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ORIGINAL PAPER

On an extension of the H^k mean curvature flow of closed convex hypersurfaces

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Abstract In this paper we prove that the H^k (k is odd and larger than 2) mean curvature flow of a closed convex hypersurface can be extended over the maximal time provided that the total L^p integral of the mean curvature is finite for some p.

Keywords H^k mean curvature flow · Closed convex hypersurfaces · Singularity time

Mathematics Subject Classification (2000) Primary 53C45 · 35K55

1 Introduction

Let M be a compact n-dimensional hypersurface without boundary, which is smoothly embedded into the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ by the map

$$F_0 : M \longrightarrow \mathbb{R}^{n+1}. \tag{1.1}$$

The H^k mean curvature flow, an evolution equation of the mean curvature $H(\cdot, t)$, is a smooth family of immersions $F(\cdot, t) : M \to \mathbb{R}^{n+1}$ given by

$$\frac{\partial}{\partial t} F(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t), \quad F(\cdot, 0) = F_0(\cdot), \tag{1.2}$$

where k is a positive integer and $\nu(\cdot, t)$ denotes the outer unit normal on $M_t := F(M, t)$ at $F(\cdot, t)$.

When $k = 1$ the Eq. (1.2) is the usual mean curvature flow. Huisken [1] proved that the mean curvature flow develops to singularities in finite time: Suppose that $T_{\max} < \infty$ is the first singularity time for the mean curvature flow. Then $\sup_{M_t} |A|(t) \to \infty$ as $t \to T_{\max}$. Recently, Le and Sesum [2] and Xu et al. [5] independently proved an extension theorem on

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the mean curvature flow under some curvature conditions. A natural question is whether we can extend general H^k mean curvature flow over the maximal time interval.

The short time existence of the H^k mean curvature flow has been established in [4], i.e., there is a maximal time interval $[0, T_{\max})$, $T_{\max} < \infty$, on which the flow exists. In [3], we proved an extension theorem on the H^k mean curvature flow under some curvature condition; that is, the condition (b) in Theorem 1.1 holds and the second fundamental form has a lower bound along the flow. In this paper, we give another extension theorem of the H^k mean curvature flow for convex hypersurfaces.

Theorem 1.1 *Suppose that the integers n and k are greater than or equal to 2, k is odd, and $n+1 \geq k$. Suppose that M is a compact n-dimensional hypersurface without boundary, smoothly embedded into $\mathbb{R}^{n+1}$ by a smooth function F_0. Consider the H^k mean curvature flow on M,*

$$\frac{\partial}{\partial t} F(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t), \quad F(\cdot, 0) = F_0(\cdot).$$

If

(a) $H(\cdot) > 0$ on M,
(b) for some $\alpha \geq n+k+1$,

$$||H(\cdot, t)||_{L^\alpha(M\times[0,T_{\max}))} := \left(\int_0^{T_{\max}} \int_{M_t} |H(\cdot, t)|^\alpha_{g(t)} d\mu(t) dt \right)^{\frac{1}{\alpha}} < \infty,$$

then the flow can be extended over the time $T_{\max}$. Here $d\mu(t)$ denotes the induced metric on M_t.

If the second fundamental form has a lower bound, i.e., $h_{ij}(t) \geq C g_{ij}(t)$, then $H(t) \geq nC > 0$ which satisfies condition (a). Therefore the above theorem is a weak version of that in [3].

2 Evolution equations for the H^k mean curvature flow

Let $g = \{g_{ij}\}$ be the induced metric on M obtained by the pullback of the standard metric $g_{\mathbb{R}^{n+1}}$ of $\mathbb{R}^{n+1}$. We denote by $A = \{h_{ij}\}$ the second fundamental form and $d\mu = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n$ the volume form on M, respectively, where $x^1, \ldots, x^n$ are local coordinates. The mean curvature can be expressed as

$$H = g^{ij} h_{ij}, \quad g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle_{g_{\mathbb{R}^{n+1}}}; \tag{2.1}$$

meanwhile the second fundamental forms are given by

$$h_{ij} = -\left\langle \nu, \frac{\partial^2 F}{\partial x^i \partial x^j} \right\rangle_{g_{\mathbb{R}^{n+1}}}. \tag{2.2}$$

We write $g(t) = \{g_{ij}(t)\}$, $A(t) = \{h_{ij}(t)\}$, $\nu(t)$, $H(t)$, $d\mu(t)$, ∇_t, and Δ_t the corresponding induced metric, second fundamental form, outer unit normal vector, mean curvature, volume form, induced Levi–Civita connection, and induced Laplacian operator at time t.

