MABUCHI AND AUBIN-YAU FUNCTIONALS OVER COMPLEX MANIFOLDS

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ABSTRACT. In the previous papers [2, 3] the author constructed Mabuchi and Aubin-Yau functionals over any complex surfaces and three-folds, respectively. Using the method in [3], we construct those functionals over any complex manifolds of the complex dimension bigger than or equal to 2.

CONTENTS

1. INTRODUCTION

Mabuchi and Aubin-Yau functionals play a crucial role in studying Kähler-Einstein metrics and constant scalar curvatures(see [5]). How to generalize these functionals from Kähler geometry to complex geometry is an interesting problem:

Question 1.1. *Can we define Mabuchi and Aubin-Yau functionals over compact complex manifolds so that these functionals coincide with the original definitions and satisfy the same basic properties?*

In [2, 3], the author answered this question in the complex dimension two and three, respectively, and proved similar results in the Kähler setting. By carefully

checking and using a similar method in [3], we can construct those functionals in higher dimension cases. So, now, we give an affirmative answer to Question 1.1.

1.1. **Mabuchi and Aubin-Yau functionals on Kähler manifolds.** In this subsection we review Mabuchi and Aubin-Yau functionals on Kähler manifolds, and describe some basic properties of these functionals which also hold in any complex manifolds. Let (X, ω) be a compact Kähler manifold of the complex dimension n. Then the volume

$$(1.1) \qquad V_\omega := \int_X \omega^n$$

depends only on the Kähler class of ω. Let $\mathcal{P}_\omega^{\mathrm{Kahler}}$ denote the space of Kähler potentials and define the Mabuchi functional, for any smooth functions $\varphi', \varphi'' \in \mathcal{P}_\omega^{\mathrm{Kahler}}$, by

$$(1.2) \qquad \mathcal{L}_\omega^{\mathrm{M,Kahler}}(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \dot\varphi_t \omega_{\varphi_t}^n dt$$

where φ_t is any smooth path in $\mathcal{P}_\omega^{\mathrm{Kahler}}$ from φ' to φ''. Mabuchi [4] showed that (1.2) is well-defined.

Using (1.2) we can define Aubin-Yau functionals, for any smooth function $\varphi \in \mathcal{P}_\omega^{\mathrm{Kahler}}$, as follows:

$$(1.3) \qquad \mathcal{I}_\omega^{\mathrm{AY,Kahler}}(\varphi) \;=\; \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n),$$

$$(1.4) \qquad \mathcal{J}_\omega^{\mathrm{AY,Kahler}}(\varphi) \;=\; -\mathcal{L}_\omega^{\mathrm{M,Kahler}}(0, \varphi) + \frac{1}{V_\omega} \int_X \varphi\omega^n.$$

So Aubin-Yau functionals are also well-defined. The basic and often useful inequalities, for any smooth function $\varphi \in \mathcal{P}_\omega^{\mathrm{Kahler}}$, are

$$(1.5) \qquad \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY,Kahler}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY,Kahler}}(\varphi) \;\geq\; 0,$$

$$(1.6) \qquad (n+1)\mathcal{J}_\omega^{\mathrm{AY,Kahler}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY,Kahler}}(\varphi) \;\geq\; 0.$$

An important consequence is that we will use the inequalities (1.5) and (1.6) to determine the extra terms on the definitions of $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$, which are Aubin-Yau functionals over complex manifolds.

However, if ω is not closed, then the above definitions (1.2), (1.3), and (1.4) do not make any sense. Hence we should add some extra terms on the definitions of those functionals; these extra terms should involve $\partial\omega$ and $\overline{\partial}\omega$, but, the essential question is to find the structure of the extra terms. Roughly speaking, $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$ can be written as

$$\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \;=\; \mathcal{I}_\omega^{\mathrm{AY,Kahler}}(\varphi) + \text{terms involving } \partial\omega, \overline{\partial}\varphi + \text{terms invloving } \overline{\partial}\omega, \partial\varphi,$$
$$\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \;=\; \mathcal{J}_\omega^{\mathrm{AY,Kahler}}(\varphi) + \text{terms involving } \partial\omega, \overline{\partial}\varphi + \text{terms involving } \overline{\partial}\omega, \partial\varphi.$$

In the following sections, we will explicitly determine the extra terms.

Throughout the rest part of this paper, we denote by (X, g) a compact complex manifold of the complex dimension $n \geq 2$, and ω be the associated real $(1,1)$-form. Let

$$(1.7) \qquad \mathcal{P}_\omega := \left\{ \varphi \in C^\infty(X)_\mathbb{R} \Big| \omega_\varphi := \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0 \right\}$$

be the space of all real-valued smooth functions on X whose associated real $(1, 1)$-forms are positive.

1.2. Mabuchi and Aubin-Yau functionals on complex surfaces.

In this subsection we recall the main result in [2]. Let (X, g) be a compact complex manifold of the complex dimension 2 and ω be its associated real $(1, 1)$-form. For any $\varphi', \varphi'' \in \mathcal{P}_\omega$, we define

$$(1.8) \qquad \begin{aligned} \mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'') &:= \frac{1}{V_\omega} \int_0^1 \int_X \dot{\varphi}_t \cdot \omega_{\varphi_t}^2 dt \\ &- \frac{1}{V_\omega} \int_0^1 \int_X \sqrt{-1}\partial\omega \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) dt \\ &+ \frac{1}{V_\omega} \int_0^1 \int_X \sqrt{-1}\overline{\partial}\omega \wedge (\partial\dot{\varphi}_t \cdot \varphi_t) dt, \end{aligned}$$

where $\{\varphi_t\}_{0 \leq t \leq 1}$ is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. Then in [2] we showed that the functional (1.8) is independent of the choice of the smooth path $\{\varphi_t\}_{0 \leq t \leq 1}$. If we set

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi) := \mathcal{L}_\omega^{\mathrm{M}}(0, \varphi),$$

then we have an explicit formula [2] of $\mathcal{L}_\omega^{\mathrm{M}}(\varphi)$:

$$(1.9) \qquad \begin{aligned} \mathcal{L}_\omega^{\mathrm{M}}(\varphi) &= \frac{1}{3V_\omega} \int_X \varphi(\omega^2 + \omega \wedge \omega_\varphi + \omega_\varphi^2) \\ &+ \frac{1}{2V_\omega} \int_X \varphi\left(-\sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi\right). \end{aligned}$$

Now Aubin-Yau functionals are defined by

$$(1.10) \quad \begin{aligned} \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &:= \frac{1}{V_\omega} \int_X \varphi(\omega^2 - \omega_\varphi^2) \\ &- \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi, \end{aligned}$$

$$(1.11) \quad \begin{aligned} \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) &:= -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi\omega^2 \\ &- \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{1}{V_\omega} \int_X \varphi \cdot \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi. \end{aligned}$$

Moreover they also satisfy the inequalities (1.5) and (1.6); that is

$$(1.12) \qquad \begin{aligned} \frac{2}{3}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) &\geq 0, \\ 3\mathcal{J}_\omega^{\mathrm{AY}} - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &\geq 0. \end{aligned}$$

1.3. Mabuchi and Aubin-Yau functionals on complex three-folds.

The functionals over complex three-folds are very different with these over complex

surfaces. For any $\varphi', \varphi'' \in \mathcal{P}_\omega$, we define

$$
\begin{aligned}
(1.13) \qquad \mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'') \quad := \quad & \frac{1}{V_\omega} \int_0^1 \int_X \dot{\varphi}_t \omega_{\varphi_t}^3 dt \\
& - \quad \frac{3}{V_\omega} \int_0^1 \int_X \sqrt{-1} \partial \omega \wedge \omega_{\varphi_t} \wedge (\overline{\partial} \dot{\varphi}_t \cdot \varphi_t) dt \\
& + \quad \frac{3}{V_\omega} \int_0^1 \int_X \sqrt{-1} \overline{\partial} \omega \wedge \omega_{\varphi_t} \wedge (\partial \dot{\varphi}_t \cdot \varphi_t) dt \\
& - \quad \frac{1}{V_\omega} \int_0^1 \int_X \partial \varphi_t \wedge \overline{\partial} \varphi_t \wedge \partial \omega \wedge \overline{\partial} \dot{\varphi}_t \\
& - \quad \frac{1}{V_\omega} \int_0^1 \int_X \overline{\partial} \varphi_t \wedge \partial \varphi_t \wedge \overline{\partial} \omega \wedge \partial \dot{\varphi}_t,
\end{aligned}
$$

where $\{\varphi_t\}_{0 \le t \le 1}$ is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. In [3], we proved that (1.13) is well-defined.

For any $\varphi \in \mathcal{P}_\omega$ we also set $\mathcal{L}_\omega^{\mathrm{M}}(\varphi) := \mathcal{L}_\omega^{\mathrm{M}}(0, \varphi)$. If we chose $\varphi_t = t \cdot \varphi$, then we have an explicit formula [3] of $\mathcal{L}_\omega^{\mathrm{M}}(\varphi)$:

$$
\begin{aligned}
(1.14) \qquad \mathcal{L}_\omega^{\mathrm{M}}(\varphi) \quad = \quad & \frac{1}{4V_\omega} \sum_{i=0}^3 \int_X \varphi \omega_\varphi^i \wedge \omega^{3-i} \\
& - \quad \sum_{i=0}^1 \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{1-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
& + \quad \sum_{i=0}^1 \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{1-i} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi.
\end{aligned}
$$

Now we define Aubin-Yau functionals $\mathcal{I}_\omega^{\mathrm{AY}}, \mathcal{J}_\omega^{\mathrm{AY}}$ for any compact complex three-fold (X, ω):

$$
\begin{aligned}
(1.15) \qquad \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad := \quad & \frac{1}{V_\omega} \int_X \varphi(\omega^3 - \omega_\varphi^3) \\
& - \quad \frac{3}{2V_\omega} \int_X \varphi \omega_\varphi \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi - \frac{3}{2V_\omega} \int_X \varphi \omega \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
& + \quad \frac{3}{2V_\omega} \int_X \varphi \omega_\varphi \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi + \frac{3}{2V_\omega} \int_X \varphi \omega \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi,
\end{aligned}
$$

$$
\begin{aligned}
(1.16) \qquad \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad := \quad & -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^3 \\
& - \quad \frac{3}{2V_\omega} \int_X \varphi \omega_\varphi \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi - \frac{3}{2V_\omega} \int_X \varphi \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi \\
& + \quad \frac{3}{2V_\omega} \int_X \varphi \omega_\varphi \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi + \frac{3}{2V_\omega} \int_X \varphi \omega \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(1.17) \qquad \qquad \frac{3}{4} \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad & \ge \quad 0, \\
4 \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad & \ge \quad 0.
\end{aligned}
$$

1.4. Mabuchi and Aubin-Yau functionals on complex manifolds. In this subsection we assume that the complex dimension n of the compact complex manifold (X, ω) is bigger than or equal to 3. For $\varphi', \varphi'' \in \mathcal{P}_\omega$, define

$$
(1.18) \mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'') \ := \ \frac{1}{V_\omega} \int_0^1 \int_X \dot{\varphi}_t \omega_{\varphi_t}^n dt
$$

$$
- \ \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \partial\omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) dt
$$

$$
+ \ \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \overline{\partial}\omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\partial\dot{\varphi}_t \cdot \varphi_t) dt
$$

$$
+ \ \sum_{i=1}^{n-2} \frac{1}{V_\omega} \int_0^1 \int_X (-1)^i \binom{n}{i+2} \partial\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}
$$

$$
+ \ \sum_{i=1}^{n-2} \frac{1}{V_\omega} \int_0^1 \int_X (-1)^i \binom{n}{i+2} \overline{\partial}\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \wedge \partial\varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1},
$$

where $\{\varphi_t\}_{0 \le t \le 1}$ is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. In Section 2, we prove

Theorem 1.2. *For any $n \ge 3$, the functional (1.18) is independent of the choice of the smooth path $\{\varphi_t\}_{0 \le t \le 1}$ in $\mathcal{P}_\omega$ from φ' to φ''.*

As a consequence of Theorem 1.2, by taking $\varphi_t = t \cdot \varphi$, we have, for any $\varphi \in \mathcal{P}_\omega$,

$$
\mathcal{L}_\omega^{\mathrm{M}}(\varphi) \ := \ \mathcal{L}_\omega^{\mathrm{M}}(0, \varphi) \ = \ \frac{1}{V_\omega} \sum_{i=0}^{n} \int_X \frac{1}{n+1} \varphi \omega_\varphi^i \wedge \omega^{n-i}
$$

$$
(1.19) \qquad\qquad - \ \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi
$$

$$
+ \ \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
$$

As before, we define Aubin-Yau functionals for any complex manifolds by

$$
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \ := \ \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) - \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi
$$

$$
(1.20) \qquad + \ \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi
$$

$$
= \ \frac{1}{V_\omega} \sum_{i=0}^{n-1} \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i},
$$

$$
\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \ := \ -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n - \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi
$$

$$
(1.21) \qquad + \ \frac{n}{2V_\omega} \sum_{i=0}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi
$$

$$
= \ \frac{1}{V_\omega} \sum_{i=0}^{n-1} \int_X \frac{n-i}{n+1} \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}.
$$

In Section 3, we shall show the following

Theorem 1.3. *For any $\varphi \in \mathcal{P}_\omega$, one has*

$$(1.22) \qquad\qquad \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad \geq \quad 0,$$

$$(1.23) \qquad\qquad (n+1)\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad \geq \quad 0.$$

In particular, $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi), \mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$ are nonnegative, and

$$(1.24) \qquad\qquad \frac{1}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad \leq \quad \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad \leq \quad \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi),$$

$$(1.25) \qquad\qquad \frac{n+1}{n}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad \leq \quad \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad \leq \quad (n+1)\mathcal{J}_\omega^{\mathrm{AY}}(\varphi),$$

$$(1.26) \quad \frac{1}{n}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad \leq \quad \frac{1}{n+1}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad \leq \quad \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$$

$$\leq \quad \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad \leq \quad n\mathcal{J}_\omega^{\mathrm{AY}}(\varphi).$$

1.5. **Further questions.** Up to now we consider functionals over compact complex manifolds without boundary, and we hope that the similar constructions can be achieved for compact complex manifolds with boundary.

