bound each term on the right-hand side of (2.3):

$$|w\langle \Lambda, \nabla\Psi\rangle| \leq 2\frac{w|f|}{1-f}|\nabla f||\nabla\Psi| = 2w^{3/2}|f||\nabla\Psi| \leq \Psi(1-f)w^2 + \frac{27}{16}\frac{|f|^4|\nabla\Psi|^4}{[\Psi(1-f)]^3}$$

where we used the Young's inequality that $ab \leq \epsilon a^p + b^q/(q(p\epsilon)^{q/p})$ for any $a, b, \epsilon > 0$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$. This together with Lemma 2.1 implies

$$|w\langle \Lambda, \nabla\Psi\rangle| \leq \Psi(1-f)w^2 + C_2\frac{f^4}{\rho^4(1-f)^3}. \tag{2.4}$$

Using again Lemma 2.1 we have

$$\frac{|\nabla\Psi|^2}{\Psi} w = \Psi^{1/2} w \frac{|\nabla\Psi|^2}{\Psi^{3/2}} \leq \frac{1}{8}\Psi w^2 + 2\frac{|\nabla\Psi|^4}{\Psi^3} \leq \frac{1}{8}\Psi w^2 + \frac{C_3}{\rho^4}. \tag{2.5}$$

Furthermore, by the properties of Ψ and the assumption of the Ricci curvature, one has (cf., [12,15])

$$-w\Delta\Psi \leq \frac{1}{8}\Psi w^2 + \frac{C_4}{\rho^4} + C_4 K^2. \tag{2.6}$$

The estimation for $w\Psi_t$ is given by (cf. [1])

$$|w\Psi_t| \leq \frac{1}{8}\Psi w^2 + \frac{C_5}{\tau^2} + C_5 K^2. \tag{2.7}$$

Since $f \leq 0$ it follows that

$$\begin{aligned} \left|\frac{2\langle \nabla f, \nabla q\rangle}{(1-f)^2}\Psi\right| \leq \frac{|\nabla f|^2 + |\nabla q|^2}{(1-f)^2}\Psi \quad & \leq \quad w\Psi + \gamma^2\Psi \\ & \leq \frac{1}{8}\Psi w^2 + (2+\gamma^2)\Psi, \end{aligned} \tag{2.8}$$

$$\left|2a\frac{|\nabla f|^2}{(1-f)^2}\Psi\right| = 2aw\Psi \quad \leq \quad \frac{1}{8}\Psi w^2 + 2a^2\Psi \tag{2.9}$$

and

$$
\begin{aligned}
-2\frac{|\nabla f|^2(q+af)}{(1-f)^3}\Psi &= 2\Psi w^2\left(\frac{-q}{1-f}+a\frac{-f}{1-f}\right)\\
&\le 2\Psi w^2\left(-\min_{M\times[0,T]} q + a\frac{-f}{1-f}\right)\\
&\le 2\left(\max\{a,0\}-\min_{M\times[0,T]} q\right)\Psi w^2.
\end{aligned}
\tag{2.10}
$$

Substituting (2.5)–(2.10) to the right-hand side of (2.3), we deduce that

$$
\begin{aligned}
\Psi(1-f)w^2 \le{}& C_6\frac{f^4}{\rho^4(1-f)^3}+\frac{3}{4}\Psi w^2+2\left(\max\{a,0\}-\min_{M\times[0,T]} q\right)\Psi w^2\\
&+\frac{C_6}{\tau^2}+\frac{C_6}{\rho^4}+C_6K^2+\left(2+\gamma^2+2a^2\right)
\end{aligned}
$$

at (x_0,t_0). Since $f<0$, then $f^4/(1-f)^4\le 1$. It follows that

$$
\left(\frac{1}{4}+2\min_{M\times[0,T]} q-2\max\{a,0\}\right)\Psi w^2\le\frac{2C_6}{\rho^4}+\frac{C_6}{\tau^2}+C_6K^2+\left(2+\gamma^2+2a^2\right)
$$

at (x_0,t_0). When

$$
\frac{1}{8}+\min_{M\times[0,T]} q-\max\{a,0\}>0 \tag{2.11}
$$

we can conclude that

$$
\Psi w^2\le\frac{C_7}{\frac{1}{8}+\min_{M\times[0,T]} q-\max\{a,0\}}\left(\frac{1}{\rho^4}+\frac{1}{\tau^2}+K^2+1+\gamma^2+a^2\right)
$$

