J. Math. Anal. Appl. 416 (2014) 81–98

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mabuchi and Aubin–Yau functionals over complex surfaces

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ARTICLE INFO

Article history:
Received 16 October 2013
Available online 21 February 2014
Submitted by J. Xiao

This paper is dedicated to my colleague and friend Lin Chen who has passed away in an accident.

Keywords:
Mabuchi functional
Aubin–Yau functional
Compact complex surfaces

ABSTRACT

In this note we construct Mabuchi $\mathcal{L}^{\mathrm{M}}_{\omega}$ functional and Aubin–Yau functionals $\mathcal{I}^{\mathrm{AY}}_{\omega}, \mathcal{J}^{\mathrm{AY}}_{\omega}$ on any compact complex surfaces, and establish a number of properties. Our construction coincides with the original one in the Kähler case. This method has been generalized to construct Mabuchi and Aubin–Yau functional over complex manifolds with complex dimension larger than two, see [5,6]. We will consider in [7] their applications to geometry.

1. Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension n. It is known that the volume V_ω depends only on the Kähler class of ω, namely,

$$\int_X (\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \int_X \omega^n =: V_\omega$$

for any real-valued smooth function φ with $\omega_\varphi := \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0$, because of the closedness of ω.

If (X, g) is a compact Hermitian manifold of complex dimension n, the same result does not hold in general. In Section 2 below, we consider a function to describe such a phenomena, i.e., we define

$$\mathrm{Err}_\omega(\varphi) := \int_X \omega^n - \int_X \omega^n_\varphi,$$

where ω is the associated real $(1,1)$-form of g. If $\partial\overline{\partial}(\omega^k) = 0$ for $k = 1, 2$, we can show

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http://dx.doi.org/10.1016/j.jmaa.2014.02.039
0022-247X/

$$\int_X \omega_\varphi^n = \int_X \omega^n =: V_\omega$$

for any real-valued function $\varphi \in C^\infty(X)_{\mathbf{R}}$ with $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$. Actually, this result has implicitly contained in [2,13]. We give here an alternative proof of this result.

In Kähler geometry, energy functionals, such as Mabuchi K-energy functional [9], Aubin–Yau energy functionals [10,14], and Chen–Tian energy functionals [1], play an important role in studying Kähler–Einstein metrics and constant scalar curvatures. When I was in Yau's Seminar, I asked myself that it is possible to define energy functionals on compact complex manifolds? This is one motivation to write down this note. Another motivation comes from a question in S.-T. Yau's survey [15], namely to find necessary and sufficient conditions for a complex manifold to admit a Kähler structure. When $n = 2$, it was settled by Siu [11] or see also [8]: A compact complex surface is Kähler if and only if its first Betti number is even. In the second part of this note we construct Mabuchi $\mathcal{L}_\omega^{\mathrm{M}}$ functional and Aubin–Yau functionals $\mathcal{I}_\omega^{\mathrm{AY}}, \mathcal{J}_\omega^{\mathrm{AY}}$ on any compact complex surface.

Let (X, g) be a compact Hermitian manifold of complex dimension 2 and ω be its associated real $(1,1)$-form. Let

$$\mathcal{P}_\omega := \{\varphi \in C^\infty(X)_{\mathbf{R}} \mid \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

For any $\varphi', \varphi'' \in \mathcal{P}_\omega$, we define

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'') := \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t\omega_{\varphi_t}^2\,dt - \frac{1}{V_\omega}\int_0^1\int_X \sqrt{-1}\partial\omega \wedge (\varphi_t\bar{\partial}\dot{\varphi}_t)\,dt + \frac{1}{V_\omega}\int_0^1\int_X \sqrt{-1}\bar{\partial}\omega \wedge (\varphi_t\partial\dot{\varphi}_t)\,dt$$

where φ_t is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. Also we set

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi) := \mathcal{L}_\omega^{\mathrm{M}}(0, \varphi).$$

Theorem 1.1. *The functional $\mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'')$ is independent of the choice of the smooth path $\{\varphi_t\}_{0\leqslant t\leqslant 1}$ and satisfies the 1-cocycle condition. In particular*

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi) = \frac{1}{3V_\omega}\int_X \varphi(\omega^2 + \omega\wedge\omega_\varphi + \omega_\varphi^2) + \frac{1}{2V_\omega}\int_X \varphi(-\sqrt{-1}\partial\omega\wedge\bar{\partial}\varphi + \sqrt{-1}\bar{\partial}\omega\wedge\partial\varphi).$$

Moreover, for any $\varphi \in \mathcal{P}_\omega$ and any constant $C \in \mathbf{R}$, we have

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi, \varphi + C) = C\left(1 - \frac{\mathrm{Err}_\omega(\varphi)}{V_\omega}\right);$$

for any $\varphi_1, \varphi_2 \in \mathcal{P}_\omega$ and any constant $C \in \mathbf{R}$, we have

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi_1, \varphi_2 + C) = \mathcal{L}_\omega^{\mathrm{M}}(\varphi_1, \varphi_2) + C\left(1 - \frac{\mathrm{Err}_\omega(\varphi)}{V_\omega}\right).$$

In Section 4, we construct Aubin–Yau functionals on compact complex surfaces. Let (X, g) be a compact Hermitian manifold of complex dimension 2 and ω be its associated real $(1,1)$-form. Set

$$\mathcal{A}_\omega(\varphi) := \frac{1}{2V_\omega}\int_X -\sqrt{-1}\partial\omega\wedge\varphi\bar{\partial}\varphi.$$

For convenience, we consider the complex conjugate of $\mathcal{A}_\omega(\varphi)$ given by

$$\mathcal{B}_\omega(\varphi) := \frac{1}{2V_\omega}\int_X \sqrt{-1}\,\bar\partial\omega \wedge \varphi\partial\varphi.$$

Note that $\mathcal{A}_\omega(\varphi)$ is actually equal to $\mathcal{B}_\omega(\varphi)$ so that $\mathcal{A}_\omega(\varphi)$ is real-valued. Indeed, by Stokes' theorem, we obtain

$$\begin{aligned}
\mathcal{A}_\omega(\varphi) &= \frac{-\sqrt{-1}}{2V_\omega}\int_X \omega \wedge \partial(\varphi\bar\partial\varphi) \\
&= \frac{-\sqrt{-1}}{2V_\omega}\int_X \omega \wedge (\partial\varphi \wedge \bar\partial\varphi + \varphi\partial\bar\partial\varphi) \\
&= \frac{\sqrt{-1}}{2V_\omega}\int_X \omega \wedge (\bar\partial\varphi \wedge \partial\varphi + \varphi\bar\partial\partial\varphi) \\
&= \frac{\sqrt{-1}}{2V_\omega}\int_X \omega \wedge \bar\partial(\varphi\partial\varphi) \\
&= \mathcal{B}_\omega(\varphi).
\end{aligned}$$

We now define

$$\begin{aligned}
\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) &:= \frac{1}{V_\omega}\int_X \varphi\big(\omega^2 - \omega_\varphi^2\big) + 2\mathcal{A}_\omega(\varphi) + 2\mathcal{B}_\omega(\varphi) \\
&= \frac{1}{V_\omega}\int_X \varphi\big(\omega^2 - \omega_\varphi^2\big) + 4\mathcal{A}_\omega(\varphi), \\
\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) &:= \frac{1}{V_\omega}\int_0^1\int_X \varphi\big(\omega^2 - \omega_{s\varphi}^2\big)\,ds + \mathcal{A}_\omega(\varphi) + \mathcal{B}_\omega(\varphi) \\
&= \frac{1}{V_\omega}\int_0^1\int_X \varphi\big(\omega^2 - \omega_{s\varphi}^2\big)\,ds + 2\mathcal{A}_\omega(\varphi).
\end{aligned}$$

Theorem 1.2. *For any compact Hermitian manifold (X, g) of complex dimension 2, we have*

$$\frac{1}{3}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) \leqslant \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \leqslant \frac{2}{3}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi)$$

for any $\varphi \in \mathcal{P}_\omega$, where ω is its associated real $(1,1)$-form.