The position coordinates are not explicitly written in the above symbols if there is no confusion.

The following evolution equations are obvious.

Lemma 2.1 *For the H^k mean curvature flow, we have*

$$\frac{\partial}{\partial t}H(t) = kH^{k-1}(t)\Delta_t H(t) + H^k(t)|A(t)|^2 + k(k-1)H^{k-2}(t)\,|\nabla_t H(t)|^2,$$
$$\frac{\partial}{\partial t}|A(t)|^2 = kH^{k-1}(t)\Delta_t|A(t)|^2 - 2kH^{k-1}(t)\,|\nabla_t A(t)|^2 + 2kH^{k-1}(t)|A(t)|^4 + 2k(k-1)H^{k-2}(t)|\nabla_t H(t)|^2.$$

Here and henceforth, the norm $|\cdot|$ is respect to the induced metric $g(t)$.

Corollary 2.2 *Suppose that* $\min_M H(0) > 0$. *If k is odd and larger than 2, then*

$$H(t) \geq \min_M H(0) \tag{2.3}$$

along the H^k mean curvature flow. In particular, $H(t) > 0$ is preserved by the H^k mean curvature flow.

Proof By Lemma 2.1, we have

$$\begin{aligned}\frac{\partial}{\partial t}H(t) &= kH^{k-1}(t)\Delta_t H(t) + H^k(t)|A(t)|^2 + k(k-1)H^{k-2}(t)\,|\nabla_t H(t)|^2\\ &= kH^{k-1}(t)\Delta_t H(t) + \left(H^{k-1}(t)|A(t)|^2 + k(k-1)H^{k-3}(t)\,|\nabla_t H(t)|^2\right)H(t).\end{aligned}$$

Since $k \geq 2$ and k is odd, it follows that

$$H^{k-1}(t)|A(t)|^2 + k(k-1)H^{k-3}(t)\,|\nabla_t H(t)|^2$$

is nonnegative and then (2.3) follows from the maximum principle. □

Lemma 2.3 *Suppose k is odd and larger than 2, and $H > 0$. For the H^k mean curvature flow and any positive integer ℓ, we have*

$$\begin{aligned}\left(\frac{\partial}{\partial t} - kH^{k-1}(t)\Delta_t\right)\left(\frac{|A(t)|^2}{H^{\ell+1}(t)}\right) &= \frac{k(\ell+1)}{k-1}\left\langle \nabla_t H^{k-1}(t), \nabla_t\left(\frac{|A(t)|^2}{H^{\ell+1}(t)}\right)\right\rangle\\ &\quad - \frac{2k}{H^{\ell+4-k}(t)}\left[\left(H(t)\nabla_t A(t) - \frac{\ell+1}{2}A(t)\nabla_t H(t)\right)\right]^2\\ &\quad + \frac{2k(k-1)}{H^{\ell+3-k}(t)}\,|\nabla_t H(t)|^2 + \frac{2k-\ell-1}{H^{\ell+2-k}(t)}|A(t)|^4\\ &\quad - \frac{k(\ell+1)(2k-\ell-1)}{2H^{\ell+4-k}(t)}|A(t)|^2\,|\nabla_t H(t)|^2.\end{aligned}$$

Proof In the following computation, we will always omit time t and write $\partial/\partial t$ as ∂_t. Then

$$\partial_t H = kH^{k-1}\Delta H + H^k|A|^2 + k(k-1)H^{k-2}|\nabla H|^2.$$

By Corollary 2.2, $H(t) > 0$ along the H^k mean curvature flow so that $|H(t)|^i = H^i(t)$ for each positive integer i. For any positive integer ℓ, we have