at (x_0,t_0). Because $\Psi(x,\tau)=1$ when $\mathrm{dist}_{g(\tau)}(x,x_0)<\rho/2$, we finally arrive at

$$
w^2(x,\tau)\le\frac{C_7}{\frac{1}{8}+\min_{M\times[0,T]} q-\max\{a,0\}}\left(\frac{1}{\rho^4}+\frac{1}{\tau^2}+K^2+1+\gamma^2+a^2\right)
$$

on $B_{\rho/2,T}$, which, since $\tau\in(0,T]$ was arbitrary, implies

$$
\frac{|\nabla f|}{1-f}\le\frac{C_8}{\frac{1}{8}+\min_{M\times[0,T]} q-\max\{a,0\}}\left(\frac{1}{\rho}+\frac{1}{\sqrt{t}}+\sqrt{K}+1+\sqrt{\gamma}+\sqrt{|a|}\right).
$$

We have completed the proof of Theorem 2.2.

The number $1/8$ in (2.11) is not essential, because in the above argument we can replace $1/8$ in (2.5)–(2.9) by a given positive number ϵ, and hence we need only to require that

$$
1-5\epsilon+2\min_{M\times[0,T]} q-2\max\{a,0\}>0
$$

instead of (2.11). When

$$
1+2\min_{M\times[0,T]} q-2\max\{a,0\}
$$

is positive, we choose ϵ to be this number divided by 5.

3. Gradient estimates II: general case

In this section we extend Theorem 2.2 with $\alpha = 1$ to the general case.

Lemma 3.1. *Suppose $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M, with $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some $K_1, K_2 > 0$ on $B_{\rho,T}$. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then, for given $\beta \ge 1$ and any $c, d > 0$ with $c + d = 1/\beta$, we have*

$$\begin{aligned}(\Delta - \partial_t) F \ge & -2\langle \nabla f, \nabla F\rangle - \frac{F}{t} + \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 \\ & - 2(\beta-1)t\langle \nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\ & - \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 - 2\beta t K_1|\nabla f|^2 - \frac{n\beta t}{2d}\overline{K}^2 \\ & - \beta a\alpha t f^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + af^\alpha\right) - \beta t\Delta q,\end{aligned} \tag{3.1}$$

where $\overline{K} := \max\{K_1, K_2\}$, $f := \ln u$, and $F := t(|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha)$.

Proof. The proof of this lemma was original from [1]. Now we will find a convenient bound on ΔF like the way in [16]. Notice that

$$\nabla_i F = t\left(2\nabla^j f\nabla_i\nabla_j f - \beta\nabla_i f_t - \beta\nabla_i q - \beta a\alpha f^{\alpha-1}\nabla_i f\right).$$

Then the Laplace of F equals

$$\begin{aligned}\Delta F &= \nabla^i\nabla_i F \\ &= t\Big[2|\nabla^2 f|^2 + 2\langle\nabla f, \Delta\nabla f\rangle - \beta\Delta f_t - \beta\Delta q \\ &\quad - \beta a\alpha\left((\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right)\Big].\end{aligned}$$

Using the Bochner's formula $\Delta\nabla f = \nabla\Delta f + \mathrm{Ric}(\nabla f, \cdot)$, we get

$$\begin{aligned}\Delta F &= t\Big[2|\nabla^2 f|^2 + 2\langle\nabla f, \nabla\Delta f\rangle + 2\mathrm{Ric}(\nabla f, \nabla f) - \beta\Delta f_t \\ &\quad - \beta\Delta q - \beta a\alpha\Big((\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\Big)\Big] \\ &\ge t\Big[\frac{2(\Delta f)^2}{n} + 2\langle\nabla f, \nabla\Delta f\rangle - 2K_1|\nabla f|^2 - \beta(\Delta f)_t - \beta\Delta q \\ &\quad - \beta a\alpha\Big((\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\Big) + 2\beta R_{ij}\nabla^i\nabla^j f\Big]\end{aligned}$$