We hope this exposition will give some ideas to study Yau's problem. The author [5,6] have constructed those functionals on higher dimensional compact complex manifolds. We [7] will later use those functionals to study geometric problems.

2. $\mathbf{Err}_\omega$ map on complex manifolds

Let (X, g) be a compact Hermitian manifold of complex dimension n and write the associated real $(1,1)$-form as

$$\omega = \sqrt{-1} g_{i\bar{j}}\, dz^i \wedge d\overline{z^j}.$$

Let $\mathcal{P}_\omega$ be the space of all real-valued smooth functions $\varphi \in C^\infty(X)_{\mathbf{R}}$, so that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is positive definite on X:

$$\mathcal{P}_\omega := \{\varphi \in C^\infty(X)_{\mathbf{R}} \mid \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

Also we set

$$\mathcal{P}_\omega^0 := \Big\{\varphi \in \mathcal{P}_\omega \Bigm| \sup_X \varphi = 0\Big\}.$$

2.1. Err_ω *map on compact complex manifolds*

To such a function $\varphi \in \mathcal{P}_\omega$ we associate the quantity

$$V_\omega(\varphi) := \int_X \omega_\varphi^n \tag{2.1}$$

the volume of X with respect to φ. In particular we set

$$V_\omega := V_\omega(0) = \int_X \omega^n. \tag{2.2}$$

If (X, ω) is Kähler, we have $V_\omega = V_\omega(\varphi)$ for any $\varphi \in \mathcal{P}_\omega$. In the non-Kähler case, it is not in general true. Hence it is reasonable to define

$$\mathrm{Err}_\omega(\varphi) := V_\omega - V_\omega(\varphi) = \int_X \omega^n - \int_X \omega_\varphi^n. \tag{2.3}$$

A natural question is when does $\mathrm{Err}_\omega(\varphi)$ vanish for any/a $\varphi \in \mathcal{P}_\omega$? Clearly there exists a smooth real-valued function $\varphi_0 \equiv 0 \in \mathcal{P}_\omega$ such that

$$\omega_{\varphi_0}^n = \omega^n, \qquad \sup_X \varphi_0 = 0,$$

hence

$$\mathrm{Err}_\omega(\varphi_0) = \int_X \omega^n - \int_X \omega^n = 0. \tag{2.4}$$

This gives us some information about $\mathrm{Err}_\omega(\varphi)$ and motivates us to consider

$$\sup \mathrm{Err}_\omega := \sup_{\varphi \in \mathcal{P}_\omega^0} \big(\mathrm{Err}_\omega(\varphi)\big), \qquad \inf \mathrm{Err}_\omega := \inf_{\varphi \in \mathcal{P}_\omega^0} \big(\mathrm{Err}_\omega(\varphi)\big). \tag{2.5}$$

In any case, one has

$$\inf \mathrm{Err}_\omega \leqslant 0 \leqslant \sup \mathrm{Err}_\omega \leqslant \int_X \omega^n. \tag{2.6}$$

It is interesting to find some conditions to guarantee that the equalities hold. To study this behavior of Err_ω we consider the following several natural conditions on ω:

- **Condition 1.1:**

$$\sqrt{-1}\partial\omega \wedge \bar{\partial}\omega \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}\omega \quad \text{are non-negative} \tag{2.7}$$

- **Condition 1.2:**

$$\sqrt{-1}\partial\omega \wedge \bar{\partial}\omega \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}\omega \quad \text{are non-positive} \tag{2.8}$$

- **Condition 2:**

$$\partial\bar{\partial}\big(\omega^k\big) = 0, \quad k = 1, 2. \tag{2.9}$$

- **Condition 3:**

$$d\big(\omega^{n-1}\big) = 0. \tag{2.10}$$

- **Condition 4:**

$$\partial\bar{\partial}\big(\omega^{n-1}\big) = 0. \tag{2.11}$$

Remark 2.1. Condition 2 has appeared in [4] as a sufficient condition to solving the complex Monge–Ampère equation on Hermitian manifolds. The metric satisfying the third condition is called a balanced metric, which naturally appears in string theory (V. Tosatti and B. Weinkove [12] solved the complex Monge–Ampère equation on Hermitian manifolds with balanced metrics; later, they [13] dropped off the balanced condition). When $n = 2$, this condition is indeed the Kähler condition. A metric satisfying Condition 4 is called a Gauduchon metric, and a theorem of Gauduchon [3] shows that there exists a Gauduchon metric on every compact Hermitian manifold. Notice that Condition 3 implies Condition 4, and Condition 2 is equivalent to $\partial\bar{\partial}\omega = 0 = \partial\omega \wedge \bar{\partial}\omega$. In particular, Condition 2 implies Condition 1.1 and Condition 1.2. In our case $n = 2$, Condition 2 is equivalent to Condition 4.

Theorem 2.2. (i) *If ω satisfies Condition* 1.1, *then*

$$\inf \mathrm{Err}_\omega = 0. \tag{2.12}$$

(ii) *Correspondingly, if ω satisfies Condition* 1.2, *then*

$$\sup \mathrm{Err}_\omega = 0. \tag{2.13}$$

(iii) *In particular* $\sup \mathrm{Err}_\omega = \inf \mathrm{Err}_\omega = 0$ *provided that ω satisfies Condition* 2.

