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HARNACK INEQUALITY FOR THE NEGATIVE POWER GAUSSIAN CURVATURE FLOW

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Abstract. In this paper, we study the power of Gaussian curvature flow of a compact convex hypersurface and establish its Harnack inequality when the power is negative. In the Harnack inequality, we require that the absolute value of the power is strictly positive and strictly less than the inverse of the dimension of the hypersurface.

1. Introduction

The Harnack estimate or Harnack inequality plays an important role in geometric flows. For the heat equation, P. Li and S.-T. Yau [6] obtained the corresponding Harnack inequality by using the parabolic maximum principle. Hamilton [4, 5] proved a Harnack inequality for the Ricci flow and mean curvature flow for all dimensions. For a Harnack inequality for m-power mean curvature flow, we refer to [1], [9] and [10], where m is positive. B. Chow [3] considered a Harnack inequality for m-power Gaussian curvature flow for $m > 0$.

In this paper, we consider the negative power Gaussian curvature flow of a compact convex hypersurface $F_0 : M^n \to \mathbb{R}^{n+1}$,

$$\frac{\partial}{\partial t}F(x,t) = \frac{1}{K(x,t)^b} \cdot \nu(x,t),\ 0 < b < \frac{1}{n};\ F(x,0) = F_0(x),\ x \in M^n. \tag{1.1}$$

Here K is the Gaussian curvature and ν denotes the outward unit normal vector field. Using the similar argument in [3], we obtain a Harnack inequality for the flow (1.1).

Theorem 1.1. *Suppose that $F_0 : M^n \to \mathbb{R}^{n+1}$ is a compact convex hypersurface. If $0 < b < \frac{1}{n}$, then*

$$\frac{\partial}{\partial t}\left(\frac{1}{K(x,t)^b}\right) + \left|\nabla\left(\frac{1}{K(x,t)^b}\right)\right|_h^2 - \frac{nb}{(1-nb)t}\left(\frac{1}{K(x,t)^b}\right) \le 0, \tag{1.2}$$

where the notation $|\cdot|_h$ is defined in the next section.

When $\frac{1}{n} \le b \le 1$, some interesting results have been derived in [7], where the author considered $n = 2$.

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2. Notation and evolution equations

2.1. **Notation.** Suppose that $F : M^n \to \mathbb{R}^{n+1}$ is a hypersurface. The second fundamental form is given by

$$h_{ij} = -\left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, \nu \right\rangle, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard metric on $\mathbb{R}^{n+1}$. The induced metric

$$g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle \tag{2.2}$$

on M gives us the mean curvature

$$H = g^{ij} h_{ij} \tag{2.3}$$

and the Gaussian curvature

$$K = \frac{\det(h_{ij})}{\det(g_{ij})}. \tag{2.4}$$

If $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$ are 1-forms and $s = \{s_{ij}\}$ is a symmetric positive definite covariant 2-tensor, we use the short notation

$$\langle \alpha, \beta \rangle_s \equiv \langle \alpha_i, \beta_i \rangle_s := s^{-1}_{ij} \alpha_i \beta_j,$$

where (s^{-1}_{ij}) is the inverse matrix of (s_{ij}). Similarly, if $A = \{\alpha_{ijk}\}$ and $B = \{\beta_{pqr}\}$ are covariant 3-tensors, we define

$$\langle A, B \rangle_s \equiv \langle A_{ijk}, B_{ijk} \rangle_s := s^{-1}_{ip} s^{-1}_{jq} s^{-1}_{kr} A_{ijk} B_{pqr}.$$

Finally, we define the Laplacian-type operator by

$$\Box := h^{-1}_{ij} \nabla_i \nabla_j. \tag{2.5}$$

Here ∇ denotes the Levi-Civita connection of the induced metric g on M.

Let M^n be a convex hypersurface in $\mathbb{R}^{n+1}$, $\alpha = \{\alpha_i\}$ a 1-form on M^n, and ϕ a smooth function on M. We have the following identities (see [2] or [3]):

$$R_{ijk\ell} = h_{i\ell} h_{jk} - h_{ik} h_{j\ell}, \tag{2.6}$$

$$\nabla_i h_{jk} = \nabla_j h_{ik}, \tag{2.7}$$

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_k = -R_{ijk\ell} g^{\ell p} \alpha_p, \tag{2.8}$$

$$\nabla_i \nabla_j F = -h_{ij} \nu, \tag{2.9}$$

$$\nabla_i \nu = h_{ij} g^{jk} \nabla_k F, \tag{2.10}$$

$$\nabla_i \nabla_j \nu = g^{k\ell} \nabla_k h_{ij} \cdot \nabla_\ell F - g^{\ell k} h_{i\ell} h_{kj} \nu, \tag{2.11}$$

$$\nabla_i \left(K h^{-1}_{ij} \right) = 0, \tag{2.12}$$

$$(\nabla_i \Box - \Box \nabla_i) \phi = -\langle \nabla_i h_{jk}, \nabla_j \nabla_k \phi \rangle_h - (n-1) h_{ij} g^{jk} \nabla_k \phi. \tag{2.13}$$

2.2. **Evolution equations.** Now we consider a generalized Gaussian curvature flow

$$\frac{\partial}{\partial t} F(x,t) = -f(K(x,t)) \cdot \nu(x,t), \quad F(x,0) = F_0(x), \ x \in M^n, \tag{2.14}$$

where $f : (0, +\infty) \to \mathbb{R}$ is a smooth function depending only on the Gaussian curvature K, which satisfies $f' > 0$ everywhere in order to guarantee a short time existence. Such a type of Gaussian curvature flow is called the *f-Gaussian curvature flow.*

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Remark 2.1. For convenience, in what follows, we write $f_t = f(K_t)$ and $\partial_2 = \frac{\partial}{\partial_t}$.

