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Harnack estimates for a heat-type equation under the Ricci flow ☆

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Abstract

In this paper, we consider the gradient estimates for a positive solution of the nonlinear parabolic equation $\partial_t u = \Delta_t u + hu^p$ on a Riemannian manifold whose metrics evolve under the Ricci flow. Two Harnack inequalities and other interesting results are obtained.

MSC: 53C44

Keywords: Nonlinear parabolic equation; Harnack estimate; Ricci flow

1. Introduction

We continue to consider the gradient estimates for nonlinear partial differential equations after our previous works [8,9]. Let $(M, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)}, \quad t \in [0, T], \tag{1.1}$$

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on an n-dimensional manifold M and consider a positive function $u = u(x,t)$ defined on $M \times [0,T]$ solving the equation

$$\partial_t u = \Delta_t u + hu^p, \quad t \in [0,T], \tag{1.2}$$

where Δ_t stands for the Laplacian of $g(t)$, h is a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, and p is a positive constant. When metrics are fixed, the study on the gradient estimates of (1.2) arose from [4]. If $p = 1$, Bailesteanu, Cao and Pulemotov [1] derived the gradient estimates and the Harnack inequalities for the positive solutions of the linear parabolic equation $\partial_t u = \Delta_t u$ under the Ricci flow. In this paper, we consider the general case for the nonlinear parabolic equation. Notice that the Δ_t depends on the parameter t, and we should study the equation (1.2) coupled with the Ricci flow (1.1). The formula (1.1) provides us with additional information about the coefficients of the operator Δ_t appearing in (1.2) but is itself fully independent of (1.2).

We introduce notions used throughout this paper. Let $B_{\rho,T} = \{(x,t) \in M \times [0,T] : \mathrm{dist}_t(x,x_0) < \rho\}$, where $\mathrm{dist}_t(x,x_0)$ denotes the distance between x and a fixed point x_0 with respect to $g(t)$. ∇_t and $|\cdot|_t$ stand for the Levi-Civita connection and norm with respect to $g(t)$ respectively.

Theorem 1.1. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M with $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some $K_1, K_2 > 0$ on $B_{2R,T}$, and $\overline{K} = \max\{K_1, K_2\}$. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, satisfying $\Delta_t h \ge -\theta$ and $|\nabla_t h|_t \le \gamma$ on $B_{2R,T} \times [0,T]$ for some constants θ and γ. If $u(x,t)$ is a positive smooth solution of (1.2) on $M \times [0,T]$, then*

(i) *for $0 < p < 1$, we have*

$$\begin{aligned}\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le{}& \frac{C_1}{p^2 t} + \frac{n(1-p)}{p^2}M_1 M_2 + \frac{n}{2p^2(1-p)}K_1 \\ &+ \frac{C_1}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + \overline{K} + \frac{n}{p(1-p)}\right) \\ &+ \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{n/K_1}}{p}\gamma M_2,\end{aligned} \tag{1.3}$$

where C_1 is a positive constant depending only on n and

$$M_1 := \max_{B_{2R,T}} h_-, \quad M_2 := \max_{B_{2R,T}} u^{p-1}, \quad h_- := \max(-h, 0);$$

(ii) *for $p \ge 1$, we have*

$$\begin{aligned}\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{t} \le{}& \frac{k^2 C_2}{p^2 t} + \frac{nk^2(p-1)}{p^2}M_4 M_5 + \frac{k^3 n}{k-p}M_3 M_4 \\ &+ \frac{k^2 C_2}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + \overline{K} + \frac{k^2 n}{p(k-p)}\right)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2k^3 n}{(k-p)p^2} K_1 + \frac{k^2\sqrt{n}\gamma}{p}\sqrt{M_4} \\
&+ \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4} + \frac{k^2 n}{p^2}\overline{K},
\end{aligned} \tag{1.4}
$$

where $k > p$, C_2 is a positive constant depending only on n and

$$
M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h.
$$

As an immediate consequence of the above theorem we have

Theorem 1.2. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t.*

(i) For $0 < p < 1$, assume that $h \ge 0$, $|\nabla_t h|_t \le \gamma$, $\Delta_t h \ge 0$, and $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some positive constants γ, K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$, along the Ricci flow. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then

$$
\begin{aligned}
\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \frac{C_1}{p^2 t} + \frac{C_1}{p^3(1-p)} + \frac{C_1}{p^2}\overline{K} + \frac{n}{2p^2(1-p)}K_1 \\
+ \frac{\sqrt{n/K_1}}{p}\gamma M
\end{aligned} \tag{1.5}
$$

for some positive constant C_1 depending only on n, where $M := \max_{M\times[0,T]} u^{p-1}$.

(ii) For $p = 1$, assume that $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some positive constants K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$, $h \ge 0$, $\Delta_t h \ge -\theta$ (θ is nonnegative), and $|\nabla_t h|_t \le \gamma$ (γ is nonnegative), along the Ricci flow. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then

$$
\frac{|\nabla_t u|_t^2}{u^2} + h - \frac{u_t}{u} \le \frac{C_2}{t} + C_2\left(1 + K_1 + \overline{K} + \gamma + \sqrt{\theta}\right) \tag{1.6}
$$

for some positive constant C_2 depending only on n.

(iii) For $p > 1$, assume that $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some positive constants K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$, $\Delta_t h \ge -\theta$, $|\nabla_t h|_t \le \gamma$, and $-k_1 \le h \le k_2$, where $\theta, \gamma, k_1, k_2 > 0$, along the Ricci flow. If u is a bounded smooth positive function satisfying the nonlinear parabolic equation (1.2), then

$$
\begin{aligned}
\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} &\le \left(\frac{k}{p}\right)^2 \frac{C_3}{t} + \left(\frac{k}{p}\right)^3 \frac{k}{k-p} C_3 \\
&+ \left(\frac{k}{p}\right)^2 C_3\left(\overline{K} + \frac{k}{k-p}K_1\right) + \left(\frac{k}{p}\right)^2 n(p-1)k_2 M \\
&+ \frac{k^3 n}{k-p}k_1 M + \frac{k^2\sqrt{n}}{p}\gamma\sqrt{M} + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M},
\end{aligned} \tag{1.7}
$$

for some positive constant C_3 depending only on n, where $M := \max_{M\times[0,T]} u^{p-1}$ and $k > p$. In particular, taking $k = 2p$, we get

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \frac{C_4}{t} + C_4\left(1 + K_1 + \overline{K}\right) + C_4 p^2\left[(k_1+k_2)M + \gamma\sqrt{M} + \sqrt{\theta M}\right], \tag{1.8}$$

for some positive constant C_4 depending only on n.

Another type of Harnack inequality is the following

Theorem 1.3. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M, satisfying $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some positive constants K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \ge 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \le 1$), and $0 < p \le \frac{2n}{2n-1}$ ($n \ge 3$). If u is a positive solution of (1.2), then*

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{u} \le \frac{C}{p^2 t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{4n}{p(2-p)}K_1 \tag{1.9}$$

for some positive constant C depending only on n.

We require the restriction $0 < p \le \frac{2n}{2n-1}$ and $n \ge 3$ for a technical reason. In [4], the restriction is $0 < p < \frac{n}{n-1}$. When the metric evolves by the Ricci flow, additional terms in the computation lead to the above restriction $0 < p \le \frac{2n}{2n-1} < \frac{n}{n-1}$. However, both restrictions contain the critical point $p = 1$.

This theorem has three important consequences.

Corollary 1.4. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M, satisfying $0 \le \mathrm{Ric}_{g(t)} \le K g(t)$ for some positive constant K. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \ge 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \le 1$), and $0 < p \le \frac{2n}{2n-1}$ ($n \ge 3$). If u is a positive solution of (1.2), then*

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{u} \le \frac{C}{p^2 t} + \frac{8n}{p^2}K \tag{1.10}$$

for some positive constant C depending only on n.

Corollary 1.5. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M, satisfying $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some positive constants K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \ge 0$ on $M \times [0,T]$ (where*

$C_{n,p}=\frac{p}{p-1}$ if $p>1$ and $C_{n,p}=n$ if $p\le 1$), and $0<p\le\frac{2n}{2n-1}$ ($n\ge 3$). If u is a positive solution of (1.2), then

$$\frac{u(x_2,t_2)}{u(x_1,t_1)}\ge\left(\frac{t_2}{t_1}\right)^{-C/p}\exp\Bigg[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot\gamma(t)|_t^2dt$$
$$-2n(t_2-t_1)\left(\frac{1}{p}\overline{K}+\frac{2}{p}\sqrt{\frac{2n}{p(2-p)}K_1}+\frac{1}{2-p}K_1\right)\Bigg] \tag{1.11}$$

for some positive constant C depending only on n, where $(x_1,t_1),(x_2,t_2)\in M\times[0,T]$ with $t_1<t_2$, and $\Theta(x_1,t_1,x_2,t_2)$ is the set of all the smooth paths $\gamma:[t_1,t_2]\to\mathcal{M}$ connecting x_1 to x_2.

When $K_1=0$, we have the following

Corollary 1.6. *Suppose that $(M,g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M, satisfying $0\le \mathrm{Ric}_{g(t)}\le Kg(t)$ for some positive constant K. Let $h(x,t)$ be a nonnegative function defined on $M\times[0,T]$ which is C^2 in x and C^1 in t, $\Delta_t h+h_t-2C_{n,p}p\frac{|\nabla_t h|_t^2}{h}\ge 0$ on $M\times[0,T]$ (where $C_{n,p}=\frac{p}{p-1}$ if $p>1$ and $C_{n,p}=n$ if $p\le 1$), and $0<p\le\frac{2n}{2n-1}$ ($n\ge 3$). If u is a positive solution of (1.2), then*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)}\ge\left(\frac{t_2}{t_1}\right)^{-C/p}\exp\left[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot\gamma(t)|_t^2dt-\frac{2nK}{p}(t_2-t_1)\right]$$

for some positive constant C depending only on n, where $(x_1,t_1),(x_2,t_2)\in M\times[0,T]$ with $t_1<t_2$.