Question 1.4. *Can we define Mabuchi and Aubin-Yau functionals over compact complex manifolds with boundary so that these functionals coincide with the original definitions and satisfy the same basic properties?*

There are other functionals, for example, Mabuchi $\mathcal{K}_\omega^{\mathrm{M}}$ [4] functional, Chen-Tian functionals [1], etc. We can ask the following

Question 1.5. *Can we define the analogy Mabuchi $\mathcal{K}_\omega^{\mathrm{M}}$ and Chen-Tian functionals over complex manifolds with(out) boundary?*

In the future, we will study those two questions.

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2. Mabuchi $\mathcal{L}_\omega^{\mathrm{M}}$ functional on complex manifolds

In this section we assume that (X, ω) is a compact complex manifold of the complex dimension $n \geq 3$. For any two complex forms α and β, we frequently use the following formulas: if $|\alpha| + |\beta| = 2n - 1$, then

$$(2.1) \qquad \int_X \alpha \wedge \partial\beta \quad = \quad (-1)^{|\beta|}\int_X \partial\alpha \wedge \beta \quad = \quad -(-1)^{|\alpha|}\int_X \partial\alpha \wedge \beta,$$

$$(2.2) \qquad \int_X \alpha \wedge \overline{\partial}\beta \quad = \quad (-1)^{|\beta|}\int_X \overline{\partial}\alpha \wedge \beta \quad = \quad -(-1)^{|\alpha|}\int_X \overline{\partial}\alpha \wedge \beta.$$

Another useful formula is

$$(2.3) \qquad\qquad \alpha \wedge \alpha = 0, \quad \text{if } |\alpha| \text{ is odd.}$$

By the definition of operators ∂ and $\overline{\partial}$, one has

$$(2.4) \qquad\qquad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Hence the complex conjugate of the operator $\sqrt{-1}\partial\overline{\partial}$ is itself.

2.1. **The main idea.** Similarly in [3], we consider $\psi(s,t) = s \cdot \varphi_t$ and we can show that (see (2.49))

$$(2.5) \qquad \frac{2I^0}{n(n-1)\sqrt{-1}} = \frac{I^1}{a_1} - \frac{I^2}{a_2} + c_1,$$

$$(2.6) \qquad \frac{I^3}{a_3} + \frac{I^4}{a_4} = \frac{3}{-(n-2)\sqrt{-1}}c_1 + c_2,$$

where I^i are functionals which can be determined[1], c_j are also functionals but may not be determined, and a_k are nonzero constants which can be determined later. We can use equation (2.6) to eliminate the undetermined term c_1, but there arises another undetermined term c_2. Our strategy is to find a determined expression for c_2. To achieve this we construct two sequences $\{I^{2i+1}\}_{2 \le i \le n-2}$, which can be determined, and $\{c_i\}_{2 \le i \le n-2}$, which may not be determined, satisfying

$$(2.7) \qquad \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} = \frac{i+2}{n-(i+1)}c_i + c_{i+1}, \quad 2 \le i \le n-2,$$

where $I^{2i+2} := \overline{I^{2i+1}}$, $a_{2i+2} := \overline{a_{2i+1}}$ and a_{2i+1} can be determined later. By our construction we have $c_{n-1} \equiv 0$ which gives us the determined and explicit formula for c_2 in terms of I^{2i+1} and a_{2i+1}. More precisely, setting

$$(2.8) \qquad J_i := \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}},$$

yields

$$(2.9) \qquad c_{i+1} = -\frac{i+2}{n-(i+1)}c_i + J_i, \quad 2 \le i \le n-2.$$

By induction on i we obtain

$$(2.10) \quad c_i = (-1)^{i-2}\frac{(i+1)!(n-i-1)!}{3!(n-3)!}c_2 + \sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k.$$

Since $c_{n-1} \equiv 0$ it follows that

$$c_2 = (-1)^{n-3}\frac{3!(n-3)!}{n!}\left[0 - \sum_{k=2}^{n-2}(-1)^{n-2-k}\frac{n!}{(k+2)!(n-k-2)!}J_k\right]$$

$$(2.11) \qquad = \sum_{k=2}^{n-2}(-1)^k\frac{3!(n-3)!}{(k+2)!(n-k-2)!}J_k, \quad n \ge 4.$$

When X is a three-fold, we knew that $c_2 = 0$ [3], but this can be seen from (2.11) if we take $n = 3$. Hence the formula (2.11) holds for any $n \ge 3$.

2.2. **The definitions of c_1 and c_2.** Firstly, we consider the "Kähler part" of Mabuchi functional. Let

$$(2.12) \qquad \mathcal{L}^0_\omega(\varphi', \varphi'') := \frac{1}{V_\omega}\int_0^1 \int_X \dot{\varphi}_t \omega^n_{\varphi_t} dt.$$

---

[1]A functional $\mathcal{I}$ is said to be determined if $d\Psi = \mathcal{I} \cdot dt \wedge ds$ for some 1-form Ψ on $[0,1] \times [0,1]$.

As in [2, 3], we set $\psi(s,t) := s \cdot \varphi_t$, $0 \leq t, s \leq 1$, and consider the corresponding 1-form on $[0,1] \times [0,1]$,

$$(2.13) \qquad \Psi^0 = \left(\int_X \frac{\partial \psi}{\partial s} \omega_\psi^n \right) ds + \left(\int_X \frac{\partial \psi}{\partial t} \omega_\psi^n \right) dt.$$

Taking the differential on both sides implies

$$(2.14) \qquad d\Psi^0 = I^0 \cdot dt \wedge ds$$

where

$$(2.15) \qquad I^0 := \int_X \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial s} \omega_\psi^n \right) - \int_X \frac{\partial}{\partial s} \left(\frac{\partial \psi}{\partial t} \omega_\psi^n \right).$$

The explicit expression of I^0 can be determined as follows:

$$
\begin{aligned}
I^0 &= \int_X \left[\frac{\partial \psi}{\partial s} n \omega_\psi^{n-1} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) - \frac{\partial \psi}{\partial t} n \omega_\psi^{n-1} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \right] \\
&= \int_X n \frac{\partial \psi}{\partial s} \omega_\psi^{n-1} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) - \int_X n \frac{\partial \psi}{\partial t} \omega_\psi^{n-1} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \\
&= \int_X n \frac{\partial \psi}{\partial s} \omega_\psi^{n-1} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) + \int_X n \frac{\partial \psi}{\partial t} \omega_\psi^{n-1} \wedge \sqrt{-1} \overline{\partial} \partial \left(\frac{\partial \psi}{\partial s} \right) \\
&= -\int_X n \frac{\partial \psi}{\partial s} \omega_\psi^{n-1} \wedge \sqrt{-1} \overline{\partial} \partial \left(\frac{\partial \psi}{\partial t} \right) - \int_X n \frac{\partial \psi}{\partial t} \omega_\psi^{n-1} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right).
\end{aligned}
$$

Here we have two slightly different expressions of I^0, and in the following we will use those expressions to find $c_1 = A_1 + B_1 - (\overline{A_1} + \overline{B_1})$, where $A_1, B_1, \overline{A_1}$ and $\overline{B_1}$ are determined later. This technique will be frequently used in many places. Hence

$$
\begin{aligned}
I^0 &= \int_X -n\sqrt{-1} \partial \left(\frac{\partial \psi}{\partial s} \omega_\psi^{n-1} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) + \int_X -n\sqrt{-1} \partial \left(\frac{\partial \psi}{\partial t} \omega_\psi^{n-1} \right) \wedge \partial \left(\frac{\partial \psi}{\partial s} \right) \\
&= \int_X -n\sqrt{-1} \left[\partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \omega_\psi^{n-1} + (n-1) \frac{\partial \psi}{\partial s} \omega_\psi^{n-2} \wedge \partial \omega \right] \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \\
&\quad + \int_X -n\sqrt{-1} \left[\overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \omega_\psi^{n-1} + (n-1) \frac{\partial \psi}{\partial t} \omega_\psi^{n-2} \wedge \overline{\partial} \omega \right] \wedge \partial \left(\frac{\partial \psi}{\partial s} \right) \\
&= \int_X -n(n-1)\sqrt{-1} \frac{\partial \psi}{\partial s} \omega_\psi^{n-2} \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \\
&\quad + \int_X -n(n-1)\sqrt{-1} \frac{\partial \psi}{\partial t} \omega_\psi^{n-2} \wedge \overline{\partial} \omega \wedge \partial \left(\frac{\partial \psi}{\partial s} \right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
I^0 &= \int_X -n(n-1)\sqrt{-1} \frac{\partial \psi}{\partial s} \omega_\psi^{n-2} \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \\
(2.16) &\quad + \int_X -n(n-1)\sqrt{-1} \frac{\partial \psi}{\partial t} \omega_\psi^{n-2} \wedge \overline{\partial} \omega \wedge \partial \left(\frac{\partial \psi}{\partial s} \right).
\end{aligned}
$$

Using another expression of I^0, or taking the complex conjugate on both sides of (2.16) since I^0 is real, one has

$$
\begin{aligned}
I^0 &= \int_X n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial s}\omega_\psi^{n-2}\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\\
(2.17)\qquad &+ \int_X n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial t}\omega_\psi^{n-2}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right).
\end{aligned}
$$

Hence, adding (2.17) to (2.16) and dividing by $n(n-1)\sqrt{-1}$ on both sides, we deduce

$$
\begin{aligned}
\frac{2I^0}{n(n-1)\sqrt{-1}} &= \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial s}\wedge\omega_\psi^{n-2}\wedge\partial\omega + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\frac{\partial\psi}{\partial t}\wedge\omega_\psi^{n-2}\wedge\overline{\partial}\omega\\
(2.18)\qquad &- \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\frac{\partial\psi}{\partial t}\wedge\omega_\psi^{n-2}\wedge\partial\omega - \int_X \partial\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial s}\wedge\omega_\psi^{n-2}\wedge\overline{\partial}\omega.
\end{aligned}
$$

According to the expression (2.18), we introduce two functionals

$$
(2.19)\qquad \mathcal{L}_\omega^1(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_1\partial\omega\wedge\omega_{\varphi_t}^{n-2}\wedge(\overline{\partial}\dot\varphi_t\cdot\varphi_t)dt,
$$

$$
(2.20)\qquad \mathcal{L}_\omega^2(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_2\overline{\partial}\omega\wedge\omega_{\varphi_t}^{n-2}\wedge(\partial\dot\varphi_t\cdot\varphi_t)dt.
$$

Here a_1, a_2 are non-zero constants determined later and we require $\overline{a_1} = a_2$. Satisfying this condition, a_1 and a_2 have lots of solutions. In the following we will see that we take only two special cases: a_i are purely complex numbers; a_i are real numbers. Consider the corresponding two 1-forms on $[0,1]\times[0,1]$,

$$
\begin{aligned}
(2.21)\qquad \Psi^1 &= \left[\int_X a_1\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\cdot\psi\right)\right]ds\\
&+ \left[\int_X a_1\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\cdot\psi\right)\right]dt,\\
(2.22)\qquad \Psi^2 &= \left[\int_X a_2\overline{\partial}\omega\wedge\omega_\psi^{n-2}\wedge\left(\partial\left(\frac{\partial\psi}{\partial s}\right)\cdot\psi\right)\right]ds\\
&+ \left[\int_X a_2\overline{\partial}\omega\wedge\omega_\psi^{n-2}\wedge\left(\partial\left(\frac{\partial\psi}{\partial t}\right)\cdot\psi\right)\right]dt.
\end{aligned}
$$

Firstly, we compute the differential of Ψ^1, and the differential $d\Psi^2$ can be easily written down only by taking the complex conjugate on both sides. Calculate

$$
(2.23)\qquad d\Psi^1 = I^1\cdot dt\wedge ds
$$

where

$$
\begin{aligned}
(2.24)\qquad I^1 &= \int_X a_1\frac{\partial}{\partial t}\left[\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\cdot\psi\right)\right]\\
&- \int_X a_1\frac{\partial}{\partial s}\left[\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\cdot\psi\right)\right].
\end{aligned}
$$

Dividing by a_1 on both sides of (2.24), we have

$$
\begin{aligned}
\frac{I^1}{a_1} &= \int_X -\frac{\partial}{\partial t}\left[\left(\psi \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right)\right) \wedge \omega_\psi^{n-2} \wedge \partial\omega\right] \\
&\quad + \int_X \frac{\partial}{\partial s}\left[\left(\psi \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right)\right) \wedge \omega_\psi^{n-2} \wedge \partial\omega\right] \\
&= \int_X -\left[\frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) + \psi \cdot \overline{\partial}\left(\frac{\partial^2 \psi}{\partial t \partial s}\right)\right] \wedge \omega_\psi^{n-2} \wedge \partial\omega \\
&\quad + \int_X \left[\frac{\partial \psi}{\partial s} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) + \psi \cdot \overline{\partial}\left(\frac{\partial^2 \psi}{\partial s \partial t}\right)\right] \wedge \omega_\psi^{n-2} \wedge \partial\omega \\
&\quad + \int_X \psi \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge (n-2)\omega_\psi^{n-3} \wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \partial\omega \\
&\quad + \int_X \psi \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge (n-2)\omega_\psi^{n-3} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial\omega \\
&= \int_X -\frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \omega_\psi^{n-2} \wedge \partial\omega + \int_X \frac{\partial \psi}{\partial s} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \omega_\psi^{n-2} \wedge \partial\omega \\
&\quad + \int_X \psi \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge (n-2)\omega_\psi^{n-3} \wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \partial\omega \\
&\quad + \int_X \psi \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge (n-2)\omega_\psi^{n-3} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial\omega.
\end{aligned}
$$

To simplify the notation, we set

$$
(2.25) \quad A_1 := \int_X \psi(n-2)\overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \omega_\psi^{n-3} \wedge \partial\omega \wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial t}\right),
$$

$$
(2.26) \quad B_1 := \int_X \psi(n-2)\overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \omega_\psi^{n-3} \wedge \partial\omega \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial s}\right).
$$

Consequently, I^1/a_1 can be written as

$$
\begin{aligned}
\frac{I^1}{a_1} &= \int_X -\frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \omega_\psi^{n-2} \wedge \partial\omega + \int_X \frac{\partial \psi}{\partial s} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \omega_\psi^{n-2} \wedge \partial\omega \\
(2.27) \quad &+ A_1 + B_1.
\end{aligned}
$$