Under the f-Gaussian curvature flow, it is easy to verify the following evolution equations (compared with Lemma 3.1 in [3]), where $h_t = \{(h_t)_{ij}\}$:

$$\partial_t (g_t)_{ij} = -2 f_t (h_t)_{ij}, \tag{2.15}$$

$$\partial_t \nu_t = \nabla f_t = f'_t \cdot \nabla K_t, \tag{2.16}$$

$$\partial_t (h_t)_{ij} = \nabla_i \nabla_j f_t - f_t (g_t)^{k\ell} (h_t)_{ik} (h_t)_{\ell j}, \tag{2.17}$$

$$\partial_t K_t = f'_t K_t \cdot \left(\Box_t K_t + \frac{f''_t}{f'_t} |\nabla K_t|^2_{h_t} + \frac{f_t}{f'_t K_t} H_t K_t \right), \tag{2.18}$$

$$\partial_t f_t = f'_t K_t \cdot \left[\Box_t f_t + H_t f_t \right], \tag{2.19}$$

$$\partial_t H_t = \Delta_t f_t + f_t |h_t|^2_{g_t}, \tag{2.20}$$

$$\partial_t \Box_t = -\langle \nabla\nabla f_t, \nabla\nabla \rangle_{h_t} + f_t \Delta_t + \left(2 - n + \frac{f_t}{f'_t K_t} \right) \langle \Delta_t f_t, \Delta_t \rangle_g. \tag{2.21}$$

Remark 2.2. If M^n is compact and convex, then $H = H_0 > 0$; using the evolution equation (2.20), we see that $H(x,t) = H_t(x) > 0$ under the f-Gaussian curvature flow. According to (2.4), we conclude that $K(x,t) > 0$ along the f-Gaussian curvature flow. Therefore $\frac{1}{K_t}$ is well-defined.

3. Harnack inequality

Motivated by the self-similar solutions in [3], we define a time-dependent tensor field $P_t = \{(P_t)_{ij}\}$ by

$$(P_t)_{ij} = \nabla_i \nabla_j f_t - (h_t)^{-1}_{k\ell} \nabla_k (h_t)_{ij} \cdot \nabla_\ell f_t + f_t (g_t)^{k\ell} (h_t)_{ik} (h_t)_{\ell j}. \tag{3.1}$$

Taking the trace of $(P_t)_{ij}$ with respect to $(h_t)_{ij}$, we set

$$\mathbf{P}_t = (h_t)^{-1}_{ij} (P_t)_{ij}. \tag{3.2}$$

Since $\nabla K_t = K_t (h_t)^{-1}_{pq} \nabla (h_t)_{pq}$ by (2.12), we can rewrite $\mathbf{P}_t$ as

$$\mathbf{P}_t = \Box_t f_t + f_t H_t - (h_t)^{-1}_{ij} (h_t)^{-1}_{k\ell} \nabla_k (h_t)_{ij} \cdot \nabla_\ell f_t = \Box_t f_t + f_t H_t - \frac{|\nabla f_t|^2_{h_t}}{f'_t K_t}. \tag{3.3}$$

3.1. Evolution equation for $\mathbf{P}_t$.

In this subsection our task is to find the evolution equation for $\mathbf{P}_t$. Before doing this, we first write down some elementary formulas which will be used in our complicated and tedious computation. Since f is smooth depending only on K_t, we have $\nabla f_t = f'_t \nabla K_t$ and

$$\begin{aligned} \Box_t f_t &= (h_t)^{-1}_{ij} \nabla_i \nabla_j f_t = (h_t)^{-1}_{ij} \nabla_i \left(f'_t \nabla_j K_t \right) \\ &= (h_t)^{-1}_{ij} \left[f''_t \nabla_i K_t \cdot \nabla_j K_t + f'_t \nabla_i \nabla_j K_t \right] = f'_t \Box_t f_t + f''_t |\nabla K_t|^2_{h_t}. \end{aligned} \tag{3.4}$$

Using $\nabla_i K_t = \nabla_i f_t / f'_t$, we obtain

$$\nabla_i \nabla_j K_t = \nabla_i \left(f'^{-1}_t \nabla_i f_t \right) = -f'^{-3}_t f''_t \nabla_i f_t \cdot \nabla_j f_t + f'^{-1}_t \nabla_i \nabla_j f_t. \tag{3.5}$$

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The next useful formula is

$$\begin{aligned}\Box_t(f_t')K_t) &= (h_t)^{-1}_{ij}\nabla_i\left(f_t''\nabla_j K_t\cdot K_t + f_t'\cdot\nabla_j K_t\right)\\
&= (h_t)^{-1}_{ij}\left[f_t''K_t\nabla_i\nabla_j K_t + (f_t'''K_t+2f_t'')\nabla_i K_t\cdot\nabla_j K_t + f_t'\nabla_i\nabla_j K_t\right]\\
&= (h_t)^{-1}_{ij}\left[f_t'+f_t''K_t\right]\left(\frac{1}{f_t'}\nabla_i\nabla_j f_t - \frac{f_t''}{f_t'^3}\nabla_i f_t\cdot\nabla_j f_t\right)\\
&+ (h_t)^{-1}_{ij}\left[f_t'''K_t+2f_t''\right]\frac{\nabla_i f_t\cdot\nabla_j f_t}{f_t'^2}\\
&= \left(1+\frac{f_t''K_t}{f_t'}\right)\Box_t f_t + \left(\frac{f_t'''K_t}{f_t'^2}+\frac{f_t''}{f_t'^2}-\frac{f_t''^2K_t}{f_t'^3}\right)|\nabla f_t|^2_{h_t}. \quad (3.6)\end{aligned}$$

Lemma 3.1. *Under the f-Gaussian curvature flow, we have*

$$\begin{aligned}\partial_t\left(\Box_t f_t\right) &= f_t'K_t\cdot\Box_t(\Box_t f_t) + 2\left(1+\frac{f_t''K_t}{f_t'}\right)\langle\nabla f_t,\nabla(\Box_t f_t\rangle_{h_t}\\
&+ \left(1+\frac{f_t''K_t}{f_t'}\right)(\Box_t f_t)^2 + f_t'K_t\cdot\Box_t(H_t f(K_t))\\
&+ 2\left(1+\frac{f_t''K_t}{f_t'}\right)\langle\nabla f_t,\nabla(H_t f_t)\rangle_{h_t} + \left(1+\frac{f_t''K_t}{f_t'}\right)H_t f_t\cdot\Box_t f_t \quad (3.7)\\
&+ \left(\frac{f_t'''K_t}{f_t'^2}+\frac{f_t''}{f_t'^2}-\frac{f_t''^2K_t}{f_t'^3}\right)|\nabla f_t|^2_{h_t}\left[\Box_t f_t + H_t f_t\right]\\
&- |\nabla\nabla f_t|^2_{h_t} + f_t\Delta_t f_t + \left(2-n+\frac{f_t}{f_t'K_t}\right)|\nabla f_t|^2_{g_t}.\end{aligned}$$