$$\begin{aligned}\partial_t |H|^{\ell+1} &= (\ell+1)H^\ell \partial_t H\\ &= (\ell+1)H^\ell \left(kH^{k-1}\Delta H + H^k|A|^2 + k(k-1)H^{k-2}|\nabla H|^2\right)\\ &= k(\ell+1)H^{k+\ell-1}\Delta H + (\ell+1)H^{k+\ell}|A|^2\\ &\quad + k(k-1)(\ell+1)H^{k+\ell-2}|\nabla H|^2,\\ \Delta|H|^{\ell+1} &= \Delta H^{\ell+1} \;=\; (\ell+1)\nabla\left(H^\ell \nabla H\right)\\ &= (\ell+1)\left(\ell H^{\ell-1}|\nabla H|^2 + H^\ell \Delta H\right)\\ &= (\ell+1)H^\ell \Delta H + \ell(\ell+1)H^{\ell-1}|\nabla H|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\partial_t H^{\ell+1} &= kH^{k-1}\Delta H^{\ell+1} - k\ell(\ell+1)H^{k+\ell-2}|\nabla H|^2\\ &\quad + (\ell+1)H^{k+\ell}|A|^2 + k(k-1)(\ell+1)H^{k+\ell-2}|\nabla H|^2\\ &= kH^{k-1}\Delta H^{\ell+1} + (\ell+1)H^{k+\ell}|A|^2\\ &\quad + k(k-\ell-1)(\ell+1)H^{k+\ell-2}|\nabla H|^2. \qquad (2.4)\end{aligned}$$

Recall from Lemma 2.1 that

$$\partial_t|A|^2 = kH^{k-1}\Delta|A|^2 - 2kH^{k-1}|\nabla A|^2 + 2kH^{k-1}|A|^4 + 2k(k-1)H^{k-2}|\nabla H|^2.$$

Calculate, using (2.4),

$$\begin{aligned}&\partial_t\left(\frac{|A|^2}{|H|^{\ell+1}}\right)\\ &= \frac{\partial_t|A|^2}{|H|^{\ell+1}} - \frac{|A|^2}{|H|^{2\ell+2}}\partial_t|H|^{\ell+1}\\ &= \frac{kH^{k-1}\Delta|A|^2 - 2kH^{k-1}|\nabla A|^2 + 2kH^{k-1}|A|^4 + 2k(k-1)H^{k-2}|\nabla H|^2}{H^{\ell+1}}\\ &\quad - \frac{|A|^2\left[kH^{k-1}\Delta H^{\ell+1} + (\ell+1)H^{k+\ell}|A|^2 + k(k-\ell-1)(\ell+1)H^{k+\ell-2}|\nabla H|^2\right]}{H^{2\ell+2}}\\ &= kH^{k-1}\frac{1}{H^{\ell+1}}\Delta|A|^2 - \frac{2k}{H^{\ell+2-k}}|\nabla A|^2 + \frac{2k}{H^{\ell+2-k}}|A|^4 + \frac{2k(k-1)}{H^{\ell+3-k}}|\nabla H|^2\\ &\quad - \frac{k|A|^2}{H^{2\ell+3-k}}\Delta H^{\ell+1} - \frac{\ell+1}{H^{\ell+2-k}}|A|^4 - \frac{k(k-\ell-1)(\ell+1)}{H^{\ell+4-k}}|A|^2|\nabla H|^2,\end{aligned}$$

and

$$\begin{aligned}\Delta\left(\frac{|A|^2}{H^{\ell+1}}\right) &= \frac{1}{H^{\ell+1}}\Delta|A|^2 + \Delta\left(\frac{1}{H^{\ell+1}}\right)|A|^2 + 2\left\langle\nabla|A|^2, \nabla\left(\frac{1}{H^{\ell+1}}\right)\right\rangle,\\ \nabla\left(\frac{1}{H^{\ell+1}}\right) &= \frac{-(\ell+1)H^\ell\nabla H}{H^{2\ell+2}} \;=\; \frac{-(\ell+1)\nabla H}{H^{\ell+2}},\\ \Delta\left(\frac{1}{H^{\ell+1}}\right) &= \nabla\left(\frac{-(\ell+1)\nabla H}{H^{\ell+2}}\right)\\ &= -(\ell+1)\frac{H^{\ell+2}\Delta H - \nabla H(\ell+2)H^{\ell+1}\nabla H}{H^{2\ell+4}}\end{aligned}$$

$$
\begin{aligned}
&= -(\ell+1)\left[\frac{\Delta H}{H^{\ell+2}} - (\ell+2)\frac{|\nabla H|^2}{H^{\ell+3}}\right],\\
\Delta H^{\ell+1} &= \nabla\left[(\ell+1)H^{\ell}\nabla H\right] \;=\; (\ell+1)\left[\ell H^{\ell-1}|\nabla H|^2 + H^{\ell}\Delta H\right]\\
&= \ell(\ell+1)H^{\ell-1}|\nabla H|^2 + (\ell+1)H^{\ell}\Delta H.
\end{aligned}
$$