Proof. (i) We knew that $\mathrm{Err}_\omega(\varphi_0) = 0$ for some $\varphi_0 \in \mathcal{P}_\omega^0$. To prove the result, we need only to show that $\mathrm{Err}_\omega(\varphi) \geqslant 0$ for each function $\varphi \in \mathcal{P}_\omega^0$. By definition we have

$$\begin{aligned}
\mathrm{Err}_\omega(\varphi) &= -\int_X \omega_\varphi^n + \int_X \omega^n = \int_X -\sqrt{-1}\partial\bar\partial\varphi \wedge \sum_{0\leqslant i\leqslant n-1} \omega_\varphi^i \wedge \omega^{n-1-i} \\
&= \sum_{0\leqslant i\leqslant n-1} \int_X \omega_\varphi^i \wedge \omega^{n-1-i} \wedge \big(-\sqrt{-1}\partial\bar\partial\varphi\big) \\
&= \sum_{0\leqslant i\leqslant n-1} \int_X \sqrt{-1}\partial\big(\omega_\varphi^i \wedge \omega^{n-1-i}\big) \wedge \bar\partial\varphi \\
&= \sum_{0\leqslant i\leqslant n-1} \int_X \big[i\omega + (n-1-i)\omega_\varphi\big] \wedge \omega_\varphi^{i-1} \wedge \partial\omega \wedge \omega^{n-2-i} \wedge \sqrt{-1}\bar\partial\varphi \\
&= \sum_{0\leqslant i\leqslant n-1} \int_X \big[i\omega + (n-1-i)\omega_\varphi\big] \wedge \omega_\varphi^{i-1} \wedge \omega^{n-2-i} \wedge \partial\omega \wedge \sqrt{-1}\bar\partial\varphi \\
&= \sum_{0\leqslant i\leqslant n-1} \int_X \sqrt{-1}\big[\varphi\big(i\omega_\varphi^{i-1} \wedge \omega^{n-1-i} + (n-1-i)\omega_\varphi^i \wedge \omega^{n-2-i}\big) \wedge \bar\partial\partial\omega \\
&\quad + \varphi\bar\partial\big(i\omega_\varphi^{i-1} \wedge \omega^{n-1-i} + (n-1-i)\omega_\varphi^i \wedge \omega^{n-2-i}\big) \wedge \partial\omega\big] \\
&= \sum_{0\leqslant i\leqslant n-1} (\mathrm{I}_i + \mathrm{II}_i),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_i &= \int_X \varphi\big[i\omega_\varphi^{i-1} \wedge \omega^{n-1-i} + (n-1-i)\omega_\varphi^i \wedge \omega^{n-2-i}\big] \wedge \big(-\sqrt{-1}\partial\bar\partial\omega\big), \\
\mathrm{II}_i &= \int_X \varphi\sqrt{-1}\,\bar\partial\big[i\omega_\varphi^{i-1} \wedge \omega^{n-1-i} + (n-1-i)\omega_\varphi^i \wedge \omega^{n-2-i}\big] \wedge \partial\omega.
\end{aligned}$$

Since $\sqrt{-1}\partial\bar\partial\omega \geqslant 0$ and $\varphi \leqslant 0$ on X, the first term I_i is non-negative. Applying the integration by parts to II_i, we deduce

$$\begin{aligned}
\mathrm{II}_i &= \int_X \varphi\big[i(i-1)\omega_\varphi^{i-2} \wedge \bar\partial\omega \wedge \omega^{n-1-i} + i(n-1-i)\omega_\varphi^{i-1} \wedge \omega^{n-2-i} \wedge \bar\partial\omega \\
&\quad + i(n-1-i)\omega_\varphi^{i-1} \wedge \bar\partial\omega \wedge \omega^{n-2-i} + (n-1-i)(n-2-i)\omega_\varphi^i \wedge \omega^{n-3-i} \wedge \bar\partial\omega\big] \wedge \sqrt{-1}\partial\omega \\
&= \int_X \varphi\omega_\varphi^{i-2} \wedge \omega^{n-3-i} \wedge \big[i(i-1)\omega^2 + 2i(n-1-i)\omega_\varphi \wedge \omega + (n-1-i)(n-2-i)\omega_\varphi^2\big] \\
&\quad \wedge \big(-\sqrt{-1}\partial\omega \wedge \bar\partial\omega\big).
\end{aligned}$$

Since $\sqrt{-1}\partial\omega \wedge \bar\partial\omega$ is non-negative and φ is non-positive, it follows that $\mathrm{II}_i \geqslant 0$. Thus $\mathrm{Err}_\omega(\varphi) \geqslant 0$ for each $\varphi \in \mathcal{P}_\omega^0$ and therefore $\inf \mathrm{Err}_\omega = 0$.

(ii) If ω satisfies Condition 1.2, the above reasoning gives that $\mathrm{Err}_\omega(\varphi) \leqslant 0$ for each $\varphi \in \mathcal{P}_\omega^0$, i.e., $\sup \mathrm{Err}_\omega \leqslant 0$. Hence $\sup \mathrm{Err}_\omega = 0$.

(iii) It is an immediate consequence of (i) and (ii). □

Corollary 2.3. *If ω satisfies Condition 2, then $\mathrm{Err}_\omega(\varphi) = 0$ for any $\varphi \in \mathcal{P}^0_\omega$. Equivalently, in this case, the number $V_\omega(\varphi) = \int_X \omega^n_\varphi$ does not depend on the choice of $\varphi \in \mathcal{P}_\omega$ and equals $V_\omega = \int_X \omega^n$.*

2.2. Vanishing property of Err_ω map on compact complex surface

Let (X, g) be a Hermitian manifold of complex dimension n and let ω_g be its associated real $(1,1)$-form. We say that g is a Gauduchon metric if $\partial\bar{\partial}(\omega_g^{n-1}) = 0$.

We recall a theorem of Gauduchon or see Remark 2.1.

Theorem 2.4. *(See Gauduchon [3].) If X is a compact complex manifold of complex dimension n, then in the conformal class of every Hermitian metric g there exists a Gauduchon metric g_{G}, i.e., there is a positive function $\varphi \in C^\infty(X)_{\mathbf{R}}$ such that $g_{\mathrm{G}} := \varphi g$ is Gauduchon. If X is connected and $n \geqslant 2$, then g_{G} is unique up to a positive constant.*

Suppose now that (X, g_0) be a compact complex surface with Hermitian metric g. By Theorem 2.4, we have a Gauduchon metric g and its associated $(1,1)$-form ω. Since ω is a Gauduchon metric, it follows that $\partial\bar{\partial}\omega = 0$ and that

$$\begin{aligned}\mathrm{Err}_\omega(\varphi) &= \int_X (\omega - \omega_\varphi)(\omega + \omega_\varphi) = \int_X (\omega + \omega_\varphi) \wedge -\sqrt{-1}\partial\bar{\partial}\varphi \\ &= \int_X -\sqrt{-1}\partial\bar{\partial}(\omega + \omega_\varphi) \cdot \varphi = \int_X -2\sqrt{-1}\partial\bar{\partial}\omega \cdot \varphi = 0.\end{aligned}$$

Corollary 2.5. *Let (X, g) be a compact complex surface with Hermitian metric g and let ω_{G} be its associated Gauduchon metric. Then*

$$\mathrm{Err}_{\omega_{\mathrm{G}}}(\varphi) = 0 \tag{2.14}$$

for all $\varphi \in \mathcal{P}_{\omega_{\mathrm{G}}}$.

3. Mabuchi $\mathcal{L}^{\mathrm{M}}_\omega$ functional on complex surfaces

In this section we construct the Mabuchi functional on a complex surface.