Proof. From $\partial_t(\Box_t f_t) = (\partial_t\Box_t)\,f_t + \Box_t\,(\partial_t f_t)$, we get

$$\partial_t(\Box_t f_t) = -|\nabla\nabla f_t|^2_{h_t} + f_t\Delta_t f_t + \left(2-n+\frac{f_t}{f_t'K_t}\right)|\nabla f_t|^2_{g_t} + \Box_t\left[f_t'K_t(\Box_t f_t + H_t f_t)\right].$$

Now we evaluate the last term:

$$\begin{aligned}\Box_t\left[f_t'K_t(\Box_t f_t + H_t f_t)\right] &= \Box_t(f_t'K_t)\cdot(\Box_t f_t + H_t f_t)\\
&+ f_t'K_t\cdot\Box_t(\Box_t f_t + H_t f_t) + 2\langle\nabla(f_t'K_t),\nabla(\Box_t f_t + H_t f_t)\rangle_{h_t}\\
&= \left(\left(1+\frac{f_t''K}{f_t'}\right)\Box_t f_t + \left(\frac{f_t'''K_t}{f_t'^2}+\frac{f_t''}{f_t'^2}-\frac{f_t''^2K_y}{f_t'^3}\right)|\nabla f_t|^2_{h_t}\right)(\Box_t f_t + H_t f_t)\\
&+ f_t'K_t\cdot\Box_t(\Box_t f_t) + f_t'K_t\cdot\Box_t(H_t f_t)\\
&+ 2\left\langle\left(1+\frac{f_t''K_t}{f_t'}\right)\nabla f_t,\nabla(\Box_t f_t)+\nabla(H_t f_t)\right\rangle_{h_t}.\end{aligned}$$

Simplifying the above and plugging into the expression of $\partial_t(\Box_t f_t)$, we obtain the required result. □

Lemma 3.2. *Under the f-Gaussian curvature flow, we have*

$$\begin{aligned}\partial_t\left(\frac{|\nabla f_t|^2_h}{f_t'K_t}\right) &= (f_t'K_t)\Box_t\left(\frac{|\nabla f_t|^2_{h_t}}{f_t'K_t}\right) + 2\left(1+\frac{f_t''K_t}{f_t'}\right)\left\langle\nabla f_t,\nabla\left(\frac{|\nabla f_t|^2_{h_t}}{f_t'K_t}\right)\right\rangle_{h_t}\\
&- 2|\nabla\nabla f_t|^2_{h_t} + 2\langle\nabla_i(h_t)_{jk},\nabla_i f_t\cdot\nabla_j\nabla_k f_t\rangle_{h_t}\\
&+ \left(1+\frac{f_t''K_t}{f_t'}\right)\frac{2\Box_t f_t|\nabla f_t|^2_g}{f_t'K_t}\end{aligned}$$

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$$
\begin{aligned}
&+ \quad \left[1+\left(1+\frac{f_t''K_t}{f_t'}\right)\frac{f_t}{f_t'K_t}\right]H_t|\nabla f_t|^2_{h_t}\\
&- \quad \langle \nabla_i f\cdot\nabla_j h_{kl}, \nabla_j f\cdot \nabla_i h_{kl}\rangle_h + \left(\frac{f}{f'K}+2-n\right)|\nabla f|^2_g\\
&+ \quad 2f_t\langle \nabla H_t, \nabla f_t\rangle_{h_t} - \left[\frac{f_t''^2}{f_t'^4}-\frac{f_t'''}{f_t'^3}+\frac{1}{(f_t'K_t)^2}\right]\nabla f_t|^2_{h_t}.
\end{aligned}
$$

Proof. The proof is similar to that in [3]. We observe first that

$$
\begin{aligned}
\partial_t\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right) \quad &= \quad \partial_t\left(f_t'^{-1}K_t^{-1}(h_t)^{-1}_{ij}\nabla_i f_t\nabla_j f_t\right)\\
= -\frac{1}{f_t'^2}&\left(\frac{f_t''}{f_t'K_t}+\frac{1}{K_t^2}\right)f_t'K_t(\Box_t f_t + H_t f_t)|\nabla f_t|^2_{h_t}\\
&+2(f_t'K_t)^{-1}\langle \nabla(f_t'K_t(\Box_t f_t+H_t f_t)), \nabla f_t\rangle_{h_t}\\
&-(f_t'K_t)^{-1}(h_t)^{-1}_{ik}(h_t)^{-1}_{j\ell}\left(\nabla_k\nabla_\ell f_t - f_t g^{pq}h_{kp}h_{q\ell}\right)\nabla_i f_t\nabla_j f_t\\
= -\frac{1}{f_t'^2}&\left(\frac{f_t''}{f_t'K_t}+\frac{1}{K_t^2}\right)f_t'K_t\left(\Box_t f_t + H_t f_t\right)|\nabla f_t|^2_{h_t}\\
&-(f_t'K_t)^{-1}\langle \nabla_i\nabla_j f_t, \nabla_i f_t\nabla_j f_t\rangle_{h_t} + (f_t'K_t)^{-1}f_t|\nabla f_t|^2_{g_t}\\
&+2\langle \nabla(\Box_t f_t + H_t f_t), \nabla f_t\rangle_{h_t} + 2(f_t'K_t)^{-1}\langle\nabla(f_t'K_t), \nabla f_t\rangle_{h_t}(\Box_t f_t + H_t f_t)\\
= \left(1+\frac{f_t''K_t}{f_t'}\right)&\frac{1}{f_t'K_t}(\Box_t f_t + H_t f_t)|\nabla f_t|^2_{h_t} - \frac{1}{f_t'K_t}\langle\nabla_i\nabla_j f_t, \nabla_i f_t\nabla_j f_t\rangle_h\\
&+\frac{f_t}{f_t'K_t}|\nabla f_t|^2_{g_t} + 2\langle \nabla(\Box_t f_t + H_t f_t), \nabla f_t\rangle_{h_t}.
\end{aligned}
$$