2. Auxiliary lemmas

Suppose u is a positive solution of (1.1). As in [4], we introduce a new function

$$W=u^{-q}, \tag{2.1}$$

where q is a positive constant to be determined later. For convenience, we always omit time variable t and write $\mathcal{Q}_t$ for the partial derivative of $\mathcal{Q}$ relative to t. For example, throughout this paper, $\Delta,\nabla,|\cdot|$ mean the correspondence quantities with respect to $g(t)$. Write

$$\square:=\Delta-\partial_t.$$

A simple computation shows that

$$\begin{aligned}\nabla W=-qu^{-q-1}\nabla u,\quad |\nabla W|^2&=q^2u^{-2q-2}|\nabla u|^2,\\ W_t=-qu^{-q-1}u_t,\quad \Delta W&=q(q+1)u^{-q-2}|\nabla u|^2-qu^{-q-1}\Delta u.\end{aligned}$$

The relation (2.1) yields

$$|\nabla u|^2 = \frac{|\nabla W|^2}{q^2 W^{2+2/q}}, \quad u_t = -\frac{W_t}{q W^{1+1/q}},$$

and hence

$$\begin{aligned}\Delta W &= q(q+1) W^{1+2/q} |\nabla u|^2 - q W^{1+1/q} \Delta u \\ &= q(q+1) W^{1+2/q} \frac{|\nabla W|^2}{q^2 W^{2+2/q}} - q W^{1+1/q} \Delta u \\ &= \frac{q+1}{q} \frac{|\nabla W|^2}{W} - q W^{1+1/q} \Delta u. \end{aligned} \tag{2.2}$$

From the equation (1.1), we have

$$\begin{aligned}\Delta W &= \frac{q+1}{q} \frac{|\nabla W|^2}{W} - q W^{1+1/q} \left(u_t - h u^p\right) \\ &= \frac{q+1}{q} \frac{|\nabla W|^2}{W} - q W^{1+1/q} \left(-\frac{W_t}{q W^{1+1/q}} - h W^{-p/q}\right) \\ &= \frac{q+1}{q} \frac{|\nabla W|^2}{W} + W_t + q h W^{1+\frac{1-p}{q}}. \end{aligned} \tag{2.3}$$

Therefore

$$\Box W = \frac{q+1}{q} \frac{|\nabla W|^2}{W} + q h W^{1+\frac{1-p}{q}}. \tag{2.4}$$

Because $|\nabla W|^2/W^2 = q^2 |\nabla u|^2/u^2$ and $h W^{(1-p)/q} = h u^{p-1}$, we may consider the same quantities as in [4]

$$F_0 := \frac{|\nabla W|^2}{W^2} + \alpha h W^{(1-p)/q} \;=\; |\nabla \ln W|^2 + \alpha h W^{(1-p)/q}, \tag{2.5}$$

$$F_1 := \frac{W_t}{W} \;=\; \partial_t \ln W, \tag{2.6}$$

$$F := F_0 + \beta F_1. \tag{2.7}$$

Here α, β are two positive constants to be fixed later.

Lemma 2.1. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M. If u is a positive solution of (1.2), then*

$$\begin{aligned}\Box F_1 &= \frac{2}{q} \langle \nabla F_1, \nabla \ln W\rangle + (1-p) h W^{\frac{1-p}{q}} \frac{W_t}{W} + q h_t W^{(1-p)/q} \\ &\quad + 2\left(1 + \frac{1}{q}\right) \mathrm{Ric}(\nabla \ln W, \nabla \ln W) - 2\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle. \end{aligned} \tag{2.8}$$

Proof. Calculate

$$\nabla F_1 = \frac{\nabla W_t}{W} - \frac{W_t \nabla W}{W^2}, \quad \partial_t F_1 = \frac{W_{tt}}{W} - \frac{W_t^2}{W^2}$$
$$\Delta F_1 = \frac{\Delta W_t}{W} - \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{W_t \Delta W}{W^2} + \frac{2|\nabla W|^2 W_t}{W^3}.$$

Then we conclude that

$$\Box F_1 = \frac{\Delta W_t - W_{tt}}{W} - \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{W_t(\Delta W - W_t)}{W^2} + \frac{2|\nabla W|^2 W_t}{W^3}. \tag{2.9}$$

Since g_{ij} evolves under the Ricci flow (1.1), it follows that

$$\begin{aligned}(\Delta W)_t = \partial_t\left(g^{ij}\nabla_i\nabla_j W\right) &= \left(\partial_t g^{ij}\right)\nabla_i\nabla_j W + g^{ij}\partial_t\left(\partial_i\partial_j W - \Gamma_{ij}^k \partial_k W\right)\\
&= 2R_{ij}\nabla^i\nabla^j W + \Delta(W_t) - g^{ij}\partial_k W \partial_t \Gamma_{ij}^k\\
&= \Delta(W_t) + 2R_{ij}\nabla^i\nabla^j W\end{aligned}$$

using the fact $g^{ij}\partial_t\Gamma_{ij}^k = 0$. Now the term $\Delta W_t - W_{tt} = (\Delta W - W_t)_t - 2R_{ij}\nabla^i\nabla^j W$ can be simplified by the above formula and (2.4) as

$$\begin{aligned}\Delta W_t - W_{tt} &= \left[\left(1+\frac{1}{q}\right)\frac{|\nabla W|^2}{W} + qhW^{1+\frac{1-p}{q}}\right]_t - 2R_{ij}\nabla^i\nabla^j W\\
&= \left(1+\frac{1}{q}\right)\left(\frac{2\langle \nabla W, \nabla W_t\rangle}{W} - \frac{|\nabla W|^2 W_t}{W^2} - \frac{2\mathrm{Ric}(\nabla W, \nabla W)}{W}\right)\\
&\quad + q\left[W^{1+\frac{1-p}{q}}h_t + h\left(1+\frac{1-p}{q}\right)W^{\frac{1-p}{q}}W_t\right] + 2R_{ij}\nabla^i\nabla^j W\\
&= 2\left(1+\frac{1}{q}\right)\frac{\langle \nabla W, \nabla W_t\rangle}{W} - \left(1+\frac{1}{q}\right)\frac{|\nabla W|^2 W_t}{W^2} + qh_t W^{1+\frac{1-p}{q}}\\
&\quad + \left(1+\frac{1}{q}\right)\frac{2\mathrm{Ric}(\nabla W, \nabla W)}{W}\\
&\quad + h(q+1-p)W^{\frac{1-p}{q}}W_t - 2R_{ij}\nabla^i\nabla^j W.\end{aligned}$$

Plugging it into (2.9) yields

$$\begin{aligned}\Box F_1 &= 2\left(1+\frac{1}{q}\right)\frac{\langle \nabla W, \nabla W_t\rangle}{W^2} - \left(1+\frac{1}{q}\right)\frac{|\nabla W|^2 W_t}{W^3}\\
&\quad + h(q+1-p)W^{\frac{1-p}{q}-1}W_t + qh_t W^{\frac{1-p}{q}}\\
&\quad - \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} + \left(1+\frac{1}{q}\right)\frac{2\mathrm{Ric}(\nabla W, \nabla W)}{W} - \frac{2R_{ij}\nabla^i\nabla^j W}{W}\end{aligned}$$

$$
\begin{aligned}
&- \frac{W_t}{W^2}\left(\frac{q+1}{q}\frac{|\nabla W|^2}{W} + qhW^{1+\frac{1-p}{q}}\right) + \frac{2|\nabla W|^2 W_t}{W^3} \\
&= \frac{2}{q}\frac{\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{2}{q}\frac{|\nabla W|^2 W_t}{W^3} + (1-p)hW^{\frac{1-p}{q}-1}W_t + qh_tW^{\frac{1-p}{q}} \\
&+ \left(1+\frac{1}{q}\right)\frac{2\mathrm{Ric}(\nabla W, \nabla W)}{W} - \frac{2R_{ij}\nabla^i\nabla^j W}{W}.
\end{aligned}
$$

The desired equation (2.8) immediately follows from $\langle \nabla F_1, \nabla \ln W\rangle = \frac{\langle \nabla W_t, \nabla W\rangle}{W^2} - \frac{|\nabla W|^2 W_t}{W^3}$. $\square$

Similarly, we can find the evolution equation of (2.5).

Lemma 2.2. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M. If u is a positive solution of (1.2), then*

$$
\begin{aligned}
\square F_0 \geq\ & 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F_0, \nabla \ln W\rangle \\
&- \frac{2\alpha p}{q}W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle + \alpha W^{\frac{1-p}{q}}(\Delta h - h_t) \\
&+ (1-p)\left(2-\frac{\alpha p}{q^2}\right)hW^{\frac{1-p}{q}}|\nabla \ln W|^2 + \alpha(1-p)h^2W^{\frac{2(1-p)}{q}},
\end{aligned}
\tag{2.10}
$$

where $\epsilon \in (0, 1]$ is any given constant.

Proof. Compute

$$
\begin{aligned}
\nabla F_0 &= \frac{\nabla|\nabla W|^2}{W^2} - \frac{2|\nabla W|^2\nabla W}{W^3} + \alpha W^{\frac{1-p}{q}}\nabla h + \alpha\left(\frac{1-p}{q}\right)hW^{\frac{1-p}{q}-1}\nabla W, \\
\Delta F_0 &= \nabla^i\left(\frac{2\nabla^j W\nabla_i\nabla_j W}{W^2} - \frac{2|\nabla W|^2\nabla_i W}{W^3}\right) + \alpha\left(\frac{1-p}{q}\right)hW^{\frac{1-p}{q}-1}\Delta W \\
&+ \alpha\left[W^{\frac{1-p}{q}}\Delta h + \left(\frac{1-p}{q}\right)W^{\frac{1-p}{q}-1}\langle \nabla h, \nabla W\rangle\right] \\
&+ \alpha\left(\frac{1-p}{q}\right)\left[h\left(\frac{1-p}{q}-1\right)W^{\frac{1-p}{q}}\frac{|\nabla W|^2}{W^2} + W^{\frac{1-p}{q}-1}\langle \nabla h, \nabla W\rangle\right].
\end{aligned}
$$

Simplifying ΔF_0 yields

$$
\begin{aligned}
\Delta F_0 =\ & \frac{2|\nabla^2 W|^2}{W^2} + \frac{2\langle \nabla W, \Delta\nabla W\rangle}{W^2} - 8\frac{\langle \nabla^2 W, \nabla W\otimes\nabla W\rangle}{W^3} - \frac{2|\nabla W|^2\Delta W}{W^3} \\
&+ \frac{6|\nabla W|^4}{W^4} + \alpha W^{\frac{1-p}{q}}\Delta h + 2\alpha\left(\frac{1-p}{q}\right)W^{\frac{1-p}{q}-1}\langle \nabla W, \nabla h\rangle \\
&+ \alpha\left(\frac{1-p}{q}\right)\left(\frac{1-p}{q}-1\right)hW^{\frac{1-p}{q}}\frac{|\nabla W|^2}{W^2} + \alpha\left(\frac{1-p}{q}\right)hW^{\frac{1-p}{q}-1}\Delta W.
\end{aligned}
\tag{2.11}
$$

On the other hand, the time derivative of F_0 equals

$$\partial_t F_0 = \frac{2\langle \nabla W, \nabla W_t\rangle}{W^2} - \frac{2|\nabla W|^2 W_t}{W^3} + \alpha h_t W^{\frac{1-p}{q}} + \alpha\left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1} W_t + \frac{2\mathrm{Ric}(\nabla W, \nabla W)}{W^2}. \tag{2.12}$$