3.1. Mabuchi $\mathcal{L}^{\mathrm{M}}_\omega$ functional on compact Kähler manifolds

Suppose that (X, ω) is a compact Kähler manifold of complex dimension n. For any pair $(\varphi', \varphi'') \in \mathcal{P}_\omega \times \mathcal{P}_\omega$ we define

$$\mathcal{L}^{\mathrm{M}}_\omega : \mathcal{P}_\omega \times \mathcal{P}_\omega \longrightarrow \mathbf{R}$$

as follows:

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi', \varphi'') := \frac{1}{V_\omega} \int_0^1 \int_X \dot{\varphi}_t \omega^n_{\varphi_t}\, dt \tag{3.1}$$

where $\{\varphi_t : 0 \leqslant t \leqslant 1\}$ is any smooth path in $\mathcal{P}_\omega$ such that $\varphi_0 = \varphi'$ and $\varphi_1 = \varphi''$. For any $\varphi \in \mathcal{P}_\omega$ we set

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi) := \mathcal{L}^{\mathrm{M}}_\omega(0, \varphi). \tag{3.2}$$

Mabuchi [9] showed that the functional (3.1) is well-defined, and hence we can explicitly write down $\mathcal{L}_\omega^{\mathrm{M}}(\varphi)$.

In this section we extend Mabuchi $\mathcal{L}_\omega^{\mathrm{M}}$ functional to any compact complex surface by adding two extra terms on the right hand side of (3.1).

3.2. Mabuchi $\mathcal{L}_\omega^{\mathrm{M}}$ functional on compact complex surfaces

Suppose now that (X,g) is a compact complex surface and ω be its associated real $(1,1)$-form. Let $\varphi',\varphi''\in\mathcal{P}_\omega$ and $\{\varphi_t\}_{0\leqslant t\leqslant 1}$ be a smooth path in $\mathcal{P}_\omega$ from φ' to φ''.

Let

$$\mathcal{L}_\omega^0(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t\omega_{\varphi_t}^2\,dt. \tag{3.3}$$

Set

$$\psi(s,t) := s\varphi_t, \quad 0\leqslant s\leqslant 1,\ 0\leqslant t\leqslant 1. \tag{3.4}$$

Consider a 1-form on $[0,1]\times[0,1]$

$$\Psi^0 := \left(\int_X \frac{\partial\psi}{\partial s}\omega_\psi^2\right)ds + \left(\int_X \frac{\partial\psi}{\partial t}\omega_\psi^2\right)dt. \tag{3.5}$$

Taking differential on Ψ^0, we have

$$d\Psi^0 = I^0\,dt\wedge ds,$$

where

$$I^0 = \int_X \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial s}\omega_\psi^2\right) - \int_X \frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial t}\omega_\psi^2\right). \tag{3.6}$$

Directly computation shows

$$\begin{aligned}
I^0 &= \int_X \left[\frac{\partial^2\psi}{\partial t\partial s}\omega_\psi^2 + 2\frac{\partial\psi}{\partial s}\omega_\psi\wedge\sqrt{-1}\partial\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\right]\int_X\left[\frac{\partial^2\psi}{\partial s\partial t}\omega_\psi^2 + 2\frac{\partial\psi}{\partial t}\omega_\psi\wedge\sqrt{-1}\partial\bar\partial\left(\frac{\partial\psi}{\partial s}\right)\right]\\
&= \int_X 2\frac{\partial\psi}{\partial s}\omega_\psi\wedge\sqrt{-1}\partial\bar\partial\left(\frac{\partial\psi}{\partial t}\right) - \int_X 2\frac{\partial\psi}{\partial t}\omega_\psi\wedge\sqrt{-1}\partial\bar\partial\left(\frac{\partial\psi}{\partial s}\right).
\end{aligned}$$

In the following we deduce two slightly different formulae of I^0. The first one is

$$\begin{aligned}
I^0 &= \int_X 2\frac{\partial\psi}{\partial s}\omega_\psi\wedge\sqrt{-1}\partial\bar\partial\left(\frac{\partial\psi}{\partial t}\right) + \int_X 2\frac{\partial\psi}{\partial t}\omega_\psi\wedge\sqrt{-1}\bar\partial\partial\left(\frac{\partial\psi}{\partial s}\right)\\
&= \int_X -2\sqrt{-1}\partial\left(\frac{\partial\psi}{\partial s}\omega_\psi\right)\wedge\bar\partial\left(\frac{\partial\psi}{\partial t}\right) + \int_x -2\sqrt{-1}\,\bar\partial\left(\frac{\partial\psi}{\partial t}\omega_\psi\right)\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\\
&= \int_X -2\sqrt{-1}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi + \frac{\partial\psi}{\partial s}\partial\omega\right]\wedge\bar\partial\left(\frac{\partial\psi}{\partial t}\right)
\end{aligned}$$

$$+\int_X -2\sqrt{-1}\left[\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi+\frac{\partial\psi}{\partial t}\bar\partial\omega\right]\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)$$
$$=\int_X -2\sqrt{-1}\frac{\partial\psi}{\partial s}\partial\omega\wedge\bar\partial\left(\frac{\partial\psi}{\partial t}\right)+\int_X -2\sqrt{-1}\frac{\partial\psi}{\partial t}\bar\partial\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)$$
$$=\int_X 2\sqrt{-1}\frac{\partial\psi}{\partial s}\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega+\int_X 2\sqrt{-1}\frac{\partial\psi}{\partial t}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\bar\partial\omega.$$

Similarly, we have

$$I^0=\int_X -2\sqrt{-1}\frac{\partial\psi}{\partial s}\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\bar\partial\omega+\int_X -2\sqrt{-1}\frac{\partial\psi}{\partial t}\bar\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega.$$

Next, we define

$$\mathcal{L}^1_\omega(\varphi',\varphi'')=\frac{1}{V_\omega}\int_0^1\int_X a_2\partial\omega\wedge(\varphi_t\bar\partial\dot\varphi_t)\,dt, \tag{3.7}$$
$$\mathcal{L}^2_\omega(\varphi',\varphi'')=\frac{1}{V_\omega}\int_0^1\int_X b_2\bar\partial\omega\wedge(\varphi_t\partial\dot\varphi_t)\,dt. \tag{3.8}$$

Here we require $\overline{a_2}=b_2$, and a_2,b_2 are determined later. As before, consider

$$\Psi^1=\left[\int_X a_2\partial\omega\wedge\left(\bar\partial\left(\frac{\partial\psi}{\partial s}\right)\psi\right)\right]ds+\left[\int_X a_2\partial\omega\wedge\left(\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\psi\right)\right]dt,$$
$$\Psi^2=\left[\int_X b_2\bar\partial\omega\wedge\left(\partial\left(\frac{\partial\psi}{\partial s}\right)\psi\right)\right]ds+\left[\int_X b_2\bar\partial\omega\wedge\left(\partial\left(\frac{\partial\psi}{\partial t}\right)\psi\right)\right]dt.$$

Therefore

$$d\Psi^1=I^1\,dt\wedge ds,$$

where

$$I^1=\int_X a_2\frac{\partial}{\partial t}\left[\partial\omega\wedge\left(\bar\partial\left(\frac{\partial\psi}{\partial s}\right)\psi\right)\right]-\int_X a_2\frac{\partial}{\partial s}\left[\partial\omega\wedge\left(\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\psi\right)\right]. \tag{3.9}$$