On the other hand, we compute the Laplacian of $(f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}$ with respect to $(h_t)_{ij}$:

$$
\begin{aligned}
\Box_t\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right) \quad &= \quad \Box_t\left(f_t'^{-1}K_t^{-1}(h_t)^{-1}_{ij}\nabla_i f_t\nabla_j f_t\right)\\
= \Box_t&\left((f_t'K_t)^{-1}\right)|\nabla f_t|^2_{h_t} + (f_t'K_t)^{-1}\cdot\Box_t(h_t)^{-1}_{ij}\cdot\nabla_i f_t\nabla_j f_t\\
&+(f_t'K_t)^{-1}(h_t)^{-1}_{ij}\left(2\Box_t(\nabla_i f_t)\cdot\nabla_j f_t + 2\left\langle \nabla_k\nabla_i f_t, \nabla_k\nabla_j f_t\right\rangle_{h_t}\right)\\
&+4\left\langle \nabla_k h^{-1}_{ij}, \nabla_k\nabla_i f\cdot\nabla_j f\right\rangle_h (f'K)^{-1}\\
&+2\left\langle \nabla_k(f_t'K_t)^{-1}, \nabla_k(h_t)^{-1}_{ij}\cdot\nabla_i f_t\nabla_j f_t\right\rangle_{h_t}\\
&+4\left\langle \nabla_k(f_t'K_t)^{-1}, (h_t)^{-1}_{ij}\nabla_k\nabla_i f_t\cdot\nabla_j f_t\right\rangle_{h_t}\\
= \Box_t&\left((f_t'K_t)^{-1}\right)|\nabla f_t|^2_{h_t} + (f_t'K_t)^{-1}\cdot\Box(h_t)^{-1}_{ij}\cdot\nabla_i f_t\nabla_j f_t\\
&+2(f_t'K_t)^{-1}\langle\Box_t(\nabla f_t), \nabla f_t\rangle_{h_t}\\
&+2(f_t'K_t)^{-1}|\nabla_i\nabla_j f_t|^2_{h_t} + 2\left\langle\nabla_k(f_t'K_t)^{-1}, \nabla_k(h_t)^{-1}_{ij}\right\rangle_{h_t}\cdot\nabla_i f_t\nabla_j f_t\\
&+4\left\langle \nabla_k(f_t'K_t)^{-1}\nabla_i f_t, \nabla_k\nabla_i f_t\right\rangle_{h_t} - 4(f_t'K_t)^{-1}\langle\nabla_k(h_t)_{ij}, \nabla_k\nabla_i f_t\cdot\nabla_j f_t\rangle_{h_t}.
\end{aligned}
$$

We compute some elementary formulas which will be used later. Note that

$$
\begin{aligned}
\nabla(K_t^{-1}) \quad &= \quad -K_t^{-2}\nabla K_t \quad = \quad -\frac{\nabla f_t}{f_t'K_t^2},\\
\nabla(f_t'^{-1}) \quad &= \quad -f_t'^{-2}\nabla f_t' \quad = \quad -\frac{f_t''}{f_t'^3}\nabla f_t.
\end{aligned}
$$

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Therefore

$$\begin{aligned}\Box_t(K_t^{-1}) &= (h_t)^{-1}_{ij}\left(2K_t^{-3}\nabla_i K_t\cdot\nabla_j K_t - K_t^{-2}\nabla_i\nabla_j K_t\right)\\ &= (h_t)^{-1}_{ij}\left[\frac{2}{K_t^3}\frac{\nabla_i f_t\cdot\nabla_j f_t}{f_2'^2} - \frac{1}{K_t^2}\left(\frac{\nabla_i\nabla_j f_f}{f_t'} - \frac{f_t''}{f_t'^3}\nabla_i f_t\nabla_j f_t\right)\right]\\ &= \frac{2}{f_t'^2K_t^3}|\nabla f_t|^2_{h_t} - \frac{1}{f_t'K_t^2}\Box_t f_t + \frac{f_t''}{f_t'^3K_t^2}|\nabla f_t|^2_{h_t}\\ &= \left(\frac{2}{K_t}+\frac{f_t''}{f_t'}\right)\frac{1}{(f_t'K_t)^2}|\nabla f_t|^2_{h_t} - \frac{1}{f_t'K_t^2}\Box_t f_t,\\ \Box_t(f_t'^{-1}) &= (h_t)^{-1}_{ij}\left(2\frac{f_t''^2}{f_t'^3}\nabla_i K_t\nabla_j K_t - \frac{f_t'''}{f_t'^2}\nabla_i K_t\nabla_j K_t - \frac{f_t''}{f_t'^2}\nabla_i\nabla_j K_t\right)\\ &= \left(\frac{3f_t''^2}{f_t'^5} - \frac{f_t'''}{f_t'^4}\right)|\nabla f_t|^2_{h_t} - \frac{f_t''}{f_t'^3}\Box_t f_t,\\ \left\langle\nabla(f_t'^{-1}),\nabla(K_t^{-1})\right\rangle_{h_t} &= \frac{f_t''}{f_t'^4K_t^2}|\nabla f_t|^2_{h_t}.\end{aligned}$$

Using these equations, we arrive at

$$\begin{aligned}\Box_t\left((f_t'K_t)^{-1}\right) &= \Box_t(f_t'^{-1})\cdot K_t^{-1} + f_t'^{-1}\cdot\Box_t(K_t^{-1}) + 2\left\langle\nabla f_t'^{-1},\nabla K_t^{-1}\right\rangle_{h_t}\\ &= \left(\frac{3f_t''^2}{K_tf_t'^5} - \frac{f_t'''}{K_tf_t'^4}\right)|\nabla f_t|^2_{h_t} - \frac{f_t''}{K_tf_t'^3}\Box_t f_t\\ &\quad + \left(\frac{2}{f_t'^3K_t^3} + \frac{f_t''}{f_t'^4K_t^2}\right)|\nabla f_t|^2_{h_t} - \frac{1}{f_t'^2K_t^2}\Box_t f_t + \frac{2f_t''}{f_t'^4K_t^2}|\nabla f_t|^2_{h_t}\\ &= -\left(1+\frac{f_t''K_t}{f_t'}\right)\frac{1}{f_t'^2K_t^2}\Box_t f_t\\ &\quad + \left(\frac{3f_t''^2}{f_t'^4} - \frac{f_t'''}{f_t'^3} + \frac{2}{f_t'^2K_t^2} + \frac{3f_t''}{f_t'^3K_t}\right)\frac{|\nabla f_t|^2_{h_t}}{f_t'K_t}.\end{aligned}$$