From (2.11), (2.12) and the Ricci identity $\Delta \nabla_i W = \nabla_i \Delta W + R_{ij}\nabla^j W$, we have

$$\begin{aligned}\Box F_0 = {} & \frac{2\langle \nabla W, \nabla(\Delta W - W_t)\rangle}{W^2} - \frac{2|\nabla W|^2(\Delta W - W_t))}{W^3} \\ & + \left(\frac{2|\nabla^2 W|^2}{W^2} - \frac{8\langle \nabla^2 W, \nabla W \otimes \nabla W\rangle}{W^3} + \frac{6|\nabla W|^4}{W^4}\right) \\ & + \alpha W^{\frac{1-p}{q}}(\Delta h - h_t) + \alpha\left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-1}(\Delta W - W_t) \\ & + 2\alpha\left(\frac{1-p}{q}\right) W^{\frac{1-p}{q}-1}\langle \nabla W, \nabla h\rangle \\ & + \alpha\left(\frac{1-p}{q}\right)\left(\frac{1-p}{q} - 1\right) h W^{\frac{1-p}{q}} \frac{|\nabla W|^2}{W^2}. \end{aligned} \tag{2.13}$$

Plugging (2.5) and

$$\begin{aligned}\langle \nabla F_0, \nabla \ln W\rangle = {} & \frac{2}{W^3}\langle \nabla^2 W, \nabla W \otimes \nabla W\rangle - \frac{2|\nabla W|^4}{W^4} \\ & + \alpha W^{\frac{1-p}{q}-1}\langle \nabla W, \nabla h\rangle + \alpha\left(\frac{1-p}{q}\right) h W^{\frac{1-p}{q}-2}|\nabla W|^2\end{aligned}$$

into (2.13), we arrive at

$$\begin{aligned}\Box F_0 - \frac{2}{q}\langle \nabla F_0, \nabla \ln W\rangle = {} & 2\left(\frac{|\nabla^2 W|^2}{W^2} - 2\frac{\langle \nabla^2 W, \nabla W \otimes \nabla W\rangle}{W^3} + \frac{|\nabla W|^4}{W^4}\right) \\ & + \frac{-2\alpha p}{q^2} W^{\frac{1-p}{q}-1}\langle \nabla W, \nabla h\rangle \\ & + \alpha W^{\frac{1-p}{q}}(\Delta h - h_t) + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} \\ & + (1-p)\left(2 - \frac{\alpha p}{q^2}\right) h W^{\frac{1-p}{q}} \frac{|\nabla W|^2}{W^2}.\end{aligned}$$

Therefore (2.10) follows by Hölder inequality. $\Box$

Combining Lemma 2.1 with Lemma 2.2, we get

Proposition 2.3. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M. If u is a positive solution of (1.2), define*

$$W = u^{-q}, \quad F = \frac{|\nabla W|^2}{W^2} + \alpha h W^{\frac{1-p}{q}} + \beta \frac{W_t}{W}.$$

Then for all $\epsilon \in (0, 1]$ we have

$$\begin{aligned}
\Box F \geq\ & 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\
& + 2\beta\left(1+\frac{1}{q}\right)\mathrm{Ric}(\nabla \ln W, \nabla \ln W) - \frac{2\alpha p}{q} W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
& + (1-p)\left(2-\frac{\alpha p}{q^2}\right) h W^{\frac{1-p}{q}} |\nabla \ln W|^2 + W^{\frac{1-p}{q}}[\alpha \Delta h + h_t(q\beta - \alpha)] \\
& + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} + \beta(1-p) h W^{\frac{1-p}{q}} \frac{W_t}{W} - 2\beta\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle. \qquad (2.14)
\end{aligned}$$

3. Two special cases

The first special case of (2.14) is to choose

$$\beta := \frac{\alpha}{q}, \quad \alpha = \frac{kq^2}{p}. \qquad (3.1)$$

Then $q\beta - \alpha = 0$ so that (2.14) becomes

$$\begin{aligned}
\Box F \geq\ & 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2\alpha(1+q)}{q^2}\mathrm{Ric}(\nabla \ln W, \nabla \ln W) \\
& + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle - \frac{2\alpha p}{q} W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
& + (1-p)\left(2-\frac{\alpha p}{q^2}\right) h W^{\frac{1-p}{q}} |\nabla \ln W|^2 + \alpha W^{\frac{1-p}{q}} \Delta h \\
& + \alpha(1-p)h^2 W^{\frac{2(1-p)}{q}} + \frac{\alpha(1-p)}{q} h W^{\frac{1-p}{q}} \frac{W_t}{W} - \frac{2\alpha}{q}\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle. \qquad (3.2)
\end{aligned}$$

Recall the inequality

$$\begin{aligned}
2\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{2\alpha}{q}\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle &= 2\left[(a+b)\frac{\alpha}{q}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{\alpha}{q}\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle\right] \\
&= 2\left[\frac{a\alpha}{q}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{\alpha}{4bq}|\mathrm{Ric}|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\alpha}{q}\left(\sqrt{b}\frac{\nabla^2 W}{W}-\frac{\mathrm{Ric}}{2\sqrt{b}}\right)^2\\
&\geq 2\left[\frac{a\alpha}{q}\left|\frac{\nabla^2 W}{W}\right|^2-\frac{\alpha}{4bq}|\mathrm{Ric}|^2\right],
\end{aligned}
\tag{3.3}
$$

for any positive real numbers a, b satisfying $a+b=\frac{q}{\alpha}$, with the equality if $\mathrm{Ric}=2b\nabla^2 W/W$. Using the inequality $|\nabla^2 W|^2\geq (\Delta W)^2/n$, we conclude from (3.2) and (3.3) that

$$
\begin{aligned}
\square F \geq\ & \frac{2}{n}\left(\frac{a\alpha}{q}-\epsilon\right)\left|\frac{\Delta W}{W}\right|^2+2\left(1-\frac{1}{\epsilon}\right)|\nabla \ln W|^4+\frac{2}{q}\langle\nabla F,\nabla\ln W\rangle\\
&+\frac{2\alpha(1+q)}{q^2}\mathrm{Ric}(\nabla\ln W,\nabla\ln W)-\frac{2\alpha p}{q}W^{\frac{1-p}{q}}\langle\nabla\ln W,\nabla h\rangle\\
&+\alpha W^{\frac{1-p}{q}}\Delta h+(1-p)\left(2-\frac{\alpha p}{q^2}\right)hW^{\frac{1-p}{q}}|\nabla\ln W|^2\\
&+\alpha(1-p)h^2W^{\frac{2(1-p)}{q}}+\frac{\alpha(1-p)}{q}hW^{\frac{1-p}{q}}\frac{W_t}{W}-\frac{\alpha}{2bq}|\mathrm{Ric}|^2.
\end{aligned}
\tag{3.4}
$$

By (2.3), we get

$$
\frac{\Delta W}{W}=\frac{q+1}{q}\frac{|\nabla W|^2}{W^2}+\frac{W_t}{W}+qhW^{\frac{1-p}{q}}=\frac{q}{\alpha}F+\left(\frac{1+q}{q}-\frac{q}{\alpha}\right)|\nabla\ln W|^2.
$$

Because of the assumption $\alpha=kq^2/p$, we arrive at

$$
\frac{\Delta W}{W}=\frac{p}{kq}F+\left(\frac{1+q-p/k}{q}\right)|\nabla\ln W|^2 \tag{3.5}
$$

Substituting (3.5) into (3.4), we obtain

Lemma 3.1. *Suppose that $(M,g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M. If u is a positive solution of (1.2), then*

$$
\begin{aligned}
\square F \geq\ & \frac{2}{q}\langle\nabla F,\nabla\ln W\rangle+\frac{2}{n}\left(\frac{akq}{p}-\epsilon\right)\frac{p^2}{k^2q^2}F^2+(1-p)hW^{\frac{1-p}{q}}F\\
&+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)F|\nabla\ln W|^2-\frac{kq}{2bp}|\mathrm{Ric}|^2\\
&+2\left[\frac{1}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2+\left(1-\frac{1}{\epsilon}\right)\right]|\nabla\ln W|^4\\
&+\frac{2k(1+q)}{p}\mathrm{Ric}(\nabla\ln W,\nabla\ln W)-2qkW^{\frac{1-p}{q}}\langle\nabla\ln W,\nabla h\rangle\\
&+\frac{kq^2}{p}W^{\frac{1-p}{q}}\Delta h+(1-p)(1-k)hW^{\frac{1-p}{q}}|\nabla\ln W|^2
\end{aligned}
$$

where ϵ is a positive real number satisfying $\epsilon \in (0,1]$, p, q, k, a, b are positive real numbers such that $a+b = p/kq$, and

$$W = u^{-q}, \quad F = \frac{|\nabla W|^2}{W^2} + \frac{kq^2}{p} h W^{\frac{1-p}{q}} + \frac{kq}{p}\frac{W_t}{W}.$$

The second special case is to choose

$$\beta := \frac{2\alpha}{q}, \quad \alpha = \frac{q^2}{p}. \tag{3.6}$$

Then (2.14) becomes

$$\begin{aligned}
\Box F \geq{}& 2(1-\epsilon)\left|\frac{\nabla^2 W}{W}\right|^2 + 2\left(1 - \frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\
&+ \frac{4(1+q)}{p}\mathrm{Ric}(\nabla \ln W, \nabla \ln W) + (1-p) h W^{\frac{1-p}{q}}|\nabla \ln W|^2 \\
&+ \frac{q^2}{p} W^{\frac{1-p}{q}}(\Delta h + h_t) + q^2\left(\frac{1}{p} - 1\right) h^2 W^{\frac{2(1-p)}{q}} - 2q W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle \\
&+ 2q\left(\frac{1}{p} - 1\right) h W^{\frac{1-p}{q}}\frac{W_t}{W} - \frac{4q}{p}\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle.
\end{aligned} \tag{3.7}$$

For any positive real numbers a, b with $a + b = q/2\alpha = p/2q$, we have

$$\begin{aligned}
2\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{4q}{p}\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle &= \frac{4(a+b)q}{p}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{4q}{p}\left\langle \mathrm{Ric}, \frac{\nabla^2 W}{W}\right\rangle \\
&= \frac{4q}{p}\left[\left(\sqrt{b}\frac{\nabla^2 W}{W} - \frac{\mathrm{Ric}}{2\sqrt{b}}\right)^2 - \frac{|\mathrm{Ric}|^2}{4b}\right] + \frac{4aq}{p}\left|\frac{\nabla^2 W}{W}\right|^2 \\
&\geq \frac{4aq}{p}\left|\frac{\nabla^2 W}{W}\right|^2 - \frac{q}{bp}|\mathrm{Ric}|^2.
\end{aligned} \tag{3.8}$$