Combining with all of them yields

$$
\begin{aligned}
&\left(\partial_t - kH^{k-1}\Delta\right)\left(\frac{|A|^2}{H^{\ell+1}}\right)\\
&= kH^{k-\ell-2}\Delta|A|^2 - \frac{2k}{H^{\ell+2-k}}|\nabla A|^2\\
&\quad + \frac{2k}{H^{\ell+2-k}}|A|^4 + \frac{2k(k-1)}{H^{\ell+3-k}}|\nabla H|^2 - \frac{k|A|^2}{H^{2\ell+3-k}}\left[\ell(\ell+1)H^{\ell-1}|\nabla H|^2 + (\ell+1)H^{\ell}\Delta H\right]\\
&\quad - \frac{\ell+1}{H^{\ell+2-k}}|A|^4 - \frac{k(k-\ell-1)(\ell+1)|A|^2}{H^{\ell-k+4}}|\nabla H|^2\\
&\quad - kH^{k-1}\left[\frac{1}{H^{\ell+1}}\Delta|A|^2 - (\ell+1)\frac{|A|^2\Delta H}{H^{\ell+2}} + (\ell+1)(\ell+2)\frac{|A|^2|\nabla H|^2}{H^{\ell+3}}\right]\\
&\quad - 2kH^{k-1}\left\langle \nabla|A|^2, \nabla\left(\frac{1}{H^{\ell+1}}\right)\right\rangle\\
&= -\frac{2k}{H^{\ell+2-k}}|\nabla A|^2 + \left(\frac{2k}{H^{\ell+2-k}} - \frac{\ell+1}{H^{\ell+2-k}}\right)|A|^4 + \frac{2k(k-1)}{H^{\ell+3-k}}|\nabla H|^2\\
&\quad - \frac{k(\ell+1)(k+\ell+1)|A|^2|\nabla H|^2}{H^{\ell+4-k}} - 2kH^{k-1}\left\langle \nabla|A|^2, \nabla\left(\frac{1}{H^{\ell+1}}\right)\right\rangle.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\left\langle \nabla|A|^2, \nabla\left(\frac{1}{H^{\ell+1}}\right)\right\rangle &= 2\left\langle \nabla A\cdot A, \frac{-(\ell+1)H^{\ell}\nabla H}{H^{2\ell+2}}\right\rangle\\
&= \frac{-2(\ell+1)}{H^{\ell+3}}\langle H\nabla A\cdot A, \nabla H\rangle.
\end{aligned}
$$

Thus, we conclude that

$$
\begin{aligned}
\left(\partial_t - kH^{k-1}\Delta\right)\left(\frac{|A|^2}{H^{\ell+1}}\right) &= -\frac{2k}{H^{\ell+2-k}}|\nabla A|^2 + \frac{2k-\ell-1}{H^{\ell+2-k}}|A|^4 + \frac{2k(k-1)}{H^{\ell+3-k}}|\nabla H|^2\\
&\quad - \frac{k(\ell+1)(k+\ell+1)|A|^2|\nabla H|^2}{H^{\ell+4-k}} + \frac{4k(\ell+1)}{H^{\ell+4-k}}\langle H\nabla A\cdot A, \nabla H\rangle.
\end{aligned}
$$

Consider the function

$$
f := \frac{-2k}{H^{\ell+2-k}}|\nabla A|^2 - \frac{k(\ell+1)(k+\ell+1)|A|^2|\nabla H|^2}{H^{\ell+4-k}} + \frac{4k(\ell+1)}{H^{\ell+4-k}}\langle H\nabla A\cdot A, \nabla H\rangle.
$$

Since

$$
\begin{aligned}
\frac{2k(\ell+1)}{H^{\ell+4-k}}\langle H\nabla A\cdot A, \nabla H\rangle &= \frac{k(\ell+1)}{H^{\ell+3-k}}\left\langle \nabla|A|^2, \nabla H\right\rangle,\\
\nabla\left(\frac{|A|^2}{H^{\ell+1}}\right) &= \frac{\nabla|A|^2}{H^{\ell+1}} - \frac{(\ell+1)|A|^2\nabla H}{H^{\ell+2}},
\end{aligned}
$$

it follows that

$$\frac{2k(\ell+1)}{H^{\ell+4-k}}\langle H\nabla A\cdot A,\nabla H\rangle = \frac{k(\ell+1)}{H^{2-k}}\nabla H\left[\nabla\left(\frac{|A|^2}{H^{\ell+1}}\right)+\frac{(\ell+1)|A|^2\nabla H}{H^{\ell+2}}\right]$$
$$= \frac{k(\ell+1)}{k-1}\left\langle \nabla H^{k-1},\nabla\left(\frac{|A|^2}{H^{\ell+1}}\right)\right\rangle + \frac{k(\ell+1)^2}{H^{\ell+4-k}}|A|^2|\nabla H|^2.$$