Similarly, we can define I^2 by

$$
(2.28) \qquad\qquad\qquad d\Psi^2 = I^2 \cdot dt \wedge ds,
$$

where

$$
\begin{aligned}
\frac{I^2}{a_2} &:= \int_X -\frac{\partial \psi}{\partial t} \cdot \partial\left(\frac{\partial \psi}{\partial s}\right) \wedge \omega_\psi^{n-2} \wedge \overline{\partial}\omega + \int_X \frac{\partial \psi}{\partial s} \cdot \partial\left(\frac{\partial \psi}{\partial t}\right) \wedge \omega_\psi^{n-2} \wedge \overline{\partial}\omega \\
(2.29) \quad &+ \overline{A_1} + \overline{B_1}.
\end{aligned}
$$

Consequently, from (2.18), (2.27) and (2.29),

$$
(2.30) \qquad\qquad \frac{2I^0}{n(n-1)\sqrt{-1}} = \frac{I^1}{a_1} - \frac{I^2}{a_2} + (A_1 + B_1) - (\overline{A_1} + \overline{B_1}).
$$

By a direct computation and using (2.1) and (2.3), one can show that the sum $A_1 + B_1$ has a nice form:

$$
\begin{aligned}
A_1 &= \int_X \sqrt{-1}\partial\left[(n-2)\psi \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-3} \wedge \partial\omega\right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-2)\left[\partial\left(\psi \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right) \wedge \omega_\psi^{n-3} \wedge \partial\omega\right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-2)\left[\partial\psi \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) + \psi \cdot \partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right] \wedge \omega_\psi^{n-3} \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X (n-2)\psi \cdot \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-3} \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&\quad + \int_X (n-2)\sqrt{-1}\partial\psi \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-3} \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X -(n-2)\psi \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-3} \wedge \partial\omega \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \\
&\quad + \int_X -(n-2)\sqrt{-1}\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-3} \\
&= -B_1 - \int_X (n-2)\sqrt{-1}\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-3}.
\end{aligned}
$$

Adding the term B_1 on both sides gives

$$
(2.31) \qquad A_1 + B_1 = \int_X -(n-2)\sqrt{-1}\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-3}.
$$

According to (2.31), we define

$$
(2.32) \qquad \mathcal{L}_\omega^3(\varphi', \varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_3\partial\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-3},
$$

$$
(2.33) \qquad \mathcal{L}_\omega^4(\varphi', \varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_4\overline{\partial}\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \wedge \partial\varphi_t \wedge \omega_{\varphi_t}^{n-3},
$$

where a_3, a_4 are nonzero constants determined later and we require $\overline{a_3} = a_4$. Consider

$$
(2.34) \qquad
\begin{aligned}
\Psi^3 &= \left[\int_X a_3\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}\right] ds \\
&\quad + \left[\int_X a_3\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}\right] dt,
\end{aligned}
$$

$$
(2.35) \qquad
\begin{aligned}
\Psi^4 &= \left[\int_X a_4\overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\psi \wedge \omega_\psi^{n-3}\right] ds \\
&\quad + \left[\int_X a_4\overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\psi \wedge \omega_\psi^{n-3}\right] dt.
\end{aligned}
$$

Calculate

$$
(2.36) \qquad\qquad\qquad d\Psi^3 = I^3 \cdot dt \wedge ds,
$$

where

$$(2.37) \qquad I^3 \quad := \quad \int_X a_3 \frac{\partial}{\partial t} \left[\partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3} \right]$$
$$- \quad \int_X a_3 \frac{\partial}{\partial s} \left[\partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3} \right].$$

Also, we can calculate

$$(2.38) \qquad\qquad d\Psi^4 = I^4 \cdot dt \wedge ds,$$

where

$$(2.39) \qquad I^4 \quad := \quad \int_X a_4 \frac{\partial}{\partial t} \left[\overline{\partial} \psi \wedge \overline{\partial} \omega \wedge \partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \omega_\psi^{n-3} \right]$$
$$- \quad \int_X a_4 \frac{\partial}{\partial s} \left[\overline{\partial} \psi \wedge \overline{\partial} \omega \wedge \partial \left(\frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \omega_\psi^{n-3} \right].$$

Hence

$$\frac{I^3}{a_3} \quad = \quad \int_X \left[\partial \left(\frac{\partial \psi}{\partial t} \right) \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3} \right.$$
$$+ \quad \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial^2 \psi}{\partial t \partial s} \right) \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3} + \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \omega_\psi^{n-3}$$
$$+ \quad \left. \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \psi \wedge (n-3)\omega_\psi^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \right]$$
$$- \quad \int_X \left[\partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3} \right.$$
$$+ \quad \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial^2 \psi}{\partial s \partial t} \right) \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3} + \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \omega_\psi^{n-3}$$
$$+ \quad \left. \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \psi \wedge (n-3)\omega_\psi^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \right].$$

The second term and the sixth term cancel with each other, and the third term and the seventh term are the same, so we have

$$\frac{I^3}{a_3} \quad = \quad \int_X -\partial \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3}$$
$$+ \quad \int_X \partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \partial \omega \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-3}$$
$$+ \quad 2 \int_X \partial \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \omega_\psi^{n-3}$$
$$+ \quad \int_X (n-3) \partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \omega_\psi^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right)$$
$$- \quad \int_X (n-3) \partial \psi \wedge \overline{\partial} \psi \wedge \partial \omega \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \omega_\psi^{n-4} \wedge \sqrt{-1} \partial \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right)$$
$$= \quad H_1 + \frac{2}{-(n-2)\sqrt{-1}} (A_1 + B_1) + A_2 + B_2,$$

where

$$(2.40) \qquad H_1 \; := \; \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}$$

$$+ \; \int_X \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}$$

$$(2.41) \qquad A_2 \; := \; \int_X (n-3)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)$$

$$\wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)$$

$$(2.42) \qquad B_2 \; := \; -\int_X (n-3)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)$$

$$\wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right).$$

Similarly,

$$(2.43) \qquad \frac{I^4}{a_4} = \overline{H_1} + \frac{2}{(n-2)\sqrt{-1}}(\overline{A_1} + \overline{B_1}) + \overline{A_2} + \overline{B_2}.$$

The hard part is to find some suitable expression of H_1. In the following we will see that $H_1 + \overline{H_1}$ has a nice form which contains only $A_1, B_1, \overline{A_1}$, and $\overline{B_1}$.

Now we compute H_1, using (2.3) and (2.4):

$$
\begin{aligned}
H_1 \; = \; & \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}\right] \frac{\partial\psi}{\partial t} \\
+ \; & \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}\right] \frac{\partial\psi}{\partial t} \\
= \; & \int_X \frac{\partial\psi}{\partial t} \left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3} - \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\left(\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}\right)\right] \\
+ \; & \int_X \frac{\partial\psi}{\partial t} \left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3} - \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}\right)\right] \\
= \; & \int_X \frac{\partial\psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \left(\partial\overline{\partial}\psi \wedge \omega_\psi^{n-3} - \overline{\partial}\psi \wedge (n-3)\omega_\psi^{n-4} \wedge \partial\omega\right) \\
+ \; & \int_X -\frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3} \\
+ \; & \int_X -\frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge (n-3)\omega_\psi^{n-4} \wedge \overline{\partial}\omega \\
= \; & \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \omega_\psi^{n-3} \wedge \partial\psi \\
+ \; & \int_X \frac{\partial\psi}{\partial t} \cdot -\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \left[\overline{\partial}\partial\omega \wedge \omega_\psi^{n-3} - \partial\omega \wedge (n-3)\omega_\psi^{n-4} \wedge \overline{\partial}\omega\right] \wedge \partial\psi \\
- \; & \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3} \\
- \; & \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge (n-3)\omega_\psi^{n-4} \wedge \overline{\partial}\omega
\end{aligned}
$$

Therefore we obtain

$$(2.44) \quad H_1 \; = \; \int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-3}$$

$$- \int_X \frac{\partial\psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\partial\omega \wedge \partial\psi \wedge \omega_\psi^{n-3}$$

$$+ \int_X \frac{\partial\psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge (n-3)\omega_\psi^{n-4} \wedge \overline{\partial}\omega \wedge \partial\psi$$

$$- \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}$$

$$+ \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge (n-3)\omega_\psi^{n-4} \wedge \overline{\partial}\omega \wedge \overline{\partial}\psi,$$

and, taking the complex conjugate yields, using (2.3) and (2.4)

$$(2.45) \quad \overline{H_1} \; = \; \int_X \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-3}$$

$$- \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\overline{\partial}\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-3}$$

$$- \int_X \frac{\partial\psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\overline{\partial}\omega \wedge \partial\psi \wedge \omega_\psi^{n-3}$$

$$+ \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\omega \wedge (n-3)\omega_\psi^{n-4} \wedge \partial\omega \wedge \overline{\partial}\psi$$

$$+ \int_X \frac{\partial\psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\omega \wedge (n-3)\omega_\psi^{n-4} \wedge \partial\omega \wedge \partial\psi.$$

Adding (2.45) to (2.45), it follows that

$$(2.46) \qquad H_1 + \overline{H_1} = \frac{A_1 + B_1}{-(n-2)\sqrt{-1}} + \frac{\overline{A_1} + \overline{B_1}}{(n-2)\sqrt{-1}}.$$

and, hence,

$$\frac{I^3}{a_3} + \frac{I^4}{a_4} \; = \; H_1 + \overline{H_1} + \frac{2}{-(n-2)\sqrt{-1}}[(A_1+B_1) - (\overline{A_1} + \overline{B_1})]$$

$$+ \; A_2 + B_2 + \overline{A_2} + \overline{B_2}$$

$$(2.47) \qquad = \; \frac{3}{-(n-2)\sqrt{-1}}[(A_1+B_1) - (\overline{A_1} + \overline{B_1})] + (A_2+B_2) + (\overline{A_2} + \overline{B_2}).$$

Set

$$(2.48) \qquad c_1 := A_1 + B_1 - (\overline{A_1} + \overline{B_1}), \quad c_2 := A_2 + B_2 + \overline{A_2} + \overline{B_2}.$$

we deduce

$$(2.49) \qquad \frac{2I^0}{n(n-1)\sqrt{-1}} = \frac{I^1}{a_1} - \frac{I^2}{a_2} + c_1, \qquad \frac{I^3}{a_3} + \frac{I^4}{a_4} = \frac{3}{-(n-2)\sqrt{-1}}c_1 + c_2.$$

2.3. **The constructions of I^5 and I^6.** To give the general construction of I^{2i+1} and I^{2i+2}, we firstly consider some special cases. In this subsection we give the construction of I^5 and I^6, and in the next subsection the construction of I^7 and I^8. Finally, we give the general construction.

From (2.41), it follows that

$$
\begin{aligned}
A_2 &= \int_X (n-3)\partial\psi \wedge \overline{\partial}\psi \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-4} \wedge \partial\omega \wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}\partial\left[(n-3)\partial\psi \wedge \overline{\partial}\psi \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-4} \wedge \partial\omega\right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-3)\left[\partial\left(\partial\psi \wedge \overline{\partial}\psi \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right) \wedge \omega_\psi^{n-4} \wedge \partial\omega\right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X -\sqrt{-1}(n-3)\left[\partial\psi \wedge \left(\partial\overline{\partial}\psi \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) - \overline{\partial}\psi \wedge \partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right)\right] \\
&\qquad \wedge \omega_\psi^{n-4} \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X (n-3)\partial\psi \wedge -\sqrt{-1}\partial\overline{\partial}\psi \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-4} \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&\quad + \int_X (n-3)\partial\psi \wedge \overline{\partial}\psi \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-4} \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X (n-3)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi - B_2;
\end{aligned}
$$

hence, by the definition (2.42), we have

$$(2.50) \quad A_2 + B_2 = \int_X (n-3)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi.$$

Motivated by (2.50), we set

$$(2.51) \quad \mathcal{L}_\omega^5(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_5 \partial\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\varphi_t,$$

$$(2.52) \quad \mathcal{L}_\omega^6(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_6 \overline{\partial}\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \wedge \partial\varphi_t \wedge \omega_{\varphi_t}^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\varphi_t.$$

Consider again the 1-forms

$$
\begin{aligned}
\Psi^5 &= \left[\int_X a_5 \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] ds \\
(2.53) &\quad + \left[\int_X a_5 \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] dt, \\
\Psi^6 &= \left[\int_X a_6 \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] ds \\
(2.54) &\quad + \left[\int_X a_6 \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] dt.
\end{aligned}
$$

The differential of Ψ^5 is given by

$$(2.55) \qquad\qquad d\Psi^5 = I^5 \cdot dt \wedge ds,$$

where

$$
\begin{aligned}
\frac{I^5}{a_5} \quad := \quad & \int_X \frac{\partial}{\partial t}\left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] \\
(2.56) \qquad - \quad & \int_X \frac{\partial}{\partial s}\left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right].
\end{aligned}
$$

Hence, using (2.50),

$$
\begin{aligned}
\frac{I^5}{a_5} \quad = \quad & \int_X \left[\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \right. \\
+ \quad & \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \left. \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right] \\
- \quad & \int_X \left[\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \right. \\
+ \quad & \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \left. \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right] \\
= \quad & \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \int_X \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & 2\int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
- \quad & \int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \frac{A_2}{n-3} + \frac{B_2}{n-3} \\
= \quad & H_2 + \frac{3}{n-3}(A_2 + B_2) + A_3 + B_3,
\end{aligned}
$$

where

$$H_2 \quad := \quad \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi$$

$$(2.57) \qquad + \quad \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi,$$

$$A_3 \quad := \quad \int_X (n-4)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)$$

$$(2.58) \qquad \wedge\sqrt{-1}\partial\overline{\partial}\psi,$$

$$B_3 \quad := \quad \int_X (n-4)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-5}\wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)$$

$$(2.59) \qquad \wedge\sqrt{-1}\partial\overline{\partial}\psi.$$

Similarly, for

$$(2.60) \qquad\qquad\qquad d\Psi^6 =: I^6 \cdot dt \wedge ds,$$

we have

$$(2.61) \qquad\qquad\qquad \frac{I^6}{a_6} = \overline{H_2} + \frac{3}{n-3}(\overline{A_2}+\overline{B_2}) + \overline{A_3} + \overline{B_3}.$$