(3.7), (3.8) together with $|\nabla^2 W|^2 \geq (\Delta W)^2/n$ imply

$$\begin{aligned}
\Box F \geq{}& \frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left|\frac{\Delta W}{W}\right|^2 + 2\left(1 - \frac{1}{\epsilon}\right)|\nabla \ln W|^4 + \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle \\
&+ \frac{4(1+q)}{p}\mathrm{Ric}(\nabla \ln W, \nabla \ln W) + (1-p) h W^{\frac{1-p}{q}}|\nabla \ln W|^2 - \frac{q}{bp}|\mathrm{Ric}|^2 \\
&+ \frac{q^2}{p} W^{\frac{1-p}{q}}(\Delta h + h_t) + q^2\left(\frac{1}{p} - 1\right) h^2 W^{\frac{2(1-p)}{q}} + 2q\left(\frac{1}{p} - 1\right) h W^{\frac{1-p}{q}}\frac{W_t}{W} \\
&- 2q W^{\frac{1-p}{q}}\langle \nabla \ln W, \nabla h\rangle.
\end{aligned} \tag{3.9}$$

By (2.3), we get

$$\frac{\Delta W}{W}=\frac{p}{2q}F+\frac{q}{2}hW^{\frac{1-p}{q}}+\left(\frac{1+q-p/2}{q}\right)|\nabla\ln W|^2.$$

Substituting this identity into (3.9) yields

$$\begin{aligned}\Box F\geq{}&\frac{1}{2n}\left(\frac{2aq}{p}-\epsilon\right)\frac{p^2}{q^2}F^2+\frac{2p}{nq^2}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)F|\nabla\ln W|^2\\&+\left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2+2\left(1-\frac{1}{\epsilon}\right)\right]|\nabla\ln W|^4-\frac{q}{bp}|\mathrm{Ric}|^2\\&+\frac{2}{q}\langle\nabla F,\nabla\ln W\rangle+\frac{4(1+q)}{p}\mathrm{Ric}(\nabla\ln W,\nabla\ln W)+\frac{q^2}{p}W^{\frac{1-p}{q}}(\Delta h+h_t)\\&+\frac{q^2}{2n}\left(\frac{2aq}{p}-\epsilon\right)h^2W^{\frac{2(1-p)}{q}}+\left[\frac{p}{n}\left(\frac{2aq}{p}-\epsilon\right)+(1-p)\right]hW^{\frac{1-p}{q}}F\\&+\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)hW^{\frac{1-p}{q}}|\nabla\ln W|^2-2qW^{\frac{1-p}{q}}\langle\nabla\ln W,\nabla h\rangle.\end{aligned}$$

The last term is bounded from above by (where we assume that h is nonnegative)

$$\eta hW^{\frac{1-p}{q}}|\nabla\ln W|^2+\frac{q^2}{\eta}W^{\frac{1-p}{q}}\frac{|\nabla h|^2}{h}$$

for any given $\eta>0$. Therefore

$$\begin{aligned}\Box F\geq{}&\frac{1}{2n}\left(\frac{2aq}{p}-\epsilon\right)\frac{p^2}{q^2}F^2+\frac{2p}{nq^2}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)F|\nabla\ln W|^2\\&+\left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2+2\left(1-\frac{1}{\epsilon}\right)\right]|\nabla\ln W|^4-\frac{q}{bp}|\mathrm{Ric}|^2\\&+\frac{2}{q}\langle\nabla F,\nabla\ln W\rangle+\frac{4(1+q)}{p}\mathrm{Ric}(\nabla\ln W,\nabla\ln W)\\&+\frac{q^2}{p}W^{\frac{1-p}{q}}\left(\Delta h+h_t-\frac{p}{\eta}\frac{|\nabla h|^2}{h}\right)\\&+\frac{q^2}{2n}\left(\frac{2aq}{p}-\epsilon\right)h^2W^{\frac{2(1-p)}{q}}+\left[\frac{p}{n}\left(\frac{2aq}{p}-\epsilon\right)+(1-p)\right]hW^{\frac{1-p}{q}}F\\&+\left[\frac{2}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)-\eta\right]hW^{\frac{1-p}{q}}|\nabla\ln W|^2.\end{aligned}\tag{3.10}$$

We impose conditions on positive real numbers p, q, a, b, ϵ so that a desired inequality will be obtained. Recall that $a + b = p/2q$. Firstly, we assume

$$\frac{p}{n}\left(\frac{2aq}{p} - \epsilon\right) + (1 - p) \geq 0 \tag{3.11}$$

which implies

$$0 < \epsilon \leq \frac{2aq - n(p-1)}{p}. \tag{3.12}$$

However, the inequality makes sense only when $2aq - n(p-1) > 0$, i.e.,

$$0 < p < 1 + \frac{2aq}{n}. \tag{3.13}$$

We have two cases:

$$1 < p < 1 + \frac{2aq}{n} \tag{3.14}$$

and

$$0 < p \leq 1. \tag{3.15}$$

For the first case, from (3.11), we have

$$\frac{2aq}{p} - \epsilon \geq \frac{n(p-1)}{p} > 0;$$

from (3.13), we have

$$p < 1 + \frac{p - 2bq}{n} < 1 + \frac{p}{n} \Longrightarrow 0 < p < \frac{n}{n-1}$$

from which we get

$$\frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right) - \eta \geq \frac{p-1}{p}\frac{n-2}{n-1} - \eta \geq \frac{p-1}{2p} - \eta$$

using the fact that $\frac{n-2}{n-1} \geq \frac{1}{2}$ for any $n \geq 3$. Therefore, we can conclude that

$$\text{(3.12) and (3.14)} \Longrightarrow \left(\begin{array}{c}\text{the coefficients of the last three terms on the}\\ \text{right-hand side of (3.10) are all nonnegative}\end{array}\right) \tag{3.16}$$

provided that

$$\eta \leq \frac{p-1}{2p}.$$

Finally, we consider the second case where $0 < p \le 1$. In this case, we require the condition

$$\frac{2aq}{p} - \epsilon > 0 \tag{3.17}$$

instead of (3.11). Then

$$\frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right) - \eta \ge \frac{1}{n}\left(\frac{2aq}{p} - \epsilon\right) - \eta$$

We similarly obtain

$$\text{(3.15) and (3.17)} \Longrightarrow \begin{pmatrix} \text{the coefficients of the last three terms on the} \\ \text{right-hand side of (3.10) are all nonnegative} \end{pmatrix} \tag{3.18}$$

provided that

$$\eta \le \frac{1}{n}\left(\frac{2aq}{p} - \epsilon\right).$$

The statements (3.16) and (3.18) immediately imply the following

Lemma 3.2. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, and $\Delta h + h_t - \frac{p}{\eta}\frac{|\nabla_t h|_t^2}{h} \ge 0$ on $M \times [0,T]$ for some $p, \eta > 0$. Let p, q, a, b, ϵ be positive real numbers satisfying*

(i) *q is a priori given positive real number;*
(ii) *$0 < \epsilon \le 1$;*
(iii) *$a + b = p/2q$;*
(iv) *either (3.12) and (3.14) (then we choose $0 < \eta \le \frac{p-1}{2p}$), or (3.15) and (3.17) (then we choose $0 < \eta \le \frac{1}{n}(\frac{2aq}{p} - \epsilon)$).*

If u is a positive solution of (1.2), $F(x_0, t_0) > 0$ for some point $(x_0, t_0) \in M \times [0, T]$, where

$$F = \frac{|\nabla W|^2}{W^2} + \frac{q^2}{p} h W^{\frac{1-p}{q}} + \frac{2q}{p}\frac{W_t}{W},$$

then at the point (x_0, t_0) we have

$$\begin{aligned}\Box F \ge{}& \frac{1}{2n}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2}F^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)F|\nabla \ln W|^2 \\ &+ \left[\frac{2}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2 + 2\left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{q}{bp}|\mathrm{Ric}|^2 \\ &+ \frac{2}{q}\langle \nabla F, \nabla \ln W\rangle + \frac{4(1+q)}{p}\mathrm{Ric}(\nabla \ln W, \nabla \ln W).\end{aligned} \tag{3.19}$$

4. Gradient estimates and some relative results

In this section, we will use previous lemmas to get the gradient estimates for the positive solution of the equation (1.2) under the Ricci flow.

Theorem 4.1. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M with $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ for some $K_1, K_2 > 0$ on $B_{2R,T}$, and $\overline{K} = \max\{K_1, K_2\}$. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, satisfying $\Delta_t h \ge -\theta$ and $|\nabla_t h|_t \le \gamma$ on $B_{2R,T} \times [0,T]$ for some constants θ and γ. If $u(x,t)$ is a positive smooth solution of (1.2) on $M \times [0,T]$, then*

(i) *for $0 < p < 1$, we have*

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \frac{C_1}{p^2 t} + \frac{n(1-p)}{p^2}M_1M_2 + \frac{n}{2p^2(1-p)}K_1 + \frac{C_1}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + \overline{K} + \frac{n}{p(1-p)}\right) + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{n/K_1}}{p}\gamma M_2, \tag{4.1}$$

where C_1 is a positive constant depending only on n and

$$M_1 := \max_{B_{2R,T}} h_-, \quad M_2 := \max_{B_{2R,T}} u^{p-1}, \quad h_- := \max(-h, 0);$$

(ii) *for $p \ge 1$, we have*

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{t} \le \frac{k^2C_2}{p^2 t} + \frac{nk^2(p-1)}{p^2}M_4M_5 + \frac{k^3 n}{k-p}M_3M_4 + \frac{k^2C_2}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + \overline{K} + \frac{k^2 n}{p(k-p)}\right) + \frac{2k^3 n}{(k-p)p^2}K_1 + \frac{k^2\sqrt{n}\gamma}{p}\sqrt{M_4} + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4} + \frac{k^2 n}{p^2}\overline{K}, \tag{4.2}$$

where $k > p$, C_2 is a positive constant depending only on n and

$$M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h.$$

Proof. The proof is along the outline in [1,4,5]. Firstly, we introduce a cut-off function (see [3,1,5–7]) on $B_{\rho,T} := \{(x,t) \in M \times [0,T] : \mathrm{dist}_t(x, x_0) < \rho\}$, where $\mathrm{dist}_t(x, x_0)$ stands for the distance between x and x_0 with respect to the metric $g(t)$, which satisfies a basic analytical result stated in the following lemma.