Consequently,

$$\begin{aligned} f &= \frac{-2k}{H^{\ell+2-k}}|\nabla A|^2 - \frac{k^2(\ell+1)}{H^{\ell+4-k}}|A|^2|\nabla H|^2 \\ &\quad + \frac{k(\ell+1)}{k-1}\left\langle \nabla H^{k-1},\nabla\left(\frac{|A|^2}{H^{\ell+1}}\right)\right\rangle + \frac{2k(\ell+1)}{H^{\ell+4-k}}\langle H\nabla A\cdot A,\nabla H\rangle \\ &= \frac{-2k}{H^{\ell+4-k}}\left[\left(H\nabla A - \frac{\ell+1}{2}A\cdot\nabla H\right)^2\right] - \frac{2k(\ell+1)(2k-\ell-1)}{4H^{\ell+4-k}}|A|^2|\nabla H|^2 \\ &\quad + \frac{k(\ell+1)}{k-1}\left\langle \nabla H^{k-1},\nabla\left(\frac{|A|^2}{H^{\ell+1}}\right)\right\rangle. \end{aligned}$$

Finally, we complete the proof. □

Corollary 2.4 *Suppose k is odd and larger than 2, and $H>0$. For the H^k mean curvature flow, we have*

$$\left(\frac{\partial}{\partial t} - kH^{k-1}(t)\Delta_t\right)\left(\frac{|A(t)|^2}{H^{2k}(t)}\right) = \frac{2k^2}{k-1}\left\langle \nabla_t H^{k-1}(t),\nabla_t\left(\frac{|A(t)|^2}{H^{2k}(t)}\right)\right\rangle + \frac{2k(k-1)}{H^{k+2}(t)}|\nabla_t H(t)|^2$$
$$- \frac{2k}{H^{k+3}(t)}\left[H(t)\cdot\nabla_t A(t) - kA(t)\cdot\nabla_t H(t)\right]^2.$$

3 Proof of the main theorem

In this section we give a proof of theorem 1.1. For any positive constant C_0, consider the quantity

$$Q(t) := \frac{|A(t)|^2}{H^{2k}(t)} + C_0 H^{\ell+1}(t), \tag{3.1}$$

where the integer ℓ is determined later. By (2.4) and Corollary 2.4, we have

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - kH^{k-1}(t)\Delta_t\right)Q(t) \\ &\le \frac{2k^2}{k-1}\left\langle \nabla_t H^{k-1}(t),\nabla_t Q(t) - C_0\nabla_t H^{\ell+1}(t)\right\rangle \\ &\quad + \frac{2k(k-1)}{H^{k+2}(t)}|\nabla_t H(t)|^2 + C_0\Big[(\ell+1)H^{k+\ell}(t)|A(t)|^2 \\ &\quad + k(k-\ell-1)(\ell+1)H^{k+\ell-2}(t)\,|\nabla_t H(t)|^2\Big] \\ &= \frac{2k^2}{k-1}\left\langle \nabla_t H^{k-1}(t),\nabla_t Q(t)\right\rangle - \frac{2k^2}{k-1}C_0(k-1)(\ell+1)H^{k+\ell-2}(t)\,|\nabla_t H(t)|^2 \end{aligned}$$

$$
\begin{aligned}
&+ \frac{2k(k-1)}{H^{k+2}(t)} |\nabla_t H(t)|^2 + C_0 k(k-\ell-1)(\ell+1) H^{k+\ell-2}(t) |\nabla_t H(t)|^2 \\
&+ C_0(\ell+1) H^{k+\ell}(t) \left[Q(t) - C_0 H^{\ell+1}(t) \right] H^{2k}(t) \\
= &\frac{2k^2}{k-1} \left\langle \nabla_t H^{k-1}(t), \nabla_t Q(t) \right\rangle \\
&+ |\nabla_t H(t)|^2 \left[\frac{2k(k-1)}{H^{k+2}(t)} - C_0 k(\ell+1)(k+\ell+1) H^{k+\ell-2}(t) \right] \\
&+ C_0(\ell+1) H^{3k+\ell}(t) Q(t) - C_0^2(\ell+1) H^{3k+2\ell+1}(t).
\end{aligned}
$$