As (2.44), the hard part H_2 is calculated as follows:

$$H_2 \quad = \quad \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]\frac{\partial\psi}{\partial t}$$

$$+ \quad \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]\frac{\partial\psi}{\partial t}$$

$$= \quad \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right.$$

$$- \quad \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right)\Big]$$

$$+ \quad \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right.$$

$$- \quad \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right)\Big]$$

$$= \quad \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi$$

$$- \quad \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi$$

$$- \quad \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\overline{\partial}\omega\wedge\sqrt{-1}\partial\overline{\partial}\psi$$

$$= \quad \int_X \sqrt{-1}\partial\left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-4}\right]\wedge\partial\psi$$

$$- \quad \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi$$

$$- \quad \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\overline{\partial}\omega\wedge\sqrt{-1}\partial\overline{\partial}\psi.$$

Hence

$$(2.62) \quad H_2 = \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\psi \wedge \partial\omega \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi$$

$$- \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\psi \wedge \overline{\partial}\partial\omega \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi$$

$$+ \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\omega \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi \wedge \partial\psi$$

$$- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \overline{\partial}\partial\omega \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi$$

$$+ \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\omega \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi \wedge \overline{\partial}\psi$$

and, consequently, after taking the complex conjugate on both sides,

$$(2.63) \quad \overline{H_2} = \int_X \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi$$

$$- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \partial\overline{\partial}\omega \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi$$

$$+ \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\omega \wedge \partial\omega \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi \wedge \overline{\partial}\psi$$

$$- \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\psi \wedge \partial\overline{\partial}\omega \wedge \omega_\psi^{n-4} \wedge \sqrt{-1}\partial\overline{\partial}\psi$$

$$+ \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\omega \wedge \partial\omega \wedge (n-4)\omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi \wedge \partial\psi.$$

Therefore

$$(2.64) \quad H_2 + \overline{H_2} = \frac{A_2 + B_2}{n-3} + \frac{\overline{A_2} + \overline{B_2}}{n-3},$$

$$(2.65) \quad \frac{I^5}{a_5} + \frac{I^6}{a_6} = \frac{4}{n-3}(A_2 + B_2 + \overline{A_2} + \overline{B_2}) + (A_3 + B_3 + \overline{A_3} + \overline{B_3}).$$

If we set

$$(2.66) \quad c_3 := A_3 + B_3 + \overline{A_3} + \overline{B_3}$$

we can rewrite (2.64) and (2.65) as

$$(2.67) \quad H_2 + \overline{H_2} = \frac{c_2}{n-3}, \quad \frac{I^5}{a_5} + \frac{I^6}{a_6} = \frac{4}{n-3}c_2 + c_3.$$

2.4. The constructions of I^7 and I^8. Recall

$$A_3 = \int_X \left[(n-4)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right).$$

By a direct computation, one deduces that

$$
\begin{aligned}
A_3 &= \int_X -\sqrt{-1}\partial\left[(n-4)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] \\
&\quad \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-4)\partial\psi \wedge \partial\left(\overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right) \\
&\quad \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-4)\partial\psi \wedge \left[\partial\overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right. \\
&\quad + \left. \overline{\partial}\psi \wedge \partial\omega \wedge \partial\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right)\right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-4)\partial\psi \wedge \left[\partial\overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right. \\
&\quad + \left. \overline{\partial}\psi \wedge \partial\omega \wedge \left(\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right)\right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X (n-4)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&\quad + \int_X (n-4)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi \\
&= -B_3 + \int_X (n-4)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2.
\end{aligned}
$$

Consequently, it follows that

$$
(2.68) \quad A_3 + B_3 = \int_X (n-4)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2.
$$

Now we introduce the corresponding functionals

$$
(2.69) \quad \mathcal{L}_\omega^7(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_7 \partial\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^2,
$$

$$
(2.70) \quad \mathcal{L}_\omega^8(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X a_8 \overline{\partial}\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \wedge \partial\varphi_t \wedge \omega_{\varphi_t}^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^2
$$

and consider the 1-forms

$$
\begin{aligned}
\Psi^7 &= \left[\int_X a_7 \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right] ds \\
(2.71) \quad &\quad + \left[\int_X a_7 \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right] dt,
\end{aligned}
$$

$$
\begin{aligned}
\Psi^8 &= \left[\int_X a_8 \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right] ds \\
(2.72) \quad &\quad + \left[\int_X a_8 \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right] dt.
\end{aligned}
$$

So, we have the expression,

$$(2.73) \qquad d\Psi^7 = I^7 \cdot dt \wedge ds, \quad d\Psi^8 = I^8 \cdot dt \wedge ds,$$

where

$$
\begin{aligned}
\frac{I^7}{a_7} \quad :=& \quad \int_X \frac{\partial}{\partial t} \left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \right] \\
-& \quad \int_X \frac{\partial}{\partial s} \left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \right] \\
=& \quad \int_X \left[\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \right. \\
+& \quad \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \left. \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge 2\sqrt{-1}\partial\overline{\partial}\psi \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \right] \\
-& \quad \int_X \left[\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \right. \\
+& \quad \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial\psi} \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \left. \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge 2\sqrt{-1}\partial\overline{\partial}\psi \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \right] \\
=& \quad \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \int_X \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad 2\int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
-& \quad \int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
+& \quad \frac{2}{n-4}(A_3 + B_3) \\
=& \quad H_3 + \frac{4}{n-4}(A_3 + B_3) + A_4 + B_4
\end{aligned}
$$

where

$$H_3 \quad := \quad \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

(2.74)
$$+ \quad \int_X \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

and

$$A_4 \quad := \quad \int_X (n-5)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-6}$$

(2.75)
$$\wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2,$$

$$B_4 \quad := \quad \int_X (n-5)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-6}$$

(2.76)
$$\wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2.$$

Hence

(2.77)
$$\frac{I^7}{a_7} = H_3 + \frac{4}{n-4}(A_3 + B_3) + A_4 + B_4.$$

Similarly

(2.78)
$$\frac{I^8}{a_8} = \overline{H_3} + \frac{4}{n-4}(\overline{A_3} + \overline{B_3}) + \overline{A_4} + \overline{B_4}.$$

Calculate

$$H_3 \quad = \quad \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right]\frac{\partial\psi}{\partial t}$$

$$+ \quad \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right]\frac{\partial\psi}{\partial t}$$

$$= \quad \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right.$$

$$- \quad \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\left(\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right)\right]$$

$$+ \quad \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right.$$

$$- \quad \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right)\right]$$

$$= \quad \int_X \frac{\partial\psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \partial\overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

$$- \quad \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

$$- \quad \int_X \frac{\partial\psi}{\partial t} \cdot \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \overline{\partial}\omega \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2.$$

Since

$$(2.79) \qquad \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial \omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

$$= \int_X \left[\frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial \omega \wedge \partial\overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right] \wedge -\sqrt{-1}\partial\partial\psi$$

$$= \int_X \sqrt{-1}\overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial \omega \wedge \partial\overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi \wedge \partial\psi$$

$$- \int_X \sqrt{-1}\frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \left(\overline{\partial}\partial\omega \wedge \partial\overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge \sqrt{-1}\partial\overline{\partial}\psi\right.$$

$$- \left.\partial\omega \wedge \partial\overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \overline{\partial}\omega \wedge \sqrt{-1}\partial\overline{\partial}\psi\right) \wedge \partial\psi,$$

it follows that

$$H_3 = \int_X \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \partial\psi \wedge \partial\omega \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

$$(2.80) \qquad - \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial\psi \wedge \overline{\partial}\partial\omega \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

$$+ \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\omega \wedge (n-5)\omega_\psi^{n-6} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \wedge \partial\psi$$

$$- \int_X \frac{\partial \psi}{\partial t} \cdot \partial\left(\frac{\partial \psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \overline{\partial}\partial\omega \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2$$

$$+ \int_X \frac{\partial \psi}{\partial t} \cdot \partial\left(\frac{\partial \psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\omega \wedge (n-5)\omega_\psi^{n-6} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \wedge \overline{\partial}\psi.$$

Hence

$$(2.81) \quad H_3 + \overline{H_3} = \frac{A_3 + B_3}{n-4} + \frac{\overline{A_3} + \overline{B_3}}{n-4},,$$

$$(2.82) \quad \frac{I^7}{a_7} + \frac{I^8}{a_8} = \frac{5}{n-4}(A_3 + B_3 + \overline{A_3} + \overline{B_3}) + (A_4 + B_4 + \overline{A_4} + \overline{B_4}).$$

Set

$$(2.83) \qquad\qquad\qquad c_4 := A_4 + B_4 + \overline{A_4} + \overline{B_4}.$$

Then

$$(2.84) \qquad\qquad\qquad \frac{I^7}{a_7} + \frac{I^8}{a_8} = \frac{5}{n-4}c_3 + c_4.$$

2.5. **Recursion formula.** Suppose now $n \geq 4$. we define, for $2 \leq i \leq n-2$,

$$A_i := \int_X \left[(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial \psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right]$$

$$(2.85) \qquad \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial t}\right),$$

$$B_i := \int_X \left[(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial \psi}{\partial t}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right]$$

$$(2.86) \qquad \wedge -\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial \psi}{\partial s}\right).$$

So

$$
\begin{aligned}
A_i &= \int_X \left[(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \right] \\
&\quad \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X -\sqrt{-1}\partial\left[(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \right] \\
&\quad \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-i-1)\partial\psi \wedge \left[\partial\overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \right. \\
&\quad - \left. \overline{\partial}\psi \wedge \partial\left(\partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \right) \right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-i-1)\partial\psi \wedge \partial\overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \\
&\quad \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&\quad + \int_X \sqrt{-1}(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \partial\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \right) \\
&\quad \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X (n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \\
&\quad \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \\
&\quad + \int_X (n-i-1)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1};
\end{aligned}
$$

thus

$$
(2.87) \quad A_i + B_i = \int_X (n-i-1)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}.
$$

Define, where a_{2i+1} and a_{2i+2} are nonzero constants and we require $\overline{a_{2i+1}} = a_{2i+2}$,

$$
\begin{aligned}
\mathcal{L}_\omega^{2i+1}(\varphi', \varphi'') &:= \frac{1}{V_\omega} \int_0^1 \int_X a_{2i+1}\partial\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}, \\
\mathcal{L}_\omega^{2i+2}(\varphi', \varphi'') &:= \frac{1}{V_\omega} \int_0^1 \int_X a_{2i+2}\overline{\partial}\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \wedge \partial\varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \\
(2.88) &\qquad \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}.
\end{aligned}
$$

Consider

$$
\begin{aligned}
\Psi^{2i+1} &= \left[\int_X a_{2i+1}\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right] ds \\
(2.89) &\quad + \left[\int_X a_{2i+1}\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right] dt,
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi^{2i+2} &= \left[\int_X a_{2i+2} \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right] ds \\
(2.90) \qquad &+ \left[\int_X a_{2i+2} \overline{\partial}\psi \wedge \overline{\partial}\omega \wedge \partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right] dt.
\end{aligned}
$$

So

$$
(2.91) \qquad d\Psi^{2i+1} = I^{2i+1} \cdot dt \wedge ds, \quad d\Psi^{2i+2} = I^{2i+2} \cdot dt \wedge ds,
$$

where

$$
\begin{aligned}
\frac{I^{2i+1}}{a_{2i+1}} \ :=\ & \int_X \frac{\partial}{\partial t}\left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right] \\
&- \int_X \frac{\partial}{\partial s}\left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right] \\
=\ & \int_X \left[\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right. \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial t \partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge (n-i-2)\omega_\psi^{n-i-3} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&\quad \wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (i-1)(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \\
&\quad \left. \wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \right] \\
&- \int_X \left[\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \right. \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial s \partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge (n-i-2)\omega_\psi^{n-i-3} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \\
&\quad \wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (i-1)(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \\
&\quad \left. \wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{I^{2i+1}}{a_{2i+1}} &= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&\quad + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&\quad + 2\int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&\quad + \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&\quad\quad \wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&\quad - \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\\
&\quad\quad \wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}+(i-1)(A_i+B_i)/(n-i-1)
\end{aligned}
$$

$$
(2.92)\qquad = H_i + \frac{i+1}{n-(i+1)}(A_i+B_i)+A_{i+1}+B_{i+1}.
$$

Here

$$
\begin{aligned}
H_i &:= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
(2.93)\qquad &\quad + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}.
\end{aligned}
$$

Similarly

$$
(2.94)\qquad \frac{I^{2i+2}}{a_{2i+2}} = \overline{H_i} + \frac{i+1}{n-(i+1)}(\overline{A_i}+\overline{B_i})+\overline{A_{i+1}}+\overline{B_{i+1}}.
$$

Calculate

$$
\begin{aligned}
H_i &= \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]\frac{\partial\psi}{\partial t}\\
&\quad + \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]\frac{\partial\psi}{\partial t}\\
&= \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right.\\
&\quad - \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right)\Big]\\
&\quad + \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right.\\
&\quad - \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right)\Big]\\
&= \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&\quad - \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&\quad - \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\overline{\partial}\omega\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}.
\end{aligned}
$$