Lemma 4.2. *(See [1].) Given $\tau \in (0, T]$, there exists a smooth function $\overline{\Psi} : [0, \infty) \times [0, T] \to R$ satisfying the following requirements:*

(1) *The support of $\overline{\Psi}(r, t)$ is a subset of $[0, \rho] \times [0, T]$, $0 \leq \overline{\Psi}(r, t) \leq 1$ in $[0, \rho] \times [0, T]$, and $\overline{\Psi}(r, t) = 1$ holds in $[0, \frac{\rho}{2}] \times [\tau, T]$.*
(2) *$\overline{\Psi}$ is decreasing as a radial function in the spatial variables.*
(3) *The estimate $|\partial_t \overline{\Psi}| \leq \frac{\overline{C}}{\tau} \overline{\Psi}^{1/2}$ is satisfied on $[0, \infty) \times [0, T]$ for some $\overline{C} > 0$.*
(4) *The inequalities $-\frac{C_\alpha}{\rho} \overline{\Psi}^\alpha \leq \partial_r \overline{\Psi} \leq 0$ and $|\partial_r^2 \overline{\Psi}| \leq \frac{C_\alpha}{\rho^2} \overline{\Psi}^\alpha$ hold on $[0, \infty) \times [0, T]$ for every $\alpha \in (0, 1)$ with some constant C_α dependent on α.*

For the fixed $\tau \in (0, T]$, choose the above cut-off function $\overline{\Psi}$. Define $\Psi : M \times [0, T] \to \mathbf{R}$ by setting

$$\Psi(x, t) := \overline{\Psi}\left(\mathrm{dist}_{g(t)}(x, x_0), t\right)$$

with $\rho := 2R$ in Lemma 4.2. Consider the function $\varphi(x, t) = tF(x, t)$. Using the argument of Calabi [2], we may assume that the function $G(x, t) := \varphi(x, t)\Psi(x, t)$ with support in $B_{2R,T}$ is smooth. Let (x_0, t_0) be the point where G achieves its maximum in the set $\{(x, t) : 0 \leq t \leq \tau,\ d_t(x, x_0) \leq \rho\}$. Now we will use the powerful tool maximum principle to continue our discussion. Without loss of generality, assuming $G(x_0, t_0) > 0$, we have

$$\nabla G = 0, \quad \partial_t G \geq 0, \quad \Delta G \leq 0$$

at (x_0, t_0). Now applying Lemma 4.2 and the Laplacian comparison theorem, we have

$$\begin{aligned}
\frac{|\nabla \Psi|^2}{\Psi} &\leq \frac{C_{1/2}^2}{\rho^2}, \\
\Delta \Psi &\geq -\frac{C_{1/2}\Psi^{1/2}}{\rho^2} - \frac{C_{1/2}\Psi^{1/2}}{\rho}(n-1)\sqrt{K_1}\coth(\sqrt{k_1}\rho) \\
&\geq -\frac{d_1}{\rho^2} - \frac{d_1 \Psi^{1/2}}{\rho}\sqrt{K_1}, \\
-\partial_t \Psi &\geq -\frac{\overline{C}\Psi^{1/2}}{\tau} - C_{1/2}\overline{K}\Psi^{1/2}
\end{aligned}$$

where $C_{1/2}$, $\overline{C}$ and d_1 are positive constants depending only on n. It is easy to show that

$$0 \geq \Box G = \varphi \Box \Psi + 2\langle \nabla \varphi, \nabla \Psi \rangle + \Psi \Box \varphi \tag{4.3}$$

at (x_0, t_0). Setting $p \in (0, 1)$ and $k = 1$ in Lemma 3.1, we obtain from $\Box \varphi = t\Box F - \varphi/t$ that

$$\begin{aligned}
\Box \varphi \geq\ & \frac{2p^2}{ntq^2}\left(\frac{aq}{p} - \epsilon\right)\varphi^2 + \frac{4p}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2 - \frac{q^2\theta t}{p} W^{\frac{1-p}{q}} \\
& + 2t\left[\frac{1}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{1+q-p}{q}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{nq\overline{K}^2}{2bp}t
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2(1+q)K_1 t}{p}|\nabla \ln W|^2 + \frac{2}{q}\langle \nabla\varphi, \nabla \ln W\rangle + (1-p)hW^{\frac{1-p}{q}}\varphi - \frac{\varphi}{t} \\
&- 2q\gamma t W^{\frac{1-p}{q}}|\nabla \ln W|, \qquad (4.4)
\end{aligned}
$$

where γ is an upper bound for $|\nabla_t h|$ defined in Theorem 4.1. According to Hölder's inequality,

$$
2q\gamma t W^{\frac{1-p}{q}}|\nabla \ln W| \le \frac{(1+q)K_1 t}{p}|\nabla \ln W|^2 + \frac{pq^2\gamma^2}{(1+q)K_1} t W^{2\frac{1-p}{q}}
$$

and

$$
\begin{aligned}
\frac{(1+q)K_1 t}{p}|\nabla \ln W|^2 \le{}& \frac{1}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{1-p}{q}\right)^2 2t|\nabla \ln W|^4 \\
&+ \frac{n(1+q)^2K_1^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2. \qquad (4.5)
\end{aligned}
$$

Substituting (4.5) into (4.4) yields

$$
\begin{aligned}
\Box\varphi \ge{}& \frac{2p^2}{ntq^2}\left(\frac{aq}{p} - \epsilon\right)\varphi^2 + \frac{4p}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2 \\
&+ 2t\left[\frac{1}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{2-2p+q}{q}\right) + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 - \frac{nq\overline{K}^2}{2bp}t \\
&- \frac{n(1+q)^2K_1^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 + \frac{2}{q}\langle \nabla\varphi, \nabla \ln W\rangle + (1-p)hW^{\frac{1-p}{q}}\varphi - \frac{\varphi}{t} \\
&- \frac{pq^2\gamma^2 t}{(1+q)K_1}W^{2\frac{1-p}{q}} - \frac{q^2\theta t}{p}W^{\frac{1-p}{q}}.
\end{aligned}
$$

Take $\epsilon \in (0, 1/4)$ and choose q so that $1/q \ge n(1-\epsilon)/2\epsilon^2(1-p)$. For such a pair (p, q), we may choose a positive real number a such that $aq/p \ge 2\epsilon$ and then the condition $a + b = p/q$ holds for some $b > 0$ (because in this case $0 < aq/p < 1$). Under the above assumption, we have

$$
\frac{1}{q} \ge \frac{\frac{n(1-\epsilon)}{\epsilon}}{2\epsilon(1-p)} \ge \frac{n \cdot \frac{1-\epsilon}{\frac{aq}{p}-\epsilon}}{2\epsilon(1-p)} > \frac{n\frac{1-\epsilon}{\frac{aq}{p}-\epsilon} - 1}{2\epsilon(1-p)}
$$

which implies the following inequality

$$
\frac{1}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{2-2p+q}{q}\right) + \left(1 - \frac{1}{\epsilon}\right) \ge 0.
$$

The mentioned choices of ϵ, p, q, a, b now imply

$$\begin{aligned}\Box\varphi \geq& \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\varphi^2 + \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi|\nabla \ln W|^2 \\ &- \frac{nq\overline{K}^2}{2bp}t - \frac{n(1+q)^2K_1^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 - \frac{pq^2\gamma^2 t}{(1+q)K_1}W^{2\frac{1-p}{q}} - \frac{q^2\theta t}{p}W^{\frac{1-p}{q}} \\ &+ \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle + (1-p)hW^{\frac{1-p}{q}}\varphi - \frac{\varphi}{t}. \end{aligned} \tag{4.6}$$

Plugging (4.6) into (4.3) and using the estimate for $\Box\Psi$ and the equation $0 = \nabla G = \Psi\nabla\varphi + \varphi\nabla\Psi$ at (x_0, t_0), we arrive at, where $\rho := 2R$,

$$\begin{aligned} 0 \geq& \varphi\Box\Psi - \frac{2\varphi}{\Psi}|\nabla\Psi|^2 + \Psi\Box\varphi \\ \geq& \varphi d_1\left(-\frac{1}{\rho^2} - \frac{\sqrt{K_1}}{\rho} - \frac{1}{\tau} - \overline{K}\right) - \frac{2d_1}{\rho^2}\varphi + \Psi\Box\varphi \\ \geq& \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)\Psi\varphi^2 + \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)\varphi\Psi|\nabla \ln W|^2 \\ &- \frac{n(1+q)^2K_1^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 - \frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle\varphi - \left(\frac{\theta M_2}{p} + \frac{p\gamma^2 M_2^2}{(1+p)K_1}\right)q^2 t\Psi \\ &- (1-p)M_1M_2\varphi\Psi - \frac{nq\overline{K}^2}{2bp}t\Psi - \frac{\varphi\Psi}{t} + \varphi d_2\left(-\frac{1}{\rho^2} - \frac{\sqrt{K_1}}{\rho} - \frac{1}{\tau} - \overline{K}\right) \end{aligned}$$

where d_1, d_2 are positive constants depending only on n, and

$$M_1 := \sup_{B_{2R,T}} h_-, \quad M_2 := \sup_{B_{2R,T}} u^{p-1}.$$

Multiplying the above inequality by Ψ on both sides, we get, where $G = \varphi\Psi$

$$\begin{aligned} 0 \geq& \frac{2p^2}{ntq^2}\left(\frac{aq}{p}-\epsilon\right)G^2 + \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)G\Psi|\nabla \ln W|^2 \\ &- \frac{n(1+q)^2K_1^2 t}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 - \left(\frac{\theta M_2}{p} + \frac{p\gamma^2 M_2^2}{(1+p)K_2}\right)tq^2 - (1-p)M_1M_2G \\ &- \frac{nq\overline{K}^2}{2bp}t - \frac{G}{t} + Gd_2\left(-\frac{1}{\rho^2} - \frac{\sqrt{K_1}}{\rho} - \frac{1}{\tau} - \overline{K}\right) - \frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle G. \end{aligned} \tag{4.7}$$

Using Hölder's inequality

$$\begin{aligned} \frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle G \leq& \frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)G\Psi|\nabla \ln W|^2 \\ &+ \frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p}-\epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G, \end{aligned}$$

the inequality (4.7) gives us the estimate (because $t \le \tau$)

$$0 \ge \frac{2p^2}{nq^2}\left(\frac{aq}{p} - \epsilon\right) G^2 - (1-p) M_1 M_2 G t - d_3 G$$
$$- t\left[\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \overline{K} + \frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\right] d_3 G$$
$$- t^2\left[\frac{n(1+q)^2 K_1^2}{8p(aq - p\epsilon)}\left(\frac{q}{1-p}\right)^2 + \frac{q^2}{p} M_2\theta + \frac{q^2 p}{(1+p)K_1}(M_2\gamma)^2 + \frac{nq}{2bp}\overline{K}^2\right] \quad (4.8)$$

for some positive constant d_3 depending only on n. The elementary inequality

$$aG^2 - bG - c \le 0 \quad (a, b, c > 0) \Longrightarrow G \le \frac{b}{a} + \sqrt{\frac{c}{a}},$$

implies

$$G \le \frac{d_3 + (1-p)M_1 M_2 t + t d_3\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \overline{K} + \frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{aq}{p} - \epsilon\right)\left(\frac{1+q-p}{q^2}\right)}\right)}{\frac{2p^2}{nq^2}\left(\frac{aq}{p} - \epsilon\right)}$$
$$+ t\sqrt{\frac{\frac{n(1+q)^2 K_1^2}{8p(aq-p\epsilon)}\left(\frac{q}{1-p}\right)^2 + \frac{q^2}{p} M_2\theta + \frac{q^2 p}{(1+p)K_1}(M_2\gamma)^2 + \frac{nq}{2bp}\overline{K}^2}{\frac{2p^2}{nq^2}\left(\frac{aq}{p} - \epsilon\right)}}.$$