since $|\nabla^2 f|^2 \ge \frac{1}{n}(\Delta f)^2$ and $\Delta f_t = (\Delta f)_t - 2R_{ij}\nabla^i\nabla^j f$. Recalling from the result

$$\Delta f = -|\nabla f|^2 + q + f_t + af^\alpha = -\frac{F}{t} - (\beta-1)\left(q + f_t + af^\alpha\right),$$

we arrive at

$$
\begin{aligned}
\Delta F \geq\ & \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 + \left(\frac{2d\beta t}{n}(\Delta f)^2 + 2t\beta R_{ij}\nabla^i\nabla^j f\right) \\
& - 2t\left\langle \nabla f, \nabla\left(\frac{F}{t} + (\beta-1)(q + f_t + af^\alpha)\right)\right\rangle \\
& - 2K_1 t|\nabla f|^2 - t\beta\left(-\frac{F}{t} - (\beta-1)(q + f_t + af^\alpha)\right)_t - \beta t\Delta q \\
& - \beta a\alpha t\left[(\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right]
\end{aligned}
\tag{3.2}
$$

in the set $B_{\rho,T}$. Because

$$
\begin{aligned}
\frac{2d\beta t}{n}(\Delta f)^2 + 2t\beta R_{ij}\nabla^i\nabla^j f &= \frac{2d\beta t}{n}\left[(\Delta f)^2 + \frac{n}{d}R_{ij}\nabla^i\nabla^j f\right] \\
&= \frac{2d\beta t}{n}\left|\nabla^2 f + \frac{n}{2d}\mathrm{Ric}\right|^2 - \frac{n\beta t}{2d}|\mathrm{Ric}|^2 \\
&\geq -\frac{n\beta t}{2d}|\mathrm{Ric}|^2,
\end{aligned}
$$

the inequality (3.2) can be written as

$$
\begin{aligned}
\Delta F \geq\ & \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 - \frac{n\beta t}{2d}|\mathrm{Ric}|^2 \\
& - 2t\left\langle \nabla f, \nabla\left(\frac{F}{t} + (\beta-1)(q + f_t + af^\alpha)\right)\right\rangle \\
& - 2K_1 t|\nabla f|^2 - t\beta\left(-\frac{F}{t} - (\beta-1)(q + f_t + af^\alpha)\right)_t - \beta t\Delta q \\
& - \beta a\alpha t\left[(\alpha-1)f^{\alpha-2}|\nabla f|^2 + f^{\alpha-1}\Delta f\right].
\end{aligned}
\tag{3.3}
$$

To get the time derivative of F, we shall use the identity

$$
F_t = \frac{F}{t} + t\Big(|\nabla f|^2 - \beta f_t - \beta q - a\beta f^\alpha\Big)_t.
$$

Subtracting this from (3.3), we get

$$
\begin{aligned}
(\Delta - \partial_t)F \geq\ & -2\langle\nabla f, \nabla F\rangle - \frac{F}{t} + \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 \\
& - \beta t\Delta q - 2(\beta-1)t\langle\nabla f, \nabla q\rangle - \frac{n\beta t}{2d}|\mathrm{Ric}|^2 \\
& - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 - \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 \\
& - \beta a\alpha t f^{\alpha-1}\Big(-|\nabla f|^2 + f_t + q + af^\alpha\Big) - 2\beta K_1 t|\nabla f|^2.
\end{aligned}
$$

Now the inequality (3.1) follows immediately by noting that $|\mathrm{Ric}_{g(t)}|^2_{g(t)} \leq \overline{K}^2$. $\square$

Now we can consider the local space–time gradient estimate with Lemma 2.3. In the following part, n is the dimension of M.

