Higher Algebra over the Leibniz Operad

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Abstract

We study oidification of Leibniz algebras and introduce two subclasses of classical Leibniz algebroids, Loday algebroids and symmetric Leibniz algebroids. The algebroids of the first subclass have true differential geometric brackets and the ones of the second are the main ingredients of generalized Courant algebroids, a broader category that we define and investigate, proving in particular that it admits free objects. Regarding homotopyfication of Leibniz algebras, we review and introduce five concepts of homotopy between Leibniz infinity morphisms – in particular an explicit notion of operadic homotopy – and show that they are all equivalent. Further, we prove that the category of Leibniz infinity algebras carries an ∞-category structure. The latter projects onto the strict 2-category structure obtained on 2-term Leibniz infinity algebras via transport of the canonical 2-category structure on categorified Leibniz algebras.

Keywords: Leibniz algebroid, Courant algebroid, homological vector field, free object, derived bracket, homotopy algebra, higher category, Maurer-Cartan element, operad

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1 Introduction

This paper is based on talks given by the author at the conference 'Glances@Manifolds II', held from 8 to 13 August 2016 at the Jagiellonian University of Krakow, Poland, at the conference ' 50th Seminar "Sophus Lie"', organized from 25 September to 1st October 2016 at the Bedlewo Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences, as well as at the workshop 'Leibniz Algebras and Higher Structures', which took place at the University of Luxembourg from 13-16 December 2016. Both, the present text and the underlying lectures, report on the joint works [5], [10], [11], [13] with Vladimir Dotsenko, Janusz Grabowski, Benoit Jubin, David Khudaverdian, Jian Qiu, and Kyosuke Uchino.

We start considering the horizontal categorification (oidification) of Leibniz algebras and define two subclasses of classical Leibniz algebroids: Loday algebroids and symmetric Leibniz algebroids. Whereas standard Leibniz algebroids carry only a left anchor, Loday algebroids

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are equipped with a standard left and a generalized right anchor, so that their brackets satisfy a differential operator condition on both arguments. Symmetric Leibniz algebroids are characterized by two weak locality conditions that affect both arguments and are formulated in terms of the symmetrized Leibniz bracket. These two subclasses contain most of the Leibniz brackets that appear in the literature, but no one of these classes is included in the other. Loday algebroids admit a supergeometric interpretation. Such interpretations are known for Lie algebroids, homotopy Lie algebroids, and homotopy algebras over any quadratic Koszul operad P. Symmetric Leibniz algebroids are the underlying object of generalized Courant algebroids, a new category that contains standard Courant algebroids and has free objects over anchored vector bundles. The construction of a free generalized Courant algebroid allows to show that not only derived brackets are Leibniz brackets, but that, conversely, symmetric Leibniz algebroid brackets can be (universally) represented by derived brackets. Finally we pass to the vertical categorification (homotopyfication) of P-algebras and investigate the categorical structure of homotopy P-algebras. Whereas the objects and morphisms of this category are well-understood, their homotopies are not. At least 5 candidates do exist. We explain that they are all equivalent, define higher homotopies and show that homotopy P-algebras form, not a 2-, but an ∞-category. This holds in particular for homotopy Leibniz algebras. A concrete application of Getzler's integration technique for nilpotent Lie infinity algebras allows to prove that the known (but quite mysterious) 2-categorical structure on 2-term homotopy Leibniz algebras is in fact the shadow of the ∞-categorical structure on all homotopy Leibniz algebras.

2 The supergeometry of Loday algebroids

2.1 Definition and examples of Loday algebroids

Let us start with the observation that the double $g \oplus g^\star$ of a Lie bialgebra g is of course a Lie algebra, but that the double of a Lie bialgebroid is not a Lie algebroid – but a Courant algebroid. Courant brackets are Leibniz brackets, i.e., Lie brackets without the antisymmetry property. Leibniz brackets often appear as derived brackets (to which we will come back later on). On the other hand, most Leibniz brackets that one meets in the literature are defined on sections of a vector bundle, so that it is natural to ask about Leibniz algebroids.

Classical Leibniz algebroids are, again, defined just as Lie algebroids, except that their bracket is not skew-symmetric. This has an important consequence. In fact, if $E \to M$ is a vector bundle equipped with a classical Leibniz algebroid structure $([-,-],\lambda)$, the anchor property

$$[X, fY] = f[X,Y] + \lambda(X)f\,Y$$

$(X, Y \in \Gamma(E), f \in C^\infty(M))$ should be thought of as a first order differential operator condition for the action of the left argument X. However, since the bracket $[-,-]$ is no longer antisymmetric, there is no similar locality condition for the action of the right argument. However, Differential Geometry, which is actually sheaf theoretic, can be presented, as usual, via global sections, exactly because most differential geometric operators are local or are differential operators in all their arguments, so that they can be restricted. Hence, the fact that a classical Leibniz algebroid bracket is a priori *not* local in both arguments, is clearly suboptimal.