Dividing I^1 by a_2 yields

$$\frac{I^1}{a_2}=\int_X -\frac{\partial}{\partial t}\left[\left(\bar\partial\left(\frac{\partial\psi}{\partial s}\right)\psi\right)\wedge\partial\omega\right]+\int_X\frac{\partial}{\partial s}\left[\left(\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\psi\right)\wedge\partial\omega\right]$$
$$=\int_X -\left[\bar\partial\left(\frac{\partial^2\psi}{\partial t\partial s}\right)\psi+\bar\partial\left(\frac{\partial\psi}{\partial s}\right)\frac{\partial\psi}{\partial t}\right]\wedge\partial\omega+\int_X\left[\bar\partial\left(\frac{\partial^2\psi}{\partial s\partial t}\right)\psi+\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial s}\right]\wedge\partial\omega$$
$$=\int_X -\frac{\partial\psi}{\partial t}\bar\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega+\int_X\frac{\partial\psi}{\partial s}\bar\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega.$$

In the same way, one deduces

$$d\Psi^2 = I^2\, dt \wedge ds,$$

and

$$\frac{I^2}{b_2} = \int\limits_X -\frac{\partial \psi}{\partial t}\partial\left(\frac{\partial \psi}{\partial s}\right) \wedge \bar{\partial}\omega + \int\limits_X \frac{\partial \psi}{\partial s}\partial\left(\frac{\partial \psi}{\partial t}\right) \wedge \bar{\partial}\omega.$$

Combining above formulas, we have

$$-\frac{I^1}{a_2} + \frac{I^2}{b_2} = -\frac{I^0}{\sqrt{-1}}. \tag{3.10}$$

Setting $a_2 = -\sqrt{-1}$ and $b_2 = \sqrt{-1}$, we get

$$I^0 + I^1 + I^2 = 0.$$

Thus

$$d\Psi = 0, \tag{3.11}$$

where

$$\Psi := \Psi^0 + \Psi^1 + \Psi^2.$$

The following theorem is an immediate consequence of the above discussion.

Theorem 3.1. *Let (X, g) be a compact complex surface and ω be its associated real $(1,1)$-form. The functional*

$$\begin{aligned}\mathcal{L}^{\mathrm{M}}_\omega\left(\varphi', \varphi''\right) := {} & \frac{1}{V_\omega}\int\limits_0^1\int\limits_X \dot{\varphi}_t \omega^2_{\varphi_t}\, dt - \frac{1}{V_\omega}\int\limits_0^1\int\limits_X \sqrt{-1}\partial\omega \wedge (\varphi_t \bar{\partial}\dot{\varphi}_t)\, dt \\ & + \frac{1}{V_\omega}\int\limits_0^1\int\limits_X \sqrt{-1}\,\bar{\partial}\omega \wedge (\varphi_t \partial\dot{\varphi}_t)\, dt\end{aligned} \tag{3.12}$$

is independent of the choice of the smooth path $\{\varphi_t\}_{0\leqslant t\leqslant 1}$. In particular,

$$\begin{aligned}\mathcal{L}^{\mathrm{M}}_\omega(\varphi) &:= \mathcal{L}_\omega(0, \varphi) \\ &= \frac{1}{3V_\omega}\int\limits_X \varphi\big(\omega^2 + \omega\wedge\omega_\varphi + \omega_\varphi^2\big) + \frac{1}{2V_\omega}\int\limits_X \varphi[-\sqrt{-1}\partial\omega \wedge \bar{\partial}\varphi + \sqrt{-1}\,\bar{\partial}\omega\wedge\partial\varphi].\end{aligned} \tag{3.13}$$

Proof. Applying Stokes' theorem to the region $\Delta = \{(s,t) \in \mathbf{R}^2\colon\ 0 \leqslant s, t \leqslant 1\}$ and using Eq. (3.9), we have

$$0 = \int\limits_\Delta d\Psi = \int\limits_{\partial\Delta} \Psi = \int\limits_{\partial\Delta} \big(\Psi^0 + \Psi^1 + \Psi^2\big)$$

$$= \int_0^1 \dot{\varphi}_t \omega_{\varphi_t}^2 \, dt - \int_0^1 \int_X \frac{\partial \psi}{\partial s} \omega_\psi^2 \, ds \Big|_{t=0}^{t=1} + \int_0^1 \int_X -\sqrt{-1} \partial \omega \wedge (\varphi_t \bar{\partial} \dot{\varphi}_t) \, dt$$

$$- \int_0^1 \int_X -\sqrt{-1} \partial \omega \wedge \left(\bar{\partial} \left(\frac{\partial \psi}{\partial s} \right) \psi \right) ds \Big|_{t=0}^{t=1}$$

$$+ \int_0^1 \int_X \sqrt{-1} \, \bar{\partial} \omega \wedge (\varphi_t \partial \dot{\varphi}_t) \, dt - \int_0^1 \int_X \sqrt{-1} \, \bar{\partial} \omega \wedge \left(\partial \left(\frac{\partial \psi}{\partial s} \right) \psi \right) ds \Big|_{t=0}^{t=1}.$$

Equivalently,

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'') = \int_0^1 \int_X \frac{\partial \psi}{\partial s} \omega_\psi^2 \, ds \Big|_{t=0}^{t=1} + \int_0^1 \int_X -\sqrt{-1} \partial \omega \wedge \left(\bar{\partial} \left(\frac{\partial \psi}{\partial s} \right) \psi \right) ds \Big|_{t=0}^{t=1}$$

$$+ \int_0^1 \int_X \sqrt{-1} \, \bar{\partial} \omega \wedge \left(\partial \left(\frac{\partial \psi}{\partial s} \right) \psi \right) ds \Big|_{t=0}^{t=1}.$$

It turns out that $\mathcal{L}_\omega^{\mathrm{M}}(\varphi', \varphi'')$ is well-defined. For the second argument, we can choose the smooth path $\varphi_t = t \cdot \varphi$, $0 \leqslant t \leqslant 1$. $\square$

Remark 3.2. When (X, g) is a compact Kähler surface, the functional (3.12) or (3.13) coincides with the original one.

Suppose that S is a non-empty set and A is an additive group. A mapping $\mathcal{N} : S \times S \to A$ is said to satisfy the 1**-cocycle condition** if

(i) $\mathcal{N}(\sigma_1, \sigma_2) + \mathcal{N}(\sigma_2, \sigma_1) = 0$;
(ii) $\mathcal{N}(\sigma_1, \sigma_2) + \mathcal{N}(\sigma_2, \sigma_3) + \mathcal{N}(\sigma_3, \sigma_1) = 0$.