Since

$$\int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1}$$

$$= \int_X \left[\frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-2} \right] \wedge (-\sqrt{-1}\partial \psi)$$

$$= \int_X \sqrt{-1}\partial \left[\frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-2} \right] \wedge \partial \psi$$

$$= \int_X \sqrt{-1} \left[\overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-2} \right.$$

$$- \left. \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left(\partial \omega \wedge \partial \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-2} \right) \right] \wedge \partial \psi$$

$$= \int_X \sqrt{-1}\partial \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \partial \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-2} \wedge \partial \psi$$

$$- \int_X \sqrt{-1} \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \left(\overline{\partial} \partial \omega \wedge \partial \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-2} \right.$$

$$- \left. \partial \omega \wedge \partial \overline{\partial} \psi \wedge (n-i-2)\omega_\psi^{n-i-3} \wedge \overline{\partial} \omega \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-2} \right) \wedge \partial \psi,$$

it follows that

$$H_i = \int_X \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \left(\frac{\partial \psi}{\partial t} \right) \wedge \partial \psi \wedge \partial \omega \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1}$$

$$(2.95) - \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \overline{\partial} \partial \omega \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1}$$

$$+ \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \omega \wedge (n-i-2)\omega_\psi^{n-i-3} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1} \wedge \partial \psi$$

$$- \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \partial \omega \wedge \overline{\partial} \psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1}$$

$$+ \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \omega \wedge \overline{\partial} \omega \wedge (n-i-2)\omega_\psi^{n-i-3} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1} \wedge \overline{\partial} \psi,$$

and, similarly,

$$\overline{H_i} = \int_X \partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \left(\frac{\partial \psi}{\partial t} \right) \wedge \overline{\partial} \psi \wedge \overline{\partial} \omega \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1}$$

$$(2.96) - \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \psi \wedge \partial \overline{\partial} \omega \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1}$$

$$+ \int_X \frac{\partial \psi}{\partial t} \cdot \partial \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \omega \wedge \partial \omega \wedge (n-i-2)\omega_\psi^{n-i-3} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1} \wedge \overline{\partial} \psi$$

$$- \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \partial \psi \wedge \partial \overline{\partial} \omega \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1}$$

$$+ \int_X \frac{\partial \psi}{\partial t} \cdot \overline{\partial} \left(\frac{\partial \psi}{\partial s} \right) \wedge \overline{\partial} \omega \wedge \partial \omega \wedge (n-i-2)\omega_\psi^{n-i-3} \wedge (\sqrt{-1}\partial \overline{\partial}\psi)^{i-1} \wedge \partial \psi.$$

So, for $2 \leq i \leq n - 2$,

$$(2.97) \qquad H_i + \overline{H_i} \;\;=\;\; \frac{A_i + B_i}{n - i - 1} + \frac{\overline{A_i} + \overline{B_i}}{n - i - 1},$$

$$\frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} \;\;=\;\; H_i + \overline{H_i} + \frac{i+1}{n - (i+1)}(A_i + B_i + \overline{A_i} + \overline{B_i})$$

$$+ \;\; (A_{i+1} + B_{i+1} + \overline{A_{i+1}} + \overline{B_{i+1}})$$

$$=\;\; \frac{i+2}{n - (i+1)}(A_i + B_i + \overline{A_i} + \overline{B_i})$$

$$(2.98) \qquad\qquad + \;\; (A_{i+1} + B_{i+1} + \overline{A_{i+1}} + \overline{B_{i+1}}).$$

Recall, see (2.49),

$$\frac{I^3}{a_3} + \frac{I^4}{a_4} \;\;=\;\; \frac{3\sqrt{-1}}{n-2}\left((A_1 + B_1) - (\overline{A_1} + \overline{B_1})\right) + (A_2 + B_2 + \overline{A_2} + \overline{B_2})$$

$$\frac{2I^0}{n(n-1)\sqrt{-1}} \;\;=\;\; \frac{I^1}{a_1} - \frac{I^2}{a_2} + (A_1 + B_1) - (\overline{A_1} + \overline{B_1}).$$

Let

$$(2.99) \qquad\qquad c_i \;\;:=\;\; A_i + B_i + \overline{A_i} + \overline{B_i}, \quad 2 \leq i \leq n - 1,$$

$$(2.100) \qquad\qquad c_1 \;\;:=\;\; A_1 + B_1 - (\overline{A_1} + \overline{B_1}).$$

Notice that $c_{n-1} = 0$. So

$$(2.101) \qquad \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} \;\;=\;\; \frac{i+2}{n-(i+1)}c_i + c_{i+1}, \quad 2 \leq i \leq n-2,$$

$$(2.102) \qquad \frac{I^3}{a_3} + \frac{I^4}{a_4} \;\;=\;\; \frac{3}{-(n-2)\sqrt{-1}}c_1 + c_2,$$

and

$$\frac{2I^0}{n(n-1)\sqrt{-1}} \;\;=\;\; \frac{I^1}{a_1} - \frac{I^2}{a_2} + c_1$$

$$=\;\; \frac{I^1}{a_1} - \frac{I^2}{a_2} + \left(\frac{I^3}{a_3} + \frac{I^4}{a_4} - c_2\right)\frac{n-2}{3\sqrt{-1}}$$

$$=\;\; \left(\frac{I^1}{a_1} - \frac{I^2}{a_2}\right) + \frac{n-2}{3\sqrt{-1}}\left(\frac{I^3}{a_3} + \frac{I^4}{a_4}\right) - \frac{n-2}{3\sqrt{-1}}c_2.$$

It is sufficient to determine c_2. Let

$$(2.103) \qquad\qquad J_i := \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}}.$$

Then

$$(2.104) \qquad\qquad c_{i+1} = -\frac{i+2}{n-(i+1)}c_i + J_i, \quad 2 \leq i \leq n-2.$$

To completely determine c_2, it is sufficient to solve (2.104). A direct calculation shows

$$
\begin{aligned}
c_3 \;&=\; -\frac{4}{n-3}c_2 + J_2, \\
c_4 \;&=\; -\frac{5}{n-4}c_3 + J_3 \;=\; -\frac{5}{n-4}\left(-\frac{4}{n-3}c_2 + J_2\right) + J_3 \\
&=\; (-1)^2 \frac{5\times 4}{(n-4)(n-3)}c_2 - \frac{5}{n-4}J_2 + J_3, \\
c_5 \;&=\; -\frac{6}{n-5}c_4 + J_4 \\
&=\; -\frac{6}{n-5}\left[(-1)^2\frac{5\times 4}{(n-4)(n-3)}c_2 - \frac{5}{n-4}J_2 + J_3\right] + J_4 \\
&=\; (-1)^3 \frac{6\times 5\times 4}{(n-5)(n-4)(n-3)}c_2 + (-1)^2 \frac{6\times 5}{(n-5)(n-4)}J_2 - \frac{6}{n-5}J_3 + J_4
\end{aligned}
$$

Hence, we have

$$
(2.105)\quad c_i = (-1)^{i-2}\frac{(i+1)!(n-i-1)!}{3!(n-3)!}c_2 + \sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k.
$$

By induction on i, we have

$$
\begin{aligned}
c_{i+1} \;&=\; -\frac{i+2}{n-(i+1)}c_i + J_i \\
&=\; -\frac{i+2}{n-(i+1)}\left[(-1)^{i-2}\frac{(i+1)!(n-i-1)!}{3!(n-3)!}c_2\right. \\
&\quad + \left.\sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k\right] + J_i \\
&=\; (-1)^{i+1-2}\frac{(i+2)!(n-i-2)!}{3!(n-3)!}c_2 \\
&\quad + \sum_{k=2}^{i-1}(-1)^{i-k}\frac{(i+2)!(n-i-2)!}{(k+2)!(n-k-2)!}J_k + J_i.
\end{aligned}
$$

So (2.105) holds for $2 \le i \le n-2$ and we have

$$
\begin{aligned}
(2.106)\qquad c_2 \;&=\; (-1)^{i-2}\frac{3!(n-3)!}{(i+1)!(n-i-1)!} \\
&\quad \cdot \left[c_i - \sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k\right].
\end{aligned}
$$

Setting $i = n-1$ yields

$$
\begin{aligned}
(2.107)\qquad c_2 \;&=\; (-1)^{n-3}\frac{3!(n-3)!}{n!} \\
&\quad \cdot \left[c_{n-1} - \sum_{k=2}^{n-2}(-1)^{n-2-k}\frac{n!}{(k+2)!(n-k-2)!}J_k\right], \quad n \ge 3.
\end{aligned}
$$

We deduce from (2.49) and (2.107) that

$$
\begin{aligned}
\frac{2I^0}{n(n-1)\sqrt{-1}} &= \left(\frac{I^0}{a_1} - \frac{I^2}{a_2}\right) + \frac{n-2}{3\sqrt{-1}}\left(\frac{I^3}{a_3} + \frac{I^4}{a_4}\right) \\
&\quad - \frac{n-2}{3\sqrt{-1}}(-1)^{n-3}\frac{3!(n-3)!}{n!} \\
&\quad\quad \cdot \left[c_{n-1} - \sum_{k=2}^{n-2}(-1)^{n-2-k}\frac{n!}{(k+2)!(n-k-2)!}J_k\right] \\
&= \left(\frac{I^0}{a_1} - \frac{I^2}{a_2}\right) + \frac{n-2}{3\sqrt{-1}}\left(\frac{I^3}{a_3} + \frac{I^4}{a_4}\right) \\
&\quad + \frac{2(n-2)!}{\sqrt{-1}}\sum_{k=2}^{n-2}\frac{(-1)^{k+1}}{(k+2)!(n-k-2)!}\left(\frac{I^{2k+1}}{a_{2k+1}} + \frac{I^{2k+2}}{a_{2k+2}}\right).
\end{aligned}
$$

Equivalently,

$$
(2.108) \quad \frac{2I^0}{n(n-1)\sqrt{-1}} = \left(\frac{I^1}{a_1} - \frac{I^2}{a_2}\right) + \sqrt{-1}\sum_{k=1}^{n-2}(-1)^k\frac{\binom{n}{k+2}}{\binom{n}{2}}\left(\frac{I^{2k+1}}{a_{2k+1}} + \frac{I^{2k+2}}{a_{2k+2}}\right).
$$

Set

$$
(2.109) \quad \frac{1}{a_1} = -\frac{2}{n(n-1)\sqrt{-1}}, \quad \frac{1}{a_2} = \frac{2}{n(n-1)\sqrt{-1}},
$$

$$
(2.110) \quad \frac{\sqrt{-1}(-1)^k\binom{n}{k+2}}{\binom{n}{2}a_{2k+1}} = -\frac{2}{n(n-1)\sqrt{-1}}, \quad a_{2i+1} = a_{2i+2},
$$

we obtain

$$
(2.111) \quad a_1 = -\frac{n(n-1)\sqrt{-1}}{2}, \quad a_2 = \frac{n(n-1)\sqrt{-1}}{2},
$$

$$
\begin{aligned}
a_{2k+1} &= a_{2k+2} \\
&= \frac{\sqrt{-1}(-1)^k\binom{n}{k+2}n(n-1)\sqrt{-1}}{-2\binom{n}{2}} \\
(2.112) \quad &= \frac{(-1)^k n(n-1)\binom{n}{k+2}}{2\binom{n}{2}} = (-1)^k\binom{n}{k+2}.
\end{aligned}
$$

Consequently,

$$
(2.113) \quad I^0 + \sum_{k=0}^{n-2}\left(I^{2k+1} + I^{2k+2}\right) = 0.
$$

Theorem 2.1. *The functional, for $n \geq 2$,*

$$(2.114) \qquad \mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'') \; := \; \frac{1}{V_\omega} \int_0^1 \int_X \dot{\varphi}_t \omega_{\varphi_t}^n dt$$

$$- \; \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \partial \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) dt$$

$$+ \; \frac{1}{V_\omega} \int_0^1 \int_X \frac{n(n-1)\sqrt{-1}}{2} \overline{\partial} \omega \wedge \omega_{\varphi_t}^{n-2} \wedge (\partial \dot{\varphi}_t \cdot \varphi_t) dt$$

$$+ \; \sum_{i=1}^{n-2} \frac{1}{V_\omega} \int_0^1 \int_X (-1)^i \binom{n}{i+2} \partial \varphi_t \wedge \partial \omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-i-2}$$

$$\wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}$$

$$+ \; \sum_{i=1}^{n-2} \frac{1}{V_\omega} \int_0^1 \int_X (-1)^i \binom{n}{i+2} \overline{\partial} \varphi_t \wedge \overline{\partial} \omega \wedge \partial \dot{\varphi}_t \wedge \partial \varphi_t \wedge \omega_{\varphi_t}^{n-i-2}$$

$$\wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}.$$

is independent of the choice of the smooth path $\{\varphi_t\}_{0 \leq t \leq 1}$ in $\mathcal{P}_\omega$ from φ' to φ''.

Proof. Using (2.113), we can prove Theorem 2.1 in the similar way as [2]. $\qquad \square$

Corollary 2.2. *Suppose $n \geq 2$. For any $\varphi \in \mathcal{P}_\omega$ one has*

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi) \; := \; \mathcal{L}_\omega^{\mathrm{M}}(0, \varphi) \; = \; \frac{1}{V_\omega} \sum_{i=0}^n \int_X \frac{1}{n+1} \varphi \omega_\varphi^i \wedge \omega^{n-i}$$

$$(2.115) \qquad\qquad - \; \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial \omega \wedge \overline{\partial}\varphi$$

$$+ \; \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial} \omega \wedge \partial\varphi.$$

Proof. Since $\mathcal{L}_\omega^{\mathrm{M}}(\varphi)$ is independent of the choice of smooth path, we pick $\varphi_t = t\varphi$, $0 \leq t \leq 1$. Then $\overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t = \partial \dot{\varphi}_t \wedge \partial \varphi_t = 0$, and

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi) \; = \; \frac{1}{V_\omega} \int_0^1 \int_X \varphi \omega_{t\varphi}^n dt$$

$$- \; \frac{n(n-1)}{2V_\omega} \int_0^1 \int_X \sqrt{-1}\partial \omega \wedge \omega_{t\varphi}^{n-2} \wedge (\overline{\partial}\varphi \cdot t\varphi) dt$$

$$+ \; \frac{n(n-1)}{2V_\omega} \int_0^1 \int_X \sqrt{-1}\overline{\partial} \omega \wedge \omega_{t\varphi}^{n-2} \wedge (\partial\varphi \cdot t\varphi) dt \; =: \; J_0 + J_1 + J_2.$$