Theorem 3.2. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M with $-K_1 g(t) \leq \mathrm{Ric}_{g(t)} \leq K_2 g(t)$ for some $K_1, K_2 > 0$ on $B_{\rho,T}$. If u is a smooth positive* *function satisfying the nonlinear parabolic equation (1.2), then there exists a constant C depending only on n such that, on $B_{\rho/2,T}$,*

(1) *for $a \geq 0$, we have*

$$\begin{aligned}|\nabla_{g(t)}f|^2_{g(t)} - \beta f_t - \beta q - \beta a f^\alpha \leq{}& \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3C^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2}\\ &+ \frac{n\beta[\beta K_1 + a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)}\\ &+ \frac{n\beta^2 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2c(\beta-1)(1-\epsilon)}\\ &+ \sqrt{\frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}},\end{aligned}$$

(2) *for $a \leq 0$, we have*

$$\begin{aligned}|\nabla_{g(t)}f|^2_{g(t)} - \beta f_t - \beta q - \beta a f^\alpha \leq{}& \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2(1-c\epsilon)} + \frac{n^2\beta^3C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2}\\ &+ \frac{n\beta[\beta K_1 - \frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)}\\ &+ \sqrt{\frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}.\end{aligned}$$

Here $f := \ln u$, $|f|_\infty := \max_M |f|$, $\overline{K} := \max\{K_1, K_2\}$, $\beta > 1$, $0 < \epsilon < 1$, $|\nabla_{g(t)}q|_{g(t)} \leq \gamma$, $\Delta_{g(t)}q \leq \theta$,

$$A = C\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \frac{1}{t} + \overline{K}\right)$$

and $c, d > 0$ with $c + d = 1/\beta$.

Proof. We will use the same notation $f = \ln u$ and $F = t(|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha)$ as in Lemma 3.1. For the fixed $\tau \in (0, T]$, chose the cut-off function $\overline{\Psi}$ constructed in Lemma 2.1. Define $\Psi : M \times [0, T] \to \mathbf{R}$ by setting

$$\Psi(x, t) = \overline{\Psi}\left(\mathrm{dist}_{g(t)}(x, x_0), t\right).$$

Lemma 3.1 implies that

$$\begin{aligned}(\Delta - \partial_t)(\Psi F) \geq{}& -2\langle\nabla f, \nabla(\Psi F)\rangle + 2F\langle\nabla f, \nabla\Psi\rangle\\ &+ \left[\frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 - \frac{F}{t} - \beta t\Delta q\right.\\ &- 2(\beta-1)t\langle\nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2\\ &- \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 - 2\beta tK_1|\nabla f|^2 - \frac{n\beta t}{2d}\overline{K}^2\\ &\left.- \beta a\alpha tf^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + af^\alpha\right)\right]\Psi\\ &+ F\Delta\Psi + \frac{2}{\Psi}\langle\nabla\Psi, \nabla(\Psi F)\rangle - 2\frac{|\nabla\Psi|^2}{\Psi}F - F\frac{\partial\Psi}{\partial t}.\end{aligned}$$

Let (x_0, t_0) be a maximum point for the function ΨF in the set $\{(x,t)|0 \le t \le \tau,\ d_{g(t)}(x, x_0) \le \rho\}$. Then at the point (x_0, t_0) we have

$$\begin{aligned}
0 \ge\, & 2F\langle\nabla f, \nabla\Psi\rangle + F\left(\Delta - \partial_t\right)\Psi - 2\frac{|\nabla\Psi|^2}{\Psi}F \\
& + \left[-\frac{F}{t} + \frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 - \beta t\Delta q \right.\\
& - 2(\beta-1)t\langle\nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\
& - \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 - \beta a\alpha t f^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + af^\alpha\right) \\
& \left. - 2\beta tK_1|\nabla f|^2 - \frac{n\beta t}{2d}\overline{K}^2\right]\Psi.
\end{aligned}$$

By Lemma 2.1 and the Laplacian comparison theorem, we have

$$\begin{aligned}
\frac{|\nabla\Psi|^2}{\Psi} &\le \frac{C_{1/2}^2}{\rho^2}, \\
\Delta\Psi &\ge -\frac{C_{1/2}\Psi^{1/2}}{\rho^2} - \frac{C_{1/2}\Psi^{1/2}}{\rho}(n-1)\sqrt{K_1}\coth(\sqrt{k_1}\rho) \\
&\ge -\frac{d_1}{\rho^2} - \frac{d_1\Psi^{1/2}}{\rho}\sqrt{K_1}, \\
-\partial_t\Psi &\ge -\frac{\overline{C}\Psi^{1/2}}{\tau} - C_{1/2}\overline{K}\Psi^{1/2}
\end{aligned}$$

where $C_{1/2}, \overline{C}$ and d_1 are positive constants depending only on n. Plugging those estimates into above inequality yields that