Corollary 3.3. *The functional $\mathcal{L}_\omega^{\mathrm{M}}$ satisfies the 1-cocycle condition.*

Corollary 3.4. *For any $\varphi \in \mathcal{P}_\omega$ and any constant $C \in \mathbf{R}$, we have*

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi, \varphi + C) = C \left(1 - \frac{\mathrm{Err}_\omega(\varphi)}{V_\omega} \right). \tag{3.14}$$

In particular, if $\partial \bar{\partial} \omega = 0$, then $\mathcal{L}_\omega^{\mathrm{M}}(\varphi, \varphi + C) = C$.

Proof. We choose the smooth path $\varphi_t = \varphi + tC$, $t \in [0, 1]$. So

$$\mathcal{L}_\omega^{\mathrm{M}}(\varphi, \varphi + C) = \frac{1}{V_\omega} \int_0^1 \int_X C \omega_{\varphi + Ct}^2 \, dt$$

$$= \frac{1}{V_\omega} \int_0^1 \int_X C \omega_\varphi^2 \, dt$$

$$= \frac{1}{V_\omega} \int_X C \omega_\varphi^2$$

$$= \frac{C}{V_\omega} V_\omega(\varphi)$$
$$= C\left(1 - \frac{\mathrm{Err}_\omega(\varphi)}{V_\omega}\right).$$

If furthermore $\partial\bar{\partial}\omega = 0$, then $\mathrm{Err}_\omega(\varphi) = 0$ for any $\varphi \in \mathcal{P}_\omega$. □

Corollary 3.5. *For any $\varphi_1, \varphi_2 \in \mathcal{P}_\omega$ and any constant $C \in \mathbf{R}$, we have*

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1, \varphi_2 + C) = \mathcal{L}^{\mathrm{M}}_\omega(\varphi_1, \varphi_2) + C\left(1 - \frac{\mathrm{Err}_\omega(\varphi_2)}{V_\omega}\right). \tag{3.15}$$

Proof. From Corollary 3.3, one has

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1, \varphi_2 + C) + \mathcal{L}^{\mathrm{M}}_\omega(\varphi_2, \varphi_1) = \mathcal{L}^{\mathrm{M}}_\omega(\varphi_2, \varphi_2 + C).$$

Then the conclusion follows from Corollary 3.4. □

4. Aubin–Yau functionals on compact complex surfaces

In this section we extend Aubin–Yau functionals to compact complex surfaces, including Kähler surfaces, and deduce a number of basic properties of these functionals.

4.1. Aubin–Yau functionals on compact Kähler manifolds

Suppose that (X, ω) is a compact Kähler manifold of dimension n. For $(\varphi', \varphi'') \in \mathcal{P}_\omega \times \mathcal{P}_\omega$, Aubin–Yau functionals are defined by

$$\mathcal{I}^{\mathrm{AY}}_\omega(\varphi', \varphi'') := \frac{1}{V_\omega} \int_X (\varphi'' - \varphi')\left(\omega^n_{\varphi'} - \omega^n_{\varphi''}\right), \tag{4.1}$$

$$\mathcal{J}^{\mathrm{AY}}_\omega(\varphi', \varphi'') := -\mathcal{L}^{\mathrm{M}}_\omega(\varphi', \varphi'') + \frac{1}{V_\omega} \int_X (\varphi'' - \varphi')\omega^n_{\varphi'}. \tag{4.2}$$

By definition, we have

$$\mathcal{J}^{\mathrm{AY}}_\omega(\varphi', \varphi'') + \mathcal{J}^{\mathrm{AY}}_\omega(\varphi'', \varphi') = \mathcal{I}^{\mathrm{AY}}_\omega(\varphi', \varphi'') = \mathcal{I}^{\mathrm{AY}}_\omega(\varphi'', \varphi'). \tag{4.3}$$

For any $\varphi \in \mathcal{P}_\omega$ we set

$$\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) := \frac{1}{V_\omega} \int_X \varphi\left(\omega^n - \omega^n_\varphi\right), \tag{4.4}$$

$$\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) := \int_0^1 \frac{\mathcal{I}_\omega(s\varphi)}{s}\, ds = \frac{1}{V_\omega} \int_0^1 \int_X \varphi\left(\omega^n - \omega^n_{s\varphi}\right) ds. \tag{4.5}$$

It is clear that

$$\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) = \mathcal{I}^{\mathrm{AY}}_\omega(0, \varphi), \qquad \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) = \mathcal{J}^{\mathrm{AY}}_\omega(0, \varphi). \tag{4.6}$$

By definition we have

$$
\begin{aligned}
\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega}\int_0^1 ds\int_X \varphi\big(-\sqrt{-1}\partial\bar\partial(s\varphi)\big)\wedge\sum_{i=1}^{n-1}\omega^{n-1-i}\wedge\omega_{s\varphi}^i \\
&= \frac{-\sqrt{-1}}{V_\omega}\int_0^1 s\,ds\int_X \varphi\cdot\partial\bar\partial\varphi\wedge\sum_{0\leqslant i\leqslant n-1}\omega^{n-1-i}\wedge\big[\omega+s(\omega_\varphi-\omega)\big]^i \\
&= \frac{-\sqrt{-1}}{V_\omega}\int_0^1 s\,ds\int_X \varphi\partial\bar\partial\varphi\sum_{0\leqslant i\leqslant n-1}\omega^{n-1-i}\wedge\sum_{0\leqslant j\leqslant i}\binom{i}{j}(1-s)^{i-j}s^j\omega^{i-j}\wedge\omega_\varphi^j \\
&= \frac{-\sqrt{-1}}{V_\omega}\int_X \varphi\partial\bar\partial\varphi\wedge\sum_{0\leqslant j\leqslant n-1}\sum_{j\leqslant i\leqslant n-1}\binom{i}{j}\omega^{n-1-j}\wedge\omega_\varphi^j\int_0^1(1-s)^{i-j}s^{1+j} \\
&= \frac{-\sqrt{-1}}{V_\omega}\int_X \varphi\partial\bar\partial\varphi\wedge\sum_{0\leqslant j\leqslant n-1}\omega^{n-1-j}\wedge\omega_\varphi^j\sum_{j\leqslant j\leqslant n-1}\binom{i}{j}\frac{(i-j)!(j+1)!}{(i+2)!} \\
&= \frac{-\sqrt{-1}}{V_\omega}\int_X \varphi\partial\bar\partial\varphi\wedge\sum_{j=0}^{n-1}\omega^{n-1-j}\wedge\omega_\varphi^j\sum_{i=j}^{n-1}\frac{j+1}{(i+2)(i+1)} \\
&= \frac{-\sqrt{-1}}{V_\omega}\int_X \varphi\partial\bar\partial\varphi\wedge\sum_{0\leqslant j\leqslant n-1}\frac{n-j}{n+1}\omega^{n-1-j}\wedge\omega_\varphi^j,
\end{aligned}
$$

since

$$
\sum_{j\leqslant i\leqslant n-1}\frac{1}{(i+2)(i+1)} = \sum_{j\leqslant i\leqslant n-1}\left(\frac{1}{i+1}-\frac{1}{i+2}\right) = \frac{1}{j+1}-\frac{1}{n+1}.
$$

On the other hand,

$$
\frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) = \frac{-\sqrt{-1}}{V_\omega}\int_X \varphi\partial\bar\partial\varphi\wedge\sum_{0\leqslant i\leqslant n-1}\frac{n}{n+1}\omega^{n-1-i}\wedge\omega_\varphi^i.
$$

Hence

$$
\frac{n}{n+1}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = \frac{1}{V_\omega}\int_X \varphi(-\sqrt{-1}\partial\bar\partial\varphi)\wedge\sum_{1\leqslant j\leqslant n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega_\varphi^j. \tag{4.7}
$$

Moreover,

$$
(n+1)\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) = \frac{1}{V_\omega}\int_X \varphi-\sqrt{-1}\partial\bar\partial\varphi\wedge\sum_{0\leqslant j\leqslant n-1}(n-1-j)\omega^{n-1-j}\wedge\omega_\varphi^j. \tag{4.8}
$$

Remark 4.1. Notice that formulae (4.7) and (4.8) are also valid when ω is non-Kähler.