Now we compute J_0, J_1, J_2, respectively. Using

$$\omega_{t\varphi} = \omega + t\sqrt{-1}\partial\overline{\partial}\varphi = \omega + t(\omega_\varphi - \omega) = t\omega_\varphi + (1-t)\omega,$$

it follows that

$$
\begin{aligned}
J_0 &= \frac{1}{V_\omega} \int_X \int_0^1 \varphi \sum_{i=0}^n \binom{n}{i} \omega_\varphi^i \wedge \omega^{n-i} t^i (1-t)^{n-i} dt \\
&= \sum_{i=0}^n \frac{1}{V_\omega} \int_X \varphi \binom{n}{i} \omega_\varphi^i \wedge \omega^{n-i} \cdot \int_0^1 t^i (1-t)^{n-i} dt \\
&= \sum_{i=0}^n \frac{1}{V_\omega} \int_X \binom{n}{i} \frac{\Gamma(i+1)\Gamma(n-i+1)}{\Gamma(n+2)} \varphi \omega_\varphi^i \wedge \omega^{n-i} \\
&= \sum_{i=0}^n \frac{1}{V_\omega} \int_X \frac{n!}{i!(n-i)!} \frac{i!(n-i)!}{(n+1)!} \varphi \omega_\varphi^i \wedge \omega^{n-i} = \sum_{i=0}^n \frac{1}{V_\omega} \int_X \frac{1}{n+1} \varphi \omega_\varphi^i \wedge \omega^{n-i},
\end{aligned}
$$

where $\Gamma(x)$ is the Gamma function. Similarly, we have

$$
\begin{aligned}
J_1 &= -\frac{n(n-1)}{2V_\omega} \int_X \int_0^1 \sqrt{-1}\partial\omega \wedge [t\omega_\varphi + (1-t)\omega]^{n-2} \wedge (\overline{\partial}\varphi \cdot t\varphi) dt \\
&= -\frac{n(n-1)}{2V_\omega} \int_X \int_0^1 \sqrt{-1}\partial\omega \\
&\quad \wedge \sum_{i=0}^{n-2} \binom{n-2}{i} t^{i+1} \omega_\varphi^i (1-t)^{n-2-i} \omega^{n-2-i} \wedge (\overline{\partial}\varphi \cdot \varphi) dt \\
&= -\frac{n(n-1)}{2V_\omega} \int_X \sqrt{-1}\partial\omega \wedge \sum_{i=0}^{n-2} \omega_\varphi^i \wedge \omega^{n-2-i} \wedge (\overline{\partial}\varphi \cdot \varphi) \\
&\quad \cdot \int_0^1 \binom{n-2}{i} t^{i+1} (1-t)^{n-2-i} dt \\
&= -\frac{n(n-1)}{2V_\omega} \int_X \sqrt{-1}\partial\omega \wedge \sum_{i=0}^{n-2} \omega_\varphi^i \wedge \omega^{n-2-i} \wedge (\overline{\partial}\varphi \cdot \varphi) \cdot \frac{i+1}{n(n-1)} \\
&= \sum_{i=0}^{n-2} \frac{-1}{2V_\omega} \int_X (i+1) \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi.
\end{aligned}
$$

Taking the complex conjugate gives

$$
J_2 = \sum_{i=0}^{n-2} \frac{1}{2V_\omega} \int_X (i+1) \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
$$

Together with the expressions of J_0, J_1 and J_2, we complete the proof. $\square$

Remark 2.3. *When (X, ω) is a compact Kähler manifold, the functional (2.114) or (2.115) coincides with the original one.*

Let S be a non-empty set and A an additive group. A mapping $\mathcal{N} : S \times S \to A$ is said to satisfy the 1-cocycle condition if

(i) $\mathcal{N}(\sigma_1, \sigma_2) + \mathcal{N}(\sigma_2, \sigma_1) = 0$;
(ii) $\mathcal{N}(\sigma_1, \sigma_2) + \mathcal{N}(\sigma_2, \sigma_3) + \mathcal{N}(\sigma_3, \sigma_1) = 0$.

Corollary 2.4. *(1) The functional $\mathcal{L}_\omega^{\mathrm{M}}$ satisfies the 1-cocycle condition.*
(2) For any $\varphi \in \mathcal{P}_\omega$ and any constant $C \in \mathbb{R}$, we have

$$(2.116) \qquad \mathcal{L}_\omega^{\mathrm{M}}(\varphi, \varphi + C) = C \cdot \left(1 + \frac{\mathrm{Err}_\omega(\varphi)}{V_\omega}\right), \quad \mathrm{Err}_\omega(\varphi) := \int_X \omega^n - \int_X \omega_\varphi^n.$$

In particular, if $\partial\overline{\partial}\omega = \partial\omega \wedge \overline{\partial}\omega = 0$, then $\mathcal{L}_\omega^{\mathrm{M}}(\varphi, \varphi + C) = C$.
(3) For any $\varphi_1, \varphi_2 \in \mathcal{P}_\omega$ and any constant $C \in \mathbb{R}$, we have

$$(2.117) \qquad \mathcal{L}_\omega^{\mathrm{M}}(\varphi_1, \varphi_2 + C) = \mathcal{L}_\omega^{\mathrm{M}}(\varphi_1, \varphi_2) + C \cdot \left(1 - \frac{\mathrm{Err}_\omega(\varphi_2)}{V_\omega}\right).$$

In particular, if $\partial\overline{\partial}\omega = \partial\omega \wedge \overline{\partial}\omega = 0$, then $\mathcal{L}_\omega^{\mathrm{M}}(\varphi_1, \varphi_2 + C) = \mathcal{L}_\omega^{\mathrm{M}}(\varphi_1, \varphi_2) + C$.

Proof. The proof is similar to that given in [2, 3]. $\qquad\qquad\qquad\qquad\qquad\square$

3. Aubin-Yau functionals on compact complex manifolds

3.1. **The main idea.** The strategy to construct Aubin-Yau functionals is to use the inequalities (1.5) and (1.6) to determine the extra terms. Firstly we can show that

$$\frac{n}{n+1}\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) = \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$(3.1) \qquad\qquad\qquad\qquad - \frac{\mathcal{A}_\omega(\varphi) + \mathcal{B}_\omega(\varphi)}{n+1} + \frac{\mathcal{C}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)}{2},$$

$$(n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) = \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$(3.2) \qquad\qquad\qquad\qquad + \frac{\mathcal{E}_\omega(\varphi) + \mathcal{F}_\omega(\varphi)}{2} + (n+1)\left[\mathcal{A}_\omega^1(\varphi) + \mathcal{B}_\omega^1(\varphi)\right]$$

$$\qquad\qquad\qquad\qquad - \frac{\mathcal{A}_\omega^2(\varphi) + \mathcal{B}_\omega^2(\varphi)}{n-1},$$

where $\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi), \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi), \mathcal{A}_\omega(\varphi), \mathcal{B}_\omega(\varphi), \mathcal{C}_\omega(\varphi), \mathcal{D}_\omega(\varphi), \mathcal{E}_\omega(\varphi), \mathcal{F}_\omega(\varphi), \mathcal{A}_\omega^1(\varphi), \mathcal{B}_\omega^1(\varphi),$
$\mathcal{A}_\omega^2(\varphi), \mathcal{B}_\omega^2(\varphi)$ are functionals determined in next subsection. Inspired by (3.1) and (3.2), we define Aubin-Yau functionals as follows:

$$\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad &:= \quad \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) \\
&+ \quad a_1^1\mathcal{A}_\omega^1(\varphi) + a_1^2\mathcal{A}_\omega^2(\varphi) + b_1^1\mathcal{B}_\omega^1(\varphi) + b_1^2\mathcal{B}_\omega^2(\varphi) \\
(3.3) \quad &+ \quad c_1\mathcal{C}_\omega(\varphi) + d_1\mathcal{D}_\omega(\varphi) + e_1\mathcal{E}_\omega(\varphi) + f_1\mathcal{F}_\omega(\varphi), \\
\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad &= \quad \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) \\
&+ \quad (a_2^1 - 1)\mathcal{A}_\omega^1(\varphi) + (a_2^2 - 1)\mathcal{A}_\omega^2(\varphi) + (b_2^1 - 1)\mathcal{B}_\omega^1(\varphi) + (b_2^2 - 1)\mathcal{B}_\omega^2(\varphi) \\
(3.4) \quad &+ \quad c_2\mathcal{C}_\omega(\varphi) + d_2\mathcal{D}_\omega(\varphi) + e_2\mathcal{E}_\omega(\varphi) + f_2\mathcal{F}_\omega(\varphi).
\end{aligned}$$

Here $a_j^i, b_j^i, c_k, d_k, e_k$, and f_k are constants determined by the following two inequalities:

$$\frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i} \geq 0,$$

$$(n+1)\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) = \sum_{i=1}^{n-1}\frac{n-1-i}{V_\omega}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i} \geq 0.$$

It gives us a system of linear equations of 16 parameters and eventually we evaluate these parameters:

$$a_1^1 \;=\; b_1^1 \;=\; -\frac{n}{n-1}, \quad a_2^1 \;=\; b_2^1 \;=\; -\frac{n}{n^2-1},$$

$$a_1^2 \;=\; b_1^2 \;=\; \frac{n}{(n-1)^2}, \quad a_2^2 \;=\; b_2^2 \;=\; \frac{n}{n+1}\left(1+\frac{n}{(n-1)^2}\right),$$

$$c_1 \;=\; d_1 \;=\; -\frac{n+1}{2(n-1)}, \quad e_2 \;=\; f_2 \;=\; -\frac{n}{2(n^2-1)},$$

$$c_2 \;=\; d_2 \;=\; e_1 \;=\; f_1 \;=\; -\frac{1}{2(n-1)}.$$

By a long computation we find an explicit and shorted formulae for $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$:

$$(3.5) \quad \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \;=\; \frac{1}{V_\omega}\int_X (\omega^n - \omega_\varphi^n)$$

$$- \;\frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$+ \;\frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi,$$

$$(3.6) \quad \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \;=\; -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^n$$

$$- \;\frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2} \wedge \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi$$

$$+ \;\frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.$$

3.2. The construction of Aubin-Yau functionals. Let (X,g) be a compact complex manifold of the complex dimension $n \geq 3$ and ω be its associated real $(1,1)$-form. We recall some notation in [2]. For any $\varphi \in \mathcal{P}_\omega$ we set

$$(3.7) \qquad \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) \;:=\; \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega_\varphi^n),$$

$$(3.8) \qquad \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) \;:=\; \int_0^1 \frac{\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(s\cdot\varphi)}{s}ds \;=\; \frac{1}{V_\omega}\int_0^1 \int_X \varphi(\omega^n - \omega_{s\cdot\varphi}^n)ds.$$

Two relations showed in [2] are

$$(3.9) \qquad\qquad \frac{n}{n+1}\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)$$

$$= \;\frac{1}{V_\omega}\int_X \varphi\cdot(-\sqrt{-1}\partial\overline{\partial}\varphi) \wedge \sum_{j=1}^{n-1}\frac{j}{n+1}\omega^{n-1-j} \wedge \omega_\varphi^j,$$

$$(3.10) \qquad\qquad (n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)$$

$$= \;\frac{1}{V_\omega}\int_X \varphi\cdot(-\sqrt{-1}\partial\overline{\partial}\varphi) \wedge \sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j} \wedge \omega_\varphi^j.$$

According to the expression of $\mathcal{L}_\omega^{\mathrm{M}}(\varphi)$, we set

$$(3.11) \qquad \mathcal{A}_\omega(\varphi) \;:=\; \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge -\sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi,$$

$$(3.12) \qquad \mathcal{B}_\omega(\varphi) \;:=\; \sum_{i=0}^{n-2} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \partial\varphi.$$

Using (3.9) we obtain

$$\frac{n}{n+1}\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)$$

$$= \; \frac{1}{V_\omega} \int_X \sqrt{-1}\partial\left(\varphi \sum_{j=1}^{n-1} \frac{j}{n+1}\omega^{n-1-j} \wedge \omega_\varphi^j\right) \wedge \overline{\partial}\varphi$$

$$= \; \frac{1}{V_\omega} \int_X \sqrt{-1}\left(\partial\varphi \wedge \sum_{j=1}^{n-1} \frac{j}{n+1}\omega^{n-1-j} \wedge \omega_\varphi^j\right) \wedge \overline{\partial}\varphi$$

$$+ \; \frac{1}{V_\omega} \int_X \sqrt{-1}\varphi \sum_{j=1}^{n-1} \frac{j}{n+1}\left[(n-1-j)\omega^{n-2-j} \wedge \partial\omega \wedge \omega_\varphi^j\right.$$

$$+ \; \left.\omega^{n-1-j} \wedge j\omega_\varphi^{j-1} \wedge \partial\omega\right] \wedge \overline{\partial}\varphi;$$

from the identity $i(n-1-i) + (i+1)^2 = (i+1) + in$, it follows that

$$\frac{n}{n+1}\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)$$

$$= \; \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^{n-1-i} \wedge \omega_\varphi^i$$

$$+ \; \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{i(n-1-i)}{n+1} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$+ \; \frac{1}{V_\omega} \sum_{i=0}^{n-2} \frac{(i+1)^2}{n+1} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$= \; \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$+ \; \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{i+1+in}{n+1} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$+ \; \frac{1}{(n+1)V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi.$$

To simplify the notation, we set

$$(3.13) \qquad \mathcal{C}_\omega(\varphi) := \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{in}{n+1} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi.$$

Since $n \geq 3$, the above expression is well defined. Therefore

$$(3.14) \qquad \frac{n}{n+1} \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)$$

$$= \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i} - \frac{2}{n+1} \mathcal{A}_\omega(\varphi) + \mathcal{C}_\omega(\varphi).$$

On the other hand, using the slightly different method, we obtain (see A.1)

$$(3.15) \qquad \frac{n}{n+1} \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)$$

$$= \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i} - \frac{2}{n+1} \mathcal{B}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)$$

where

$$(3.16) \qquad \mathcal{D}_\omega(\varphi) := \frac{1}{V_\omega} \sum_{i=1}^{n-2} \frac{in}{n+1} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge -\sqrt{-1}\partial\omega \wedge \partial\varphi.$$

Equations (3.14) and (3.15) implies

$$\frac{n}{n+1} \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) = \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$(3.17) \qquad\qquad - \frac{\mathcal{A}_\omega(\varphi) + \mathcal{B}_\omega(\varphi)}{n+1} + \frac{\mathcal{C}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)}{2}.$$