The Laplacian of $(h_t)^{-1}_{ij}$ with respect to $(h_t)_{ij}$ is given by

$$\begin{aligned}\Box\left((h_t)^{-1}_{ij}\right) = (h_t)^{-1}_{k\ell}\nabla_k\left(-(h_t)^{-1}_{ip}(h_t)^{-1}_{jq}\nabla_\ell(h_t)_{pq}\right) &= -(h_t)^{-1}_{ip}(h_t)^{-1}_{jq}\Box_t(h_t)_{pq}\\ &+ (h_t)^{-1}_{k\ell}(h_t)^{-1}_{ip}(h_t)^{-1}_{jr}h^{-1}_{qs}\nabla_k(h_t)_{rs}\nabla_\ell(h_t)_{pq}\\ &+ (h_t)^{-1}_{k\ell}(h_t)^{-1}_{jq}(h_t)^{-1}_{ir}(h_t)^{-1}_{ps}\nabla_k(h_t)_{rs}\nabla_\ell(h_t)_{pq}.\end{aligned}$$

So the second term in the expression of $\Box_t((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t})$ is

$$\begin{aligned}\Box_t((h_t)^{-1}_{ij})\cdot\nabla_i f_t\nabla_j f_t &= -(h_t)^{-1}_{ip}(h_t)^{-1}_{jq}\Box_t(h_t)_{pq}\cdot\nabla_i f_t\nabla_j f_t\\ &\quad + 2(h_t)^{-1}_{k\ell}(h_t)^{-1}_{ip}(h_t)^{-1}_{jr}(h_t)^{-1}_{qs}\nabla_k(h_t)_{rs}\nabla_\ell(h_t)_{pq}\cdot\nabla_i f_t\nabla_j f_t\\ &= -\left\langle\Box_t(h_t)_{ij},\nabla_i f_t\nabla_j f_t\right\rangle_{h_t}\\ &\quad + 2\left\langle\nabla_i f_t\nabla_j(h_t)_{k\ell},\nabla_j f_t\cdot\nabla_i(h_t)_{k\ell}\right\rangle_{h_t}.\end{aligned}$$

Combining those identities, we have

$$\begin{aligned}\Box_t\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right) &= -\left(1+\frac{f_t''K_t}{f_t'}\right)\frac{1}{(f_t'K_t)^2}\Box_t f_t\cdot|\nabla f_t|^2_{h_t}\\ &+ \left(\frac{3f_t''^2}{f_t'^4} - \frac{f_t'''}{f_t'^3} + \frac{2}{f_t'^2K_t^2} + \frac{3f_t''}{f_t'^3K_t}\right)\frac{|\nabla f_t|^4_{h_t}}{f_t'K_t} - (f_t'K_t)^{-1}\left\langle\Box_t(h_t)_{ij},\nabla_i f_t\nabla_j f_t\right\rangle_{h_t}\end{aligned}$$

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$$+ 2(f_t'K_t)^{-1} \langle \nabla_i f_t \nabla_i (h_t)_{k\ell}, \nabla_j f_t \cdot \nabla_i (h_t)_{k\ell} \rangle_{h_t}$$
$$+ 2(f_t'K_t)^{-1} \big\langle \nabla_i (\Box_t f_t) + \langle \nabla_i (h_t)_{jk}, \nabla_j \nabla_k f_t \rangle_{h_t} + (n-1)(h_t)_{ij}(g_t)^{jk} \nabla_k f_t, \nabla_i f_t \big\rangle_h$$
$$+ 2(f_t'K_t)^{-1} |\nabla\nabla f_t|^2_{h_t} + 2 \left\langle \nabla_k (f_t'K_t)^{-1}, \nabla_k (h_t)^{-1}_{ij} \right\rangle_{h_t} \nabla_i f_t \cdot \nabla_j f_t$$
$$+ 4 \left\langle \nabla_k (f_t'K_t)^{-1} \nabla_i f_t, \nabla_k \nabla_i f_t \right\rangle_{h_t} - 4(f_t'K_t)^{-1} \langle \nabla_k (h_t)_{ij}, \nabla_k \nabla_i f_t \cdot \nabla_j f_t \rangle_{h_t},$$

and we also have

$$\Box_t \left((f_t'K_t)^{-1} |\nabla f_t|^2_{h_t}\right) = -\left(1 + \frac{f_t''K_t}{f_t'}\right) \frac{1}{(f_t'K_t)^2} \Box_t f_t \cdot |\nabla f_t|^2_{h_t}$$
$$+ \left[\frac{3f_t''^2}{f_t'^4} - \frac{f_t'''}{f_t'^3} + \frac{2}{(f_t'K_t)^2} + \frac{3f_t''}{f_t'^3 K_t}\right] \frac{|\nabla f_t|^4_{h_t}}{f_t'K_t}$$
$$- (f_t'K_t)^{-1} \langle \Box_t (h_t)_{ij}, \nabla_i f_t \nabla_j f_t \rangle_{h_t} + 2(f_t'K_t)^{-1} \langle \nabla_i f_t \nabla_j (h_t)_{kl}, \nabla_j f_t \nabla_i (h_t)_{kl} \rangle_{h_t}$$
$$+ 2(n-1)(f_t'K_t)^{-1} |\nabla f_t|^2_{g_t} + 2(f_t'K_t)^{-1} \langle \nabla(\Box_t f_t), \nabla f_t \rangle_{h_t}$$
$$+ 2\left[\frac{f_t''}{f_t'^3 K_t} + \frac{1}{(f_t'K_t)^2}\right] \langle \nabla_i (h_t)_{jk}, \nabla_i f_t \nabla_j f_t \nabla_k f_t \rangle_{h_t}$$
$$- 4\left[\frac{f_t''}{f_t'^3 K_t} + \frac{1}{(f_t'K_t)^2}\right] \langle \nabla_i \nabla_j f_t, \nabla_i f_t \nabla_j f_t \rangle_{h_t}$$
$$- 2(f_t'K_t)^{-1} \langle \nabla_i (h_t)_{jk}, \nabla_i f_t \nabla_j \nabla_k f_t \rangle_{h_t} + 2(f_t'K_t)^{-1} |\nabla\nabla f_t|^2_{h_t},$$

where we use the identities

$$\nabla\left((f_t'K_t)^{-1}\right) = -\left(\frac{f_t''}{f_t'^3 K_t} + \frac{1}{f_t'^2 K_t^2}\right) \nabla f_t,$$
$$\nabla_k (h_t)^{-1}_{ij} = -(h_t)^{-1}_{ip} (h_t)^{-1}_{jq} \nabla_k (h_t)_{pq}.$$