$$\begin{aligned}
0 \ge\, & d_2\left(-\frac{1}{\rho^2} - \frac{\Psi^{1/2}}{\rho}\sqrt{K_1} - \frac{\Psi^{1/2}}{\tau} - \overline{K}\Psi^{1/2}\right)F - 2F|\nabla f||\nabla\Psi| \\
& + \left[\frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2 - \frac{F}{t} - \beta t\Delta q - \frac{n\beta t}{2d}\overline{K}^2\right. \\
& - 2(\beta-1)t\langle\nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 - 2\beta tk_1|\nabla f|^2 \\
& \left. - \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 - \beta a\alpha tf^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + af^\alpha\right)\right]\Psi
\end{aligned} \tag{3.4}$$

at (x_0, t_0), where d_2 is equal to $\max\{d_1, \overline{C}, C_{1/2}, 2C_{1/2}^2\}$. Introduce a function

$$A := d_2\left(\frac{1}{\rho^2} + \frac{\Psi^{1/2}}{\rho}\sqrt{K_1} + \frac{\Psi^{1/2}}{\tau} + \overline{K}\Psi^{1/2}\right).$$

If one multiplies by $t\Psi$ and makes a few elementary manipulations, one will obtain

$$\begin{aligned}
0 \ge\, & -2F|\nabla f||\nabla\Psi|\Psi t - AF\Psi t + \left[\frac{2c\beta t}{n}\left(|\nabla f|^2 - q - f_t - af^\alpha\right)^2\right. \\
& - \beta t\Delta q - 2(\beta-1)t\langle\nabla f, \nabla q\rangle - 2(\beta-1)ta\alpha f^{\alpha-1}|\nabla f|^2 \\
& - \beta ta\alpha(\alpha-1)f^{\alpha-2}|\nabla f|^2 - \beta a\alpha tf^{\alpha-1}\left(-|\nabla f|^2 + f_t + q + af^\alpha\right) \\
& \left. - 2\beta tK_1|\nabla f|^2 - \frac{n\beta t}{2d}\overline{K}^2\right]\Psi^2 t - F\Psi^2
\end{aligned} \tag{3.5}$$

at (x_0, t_0). As in [3,14], we set

$$\mu := \frac{|\nabla f|^2(x_0,t_0)}{F(x_0,t_0)} \geq 0.$$

Because $|\nabla f| = \mu^{1/2}F^{1/2}$ and

$$\begin{aligned} |\nabla f|^2 - f_t - q - af^\alpha = F\left(\mu - \frac{\mu t - 1}{\beta t}\right), \\ \langle \nabla f, \nabla \Psi\rangle \leq |\nabla f||\nabla \Psi| \quad \leq \quad \frac{C_1}{\rho}\Psi^{1/2}|\nabla f| \end{aligned}$$

we can simplify (3.5) into the following inequality

$$\begin{aligned} AFt\Psi \geq & -\frac{2C_1 t}{\rho}\Psi^{3/2}\mu^{1/2}F^{3/2} - \Psi^2 F + \frac{2c\Psi^2}{n\beta}[1+(\beta-1)\mu t]^2F^2 - \frac{n\beta}{2d}\overline{K}^2(\Psi t)^2 \\ & - 2(t\Psi)^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu F + at\Psi^2\alpha f^{\alpha-1}[1+(\beta-1)t\mu]F \\ & - \beta(t\Psi)^2\theta - 2(\beta-1)(t\Psi)^2\gamma(\mu F)^{1/2} - \beta(t\Psi)^2 a\alpha(\alpha-1)f^{\alpha-2}\mu F \end{aligned} \tag{3.6}$$

at (x_0, t_0). If we set $G := \Psi F$, then at the point (x_0, t_0) the inequality (3.6) becomes

$$\begin{aligned} AtG \geq & -\frac{2C_1 t}{\rho}\mu^{1/2}G^{3/2} - \Psi G + \frac{2c}{n\beta}[1+(\beta-1)\mu t]^2G^2 - \frac{n\beta}{2d}\overline{K}^2(\Psi t)^2 \\ & - 2\Psi t^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu G + a\Psi t\alpha f^{\alpha-1}[1+(\beta-1)\mu t]G \\ & - \beta(\Psi t)^2\theta - 2(\beta-1)t^2\Psi^{3/2}\gamma\mu^{1/2}G^{1/2} - \beta t^2\Psi a\alpha(\alpha-1)f^{\alpha-2}\mu G \end{aligned} \tag{3.7}$$