By the definition we have

$$\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) = \frac{1}{V_\omega} \int_0^1 \int_X (\varphi \omega^n - \varphi \omega_{s\varphi}^n) ds = \frac{1}{V_\omega} \int_X \varphi \omega^n - \frac{1}{V_\omega} \int_0^1 \int_X \varphi \omega_{t\varphi}^n dt$$

$$= \frac{1}{V_\omega} \int_X \varphi \omega^n - (\mathcal{L}^{\mathrm{M}}_\omega(\varphi) - \mathcal{A}_\omega(\varphi) - \mathcal{B}_\omega(\varphi))$$

$$= \frac{1}{V_\omega} \int_X \varphi \omega^n - \mathcal{L}^{\mathrm{M}}_\omega(\varphi) + \mathcal{A}_\omega(\varphi) + \mathcal{B}_\omega(\varphi).$$

If we define

$$(3.18) \qquad \mathcal{E}_\omega(\varphi) := \sum_{i=0}^{n-3} \frac{n^2}{V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi,$$

$$(3.19) \qquad \mathcal{A}^1_\omega(\varphi) := \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge -\sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$(3.20) \qquad \mathcal{A}^2_\omega(\varphi) := \frac{n-1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge -\sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi,$$

then $\mathcal{A}^1_\omega(\varphi) + \mathcal{A}^2_\omega(\varphi) = \mathcal{A}_\omega(\varphi)$ and it follows that (see A.1)

$$(n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet} - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) = \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-i) \int_X \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$(3.21) \qquad\qquad + \mathcal{E}_\omega(\varphi) + 2(n+1)\mathcal{A}^1_\omega(\varphi) - \frac{2}{n-1}\mathcal{A}^2_\omega(\varphi).$$

Introduce the corresponding functionals

$$(3.22) \qquad \mathcal{F}_\omega(\varphi) \quad := \quad \sum_{i=0}^{n-3} \frac{n^2}{V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge -\sqrt{-1}\partial\omega \wedge \partial\varphi,$$

$$(3.23) \qquad \mathcal{B}_\omega^1(\varphi) \quad := \quad \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \partial\varphi$$

$$(3.24) \qquad \mathcal{B}_\omega^2(\varphi) \quad := \quad \frac{n-1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \partial\varphi.$$

Then $\mathcal{B}_\omega^1(\varphi) + \mathcal{B}_\omega^2(\varphi) = \mathcal{B}_\omega(\varphi)$ and hence (see A.2)

$$(n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) \quad = \quad \frac{1}{V_\omega} \sum_{i=0}^{n-1}(n-1-i) \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$(3.25) \qquad\qquad\qquad + \quad \mathcal{F}_\omega(\varphi) + 2(n+1)\mathcal{B}_\omega^1(\varphi) - \frac{2}{n-1}\mathcal{B}_\omega^2(\varphi).$$

The equations (3.21) and (3.25) together gives

$$(n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) \quad = \quad \frac{1}{V_\omega} \sum_{i=0}^{n-1}(n-1-i) \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$(3.26) \qquad\qquad\qquad + \quad \frac{\mathcal{E}_\omega(\varphi) + \mathcal{F}_\omega(\varphi)}{2} + (n+1)(\mathcal{A}_\omega^1(\varphi) + \mathcal{B}_\omega^1(\varphi))$$

$$\qquad\qquad\qquad - \quad \frac{\mathcal{A}_\omega^2(\varphi) + \mathcal{B}_\omega^2(\varphi)}{n-1}.$$

Now, we define Aubin-Yau functionals over any compact complex manifolds as follows:

$$\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \quad := \quad \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)$$

$$\qquad\qquad + \quad a_1^1 \mathcal{A}_\omega^1(\varphi) + a_1^2 \mathcal{A}_\omega^2(\varphi) + b_1^1 \mathcal{B}_\omega^1(\varphi) + b_1^2 \mathcal{B}_\omega^2(\varphi)$$

$$(3.27) \qquad\qquad + \quad c_1 \mathcal{C}_\omega(\varphi) + d_1 \mathcal{D}_\omega(\varphi) + e_1 \mathcal{E}_\omega(\varphi) + f_1 \mathcal{F}_\omega(\varphi),$$

$$\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \quad := \quad -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n$$

$$\qquad\qquad + \quad a_2^1 \mathcal{A}_\omega^1(\varphi) + a_2^2 \mathcal{A}_\omega^2(\varphi) + b_2^1 \mathcal{B}_\omega^1(\varphi) + b_2^2 \mathcal{B}_\omega^2(\varphi)$$

$$\qquad\qquad + \quad c_2 \mathcal{C}_\omega(\varphi) + d_2 \mathcal{D}_\omega(\varphi) + e_2 \mathcal{E}_\omega(\varphi) + f_2 \mathcal{F}_\omega(\varphi),$$

$$\qquad\qquad = \quad \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)$$

$$\qquad\qquad + \quad (a_2^1 - 1)\mathcal{A}_\omega^1(\varphi) + (a_2^2 - 1)\mathcal{A}_\omega^2(\varphi) + (b_2^1 - 1)\mathcal{B}_\omega^1(\varphi) + (b_2^2 - 1)\mathcal{B}_\omega^2(\varphi)$$

$$(3.28) \qquad\qquad + \quad c_2 \mathcal{C}_\omega(\varphi) + d_2 \mathcal{D}_\omega(\varphi) + e_2 \mathcal{E}_\omega(\varphi) + f_2 \mathcal{F}_\omega(\varphi).$$

Plugging (3.27) and (3.28) into (3.26) and (3.17), we obtain

$$(3.29) \quad \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = \frac{1}{V_\omega} \sum_{i=1}^{n-1} \frac{i}{n+1} \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i} \geq 0,$$

and

$$(3.30) \quad (n+1)\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) = \sum_{i=0}^{n-1} \frac{n-1-i}{V_\omega} \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i} \geq 0,$$

where we require that constants satisfy the following linear equations system

$$(3.31)\quad \frac{n}{n+1}a_1^1 - (a_2^1 - 1) = \frac{1}{n+1}, \quad \frac{n}{n+1}a_1^2 - (a_2^2 - 1) = \frac{1}{n+1},$$

$$(3.32)\quad \frac{n}{n+1}b_1^1 - (b_2^1 - 1) = \frac{1}{n+1}, \quad \frac{n}{n+1}b_1^2 - (b_2^2 - 1) = \frac{1}{n+1},$$

$$(3.33)\quad \frac{n}{n+1}c_1 - c_2 = -\frac{1}{2}, \quad \frac{n}{n+1}d_1 - d_2 = -\frac{1}{2},$$

$$(3.34)\quad \frac{n}{n+1}e_1 - e_2 = 0, \quad \frac{n}{n+1}f_1 - f_2 = 0,$$

$$(3.35)\quad (n+1)(a_2^1 - 1) - a_1^1 = -(n+1), \quad (n+1)(a_2^2 - 1) - a_1^2 = \frac{1}{n-1},$$

$$(3.36)\quad (n+1)(b_2^1 - 1) - b_1^1 = -(n+1), \quad (n+1)(b_2^2 - 1) - b_1^2 = \frac{1}{n-1},$$

$$(3.37)\quad (n+1)c_2 - c_1 = 0, \quad (n+1)d_2 - d_1 = 0,$$

$$(3.38)\quad (n+1)e_2 - e_1 = -\frac{1}{2}, \quad (n+1)f_2 - f_1 = -\frac{1}{2}.$$

The constants $a_i^j, b_i^j, c_i, d_i, e_i$ and f_i , calculated in Appendix B, are

$$(3.39)\quad a_1^1 = b_1^1 = -\frac{n}{n-1}, \quad a_2^1 = b_2^1 = -\frac{n}{n^2-1},$$

$$(3.40)\quad a_1^2 = b_1^2 = \frac{n}{(n-1)^2}, \quad a_2^2 = b_2^2 = \frac{n}{n+1}\left(1 + \frac{n}{(n-1)^2}\right)$$

$$(3.41)\quad c_1 = d_1 = -\frac{n+1}{2(n-1)}, \quad e_2 = f_2 = -\frac{n}{2(n^2-1)},$$

$$(3.42)\quad c_2 = d_2 = e_1 = f_1 = -\frac{1}{2(n-1)}.$$

The explicit formulas for $\mathcal{I}_\omega^{AY}(\varphi)$ and $\mathcal{J}_\omega^{AY}(\varphi)$ are given in Proposition C.1 and C.2 respectively. Namely,

$$\mathcal{I}_\omega^{AY}(\varphi) = \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega_\varphi^n) - \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$+ \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi,$$

$$(3.43)\quad = \frac{1}{V_\omega}\sum_{i=0}^{n-1}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i},$$

$$\mathcal{J}_\omega^{AY}(\varphi) = -\mathcal{L}_\omega^M(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^n - \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$+ \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi$$

$$(3.44)\quad = \frac{1}{V_\omega}\sum_{i=0}^{n-1}\int_X \frac{n-i}{n+1}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-i-1}.$$

Here the formulas (3.43) and (3.44) come from the solution of the system of linear equations (3.29) and (3.30).

From (3.29), (3.30), (3.37) and (3.44), we deduce the following

Theorem 3.1. *For any $\varphi \in \mathcal{P}_\omega$, one has*

$$(3.45) \qquad \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \;\geq\; 0,$$

$$(3.46) \qquad (n+1)\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \;\geq\; 0.$$

In particular

$$(3.47) \qquad \frac{1}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \;\leq\; \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \;\leq\; \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi),$$

$$(3.48) \qquad \frac{n+1}{n}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \;\leq\; \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \;\leq\; (n+1)\mathcal{J}_\omega^{\mathrm{AY}}(\varphi),$$

$$(3.49) \quad \frac{1}{n}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \;\leq\; \frac{1}{n+1}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \;\leq\; \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$$

$$(3.50) \qquad\qquad\qquad\qquad \leq\; \frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \;\leq\; n\mathcal{J}_\omega^{\mathrm{AY}}(\varphi).$$

APPENDIX A. PROOF THE IDENTITIES (3.14), (3.21) AND (3.25)

In Appendix A we verify the identities (3.14), (3.21) and (3.25).

$$(A.1) \qquad \frac{n}{n+1}\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)$$

$$= \; \frac{1}{V_\omega}\int_X \left(\varphi \cdot \sum_{j=1}^{n-1} \frac{j}{n+1}\omega^{n-1-j}\wedge\omega_\varphi^j \right) \wedge \sqrt{-1}\overline{\partial}\partial\varphi$$

$$= \; \frac{1}{V_\omega}\int_X -\sqrt{-1}\overline{\partial}\left(\varphi \sum_{j=1}^{n-1} \frac{j}{n+1}\omega^{n-1-j}\wedge\omega_\varphi^j \right) \wedge \partial\varphi$$

$$= \; \frac{1}{V_\omega}\int_X -\sqrt{-1}\left(\overline{\partial}\varphi \wedge \sum_{j=1}^{n-1} \frac{j}{n+1}\omega^{n-1-j}\wedge\omega_\varphi^j \right) \wedge \partial\varphi$$

$$+ \; \frac{1}{V_\omega}\int_X -\sqrt{-1}\varphi \sum_{j=1}^{n-1} \frac{j}{n+1}\left[(n-1-j)\omega^{n-2-j}\wedge\overline{\partial}\omega\wedge\omega_\varphi^j \right.$$

$$+ \; \left. \omega^{n-1-j}\wedge j\omega_\varphi^{j-1}\wedge\overline{\partial}\omega \right]\wedge\partial\varphi$$

$$= \; \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega_\varphi^i$$

$$+ \; \frac{1}{V_\omega}\sum_{i=1}^{n-2}\frac{i(n-1-i)}{n+1}\int_X -\sqrt{-1}\varphi\omega^{n-2-i}\wedge\omega_\varphi^i\wedge\overline{\partial}\omega\wedge\partial\varphi$$

$$+ \; \frac{1}{V_\omega}\sum_{i=0}^{n-2}\frac{(i+1)^2}{n+1}\int_X -\sqrt{-1}\varphi\omega^{n-2-i}\wedge\omega_\varphi^i\wedge\overline{\partial}\omega\wedge\partial\varphi$$

$$= \; \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega_\varphi^i - \frac{2}{n+1}\mathcal{B}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)$$

which gives (3.15). Calculate

$$(n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet} - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)$$

$$= \frac{1}{V_\omega} \int_X \left(\varphi \sum_{j=0}^{n-1} (n-1-j)\omega^{n-1-j} \wedge \omega_\varphi^j \right) \wedge (-\sqrt{-1}\partial\overline{\partial}\varphi)$$

$$= \frac{1}{V_\omega} \int_X \sqrt{-1}\partial \left(\varphi \sum_{j=0}^{n-1} (n-1-j)\omega^{n-1-j} \wedge \omega_\varphi^j \right) \wedge \overline{\partial}\varphi$$

$$= \frac{1}{V_\omega} \int_X \sqrt{-1}\partial\varphi \wedge \sum_{j=0}^{n-1} (n-1-j)\omega^{n-1-j} \wedge \omega_\varphi^j \wedge \overline{\partial}\varphi$$

$$+ \frac{1}{V_\omega} \int_X \sqrt{-1}\varphi \sum_{j=0}^{n-1} \left[(n-1-j)^2\omega^{n-2-j} \wedge \partial\omega \wedge \omega_\varphi^j \right.$$

$$+ (n-1-j)j\omega^{n-1-j} \wedge \omega_\varphi^{j-1} \wedge \partial\omega \big] \wedge \overline{\partial}\varphi$$

$$= \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-j) \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$+ \frac{1}{V_\omega} \sum_{j=0}^{n-1} (n-1-j)^2 \int_X \varphi\omega^{n-2-j} \wedge \omega_\varphi^j \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$+ \frac{1}{V_\omega} \sum_{j=0}^{n-1} (n-1-j)j \int_X \varphi\omega^{n-1-j} \wedge \omega_\varphi^{j-1} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$= \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-j) \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$+ \frac{1}{V_\omega} \sum_{j=0}^{n-2} (n-1-j)^2 \int_X \varphi\omega^{n-2-j} \wedge \omega_\varphi^j \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$+ \frac{1}{V_\omega} \sum_{i=0}^{n-3} (i+1)(n-i-2) \int_X \varphi\omega^{n-2-i} \wedge \omega_\varphi^i \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

$$= \frac{1}{V_\omega} \sum_{i=0}^{n-1} (n-1-i) \int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i}$$

$$+ \frac{1}{V_\omega} \sum_{i=0}^{n-3} [n^2 - (n+1)(i+1)] \int_X \varphi\omega^{n-2-i} \wedge \omega_\varphi^i \wedge (\sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi)$$