From the above equations we obtain

$$\partial_t \left((f_t'K_t)^{-1} |\nabla f_t|^2_{h_t}\right) - (f_t'K_t) \Box_t \left((f_t'K_t)^{-1} |\nabla f_t|^2_{h_t}\right)$$
$$= \left(1 + \frac{f_t''K_t}{f_t'}\right) \frac{1}{f_t'K_t} (2\Box_t f_t + H_t f_t) |\nabla f_t|^2_{h_t}$$
$$+ \left[4\left(\frac{f_t''K_t}{f_t'} + 1\right) - 1\right] \frac{1}{f_t'K_t} \langle \nabla_i \nabla_j f_t, \nabla_i f_t \cdot \nabla_j f_t \rangle_{h_t}$$
$$+ \left[\frac{f_t}{f_t'K_t} - 2(n-1)\right] |\nabla f_t|^2_{g_t} + 2\langle \nabla(H_t f_t), \nabla f_t \rangle_{h_t}$$
$$- \left[\frac{3f_t''^2}{f_t'^4} - \frac{f_t'''}{f_t'^3} + \frac{2}{(f_t'K_t)^2} + \frac{3f_t''}{f_t'^3 K_t}\right] |\nabla f_t|^4_{h_t}$$
$$+ \langle \Box_t (h_t)_{ij}, \nabla_i f_t \cdot \nabla_j f_t \rangle_{h_t} - 2 \langle \nabla_i f_t \cdot \nabla_j (h_t)_{k\ell}, \nabla_j f_t \cdot \nabla_i (h_t)_{k\ell} \rangle_h$$
$$- 2\left(\frac{f_t''}{f_t'^2} + \frac{1}{f_t'K_t}\right) \langle \nabla_i (h_t)_{jk}, \nabla_i f_t \cdot \nabla_j f_t \cdot \nabla_k f_t \rangle_{h_t}$$
$$+ 2 \langle \nabla_i (h_t)_{jk}, \nabla_i f_t \cdot \nabla_j \nabla_k f_t \rangle_{h_t} - 2|\nabla\nabla f_t|^2_{h_t}.$$

On the other hand, from [2], we get

$$\Box_t (h_t)_{ij} = \frac{1}{f_t'K_t} \nabla_i \nabla_j f_t - \left(1 + \frac{f_t''K_t}{f_t'}\right) \frac{1}{(f_t'K_t)^2} \nabla_i f_t \nabla_j f_t$$
$$+ \langle \nabla_i (h)_{k\ell}, \nabla_j (h)_{k\ell} \rangle_{h_t} - H_t (h_t)_{ij} + n(g_t)^{k\ell} (h_t)_{ik} (h_t)_{\ell j}.$$

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Here we use the identity $\nabla_i\nabla_j f_t = f_t''\nabla_i K_t\nabla_j K_t + f_t'\nabla_i\nabla_j K_t$. Plugging it into the previous formula, we get

$$\begin{aligned}
&\partial_t\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right) - (f_t'K_t)\Box_t\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right)\\
&\quad= \left(1+\frac{f_t''K_t}{f_t'}\right)\frac{2}{f_t'K_t}\Box_t f_t\cdot|\nabla f_t|^2_{h_t} + 4\left(1+\frac{f_t''K_t}{f_t'}\right)\frac{1}{f_t'K_t}\left\langle\nabla_i\nabla_j f_t, \nabla_i f_t\nabla_j f_t\right\rangle_{h_t}\\
&\qquad+\left[\frac{f_t}{f_t'K_t} - 2(n-1)+n\right]|\nabla f_t|^2_{g_t} + \left[1+\left(1+\frac{f_t''K_t}{f_t'}\right)\frac{f_t}{f_t'K_t}\right]H_t|\nabla f_t|^2_{h_t}\\
&\qquad+2f_t\langle\nabla H_t,\nabla f_t\rangle_{h_t} - \left[\frac{3f_t''^2}{f_t'^4} - \frac{f_t'''}{f_t'^3} + \frac{3}{(f_t'K_t)^2} + \frac{4f_t''}{f_t'^3K_t}\right]|\nabla f_t|^4_{h_t}\\
&\qquad-\langle\nabla_i f_t\cdot\nabla_j(h_t)_{k\ell}, \nabla_j(f_t)\cdot\nabla_i(h_t)_{k\ell}\rangle_{h_t}\\
&\qquad-2\left(\frac{f_t''K_t}{f_t'}+1\right)\frac{1}{f_t'K_t}\left\langle\nabla_i(h_t)_{jk}, \nabla_i f_t\nabla_j f_t\nabla_k f_t\right\rangle_{h_t}\\
&\qquad+2\left\langle\nabla_i(h_t)_{jk}, \nabla_i f_t\nabla_j\nabla_k f_t\right\rangle_{h_t} - 2|\nabla\nabla f_t|^2_{h_t}.
\end{aligned}$$

The final step is to compute $\langle\nabla_i\nabla_j f_f, \nabla_i f_t\nabla_j f_t\rangle_{h_t}$. We consider

$$\begin{aligned}
\left\langle\nabla f_t, \nabla\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right)\right\rangle_{h_t} &= -\left(\frac{f_t''}{f_t'^3K_t} + \frac{1}{K_t^2 f_t'^2}\right)|\nabla f_t|^4_{h_t}\\
&\quad-\frac{1}{f_t'K_t}\left\langle\nabla_i(h_t)_{jk}, \nabla_i f_t\nabla_j f_t\nabla_k f_t\right\rangle_{h_t} + \frac{2}{f_t'K_t}\left\langle\nabla_i\nabla_j f_t, \nabla_i f_t\nabla_j f_t\right\rangle_{h_t}.
\end{aligned}$$