This motivates our quest for a better concept of Leibniz algebroid – we will refer to it as **Loday algebroid**, to distinguish it from the preceding notion of classical Leibniz algebroid – that includes a locality condition, differential operator condition, or an anchor condition, for both arguments, and that contains – the, maybe most important Leibniz algebroid – the Courant algebroid, as a special case.

The first idea is of course to define a Loday algebroid as being a Leibniz bracket $[-,-]$ on the sections of a vector bundle E, together with a left *and* a right anchor. However, it can be seen [8] that, if $\mathrm{rank}(E) = 1$, then the Leibniz bracket $[-,-]$ is necessarily antisymmetric and of 1st order, and that, if $\mathrm{rank}(E) > 1$, the Leibniz bracket $[-,-]$ is at least 'locally' a Lie algebroid bracket. In other words, the preceding definition essentially leads to Lie algebroids and not to interesting new examples. This motivates the need for an improved definition of Loday algebroids.

Let us first have a look at the local form of an anchor λ. If we denote the local basis of sections of E by e_i, we get

$$\lambda(X)f\,Y = X^i\lambda_i^a\partial_a f\;Y^j e_j\ ,$$

with self-explaining notation. This means that the considered anchor term $\lambda(X)f\,Y$ is a derivation in f, is C^∞-linear in X, and identity in Y. However, why not accept more general anchors ρ, for which the anchor term is given by

$$X^i\lambda_{ij}^{ak}\partial_a f\;Y^j e_k\ ,$$

i.e., why not *accept anchors that are a derivation in f, that are C^∞-linear in X and in Y (but no longer the identity in Y), and that are valued in sections* $(*)$? We will refer to such an anchor ρ as a generalized anchor.

It is quite easily understood that when equipped with a generalized left anchor, the new algebroids cannot have a satisfactory cohomology theory. Hence,

Definition 1. *[11] A* **Loday algebroid** *is a Leibniz bracket on sections of a vector bundle, which is endowed with a standard left and a generalized right anchor.*

Of course, Loday algebroids form a subclass of classical Leibniz algebroids. Even better, most classical Leibniz algebroids *are* Loday algebroids: in particular, Leibniz algebras, (twisted) Courant-Dorfman brackets ($\mathrm{T}\,M \oplus \mathrm{T}^*\,M$), Grassmann-Dorfman brackets ($\mathrm{T}\,M \oplus \wedge\,\mathrm{T}^*\,M$ or $E \oplus \wedge E^*$), classical Leibniz algebroids associated to Nambu-Poisson structures [9], Courant algebroids... are Loday brackets or algebroids. A precise definition of Courant algebroids will be given below. Let us at the moment just mention that a Courant algebroid is a classical Leibniz algebroid $(E \to M, [-,-], \lambda)$ together with a non-degenerate inner product $(-|-)$, such that some axioms hold. Further, it is indeed well-known that, for any Courant algebroid, one can define a derivation

$$D \in \mathrm{Der}(C^\infty(M), \Gamma(E)) .$$

When exploiting now the Courant axioms, one sees that

$$D(fX|Y) = [fX, Y] + [Y, fX]$$

($f \in C^\infty(M), X, Y \in \Gamma(E)$) – an equation that allows visibly to understand the action of Y on the product fX. One thus obtains a right anchor $D(f)(X|Y)$, which is obviously a derivation in f, C^∞-linear in X and Y, and valued in sections, i.e., that is obviously a generalized right anchor (see ($*$) above).

2.2 Supergeometric interpretation of Loday algebroids

Recall now the Vaintrop 1:1 correspondence that mentions that Lie algebroids $(E, [-,-], \lambda)$ are the same as de Rham complexes $(\Gamma(\wedge E^*), \mathrm{d})$ of vector bundle forms, or, still, the same as square 0, degree 1 derivations

$$\mathrm{d} \in \mathrm{Der}_1(\Gamma(\wedge E^*), \wedge), \mathrm{d}^2 = 0$$

of the algebra $(\Gamma(\wedge E^*), \wedge)$ of superfunctions of the supermanifold ΠE with shifted parity in the fibers. In other words, Lie algebroids are 1:1 with cohomological vector fields d on split supermanifolds ΠE.

It is natural to ask wether Loday algebroids $(E, [\bullet, \bullet], \lambda, \rho)$ admit a similar supergeometric interpretation.