$$+ \frac{1}{V_\omega} \int_X \varphi\omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi$$

where we use the elementary identity

$$(n-1-i)^2 + (i+1)(n-i-2)$$

$$= (n-1)^2 + i^2 - 2(n-1)i + (n-2)(i+1) - i(i+1)$$

$$= n^2 - 2n + 1 + i^2 - 2ni + 2i + ni + n - 2i - 2 - i^2 - i$$

$$= n^2 - n - 1 - ni - i \quad = \quad -(n+1)(i+1) + n^2.$$

Using the definitions of $\mathcal{E}_\omega(\varphi), \mathcal{A}_\omega^1(\varphi), \mathcal{A}_\omega^2(\varphi)$, we have $\mathcal{A}_\omega^1(\varphi) + \mathcal{A}_\omega^2(\varphi) = \mathcal{A}_\omega(\varphi)$ and hence (3.21) holds. Similarly, we have $\mathcal{B}_\omega^1(\varphi) + \mathcal{B}_\omega^2(\varphi) = \mathcal{B}_\omega(\varphi)$ and

$$
\begin{aligned}
&(n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) \\
&= \frac{1}{V_\omega} \int_X -\sqrt{-1}\overline{\partial}\left(\varphi \sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega_\varphi^j\right)\wedge\partial\varphi \\
&= \frac{1}{V_\omega}\int_X -\sqrt{-1}\overline{\partial}\varphi\wedge\sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega_\varphi^j\wedge\partial\varphi \\
&\quad + \frac{1}{V_\omega}\int_X -\sqrt{-1}\varphi\sum_{j=0}^{n-1}(n-1-j)((n-1-j)\omega^{n-2-j}\wedge\overline{\partial}\omega\wedge\omega_\varphi^j \\
&\quad + j\omega^{n-1-j}\wedge\omega_\varphi^{j-1}\wedge\overline{\partial}\omega)\wedge\partial\varphi \\
&= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega_\varphi^i \\
&\quad + \frac{1}{V_\omega}\sum_{j=0}^{n-1}(n-1-j)^2\int_X \varphi\omega^{n-2-j}\wedge\omega_\varphi^j\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
&\quad + \frac{1}{V_\omega}\sum_{j=0}^{n-1}(n-1-j)j\int_X \varphi\omega^{n-1-j}\wedge\omega_\varphi^{j-1}\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
&= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega_\varphi^i \\
&\quad + \frac{1}{V_\omega}\sum_{i=0}^{n-2}(n-1-i)^2\int_X \varphi\omega^{n-2-i}\wedge\omega_\varphi^i\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
&\quad + \frac{1}{V_\omega}\sum_{i=0}^{n-3}(n-i-2)(i+1)\int_X \varphi\omega^{n-2-i}\wedge\omega_\varphi^i\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
&= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega_\varphi^i \\
&\quad + \frac{1}{V_\omega}\sum_{i=0}^{n-3}[n^2-(n+1)(i+1)]\int_X \varphi\omega^{n-2-i}\wedge\omega_\varphi^i\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
&\quad + \frac{1}{V_\omega}\int_X \varphi\omega_\varphi^{n-2}\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi).
\end{aligned}
$$

and hence

$$
\begin{aligned}
(n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) - \mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{n-1-i} \\
(\mathrm{A.2}) \qquad &\quad + \mathcal{F}_\omega(\varphi) + 2(n+1)\mathcal{B}_\omega^1(\varphi) - \frac{2}{n-1}\mathcal{B}_\omega^2(\varphi).
\end{aligned}
$$

Therefore (3.21) and (3.25) together gives

$$
\begin{aligned}
(n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \;=\; & \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i} \\
(\mathrm{A.3}) \qquad\qquad + & \frac{\mathcal{E}_\omega(\varphi)+\mathcal{F}_\omega(\varphi)}{2} + (n+1)(\mathcal{A}^1_\omega(\varphi)+\mathcal{B}^1_\omega(\varphi)) \\
- & \frac{\mathcal{A}^2_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)}{n-1}.
\end{aligned}
$$

APPENDIX B. SOLVE THE SYSTEM OF THE LINEAR EQUATIONS

In this section we try to solve the system of the linear equations (3.31)-(3.38). Firstly we solve (3.31) and (3.35) as follows: (3.31) and (3.35) gives us the following equations

$$
\text{(B.1)} \qquad \frac{n}{n+1}a^1_1 - \frac{1}{n+1} \;=\; a^1_2 - 1,
$$
$$
(n+1)(a^1_2 - 1) + (n+1) \;=\; a^1_1,
$$
$$
\text{(B.2)} \qquad \frac{n}{n+1}a^2_1 - \frac{1}{n+1} \;=\; a^2_2 - 1,
$$
$$
(n+1)(a^2_2 - 1) - \frac{1}{n-1} \;=\; a^2_1.
$$

Plugging the first equation into second equation in (B.1), we have

$$
(n+1)\left(\frac{n}{n+1}a^1_1 - \frac{1}{n+1}\right) + (n+1) = a^1_1
$$

which implies

$$
\text{(B.3)} \qquad a^1_1 = -\frac{n}{n-1}, \quad a^1_2 = -\frac{n}{n^2-1}.
$$

Similarly,

$$
(n+1)\left(\frac{n}{n+1}a^2_1 - \frac{1}{n+1}\right) - \frac{1}{n+1} = a^2_1,
$$

therefore

$$
\text{(B.4)} \qquad a^2_1 = \frac{n}{(n-1)^2}, \quad a^2_2 = \frac{n}{n+1}\left(1 + \frac{n}{(n-1)^2}\right) = \frac{n^3-n^2+n}{n^3-n^2-n+1}.
$$

Secondly, (3.32) and (3.36) implies

$$
\text{(B.5)} \qquad \frac{n}{n+1}b^1_1 - \frac{1}{n+1} \;=\; b^1_2 - 1,
$$
$$
(n+1)(b^1_2 - 1) \;=\; \frac{1}{1} - (n+1),
$$
$$
\text{(B.6)} \qquad \frac{n}{n+1}b^2_1 - \frac{1}{n+1} \;=\; b^2_2 - 1,
$$
$$
(n+1)(b^2_2 - 1) \;=\; b^2_1 + \frac{1}{n-1}.
$$

The above linear equations system gives

$$
(n+1)\left(\frac{n}{n+1}b^1_1 - \frac{1}{n+1}\right) = b^1_1 - (n+1)
$$

and

$$(n+1)\left(\frac{n}{n+1}b_1^2 - \frac{1}{n+1}\right) = b_1^2 + \frac{1}{n-1},$$

respectively. Hence

(B.7)
$$b_1^1 = -\frac{n}{n-1},$$

$$b_2^1 = -\frac{n}{n^2-1},$$

(B.8)
$$b_1^2 = \frac{n}{(n-1)^2},$$

$$b_2^2 = \frac{n}{n+1}\left(1 + \frac{n}{(n-1)^2}\right).$$

Continuously, equations (3.33) and (3.37) shows that

$$\frac{n}{n+1}c_1 - c_2 = -\frac{1}{2}, \quad (n+1)c_2 - c_1 = 0,$$

$$\frac{n}{n+1}d_1 - d_2 = -\frac{1}{2}, \quad (n+1)d_2 - d_1 = 0.$$

Eliminating c_2 and d_2 respectively, we have

$$(n+1)\left(\frac{n}{n+1}c_1 + \frac{1}{2}\right) - c_1 = 0,$$

$$(n+1)\left(\frac{n}{n+1}d_1 + \frac{1}{2}\right) - d_1 = 0.$$

Thus

(B.9)
$$c_1 = -\frac{n+1}{2(n-1)},$$

$$c_2 = -\frac{1}{2(n-1)},$$

(B.10)
$$d_1 = -\frac{n+1}{2(n-1)},$$

$$d_2 = -\frac{1}{2(n-1)}.$$

Similarly, from (3.34) and (3.38) we obtain

$$\frac{n}{n+1}e_1 - e_2 = 0, \quad (n+1)e_2 - e_1 = -\frac{1}{2},$$

$$\frac{n}{n+1}f_1 - f_2 = 0, \quad (n+1)f_2 - f_1 = -\frac{1}{2},$$

and hence

(B.11)
$$e_1 = f_1 = -\frac{1}{2(n-1)},$$

(B.12)
$$e_2 = f_2 = -\frac{n}{2(n^2-1)}.$$

APPENDIX C. EXPLICIT FORMULAS OF $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ AND $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$

In this section we give the explicit formulas of $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$. In what follows, we assume that $n \geq 3$. Using the constants determined in Appendix B, we have

$$
\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) \\
&= \frac{n}{n-1} \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&\quad - \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&\quad - \frac{n}{n-1} \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&\quad + \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&\quad - \frac{n}{n-1} \sum_{i=1}^{n-2} \frac{i}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-3-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&\quad + \frac{n}{n-1} \sum_{i=1}^{n-2} \frac{i}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&\quad - \frac{n^2}{n-1} \sum_{i=0}^{n-3} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&\quad + \frac{n^2}{n-1} \sum_{i=0}^{n-3} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}
$$

When $n = 3$, it is easy to see that

$$
\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega} \int_X \varphi(\omega^3 - \omega_\varphi^3) \\
&\quad + \frac{3}{4V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{3}{4V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&\quad - \frac{3}{4V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{3}{4V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&\quad - \frac{3}{4V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{4V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&\quad - \frac{9}{4V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{9}{4V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&= \frac{1}{V_\omega} \int_X \varphi(\omega^3 - \omega_\varphi^3) \\
&\quad - \frac{3}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&\quad - \frac{3}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}
$$

For general $n \geq 4$, a simple computation shows

$$
\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) \\
&+ \sum_{i=1}^{n-3} \frac{1}{2V_\omega} \left[\frac{n(i+1)}{n-1} - \frac{in}{n-1} - \frac{n^2}{n-1} \right] \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&- \frac{n(n-2)}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&- \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \sum_{i=1}^{n-3} \frac{1}{2V_\omega} \left[-\frac{n(i+1)}{n-1} + \frac{in}{n-1} + \frac{n^2}{n-1} \right] \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&+ \frac{n(n-2)}{n-1} \frac{2}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&+ \frac{n}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{n^2}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{n^2}{n-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&= \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) \\
&- \sum_{i=1}^{n-3} \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \sum_{i=1}^{n-3} \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{n}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}
$$

Thus

Proposition C.1. *If $n \geq 3$, one has*

$$
\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega} \int_X \varphi(\omega^n - \omega_\varphi^n) \\
&- \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \\
= \quad & -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n \\
+ \quad & \frac{n}{n^2-1} \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
- \quad & \frac{n}{n+1}\left(n-1+\frac{n}{n-1}\right)\frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
- \quad & \frac{n}{n^2-1} \sum_{i=0}^{n-3} \frac{i+1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
+ \quad & \frac{n}{n+1}\left(n-1+\frac{n}{n-1}\right)\frac{2}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{n}{n^2-1} \sum_{i=1}^{n-2} \frac{i}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
+ \quad & \frac{n}{n^2-1} \sum_{i=1}^{n-2} \frac{i}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{n^3}{n^2-1} \sum_{i=0}^{n-3} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
+ \quad & \frac{n^3}{n^2-1} \sum_{i=0}^{n-3} \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi
\end{aligned}
$$

When $n = 3$, we have

$$
\begin{aligned}
& \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \\
= \quad & -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^3 \\
+ \quad & \frac{3}{8}\frac{1}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{3}{4}\left(2+\frac{3}{2}\right)\frac{1}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
- \quad & \frac{3}{8}\frac{1}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{3}{4}\left(2+\frac{3}{2}\right)\frac{1}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{3}{8}\frac{1}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{8}\frac{1}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{27}{8}\frac{1}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{27}{8}\frac{1}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
= \quad & -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^3 \\
- \quad & \frac{3}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_\omega} \int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{3}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_\omega} \int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi
\end{aligned}
$$

When $n \geq 4$, we have

$$\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n$$

$$+ \quad \frac{1}{2V_\omega} \sum_{i=1}^{n-3} \left[\frac{n(i+1)}{n^2-1} - \frac{in}{n^2-1} - \frac{n^3}{n^2-1} \right] \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$+ \quad \frac{n}{n^2-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$- \quad \frac{n}{n+1} \left(n-1+\frac{n}{n-1} \right) \frac{1}{2V_\omega} \int_Z \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$+ \quad \frac{1}{2V_\omega} \sum_{i=1}^{n-3} \left[-\frac{n(i+1)}{n^2-1} + \frac{in}{n^2-1} + \frac{n^3}{n^2-1} \right] \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi$$

$$- \quad \frac{n}{n^2-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi$$

$$+ \quad \frac{n}{n+1} \left(n-1+\frac{n}{n-1} \right) \frac{1}{2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi$$

$$- \quad \frac{n(n-2)}{(n^2-1)2V_\omega} \int_Z \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi + \frac{n(n-2)}{(n^2-1)2V_\omega} \int_X \varphi \omega_\varphi^{n-2} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi$$

$$- \quad \frac{n^3}{n^2-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi + \frac{n^3}{n^2-1} \frac{1}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi$$

Using the identities

$$\frac{n(i+1)}{n^2-1} - \frac{in}{n^2-1} - \frac{n^3}{n^2-1} = \frac{n-n^3}{n^2-1} = -n,$$

$$-\frac{n}{n+1} \left(n-1+\frac{n}{n-1} \right) - \frac{n(n-2)}{n^2-1} = \frac{-n[(n-1)^2+n]-n(n-2)}{n^2-1} = -n,$$

the above expression can be simplified as

$$\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n$$

$$- \quad \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi - \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$+ \quad \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi + \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi.$$

In summary,

Proposition C.2. *If $n \geq 3$, one has*

$$\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega} \int_X \varphi \omega^n$$

$$- \quad \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi - \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \partial \omega \wedge \overline{\partial} \varphi$$

$$+ \quad \frac{n}{2V_\omega} \sum_{i=1}^{n-2} \int_X \varphi \omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi + \frac{n}{2V_\omega} \int_X \varphi \omega^{n-2} \wedge \sqrt{-1} \overline{\partial} \omega \wedge \partial \varphi.$$

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