4.2. Aubin–Yau functionals over compact complex surfaces

Let (X,g) be a compact complex manifold of complex dimension n and ω be its associated real $(1,1)$-form. From Remark 4.1, we can formally use the notion $\mathcal{I}_\omega^{\mathrm{AY}}$, $\mathcal{J}_\omega^{\mathrm{AY}}$, and $\mathcal{L}_\omega^{\mathrm{M}}$, but now ω may not be Kähler. Precisely, for any $\varphi\in\mathcal{P}_\omega$ we set

$$\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)=\frac{1}{V_\omega}\int\limits_X \varphi\big(\omega^n-\omega_\varphi^n\big), \tag{4.9}$$

$$\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)=\int\limits_0^1\frac{\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(s\varphi)}{s}\,ds=\frac{1}{V_\omega}\int\limits_0^1\int\limits_X\varphi\big(\omega^n-\omega_{s\varphi}^n\big)\,ds. \tag{4.10}$$

Hence

$$\frac{n}{n+1}\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)-\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)=\frac{1}{V_\omega}\int\limits_X\varphi(-\sqrt{-1}\partial\bar\partial\varphi)\wedge\sum_{1\leqslant j\leqslant n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega_\varphi^j. \tag{4.11}$$

Moreover,

$$(n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)-\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)=\frac{1}{V_\omega}\int\limits_X\varphi-\sqrt{-1}\partial\bar\partial\varphi\wedge\sum_{0\leqslant j\leqslant n-1}(n-1-j)\omega^{n-1-j}\wedge\omega_\varphi^j. \tag{4.12}$$

Restricting to compact complex surfaces and introducing two extra functionals on $\mathcal{P}_\omega$

$$\mathcal{A}_\omega(\varphi):=\frac{1}{2V_\omega}\int\limits_X\varphi-\sqrt{-1}\partial\omega\wedge\bar\partial\varphi, \tag{4.13}$$

$$\mathcal{B}_\omega(\varphi):=\frac{1}{2V_\omega}\int\limits_X\varphi\sqrt{-1}\,\bar\partial\omega\wedge\partial\varphi, \tag{4.14}$$

(clearly $\overline{\mathcal{A}_\omega(\varphi)}=\mathcal{B}_\omega(\varphi)$), we define Aubin–Yau functionals as follows (here constants a,b,c,d are determined later, and actually $a=b=c=d=2$)

$$\mathcal{I}_\omega^{\mathrm{AY}}(\varphi):=\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)+a\mathcal{A}_\omega(\varphi)+b\mathcal{B}_\omega(\varphi), \tag{4.15}$$

$$\mathcal{J}_\omega^{\mathrm{AY}}(\varphi):=-\mathcal{L}_\omega^{\mathrm{M}}(\varphi)+\frac{1}{V_\omega}\int\limits_X\varphi\omega^2+c\mathcal{A}_\omega(\varphi)+d\mathcal{B}_\omega(\varphi). \tag{4.16}$$

Since

$$\begin{aligned}\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)&=\frac{1}{V_\omega}\int\limits_X\varphi(-\sqrt{-1}\partial\bar\partial\varphi)\wedge\sum_{0\leqslant j\leqslant 1}\frac{2-j}{3}\big(\omega^{1-j}\wedge\omega_\varphi^j\big)\\&=\frac{1}{V_\omega}\int\limits_X\varphi(\omega-\omega_\varphi)\wedge\left(\frac{2}{3}\omega+\frac{1}{3}\omega_\varphi\right)\\&=\frac{1}{V_\omega}\int\limits_X\varphi\left(\frac{2}{3}\omega^2-\frac{1}{3}\omega\wedge\omega_\varphi-\frac{1}{3}\omega_\varphi^2\right),\end{aligned}$$

it follows that

$$\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = \mathcal{J}_{\omega|_\bullet}^{\mathrm{AY}}(\varphi) + (c-1)\mathcal{A}_\omega(\varphi) + (d-1)\mathcal{B}_\omega(\varphi), \tag{4.17}$$

and that

$$\begin{aligned}
&\frac{2}{3}\big(\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - a\mathcal{A}_\omega(\varphi) - b\mathcal{B}_\omega(\varphi)\big) - \big(\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - (c-1)\mathcal{A}_\omega(\varphi) - (d-1)\mathcal{B}_\omega(\varphi)\big)\\
&= \frac{1}{V_\omega}\int_X \varphi(-\sqrt{-1}\partial\bar\partial\varphi)\sum_{1\leqslant j\leqslant 1}\frac{j}{3}\omega^{1-j}\wedge\omega_\varphi^j\\
&= \frac{1}{V_\omega}\int_X \varphi\frac{1}{3}\omega_\varphi\wedge(-\sqrt{-1}\partial\bar\partial\varphi)\\
&= \frac{1}{V_\omega}\int_X \varphi\frac{1}{3}\omega_\varphi\wedge\sqrt{-1}\,\bar\partial\partial\varphi.
\end{aligned}$$

Thus the left hand side has two slightly different expressions. If we adopt the first one, we have

$$\begin{aligned}
&\frac{2}{3}\big(\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - a\mathcal{A}_\omega(\varphi) - b\mathcal{B}_\omega(\varphi)\big) - \big(\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - (c-1)\mathcal{A}_\omega(\varphi) - (d-1)\mathcal{B}_\omega(\varphi)\big)\\
&= \frac{1}{3V_\omega}\int_X \sqrt{-1}\partial(\varphi\omega_\varphi)\wedge\bar\partial\varphi\\
&= \frac{1}{3V_\omega}\int_X \sqrt{-1}(\partial\varphi\wedge\omega_\varphi + \varphi\partial\omega)\wedge\bar\partial\varphi\\
&= \frac{1}{3V_\omega}\int_X \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\omega_\varphi - \frac{2}{3}\mathcal{A}_\omega(\varphi).
\end{aligned}$$