Notice first that the de Rham differential d is nothing but the Chevalley-Eilenberg or Lie algebra cohomology operator, but restricted, from the cochain space $\mathrm{A}(\Gamma(E), C^\infty(M))$ of all antisymmetric multilinear maps on the Lie algebra $(\Gamma(E), [-,-])$ represented by λ on $C^\infty(M)$, to the stable subspace

$$\mathrm{A}_{C^\infty(M)}(\Gamma(E), C^\infty(M)) = \Gamma(\wedge E^*)$$

of the antisymmetric $C^\infty(M)$-*multilinear* maps, i.e., to the stable subspace $\Gamma(\wedge E^*)$ of skew-symmetric covariant tensor fields or vector bundle forms – so that d actually acts on those forms or superfunctions.

In our Loday algebroid case, we should therefore consider the Leibniz cohomology operator restricted to the same space

$$\mathrm{Lin}_{C^\infty(M)}(\Gamma(E), C^\infty(M)) = \Gamma(\otimes E^*) ,$$

just omitting the antisymmetry condition. This space of $C^\infty(M)$-multilinear maps is of course the space of 0-order multidifferential operators. The Leibniz cohomology operator raises however the total degree of a multidifferential operator by 1, so that the considered space $\Gamma(\otimes E^*)$ is not closed under the Leibniz operator. Hence, the idea to replace the 'superfunctions' $\Gamma(\otimes E^*)$ made of the 0-order multidifferential operators by the 'superfunctions'

$$D_{\text{ploy}}(\Gamma(E), C^\infty(M)) =: D_{\text{poly}}(E)$$

made of all multidifferential operators from $\Gamma(E) \times \ldots \times \Gamma(E)$ to $C^\infty(M)$. It turns out that the associative multiplication of these 'superfunctions' can be chosen to be the shuffle multiplication $\pitchfork$, which is defined on multidifferential operators Δ', Δ'' by the same formula

$$(\Delta' \pitchfork \Delta'')(X_1, \ldots, X_{p+q}) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sign}(\sigma) \Delta'(X_{\sigma_1}, \ldots, X_{\sigma_p})\, \Delta''(X_{\sigma_{p+1}}, \ldots, X_{\sigma_{p+q}})$$

than the wedge product $\wedge$ on differential forms.

We can now prove [11] the following Vaintrop-type result:

Theorem 1. *There is a 1-to-1 correspondence between Loday algebroid structures* $(E, [\bullet, \bullet], \lambda, \rho)$ *and* equivalence classes *of cohomological vector fields*

$$\mathrm{d} \in \mathcal{D}er_1(\mathcal{D}_{\text{poly}}(E), \pitchfork), \mathrm{d}^2 = 0$$

of the 'noncommutative space' $(\mathcal{D}_{\text{poly}}(E), \pitchfork)$.

Remark 1. *Note that, as notation* $\mathcal{D}er_1$ *and* $\mathcal{D}_{\text{poly}}(E)$ *suggests, the precise result is significantly subtler than just* $\mathrm{d} \in \mathrm{Der}_1(D_{\text{poly}}(E), \pitchfork)$. *We will not describe it here in more detail. Further, as indicated above, the noncommutative associative algebra* $(\mathcal{D}_{\text{poly}}(E), \pitchfork)$ *can be interpreted as the 'superfunction' algebra of some noncommutative space. For these spaces, Cartan calculus can be developed.*

3 Free Courant and derived Leibniz algebroids

As mentioned in Remark 1, Vaintrop-type results are far from being simple in the algebroid context, except, of course, for Lie algebroids. In the algebraic setting however, there exists a well-understood correspondence, based on Koszul duality for operads and due to Ginzburg and Kapranov [7]:

If P denotes a quadratic Koszul operad, a P_∞-algebra structure on a (finite dimensional) graded vector space V (over a field of characteristic 0, say, over $\mathbb{R}$) is the same as a cohomological vector field

$$\mathrm{d} \in \mathrm{Der}_1\left(\mathcal{F}_{P^!}(sV^*)\right), \mathrm{d}^2 = 0$$

acting on the *free* Koszul P-dual algebra over the suspended linear dual of V.

- In the case $P = \mathsf{Lie}$, we recover the maybe better known result stating that L_∞-algebras on V are 1:1 with cohomological vector fields acting on $\wedge(sV^*)$, i.e., are 1:1 with cohomological vector fields of the formal supermanifold V.