Substituting this formula into the evolution equation, we obtain

$$\begin{aligned}
&\partial_t\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right) - (f_t'K_t)\Box_t\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right)\\
&\quad= 2\left(1+\frac{f_t''K_t}{f_t'}\right)\left\langle\nabla f_t, \nabla\left((f_t'K_t)^{-1}|\nabla f_t|^2_{h_t}\right)\right\rangle_{h_t} - 2|\nabla\nabla f_t|^2_{h_t}\\
&\qquad+2\left\langle\nabla_i(h_t)_{jk}, \nabla_i f_t\nabla_j\nabla_k f_t\right\rangle_{h_t} + \left(1+\frac{f_t''K_t}{f_t'}\right)\frac{2}{f_t'K_t}\Box_t f_t\cdot|\nabla f_t|^2_{h_t}\\
&\qquad-\left\langle\nabla_i f_t\nabla_j(h_t)_{kl}, \nabla_j f_t\nabla_i(h_t)_{kl}\right\rangle_{h_t} + \left(\frac{f_t}{f_t'K_t}+2-n\right)|\nabla f_t|^2_{g_t}\\
&\qquad+\left[1+\left(1+\frac{f_t''K_t}{f_t'}\right)\frac{f_t}{f_t'K_t}\right]H_t|\nabla f_t|^2_{h_t} + 2f_t\langle\nabla H_t,\nabla f_t\rangle_{h_t}\\
&\qquad-\left[\frac{3f_t''^2}{f_t'^4} - \frac{f_t'''}{f_t'^3} + \frac{3}{(f_t'K_t)^2} + \frac{4f_t''}{f_t'^3K}\right.\\
&\qquad\left.-2\left(1+\frac{f_t''K_t}{f_t'}\right)\left(\frac{f_t''}{f_t'^3K_t} + \frac{1}{(f_t'K_t)^2}\right)\right]|\nabla f_t|^4_{h_t}.
\end{aligned}$$

The bracket in the last term equals $\frac{f_t''^2}{f_t'^4} - \frac{f_t'''}{f_t'^3} + \frac{1}{(f_t'K_t)^2}$. □

Lemma 3.3. *Under the f-Gaussian curvature flow, we have*

$$\partial_t(f_t H_t) = f_t'K_t\cdot(\Box_t f_t + H_t f_t)H_t + f_t(\Delta_t f_t + f_t|h_t|^2_{g_t}). \tag{3.8}$$

Proof. This immediately follows from (2.19) and (2.20). □

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Lemma 3.4. *Under the f-Gaussian curvature flow, we have*

$$\begin{aligned} |P_t|^2_{h_t} &= |\nabla\nabla f_t|^2_{h_t} + \langle \nabla_i f_t \cdot \nabla_j (h_t)_{kl}, \nabla_j f_t \cdot \nabla_i (h_t)_{k\ell}\rangle_{h_t} + f_t^2 |h_t^\bullet|^2_{g_t} \\ &\quad - 2\langle \nabla_i (h_t)_{jk}, \nabla_i f_t \cdot \nabla_j \nabla_k f_t\rangle_{h_t} + 2 f_t \Delta_t f_t - 2 f_t \langle \nabla H_t, \nabla f_t\rangle_{h_t}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \mathbf{P}_t^2 &= (\Box_t f_t)^2 + \frac{|\nabla f_t|^4_{h_t}}{(f_t' K_t)^2} + f_t^2 H_t^2 - 2\frac{\Box_t f_t \cdot |\nabla f_t|^2_{h_t}}{f_t' K_t} \\ &\quad + 2 H_t f_t \cdot \Box_t f_t - 2\frac{H_t f_t |\nabla f_t|^2_{h_t}}{f_t' K_t}. \end{aligned} \tag{3.10}$$

Proof. By definition, we have

$$\begin{aligned} |P_t|^2_{h_t} \;=\;& (h_t)^{-1}_{ik}(h_t)^{-1}_{jl}\big[\nabla_i\nabla_j f_t \cdot \nabla_k \nabla_l f_t - (h_t)^{-1}_{pq}\nabla_p (h_t)_{ij}\cdot \nabla_k\nabla_l f_t \\ -\;& f_t (g_t)^{k\ell}(h_t)_{ik}(h_t)_{j\ell}\cdot \nabla_k\nabla_l f_t - (h_t)^{-1}_{pq}\nabla_p (h_t)_{k\ell}\nabla_q f_t \nabla_i \nabla_j f_t \\ +\;& (h_t)^{-1}_{pq}(h_t)^{-1}_{rs}\nabla_p (h_t)_{ij}\cdot \nabla_r (h_t)_{k\ell}\nabla_q f_t \nabla_s f_t \\ -\;& f_t (g_t)^{k\ell}(h_t)_{ik}(h_t)_{j\ell}(h_t)^{-1}_{pq}\nabla_p (h_t)_{k\ell}\nabla_q f_t + f_t (g_t)^{pq}(h_t)_{kp}(h_t)_{\ell q}\nabla_i\nabla_j f_t \\ -\;& f_t (g_t)^{rs}(h_t)_{kr}(h_t)_{\ell s}(h_t)^{-1}_{pq}\nabla_p (h_t)_{ij}\nabla_q f_t + f_t^2 (h_t)^2_{ij}(h_t)^2_{kl}\big] \\ =\;& |\nabla\nabla f_t|^2_{h_t} + \langle \nabla_i f_t \nabla_j (h_t)_{kl}, \nabla_j f_t \nabla_i (h_t)_{kl}\rangle_{h_t} + f_t^2 |h_t^\bullet|^2_{g_t} \\ -\;& 2\langle \nabla_i (h_t)_{jk}, \nabla_i f_t \nabla_j \nabla_k f_t\rangle_{h_t} + 2 f_t \Delta_t f_t - 2 f_t\langle \nabla H_t, \nabla f_t\rangle_{h_t}, \end{aligned}$$

where $(h_t)^2_{ij} := (h_t)_{ip}(h_t)_{jq}(g_t)^{pq}$. The second equation is obviously proved by using the first one. ☐

As a direct consequence, we obtain

Theorem 3.5. *Under the f-Gaussian curvature flow, we have*

$$\begin{aligned} \partial_t \mathbf{P}_t &= f_t' K_t \cdot \Box_t \mathbf{P}_t + 2\left(1 + \frac{f_t'' K_t}{f_t'}\right)\langle \nabla f_t, \nabla \mathbf{P}_t\rangle_{h_t} + |P_t|^2_{h_t} + \left(1 + \frac{f_t'' K_t}{f_t'}\right)\mathbf{P}_t^2 \\ &\quad + \left[\left(\frac{f_t'' K_t}{f_t'}\right)^2 - \frac{f_t'' K_t}{f_t'} - \frac{f_t''' K_t^2}{f_t'}\right]\frac{|\nabla f_t|^2_{h_t}}{f_t' K_t}\left(\frac{|\nabla f_t|^2_{h_t}}{f_t' K_t} - \Box_t f_t - H_t f_t\right) \\ &\quad + \left(f_t' K_t - \frac{f_t f_t'' K_t}{f_t'} - f_t\right)(H_t^2 f_t + H_t \cdot \Box_t f_t) \\ &\quad + \left[\left(1 + \frac{f_t'' K_t}{f_t'}\right)\frac{f_t}{f_t' K_t} - 1\right] H_t |\nabla f_t|^2_{h_t}. \end{aligned} \tag{3.11}$$