Recall the conditions on p, q, ϵ, a, b that

$$0 < p < 1, \quad 0 < \epsilon < \frac{1}{4}, \quad \frac{1}{q} \ge \frac{n(1-\epsilon)}{2\epsilon^2(1-p)}, \quad a + b = \frac{p}{q}, \quad a \ge 2\epsilon\frac{p}{q}.$$

Choose p, ϵ, q as above and

$$a = \left(\frac{1}{2} + 2\epsilon\right)\frac{p}{q}, \quad b = \left(\frac{1}{2} - 2\epsilon\right)\frac{p}{q}. \quad (4.9)$$

The additional condition (4.9), plugging into the inequality for G, yields

$$G \le \frac{tnq^2\left[\frac{d_3}{t} + (1-p)M_1 M_2 + d_3\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \overline{K} + \frac{n}{2(1+2\epsilon)p(1-p)}\right)\right]}{p^2(1+2\epsilon)}$$
$$+ t\sqrt{\frac{nq^4}{p^2(1+2\epsilon)}\left(\frac{n(1+q)^2 K_1^2}{4(1+2\epsilon)p^2(1-p)^2} + \frac{M_2\theta}{p} + \frac{p(M_2\gamma)^2}{(1+p)K_1} + \frac{n\overline{K}^2}{p^2(1-4\epsilon)}\right)}$$

at (x_0, t_0). Since

$$\begin{aligned} G = tF\Psi &= t\left(\frac{|\nabla W|^2}{W^2} + \frac{q^2}{p}hW^{\frac{1-p}{q}} + \frac{q}{p}\frac{W_t}{W}\right)\Psi \\ &= tq^2\left(\frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u}\right)\Psi \end{aligned}$$

and $q \le 2\epsilon^2(1-p)/n(1-\epsilon)$, it follows that, by letting $\epsilon \to 0$,

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{t} \le{}& \frac{d_4}{p^2 t} + \frac{n(1-p)}{p^2}M_1 M_2 \\ &+ \frac{d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + \overline{K} + \frac{n}{2p(1-p)}\right) \\ &+ \frac{n}{p^2}\sqrt{\frac{K_1^2}{4(1-p)^2} + \frac{p\theta}{n}M_2 + \frac{(p\gamma)^2}{nK_1}M_2^2 + \overline{K}^2} \end{aligned}$$

on $B_{R,\tau}$, for some positive constant d_4 depending only on n. Because $\tau \in (0, T]$ was arbitrary, we arrive at

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le{}& \frac{d_4}{p^2 t} + \frac{n(1-p)}{p^2}M_1 M_2 + \left(\frac{n}{p}\right)^{3/2}\sqrt{\theta M_2} + \frac{\sqrt{\frac{n}{K_1}}}{p}\gamma M_2 \\ &+ \frac{d_4}{p^2}\left(\frac{1}{R^2} + \frac{\sqrt{K_1}}{R} + \overline{K} + \frac{n}{2p(1-p)}\right) \\ &+ \frac{n}{2p^2(1-p)}K_1 + \frac{n}{p^2}\overline{K} \end{aligned}$$

on $B_{R,T}$. Arranging terms yields (4.1).

When $p \ge 1$, applying Lemma 3.1, we have

$$\begin{aligned} \Box\varphi \ge{}& \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p} - \epsilon\right)\varphi^2 + \frac{4p}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi|\nabla \ln W|^2 - \frac{\varphi}{t} \\ &- \frac{kq^2\theta t}{p}W^{\frac{1-p}{q}} + 2t\left[\frac{1}{n}\left(\frac{akq}{p} - \epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 \\ &- \frac{knq\overline{K}^2}{2bp}t - \frac{2k(1+q)K_1 t}{p}|\nabla \ln W|^2 + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle + (1-p)hW^{\frac{1-p}{q}}\varphi \\ &- 2qk\gamma tW^{\frac{1-p}{q}}|\nabla \ln W| + (1-p)(1-k)thW^{\frac{1-p}{q}}|\nabla \ln W|^2, \end{aligned}$$

where $\epsilon \in (0, 1]$ and p, q, k, a, b are positive real numbers such that $a + b = p/kq$ and $k \ge 1$. Define

$$M_3 := \max_{B_{2R,T}} h_-, \quad M_4 := \max_{B_{2R,T}} u^{p-1}, \quad M_5 := \max_{B_{2R,T}} h,$$

and

$$M_6 := \min_{q\geq 0}\min_{y\geq 0}\frac{1}{q^2}\left\{2\left[\frac{1}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{kq}\right)^2+\left(1-\frac{1}{\epsilon}\right)\right]y^2\right.$$
$$\left.-(p-1)(k-1)M_3M_4y-\frac{2k(1+q)K_1}{p}y-2qkM_4\gamma y^{1/2}\right\}.$$

Therefore, we arrive at the following inequality

$$\begin{aligned}\Box\varphi \geq{}& \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)\varphi^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi|\nabla\ln W|^2-\frac{\varphi}{t}\\ &-\frac{kq^2\theta t}{p}M_4-\frac{knq\overline{K}^2}{2bp}t+\frac{2}{q}\langle\nabla\varphi,\nabla\ln W\rangle-(p-1)M_4M_5\varphi+M_6q^2t.\end{aligned}$$

As before, using $0=\nabla G=\Psi\nabla\varphi+\varphi\nabla\Psi$ at (x_0,t_0), we arrive at, where $\rho:=2R$,

$$\begin{aligned}0 \geq{}& \varphi\Box\Psi-2\varphi\frac{|\nabla\Psi|^2}{\Psi}+\Psi\Box\varphi\\ \geq{}& \varphi d_1\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2d_1}{\rho^1}\varphi+\Psi\Box\varphi\\ \geq{}& \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)\Psi\varphi^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)\varphi\Psi|\nabla\ln W|^2\\ &+M_6q^2\Psi t-\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle\varphi-\frac{kq^2\theta M_4}{p}\Psi t-(p-1)M_4M_5\Psi\varphi-\frac{knq\overline{K}^2}{2bp}\Psi t\\ &-\frac{\Psi\varphi}{t}+\varphi d_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1}}{\rho}-\frac{1}{t}-\overline{K}\right)\end{aligned}$$

for some positive constants d_1,d_2. Multiplying the above inequality by Ψ on both sides, we get, where $G=\varphi\Psi$,

$$\begin{aligned}0 \geq{}& \frac{2p^2}{ntk^2q^2}\left(\frac{akq}{p}-\epsilon\right)G^2+\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)G\Psi|\nabla\ln W|^2\\ &+M_6q^2t-\frac{kq^2\theta t}{p}M_4-(p-1)M_4M_5G-\frac{knq\overline{K}^2}{2bp}t-\frac{G}{t}\\ &+Gd_2\left(-\frac{1}{\rho^2}-\frac{\sqrt{K_1}}{\rho}-\frac{1}{\tau}-\overline{K}\right)-\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle G.\end{aligned}\tag{4.10}$$

Using Hölder's inequality, where we choose $akq>\epsilon p$ and $k+kq>p$,

$$\begin{aligned}\frac{2}{q}\langle\nabla\Psi,\nabla\ln W\rangle G \leq{}& \frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)G\Psi|\nabla\ln W|^2\\ &+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G,\end{aligned}$$

the inequality (4.10) gives the following estimate

$$
\begin{aligned}
0 \geq\ & \frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)G^2-(p-1)M_4M_5Gt-d_3G \\
& -t\left[\frac{1}{\rho^2}+\frac{\sqrt{K_1}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\right]d_3G \\
& -t^2\left(\frac{knq\overline{K}^2}{2bp}+\frac{kq^2}{p}M_4\theta-M_6q^2\right)
\end{aligned}
\tag{4.11}
$$

that is similar to (4.8) at (x_0, t_0), where d_3 is a positive constant. Hence

$$
\begin{aligned}
G \leq\ & \frac{d_3+(p-1)M_4M_5t+td_3\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1}}{\rho}+\overline{K}+\frac{\frac{1}{q^2}}{\frac{4p}{n}\left(\frac{akq}{p}-\epsilon\right)\left(\frac{k+kq-p}{k^2q^2}\right)}\right)}{\frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)} \\
& +t\sqrt{\frac{\frac{knq\overline{K}^2}{2bp}+\frac{kq^2}{p}M_4\theta-M_6q^2}{\frac{2p^2}{nk^2q^2}\left(\frac{akq}{p}-\epsilon\right)}}
\end{aligned}
$$

at (x_0, t_0). Finally, we obtain

$$
\begin{aligned}
G \leq\ & \frac{tnk^2q^2}{p^2}\left[\frac{d_3}{t}+(p-1)M_4M_5+d_3\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1}}{\rho}+\overline{K}+\frac{k^2n}{2p(k+kq-p)}\right)\right] \\
& +tq^2\left[\frac{nk^2}{p^2}\left(\frac{k^2n\overline{K}^2}{p^2(1-2\epsilon)}+\frac{k}{p}M_4\theta-M_6\right)\right]^{1/2}
\end{aligned}
$$

by taking $a=(\epsilon+\frac{1}{2})\frac{p}{kq}$, $b=(\frac{1}{2}-\epsilon)\frac{p}{kq}$ with $\epsilon \in (0, 1/2)$ and $k \geq p$. As before, we conclude that

$$
\begin{aligned}
\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{t} \leq\ & \frac{k^2d_4}{p^2t}+\frac{nk^2(p-1)}{p^2}M_4M_5 \\
& +\frac{k^2d_4}{p^2}\left(\frac{1}{R^2}+\frac{\sqrt{K_1}}{R}+\overline{K}+\frac{k^2n}{2p(k-p)}\right) \\
& +\frac{k^2n}{p^2}\sqrt{-M_6\frac{p^2}{k^2n}+\frac{p\theta}{kn}M_4+\overline{K}^2}
\end{aligned}
$$

on $B_{R,\tau}$, for some positive constant d_4 depending only on n. Because $\tau \in (0, T]$ was arbitrary, we arrive at

$$\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{1}{p}\frac{u_t}{t}\le\frac{k^2d_4}{p^2t}+\frac{nk^2(p-1)}{p^2}M_4M_5$$
$$+\frac{k^2d_4}{p^2}\left(\frac{1}{R^2}+\frac{\sqrt{K_1}}{R}+\overline{K}+\frac{k^2n}{2p(k-p)}\right)$$
$$+\frac{k\sqrt{n}}{p}\sqrt{-M_6}+\left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M_4}+\frac{k^2n}{p^2}\overline{K}$$

on $B_{R,\tau}$. In the following we shall show that $-M_6>0$ is bounded from above by some constant. For any $q,y\ge 0$ we have

$$q^2M_6\ge\left[\frac{1}{n}\left(1+\frac{k-p}{kq}\right)^2+2\left(1-\frac{1}{\epsilon}\right)\right]y^2-Ay-By^{1/2}$$

where

$$A:=(p-1)(k-1)M_3M_4+\frac{2k(1+q)K_1}{p},\quad B:=2qkM_4\gamma.$$

Since $Ay\le\eta_1y^2+A^2/4\eta_1$ and $By^{1/2}\le\eta_2y+B^2/4\eta_2$ for any $\eta_1,\eta_2>0$, it follows that

$$q^2M_6\ge\left[\frac{1}{n}\left(1+\frac{k-p}{kq}\right)^2+2\left(1-\frac{1}{\epsilon}\right)-\eta_1\right]y^2-\eta_2y-\left(\frac{A^2}{4\eta_1}+\frac{B^2}{4\eta_2}\right).$$