Now we choose ℓ so that the following constraints

$$
\ell+1 \le 0, \quad k+\ell+1 \le 0, \quad 3k+2\ell+1 \ge 0
$$

are satisfied; that is

$$
-\frac{1}{2} - \frac{3}{2}k \le \ell \le -1-k. \tag{3.2}
$$

In particular, we can take

$$
\ell := -2-k. \tag{3.3}
$$

By our assumption on k, we have $k \ge 3$ and hence (3.3) implies (3.2). Plugging (3.3) into the above inequality yields

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - k H^{k-1}(t) \Delta_t \right) Q(t) \le &\frac{2k^2}{k-1} \left\langle \nabla H^{k-1}(t), \nabla_t Q(t) \right\rangle \\
&+ |\nabla_t H(t)|^2 \left[\frac{2k(k-1)}{H^{k+2}(t)} - \frac{C_0 k(k+1)}{H^4(t)} \right] \\
&- C_0(1+k) H^{2k-2}(t) Q(t) + C_0^2(1+k) H^{k-3}(t).
\end{aligned} \tag{3.4}
$$

Choosing

$$
C_0 := \frac{2(k-1)}{k+1} H_{\min}^{2-k} > 0 \tag{3.5}
$$

where $H_{\min} := \min_M H = \min_M H(0)$, we arrive at

$$
\frac{2k(k-1)}{C_0 k(k+1)} \le H_{\min}^{k-2} \le H^{k-2}(0) \le H^{k-2}(t)
$$

according to (2.3). Consequently,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - k H^{k-1}(t) \Delta_t \right) Q(t) \le &\frac{2k^2}{k-1} \left\langle \nabla_t H^{k-1}(t), \nabla_t Q(t) \right\rangle \\
&- C_1 H^{2k-2}(t) Q(t) + C_2 H^{k-3}(t),
\end{aligned} \tag{3.6}
$$

for $C_1 := C_0(1+k)$ and $C_2 := C_0^2(1+k)$.

Lemma 3.1 *If the solution can not be extended over $T_{\max}$, then $H(t)$ is unbounded.*

Proof By the assumption, we know that $|A(t)|$ is unbounded as $t \to T_{\max}$. We now claim that $H(t)$ is also unbounded. Otherwise, $0 < H_{\min} \leq H(t) \leq C$ for some uniform constant C. If we set

$$C_3 := C_1 H_{\min}^{2k-2}, \quad C_4 := C_2 C^{k-3},$$

then (3.6) implies that

$$\left(\frac{\partial}{\partial t} - kH^{k-1}(t)\Delta_t\right) Q(t) \leq \frac{2k^2}{k-1} \left\langle \nabla_t H^{k-1}(t), \nabla_t Q(t) \right\rangle - C_3 Q(t) + C_4. \tag{3.7}$$

By the maximum principle, we have

$$\mathcal{Q}'(t) \leq -C_3 \mathcal{Q}(t) + C_4 \tag{3.8}$$

where

$$\mathcal{Q}(t) := \max_M Q(t).$$

Solving (3.8) we find that

$$\mathcal{Q}(t) \leq \frac{C_4}{C_3} + \left(\mathcal{Q}(0) - \frac{C_4}{C_3}\right) e^{-C_3 t}.$$

Thus $Q(t) \leq C_5$ for some uniform constant C_5. By the definition (3.1) and the assumption $H(t) \leq C$, we conclude that $|A(t)| \leq C_6$ for some uniform constant C_6, which is a contradiction. □

The rest proof is similar to [3,5]. Using Lemma 3.1 and the argument in [3] or in [5], we get a contradiction and then the solution of the H^k mean curvature flow can be extended over $T_{\max}$.

References

1. Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differ. Geom. **20**(1), 237–266 (1984). MR0772132 (86j: 53097)
2. Le, N.Q., Sesum, N.: On the extension of the mean curvature flow. Math. Z. **267**(3–4), 583–604 (2011). MR2776050 (2012h: 53153)
3. Li, Y.: On an extension of the H^k mean curvature flow. Sci. China Math. **55**(1), 99–118 (2012). MR2873806
4. Smoczyk, K.: Harnack inequalities for curvature flows depending on mean curvature. New York J. Math. **3**, 103–118 (1997). MR1480081 (98i: 53056)
5. Xu, H.-W., Ye, F., Zhao, E.-T.: Extend mean curvature flow with finite integral curvature. Asian J. Math. **15**(4), 549–556 (2011). MR2853649 (2012h: 53158)