3.2. **Harnack inequality for the negative power Gaussian curvature flow.** For the sake of studying, we define three functions for $x > 0$,

$$\alpha(x) = \left(\frac{x f''(x)}{f'(x)}\right)^2 - \frac{x f''(x)}{f'(x)} - \frac{x^2 f'''(x)}{f'(x)}, \tag{3.12}$$

$$\beta(x) = x f'(x) - \frac{x f(x) f''(x)}{f'(x)} - f(x), \tag{3.13}$$

$$\gamma(x) = \left(1 + \frac{x f''(x)}{f'(x)}\right)\frac{f(x)}{x f'(x)} - 1. \tag{3.14}$$

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Using this simple notation, we can rewrite the evolution equation for $\mathbf{P}_t$ as

$$\begin{aligned}\partial_t \mathbf{P}_t &= f_t' K_t \cdot \Box_t \mathbf{P}_t + 2\left(1 + \frac{f_t'' K_t}{f_t'}\right)\langle \nabla f_f, \nabla \mathbf{P}_t\rangle_{h_t} + |P_t|^2_{h_t} + \left(1 + \frac{f_t'' K_t}{f_t'}\right)\mathbf{P}_t^2 \\ &\quad + \alpha(K_t)\frac{|\nabla f_t|^2_{h_t}}{f_t' K_t}\left(\frac{|\nabla f_t|^2_{h_t}}{f_t' K_t} - \Box_t f_t - H_t f_t\right) \\ &\quad + \beta(K_t)\left[H_t^2 f_t + H_t \cdot \Box_t f_t\right] + \gamma(K_t) H_t |\nabla f_t|^2_{h_t} \\ &= f_t' K_t \cdot \Box_t \mathbf{P}_t + 2\left(1 + \frac{f_t'' K_t}{f_t'}\right)\langle \nabla f_t, \nabla \mathbf{P}_t\rangle_{h_t} + |P_t|^2_{h_t} + \left(1 + \frac{f_t'' K_t}{f_t'}\right)\mathbf{P}_t^2 \\ &\quad + \left[H_t \beta(K_t) - \alpha(K_t)\frac{|\nabla f_t|^2_{h_t}}{f_t' K_t}\right]\mathbf{P}_t + \left[\frac{\beta(K_t)}{f_t' K_t} + \gamma(K_t)\right] H_t |\nabla f_t|^2_{h_t}.\end{aligned}$$

Observing that $\gamma(x) = -\beta(x)/xf(x)'$ and $\beta'(x) = f(x)\alpha(x)/x$, we have

$$\begin{aligned}\partial_t \mathbf{P}_t &= f_t' K_t \Box_t \mathbf{P}_t + 2\left(1 + \frac{f_t'' K_t}{f_t'}\right)\langle \nabla f_t, \nabla \mathbf{P}_t\rangle_{h_t} + |P_t|^2_{h_t} + \left(1 + \frac{f_t'' K_t}{f_t'}\right)\mathbf{P}_t^2 \\ &\quad + \left(H_t \beta_t - \frac{\beta_t'}{f_t f_t'}|\nabla f_t|^2_{h_t}\right)\mathbf{P}_t,\end{aligned} \tag{3.15}$$

where $\beta_t = \beta(K_t)$.

To obtain the Harnack inequality for the negative power Gaussian curvature flow, we should impose some natural condition on f. First we investigate some properties of the above three functions associated to the function f.

Lemma 3.6. *We have*

(a) $\alpha \equiv 0$ *if and only if* $f'(x) = ax^b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*;*
(b) $\beta \equiv 0$ *if and only if* $f(x) = ax^b$ *for* $ab > 0$.

Proof. Suppose $\alpha \equiv 0$. Then $x(f''^2 - f'f''') = f''f'$ and hence

$$x\left(\frac{f''^2 - f'f''}{f'^2}\right) = f''f',$$

which implies that $-x(f''/f')' = f''/f'$. Let $g := f''/f'$; so $xg' = -g$. Solving this ODE, we get $g = b/x$ for some constant b. For (b), putting $g = f'/f$ we get $f = ax^b$. □

When $f(x) = x^b$ for $b > 0$, B. Chow [3] derived the Harnack inequality for the f-Gaussian curvature flow. For the case $b < 0$, we give the following:

Theorem 3.7. *If* $f(x) = ax^b$ *satisfies (1)* $a > 0$ *and* $b > 0$*, or (2)* $a < 0$ *and* $-\frac{1}{n} < b < 0$*, then*

$$\mathbf{P}_t \ge -\frac{1}{\left(\frac{1}{n} + b\right)t}. \tag{3.16}$$

Consequently,

$$\frac{\partial f(K_t)}{\partial t} - |\nabla f(K_t)|^2_h + \frac{f'(K_t)K_t}{\left(\frac{1}{n} + b\right)t} \ge 0. \tag{3.17}$$

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Proof. From the above lemma, we have

$$\partial_t \mathbf{P}_t = abK_t^b \cdot \Box_t \mathbf{P}_t + 2b\langle \nabla f_t, \nabla \mathbf{P}_t \rangle_{h_t} + |P_t|^2_{h_t} + b\mathbf{P}_t^2.$$

Since $|P_t|^2_{h_t} \geq \frac{\mathbf{P}_t^2}{n}$, it follows that

$$\partial_t \mathbf{P}_t \geq abK_t^b \cdot \Box_t \mathbf{P}_t + 2b\langle \nabla f,_f \nabla \mathbf{P}_t \rangle_{h_t} + \left(\frac{1}{n} + b\right) \mathbf{P}_t^2.$$

The parabolic maximum principle tells us that

$$\mathbf{P}_t \geq -\frac{1}{\left(\frac{1}{n} + b\right) t}.$$

The last inequality is followed by $\partial_t f_t = f_t' K_t (\Box_t f_t + f_t H_t)$. □

Now, Theorem 1.1 follows from the above theorem.

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