If we choose $\eta_1=[(k-p)/kq]^2/2n$, then

$$-q^2M_6\le\frac{\eta_2^2}{\frac{2}{n}\left(\frac{k-p}{kq}\right)^2}+\frac{A^2}{\frac{2}{n}\left(\frac{k-p}{kq}\right)^2}+\frac{B^2}{4\eta_2}$$
$$=\frac{nk^2}{2(k-p)^2}q^2\eta_2^2+\frac{nk^2}{2(k-p)^2}q^2\left[(p-1)(k-1)M_3M_4+\frac{2kK_1}{p}(1+q)\right]^2$$
$$+\frac{k^2M_4^2\gamma^2}{\eta_2}q^2.$$

That is, the inequality

$$-M_6\le\frac{nk^2}{2(k-p)^2}\eta_2^2+\frac{nk^2}{2(k-p)^2}\left[(p-1)(k-1)M_3M_4+\frac{2kK_1}{p}(1+q)\right]^2$$
$$+\frac{k^2M_4^2\gamma^2}{\eta_2}\tag{4.12}$$

holds for any $q>0$. Because the right-hand side of (4.12) as a function of q is increasing, letting $q\to 0$ yields

$$-M_6 \leq \frac{nk^2}{2(k-p)^2}\eta^2 + \frac{nk^2}{2(k-p)^2}\left[(p-1)(k-1)M_3M_4 + \frac{2kK_1}{p}\right]^2 + \frac{k^2\gamma^2 M_4}{\eta} \tag{4.13}$$

where $\eta > 0$. Using (4.13). We prove (4.2). $\square$

As an immediate consequence of the above theorem we have

Theorem 4.3. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M. Let $h(x,t)$ be a function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t.*

(i) For $0 < p < 1$, assume that $h \geq 0$, $|\nabla_t h|_t \leq \gamma$, $\Delta_t h \geq 0$, and $-K_1 g(t) \leq \mathrm{Ric}_{g(t)} \leq K_2 g(t)$ for some positive constants γ, K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$, along the Ricci flow. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \leq \frac{C_1}{p^2 t} + \frac{C_1}{p^3(1-p)} + \frac{C_1}{p^2}\overline{K} + \frac{n}{2p^2(1-p)}K_1 + \frac{\sqrt{n/K_1}}{p}\gamma M \tag{4.14}$$

for some positive constant C_1 depending only on n, where $M := \max_{M\times[0,T]} u^{p-1}$.

(ii) For $p = 1$, assume that $-K_1 g(t) \leq \mathrm{Ric}_{g(t)} \leq K_2 g(t)$ for some positive constants K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$, $h \geq 0$, $\Delta_t h \geq -\theta$ (θ is nonnegative), and $|\nabla_t h|_t \leq \gamma$ (γ is nonnegative), along the Ricci flow. If u is a smooth positive function satisfying the nonlinear parabolic equation (1.2), then

$$\frac{|\nabla_t u|_t^2}{u^2} + h - \frac{u_t}{u} \leq \frac{C_2}{t} + C_2\left(1 + K_1 + \overline{K} + \gamma + \sqrt{\theta}\right) \tag{4.15}$$

for some positive constant C_2 depending only on n.

(iii) For $p > 1$, assume that $-K_1 g(t) \leq \mathrm{Ric}_{g(t)} \leq K_2 g(t)$ for some positive constants K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$, $\Delta_t h \geq -\theta$, $|\nabla_t h|_t \leq \gamma$, and $-k_1 \leq h \leq k_2$, where $\theta, \gamma, k_1, k_2 > 0$, along the Ricci flow. If u is a bounded smooth positive function satisfying the nonlinear parabolic equation (1.2), then

$$\begin{aligned}\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \leq{}& \left(\frac{k}{p}\right)^2\frac{C_3}{t} + \left(\frac{k}{p}\right)^3\frac{k}{k-p}C_3 \\ &+ \left(\frac{k}{p}\right)^2 C_3\left(\overline{K} + \frac{k}{k-p}K_1\right) + \left(\frac{k}{p}\right)^2 n(p-1)k_2 M \\ &+ \frac{k^3 n}{k-p}k_1 M + \frac{k^2\sqrt{n}}{p}\gamma\sqrt{M} + \left(\frac{kn}{p}\right)^{3/2}\sqrt{\theta M},\end{aligned} \tag{4.16}$$

for some positive constant C_3 depending only on n, where $M := \max_{M\times[0,T]} u^{p-1}$ and $k > p$. In particular, taking $k = 2p$, we get

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{1}{p}\frac{u_t}{u} \le \frac{C_4}{t} + C_5\left(1 + K_1 + \overline{K}\right) \\ + C_4 p^2\left[(k_1+k_2)M + \gamma\sqrt{M} + \sqrt{\theta M}\right], \tag{4.17}$$

for some positive constant C_4 *depending only on* n.

In Lemma 3.2, we required that

$$\Delta_t h + h_t - \frac{p}{\eta}\frac{|\nabla_t h|_t^2}{h} \ge 0$$

for some positive constant p, η. In the following proof, we shall see that when $0 < p \le \frac{2n}{2n-1}$, we need only to assume that

$$\Delta_t + h_t - 2C_{n,p}\, p\frac{|\nabla_t h|_t^2}{h} \ge 0$$

where

$$C_{n,p} = \begin{cases} n, & p \le 1, \\ \frac{p}{p-1}, & p > 1. \end{cases}$$

Theorem 4.4. *Suppose that* $(M, g(t))_{t\in[0,T]}$ *is a solution to the Ricci flow (1.1) on an* n*-dimensional compact manifold* M*, satisfying* $-K_1 g(t) \le \mathrm{Ric}_{g(t)} \le K_2 g(t)$ *for some positive constants* K_1, K_2 *with* $\overline{K} := \max\{K_1, K_2\}$*. Let* $h(x,t)$ *be a nonnegative function defined on* $M\times[0,T]$ *which is* C^2 *in* x *and* C^1 *in* t*,* $\Delta_t h + h_t - 2C_{n,p}\, p\frac{|\nabla_t h|_t^2}{h} \ge 0$ *on* $M \times [0,T]$ *(where* $C_{n,p} = \frac{p}{p-1}$ *if* $p > 1$ *and* $C_{n,p} = n$ *if* $p \le 1$*), and* $0 < p \le \frac{2n}{2n-1}$ *(*$n \ge 3$*). If* u *is a positive solution of (1.2), then*

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{t} \le \frac{C}{p^2 t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{4n}{p(2-p)}K_1 \tag{4.18}$$

for some positive constant C *depending only on* n.

Proof. As in the proof of Theorem 4.1, we have

$$\Box\varphi \ge \frac{1}{2nt}\left(\frac{2aq}{p} - \epsilon\right)\frac{p^2}{q^2}\varphi^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p} - \epsilon\right)\left(1 + q - \frac{p}{2}\right)\varphi|\nabla \ln W|^2 \\ + 2t\left[\frac{1}{n}\left(\frac{2aq}{p} - \epsilon\right)\left(\frac{1+q-p/2}{q}\right)^2 + \left(1 - \frac{1}{\epsilon}\right)\right]|\nabla \ln W|^4 \\ + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{4(1+q)}{p}K_1 t|\nabla \ln W|^2 - \frac{qnt}{bp}\overline{K}^2 - \frac{\varphi}{t},$$

where $\varphi = tF$, from Lemma 3.2. Using Hölder's inequality

$$\begin{aligned}\frac{4(1+q)K_1 t}{p}|\nabla \ln W|^2 \leq \frac{1}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1-p/2}{q}\right)^2 2t|\nabla \ln W|^4 \\ + \frac{2n(1+q)^2K_1^2 t}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2,\end{aligned}$$

we see that

$$\begin{aligned}\Box\varphi \geq \frac{1}{2nt}\left(\frac{2aq}{p}-\epsilon\right)\frac{p^2}{q^2}\varphi^2 + \frac{2p}{nq^2}\left(\frac{2aq}{p}-\epsilon\right)\left(1+q-\frac{p}{2}\right)\varphi|\nabla \ln W|^2 \\ - \frac{nq\overline{K}^2 t}{bp} - \frac{2n(1+q)^2K_1^2 t}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2 + \frac{2}{q}\langle\nabla\varphi, \nabla \ln W\rangle - \frac{\varphi}{t}.\end{aligned}$$

Writing $G := \varphi\Psi$ and using $\Box G = \varphi\Box\Psi - 2\varphi|\nabla\Psi|^2/\Psi + \Psi\Box\varphi$, as before, we arrive at

$$\begin{aligned}0 \geq \frac{p^2}{2ntq^2}\left(\frac{2aq}{p}-\epsilon\right)G^2 + \frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)G\Psi|\nabla \ln W|^2 \\ - \frac{2n(1+q)^2K_1^2 t}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2 - \frac{nq\overline{K}^2}{bp}t - \frac{G}{t} - \frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle G \\ + Gd_1\left(-\frac{1}{\rho^2} - \frac{\sqrt{K_1}}{\rho} - \frac{1}{\tau} - \overline{K}\right)\end{aligned} \tag{4.19}$$

for some positive constant d_1 depending only on n. Plugging the inequality

$$\begin{aligned}\frac{2}{q}\langle\nabla\Psi, \nabla \ln W\rangle G \leq \frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)G\Psi|\nabla \ln W|^2 \\ + \frac{\frac{1}{q^2}}{\frac{2p}{n}\left(\frac{2aq}{p}-\epsilon\right)\left(\frac{1+q-p/2}{q^2}\right)}\frac{|\nabla\Psi|^2}{\Psi}G\end{aligned}$$

into (4.19) yields

$$\begin{aligned}0 \geq \frac{p^2}{2nq^2}\left(\frac{2aq}{p}-\epsilon\right)G^2 - d_2 G \\ - t\left[\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \overline{K} + \frac{n}{2(2aq-p\epsilon)(1+q-p/2)}\right]d_2 G \\ - t^2\left[\frac{2n(1+q)^2K_1^2}{p(2aq-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2 + \frac{nq}{bp}\overline{K}^2\right]\end{aligned}$$

for some positive constant d_2 depending only on n. Hence

$$G \le \frac{d_2 + t\left(\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \overline{K} + \frac{n}{2(2aq-p\epsilon)(1+q-p/2)}\right)}{\frac{p^2}{2nq^2}\left(\frac{2aq}{p} - \epsilon\right)}$$

$$+ t\sqrt{\frac{\frac{2n(1+q)^2K_1^2}{p(2qa-p\epsilon)}\left(\frac{q}{1-p/2}\right)^2 + \frac{nq}{bp}\overline{K}^2}{\frac{p^2}{2nq^2}\left(\frac{2aq}{p} - \epsilon\right)}}.$$