On the other hand, using the second expression gives

$$\begin{aligned}
&\frac{2}{3}\big(\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - a\mathcal{A}_\omega(\varphi) - b\mathcal{B}_\omega(\varphi)\big) - \big(\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - (c-1)\mathcal{A}_\omega(\varphi) - (d-1)\mathcal{B}_\omega(\varphi)\big)\\
&= \frac{1}{3V_\omega}\int_X -\sqrt{-1}\,\bar\partial(\varphi\omega_\varphi)\wedge\partial\varphi\\
&= \frac{1}{3V_\omega}\int_X -\sqrt{-1}(\bar\partial\varphi\wedge\omega_\varphi + \varphi\bar\partial\omega)\wedge\partial\varphi\\
&= \frac{1}{3V_\omega}\int_X \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\omega_\varphi - \frac{2}{3}\mathcal{B}_\omega(\varphi).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{2}{3}\big(\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - a\mathcal{A}_\omega(\varphi) - b\mathcal{B}_\omega(\varphi)\big) - \big(\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - (c-1)\mathcal{A}_\omega(\varphi) - (d-1)\mathcal{B}_\omega(\varphi)\big)\\
&= \frac{1}{3V_\omega}\int_X \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\omega_\varphi - \frac{\mathcal{A}_\omega(\varphi)+\mathcal{B}_\omega(\varphi)}{3},
\end{aligned}$$

or, equivalently,

$$\frac{2}{3}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) = \frac{1}{3V_\omega}\int_X \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\omega_\varphi \tag{4.18}$$

where we require

$$\frac{2}{3}a-(c-1)-\frac{1}{3}=0=\frac{2}{3}b-(d-1)-\frac{1}{3}. \tag{4.19}$$

Theorem 4.2. *For any $\varphi\in\mathcal{P}_\omega$, one has*

$$\frac{2}{3}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi)-\mathcal{J}^{\mathrm{AY}}_\omega(\varphi)\geqslant 0. \tag{4.20}$$

Using (4.12) yields

$$\begin{aligned}
&3\Big(\mathcal{J}^{\mathrm{AY}}_\omega(\varphi)-\frac{1}{2}\mathcal{A}_\omega(\varphi)-\frac{1}{2}\mathcal{B}_\omega(\varphi)\Big)-\big(\mathcal{I}^{\mathrm{AY}}_\omega(\varphi)-\mathcal{A}_\omega(\varphi)-\mathcal{B}_\omega(\varphi)\big)\\
&\quad=\frac{1}{V_\omega}\int\limits_X\varphi(-\sqrt{-1}\partial\bar\partial\varphi)\wedge\sum_{0\leqslant j\leqslant 1}(1-j)\omega^{1-j}\wedge\omega_\varphi^j\\
&\quad=\frac{1}{V_\omega}\int\limits_X\varphi(-\sqrt{-1}\partial\bar\partial\varphi)\wedge\omega\\
&\quad=\frac{1}{V_\omega}\int\limits_X(\varphi\omega)\wedge(-\sqrt{-1}\partial\bar\partial\varphi)\\
&\quad=\frac{1}{V_\omega}\int\limits_X(\varphi\omega)\wedge\sqrt{-1}\,\bar\partial\partial\varphi.
\end{aligned}$$

As the proof of Theorem 4.2, we have

$$\begin{aligned}
&3\big(\mathcal{J}^{\mathrm{AY}}_\omega(\varphi)-(c-1)\mathcal{A}_\omega(\varphi)-(d-1)\mathcal{B}_\omega(\varphi)\big)-\big(\mathcal{I}^{\mathrm{AY}}_\omega(\varphi)-a\mathcal{A}_\omega(\varphi)-b\mathcal{B}_\omega(\varphi)\big)\\
&\quad=\frac{1}{V_\omega}\int\limits_X\sqrt{-1}\partial(\varphi\omega)\wedge\bar\partial\varphi=\frac{1}{V_\omega}\int\limits_X\sqrt{-1}(\partial\varphi\wedge\omega+\varphi\partial\omega)\wedge\bar\partial\varphi\\
&\quad=\frac{1}{V_\omega}\int\limits_X\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\omega-2\mathcal{A}_\omega(\varphi)
\end{aligned}$$

and

$$\begin{aligned}
&3\big(\mathcal{J}^{\mathrm{AY}}_\omega(\varphi)-(c-1)\mathcal{A}_\omega(\varphi)-(d-1)\mathcal{B}_\omega(\varphi)\big)-\big(\mathcal{I}^{\mathrm{AY}}_\omega(\varphi)-a\mathcal{A}_\omega(\varphi)-b\mathcal{B}_\omega(\varphi)\big)\\
&\quad=\frac{1}{V_\omega}\int\limits_X-\sqrt{-1}\,\bar\partial(\varphi\omega)\wedge\partial\varphi=\frac{1}{V_\omega}\int\limits_X-\sqrt{-1}(\bar\partial\varphi\wedge\omega+\varphi\bar\partial\omega)\wedge\partial\varphi\\
&\quad=\frac{1}{V_\omega}\int\limits_X\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\omega-2\mathcal{B}_\omega(\varphi).
\end{aligned}$$

Hence

$$\begin{aligned}
&3\big(\mathcal{J}^{\mathrm{AY}}_\omega(\varphi)-(c-1)\mathcal{A}_\omega(\varphi)-(d-1)\mathcal{B}_\omega(\varphi)\big)-\big(\mathcal{I}^{\mathrm{AY}}_\omega(\varphi)-a\mathcal{A}_\omega(\varphi)-b\mathcal{B}_\omega(\varphi)\big)\\
&\quad=\frac{1}{V_\omega}\int\limits_X\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\omega-\big(\mathcal{A}_\omega(\varphi)+\mathcal{B}_\omega(\varphi)\big).
\end{aligned}$$

Equivalently,

$$3\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) = \frac{1}{V_\omega}\int_X \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega, \tag{4.21}$$

where we also require

$$3(c-1) - a - 1 = 0 = 3(d-1) - b - 1. \tag{4.22}$$

Theorem 4.3. *For any $\varphi \in \mathcal{P}_\omega$, one has*

$$3\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \geqslant 0. \tag{4.23}$$

Combining (4.19) and (4.22) we obtain the value of those constants:

$$a = b = c = d = 2.$$

Corollary 4.4. *For any compact complex surface (X, g) and any real-valued smooth function $\varphi \in \mathcal{P}_\omega$, we have*

$$\begin{aligned}
&\frac{1}{3}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \leqslant \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \leqslant \frac{2}{3}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi),\\
&\frac{3}{2}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \leqslant \mathcal{I}_\omega(\varphi) \leqslant 3\mathcal{J}_\omega^{\mathrm{AY}}(\varphi),\\
&\frac{1}{2}\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \leqslant \frac{1}{3}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \leqslant \mathcal{I}_\omega^{\mathrm{AY}}(\varphi) - \mathcal{J}_\omega^{\mathrm{AY}}(\varphi) \leqslant \frac{2}{3}\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) \leqslant 2\mathcal{J}_\omega^{\mathrm{AY}}(\varphi),
\end{aligned}$$

where ω is its associated real $(1,1)$-form.

Acknowledgments

The author would like to thank Valentino Tosatti who read this note and pointed out a serious mistake in the first version. Furthermore, I also thank Chen-Yu Chi and Ming-Tao Chuan for useful discussions.

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