at (x_0, t_0). According to the Cauchy inequality, where $0 < \epsilon < 1$,

$$\begin{aligned} \frac{2C_1 t}{R}\mu^{1/2}G^{3/2} &\leq \frac{2\epsilon c}{n\beta}[1+(\beta-1)\mu t]^2G^2 + \frac{n\beta C_1^2 t^2\mu G}{2\epsilon c\rho^2[1+(\beta-1)\mu t]^2}, \\ 2\mu^{1/2}G^{1/2} &\leq 1 + \mu G, \end{aligned}$$

we can simplify (3.7) as

$$\begin{aligned} AtG \geq & \frac{2c(1-\epsilon)}{n\beta}[1+(\beta-1)\mu t]^2G^2 - \Psi G - \frac{n\beta^2 C_1^2 t^2\mu}{2\epsilon c\rho^2[1+(\beta-1)\mu t]^2}G \\ & - 2\Psi t^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu G + a\Psi t\alpha f^{\alpha-1}[1+(\beta-1)\mu t]G \\ & - \beta\Psi^2 t^2\theta - (\beta-1)t^2\Psi^{3/2}\gamma - (\beta-1)t^2\Psi^{3/2}\gamma\mu G \\ & - \beta t^2\Psi a\alpha(\alpha-1)f^{\alpha-2}\mu G - \frac{n\beta}{2d}\overline{K}^2(\Psi t)^2, \end{aligned}$$

at (x_0, t_0), or equivalently,

$$\begin{aligned} \frac{2c(1-\epsilon)[1+(\beta-1)\mu t]^2G^2}{n\beta} \leq & \Bigg[\frac{n\beta^2 C_1^2 t^2\mu}{2\epsilon c\rho^2[1+(\beta-1)\mu t]^2} + \beta t^2\Psi a\alpha(\alpha-1)f^{\alpha-2}\mu \\ & - a\Psi t\alpha f^{\alpha-1}[1+(\beta-1)\mu t] + (\beta-1)t^2\Psi^{3/2}\gamma\mu \end{aligned}$$

$$
\left. + 2\Psi t^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu + At + \Psi \right] G
$$

$$
+ \left[\beta\Psi^2\theta + (\beta-1)\Psi^{3/2}\gamma + \frac{n\beta}{2d}\overline{K}^2\Psi^2\right] t^2
$$

at (x_0, t_0). Note that $0 \le \Psi \le 1$ and $1 + (\beta-1)\mu t_0 \ge 1$. Therefore

$$
\begin{aligned}
\frac{2c(1-\epsilon)G^2}{n\beta} \le & \left[At + 1 + \frac{n\beta^2 C_1^2 t^2 \mu}{2\epsilon c\rho^2[1+(\beta-1)\mu t]} + \frac{2\Psi t^2[\beta K_1 + a(\beta-1)\alpha f^{\alpha-1}]\mu}{[1+(\beta-1)\mu t]^2}\right. \\
& \left. - \frac{a\Psi t\alpha f^{\alpha-1}}{1+(\beta-1)\mu t} + \frac{(\beta-1)\gamma t^2\mu}{1+(\beta-1)\mu t} + \frac{\beta t^2\Psi a\alpha|\alpha-1|f^{\alpha-2}\mu}{1+(\beta-1)\mu t}\right] G \\
& + \left[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2\right] t^2 \\
\le & \left[At + 1 + \frac{n\beta^2 C_1^2 t}{2\epsilon c\rho^2(\beta-1)} + \frac{2\Psi t^2[\beta K_1 + a(\beta-1)\alpha|f^{\alpha-1}|_\infty]\mu}{[1+(\beta-1)\mu t]^2}\right. \\
& \left. + \gamma t - \frac{a\Psi t\alpha f^{\alpha-1}}{1+(\beta-1)\mu t} + \frac{\beta t\Psi a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{\beta-1}\right] G \\
& + \left[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2\right] t^2.
\end{aligned} \tag{3.8}
$$