The above calculation is based on the assumption that

$$\Delta_t h + h_t - \frac{p}{\eta}\frac{|\nabla_t h|_t^2}{h} \ge 0$$

for some positive constant η, $p > 0$. We now choose appropriate constants, together with our assumption that

$$\Delta_t h + h_t - 2C_{n,p}\, p\frac{|\nabla_t h|_t^2}{h} \ge 0,$$

to verify this assumption in Lemma 3.2. Recall the conditions on p, q, ϵ, a, b that

$$q > 0, \quad 0 < \epsilon \le 1, \quad a + b = p/2q, \quad \text{either (3.12) and (3.14), or (3.15) and (3.17).}$$

First we consider the conditions (3.15) and (3.17); that is,

$$q > 0, \quad 0 < \epsilon \le 1, \quad a + b = \frac{p}{2q}, \quad 0 < p \le 1, \quad 0 < \epsilon < \frac{2aq}{p}. \tag{4.20}$$

Choose

$$a = \left(\epsilon + \frac{1}{2}\right)\frac{p}{2q}, \quad b = \left(\frac{1}{2} - \epsilon\right)\frac{p}{2q}, \quad 0 < \epsilon < \frac{1}{2}. \tag{4.21}$$

Then we can choose $\eta = \frac{1}{n}(\frac{2aq}{p} - \epsilon) = \frac{1}{2n}$ so that $p/\eta = 2np$ when $0 < p \le 1$, and furthermore

$$G \le \frac{4nq^2t}{p^2}\left[\frac{d_2}{t} + \left(\frac{1}{\rho^2} + \frac{\sqrt{K_1}}{\rho} + \overline{K} + \frac{n}{p\left(1 + q - \frac{p}{2}\right)}\right)\right]$$

$$+ \frac{4nq^2t}{p^2}\sqrt{\frac{1}{1-2\epsilon}\overline{K}^2 + \frac{(1+q)^2}{\left(1 - \frac{p}{2}\right)^2}K_1^2}.$$

Letting $\epsilon \to 0$ and $R \to \infty$ implies

$$\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{t}\right) \le \frac{d_2}{t} + \overline{K} + \frac{n}{p\left(1 + q - \frac{p}{2}\right)} + \sqrt{\overline{K}^2 + \frac{(1+q)^2}{\left(1 - \frac{p}{2}\right)^2}K_1^2}.$$

Now we minimize the above inequality for any $q > 0$ by the following observation

$$\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) \le \frac{d_2}{t}+2\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}+\frac{1+q-\frac{p}{2}+\frac{p}{2}}{1-\frac{p}{2}}K_1.$$

Hence

$$\begin{aligned}\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) &\le \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{n}{p\left(1-\frac{p}{2}\right)}K_1}+\frac{p}{2-p}K_1\\ &= \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{2n}{p(2-p)}K_1}+\frac{p}{2-p}K_1.\end{aligned}$$

Next we consider the second case; that is,

$$q>0, \quad 0<\epsilon \le 1, \quad a+b=\frac{p}{2q}, \quad 1<p<1+\frac{2aq}{n}, \quad 0<\epsilon \le \frac{2aq-n(p-1)}{p}. \tag{4.22}$$

We have proved that $1 < p < \frac{n}{n-1} \le 2$ and $1+q-\frac{p}{2} > 0$ in this case. Choose

$$a=\left(\epsilon+\frac{1}{2}\right)\frac{p}{2q}, \quad b=\left(\frac{1}{2}-\epsilon\right)\frac{p}{2q}, \quad 0<\epsilon<\frac{1}{2}, \quad 1<p\le \frac{2n}{2n-1} \tag{4.23}$$

and $\eta = \frac{p-1}{2p} \in (0, \frac{1}{4n}]$ so that $p/\eta = 2p\frac{p}{p-1}$ when $p > 1$. This choice of positive constants a, b, p, q, ϵ satisfies the mentioned condition (4.23). Then we obtain the same inequality

$$\begin{aligned}G \le{}& \frac{4nq^2t}{p^2}\left[\frac{d_2}{t}+\left(\frac{1}{\rho^2}+\frac{\sqrt{K_1}}{\rho}+\overline{K}+\frac{n}{p\left(1+q-\frac{p}{2}\right)}\right)\right]\\ &+\frac{4nq^2t}{p^2}\sqrt{\frac{1}{1-2\epsilon}\overline{K}^2+\frac{(1+q)^2}{\left(1-\frac{p}{2}\right)^2}K_1^2}.\end{aligned}$$

Letting $\epsilon \to 0$ and $R \to \infty$, and minimizing over all $q > 0$, we obtain

$$\begin{aligned}\frac{p^2}{4n}\left(\frac{|\nabla u|^2}{u^2}+\frac{h}{p}u^{p-1}-\frac{2}{p}\frac{u_t}{t}\right) &\le \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{n}{p\left(1-\frac{p}{2}\right)}K_1}+\frac{p}{2-p}K_1\\ &= \frac{d_2}{t}+2\overline{K}+2\sqrt{\frac{2n}{p(2-p)}K_1}+\frac{p}{2-p}K_1.\end{aligned}$$

In both cases, we proved (4.18). □

Corollary 4.5. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M, satisfying $0 \le \mathrm{Ric}_{g(t)} \le Kg(t)$ for some positive constant K. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and*

C^1 in t, $\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \geq 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.2), then

$$\frac{|\nabla_t u|_t^2}{u^2} + \frac{h}{p}u^{p-1} - \frac{2}{p}\frac{u_t}{t} \leq \frac{C}{p^2 t} + \frac{8n}{p^2}K \tag{4.24}$$

for some positive constant C depending only on n.

Under the hypotheses of Theorem 4.4, we let $f := \ln u$. Then

$$|\nabla_t f|_t^2 - \frac{2}{p}f_t \leq \frac{C}{p^2 t} + \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{4n}{p(2-p)}K_1 \tag{4.25}$$

on $M \times [0,T]$. For any two points $(x_1,t_1), (x_2,t_2) \in M \times [0,T]$ with $t_1 < t_2$, as in [1], we let $\Theta(x_1,t_1,x_2,t_2)$ be the set of all the smooth paths $\gamma : [t_1,t_2] \to M$ that connect x_1 to x_2. Using the same argument in the proof of Lemma 2.10 in [1] and the inequality (4.25), for any $\gamma \in \Theta(x_1,t_1,x_2,t_2)$ we have

$$\begin{aligned}
\frac{d}{dt}f(\gamma(t),t) &= \nabla_t f(\gamma(t),t)\dot{\gamma}(t) + \frac{\partial}{\partial s}f(\gamma(t),s)\Big|_{s=t} \\
&\geq -|\nabla_t f(\gamma(t),t)|_t|\dot{\gamma}(t)|_t + \frac{p}{2}\left(|\nabla_t f(\gamma(t),t)|_t^2 - \frac{C}{p^2 t} - A\right) \\
&\geq -\frac{1}{2p}|\dot{\gamma}(t)|_t^2 - \frac{p}{2}\left(\frac{C}{p^2 t} + A\right),
\end{aligned}$$

where

$$A := \frac{8n}{p^2}\overline{K} + \frac{8n}{p^2}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{4n}{p(2-p)}K_1.$$

Therefore, we arrive at

$$\begin{aligned}
f(x_2,t_2) - f(x_1,t_1) &= \int_{t_1}^{t_2} \frac{d}{dt}f(\gamma(t),t)\,dt \\
&\geq -\frac{1}{2p}\int_{t_1}^{t_2} |\dot{\gamma}(t)|_t^2 dt - \frac{pA}{2}(t_2 - t_1) - \frac{C}{2p}\ln\frac{t_2}{t_1}.
\end{aligned}$$

Corollary 4.6. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M, satisfying $-K_1 g(t) \leq \mathrm{Ric}_{g(t)} \leq K_2 g(t)$ for some positive constants K_1, K_2 with $\overline{K} := \max\{K_1, K_2\}$. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \geq 0$ on $M \times [0,T]$ (where*

$C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.2), then

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\Bigg[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot{\gamma}(t)|_t^2 dt - 2n(t_2-t_1)\left(\frac{1}{p}\overline{K} + \frac{2}{p}\sqrt{\frac{2n}{p(2-p)}K_1} + \frac{1}{2-p}K_1\right)\Bigg] \tag{4.26}$$

for some positive constant C depending only on n, where $(x_1,t_1), (x_2,t_2) \in M \times [0,T]$ with $t_1 < t_2$.

When $K_1 = 0$, we have the following

Corollary 4.7. *Suppose that $(M, g(t))_{t\in[0,T]}$ is a solution to the Ricci flow (1.1) on an n-dimensional compact manifold M, satisfying $0 \leq \mathrm{Ric}_{g(t)} \leq Kg(t)$ for some positive constant K. Let $h(x,t)$ be a nonnegative function defined on $M \times [0,T]$ which is C^2 in x and C^1 in t, $\Delta_t h + h_t - 2C_{n,p}p\frac{|\nabla_t h|_t^2}{h} \geq 0$ on $M \times [0,T]$ (where $C_{n,p} = \frac{p}{p-1}$ if $p > 1$ and $C_{n,p} = n$ if $p \leq 1$), and $0 < p \leq \frac{2n}{2n-1}$ ($n \geq 3$). If u is a positive solution of (1.2), then*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-C/p} \exp\left[-\frac{1}{2p}\min_{\gamma\in\Theta(x_1,t_1,x_2,t_2)}\int_{t_1}^{t_2}|\dot{\gamma}(t)|_t^2 dt - \frac{2nK}{p}(t_2-t_1)\right]$$

for some positive constant C depending only on n, where $(x_1,t_1), (x_2,t_2) \in M \times [0,T]$ with $t_1 < t_2$.

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