Before completing the proof, we recall a fact: if $x^2 \le ax + b$ for some $a, b, x \ge 0$, then

$$
x \le \frac{a}{2} + \sqrt{b + \left(\frac{a}{2}\right)^2} \le \frac{a}{2} + \sqrt{b} + \frac{a}{2} = a + \sqrt{b}. \tag{3.9}
$$

If $a \ge 0$ in (3.8), then from (3.8) we deduce that

$$
\begin{aligned}
G^2 \le & \left[\frac{(A+\gamma)n\beta t}{2c(1-\epsilon)} + \frac{n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3 C_1^2 t}{4\epsilon c^2(1-\epsilon)\rho^2(\beta-1)}\right. \\
& \left. + \frac{n\beta^2 a\alpha|\alpha-1||f^{\alpha-2}|_\infty t}{2c(\beta-1)(1-\epsilon)} + \frac{n\beta[\beta K_1 + a(\beta-1)\alpha|f^{\alpha-1}|_\infty]t}{c(1-\epsilon)(\beta-1)}\right] G \\
& + \frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta t^2}{2c(1-\epsilon)}.
\end{aligned} \tag{3.10}
$$

Applying (3.9) to the inequality (3.10), we get an upper bound for G:

$$
\begin{aligned}
G \le & \left[\frac{(A+\gamma)n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3 C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} + \frac{n\beta[\beta K_1 + a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)}\right. \\
& \left. + \frac{n\beta^2 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2c(\beta-1)(1-\epsilon)}\right]\tau + \sqrt{\frac{[\beta\theta + (\beta-1)\gamma + \frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}\tau + \frac{n\beta}{2c(1-\epsilon)},
\end{aligned}
$$

since $t_1 \le \tau$. By the construction of Ψ, we have $\sup_{B_{\rho/2,T}} F \le \sup_{B_{\rho,T}}(\Psi F) \le G(x_0, t_0)$ for all $t \in [0, \tau]$. Because $\tau \le T$ is arbitrary, it follows that

$$
\begin{aligned}
|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \le & \frac{n\beta}{2c(1-\epsilon)t} + \frac{(A+\gamma)n\beta}{2c(1-\epsilon)} + \frac{n^2\beta^3 C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} \\
& + \frac{n\beta[\beta K_1 + a(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)}
\end{aligned}
$$

$$+\frac{n\beta^2 a\alpha|\alpha-1||f^{\alpha-2}|_\infty}{2c(\beta-1)(1-\epsilon)}$$
$$+\sqrt{\frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}$$

where $|f|_\infty := \max_M |f|$. Similarly, when $a \leq 0$, we have

$$\begin{aligned} G^2 \leq & \left[\frac{(A+\gamma)n\beta t}{2c(1-\epsilon)}+\frac{n\beta}{2c(1-\epsilon)}+\frac{n^2\beta^3C_1^2 t}{4\epsilon c^2(1-\epsilon)\rho^2(\beta-1)}+\frac{n\beta^2K_1 t}{c(1-\epsilon)(\beta-1)}\right. \\ & \left.-\frac{n\beta at\alpha|f^{\alpha-1}|_\infty}{2c(1-\epsilon)}\right]G+\frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta t^2}{2c(1-\epsilon)}. \end{aligned} \tag{3.11}$$

From (3.9), (3.11), and above argument, an upper bound for desired quantity in this case is

$$\begin{aligned} |\nabla f|^2-\beta f_t-\beta q-\beta af^\alpha \leq & \frac{n\beta}{2c(1-\epsilon)t}+\frac{(A+\gamma)n\beta}{2c(1-\epsilon)}+\frac{n^2\beta^3C_1^2}{4\epsilon c^2(1-\epsilon)(\beta-1)\rho^2} \\ & +\frac{n\beta[\beta K_1-\frac{a}{2}(\beta-1)\alpha|f^{\alpha-1}|_\infty]}{c(1-\epsilon)(\beta-1)} \\ & +\sqrt{\frac{[\beta\theta+(\beta-1)\gamma+\frac{n\beta}{2d}\overline{K}^2]n\beta}{2c(1-\epsilon)}}. \end{aligned}$$

Hence, we complete the proof. □

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