- This particular case admits a geometric extension that allows to identify split L_∞-algebroids with cohomological vector fields of split $\mathbb{N}$-graded manifolds [3].
- The previously mentioned Vaintrop-realization of Lie algebroids as cohomological vector fields of split supermanifolds is then a special case of the preceding generalization.
- Moreover, we just added to this list the interpretation of Loday algebroids as cohomological vector fields of (some) noncommutative spaces.

We already understood that the considered cohomological vector fields are in fact cohomology operators. Conversely, the brackets of the LHS-s of these correspondences are *derived brackets* implemented by the relevant cohomological vector field.

The above catalogue highlights inter alia the importance of free objects and of derived brackets (although such evidence is not really needed). In the sequel, we will focus on both, free objects and derived brackets.

3.1 Free Courant algebroids

We start investigating the concept of free Courant algebroid. Let us first recall that a **Courant algebroid** is a classical Leibniz algebroid $(E \to M, [-,-], \lambda)$, endowed with a non-degenerate inner product $(-|-)$, called scalar product, such that the two invariance relations

$$\lambda(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z])$$

and

$$\lambda(X)(Y|Z) = (X|[Y,Z] + [Z,Y])$$

$(X, Y, Z \in \Gamma(E))$, as well as the (here trivial) compatibility condition

$$([X,Y]|Z) + (Y|[X,Z]) = (X|[Y,Z] + [Z,Y]) \ ,$$

are satisfied.

In the following, we do not consider a module $\Gamma(E)$ over a commutative algebra $C^\infty(M)$ over the (commutative) field $\mathbb{R}$ of real numbers, but our basic object will be, more generally, a module $\mathcal{E}$ over a commutative algebra $\mathcal{A}$ over a commutative ring R. Even more, our fundamental ingredient is as from now an anchored $\mathcal{A}$-module $(\mathcal{E}, \lambda)$ and we ask for the free Courant algebroid over this anchored module.

At this point a more detailed discussion is indispensable to convince the reader that the subsequent constructions and definitions are after all quite natural.

To discover the free Courant algebroid over the anchored module $(\mathcal{E}, \lambda)$, note first that a Courant algebroid contains a Leibniz bracket and that the free Leibniz algebra over the R-module $\mathcal{E}$ is the algebra $(\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}})$, whose bracket is the universal Leibniz bracket and which has been detailed by Loday and Pirashvili. Since this description will not be really important here, it will not be recalled. Being interested in the free Courant *algebroid*, it is natural to consider now the free Leibniz *algebroid* over the anchored $\mathcal{A}$-module $(\mathcal{E}, \lambda)$, which is (finally) given by $(\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}, \mathcal{F}(\lambda))$, where the anchor $\mathcal{F}(\lambda)$ on $\mathcal{F}(\mathcal{E})$ can be built from the 'ingredient-anchor' λ on $\mathcal{E}$. However, what about the inner product of the free Courant algebroid – what about the universal inner product $(-|-)_{\text{UIP}}$? Universal means of course that, for any 'another' Courant algebroid $(\mathcal{E}_0, [-,-]_0, \lambda_0, (-|-)_0)$, any map $f : \mathcal{E} \to \mathcal{E}_0$ from 'the basis' $\mathcal{E}$ to the new algebroid $\mathcal{E}_0$, uniquely factors through a Courant algebroid morphism $f_1 : \mathcal{F}(\mathcal{E}) \to \mathcal{E}_0$. Whatever this Courant algebroid morphism (or even generalized Courant algebroid morphism) will be, it should respect the inner products, i.e., for any $X, Y \in \mathcal{F}(\mathcal{E})$, we should have

$$\text{‘}(X|Y)_{\text{UIP}} =\text{’}\, (f_1(X)|f_1(Y))_0 \ .$$

It is easily seen that the RHS – which can of course not be a *definition* for the *universal* inner product $(-|-)_{\text{UIP}}$ – is a map on the cartesian product $\mathcal{F}(\mathcal{E}) \times \mathcal{F}(\mathcal{E})$ that induces a map f_2 on the symmetric tensor product $\mathcal{F}(\mathcal{E}) \odot \mathcal{F}(\mathcal{E})$, so that

$$(f_1(X)|f_1(Y))_0 = f_2(X \odot Y) \ .$$

This suggests to view $X \odot Y$ as the universal inner product of X and Y, and to think that a generalized Courant algebroid morphism will be made of two maps, f_1 and f_2, which respect the inner products $(X|Y)_{\text{UIP}} = X \odot Y$ and $(X|Y)_0$ in the sense that

$$(f_1(X)|f_1(Y))_0 = f_2(X|Y)_{\text{UIP}} \ .$$

More precisely, since a (generalized) Courant algebroid contains the afore-mentioned (and in the generalized setting no longer trivial) compatibility condition, say $\sim$, we force this condition by eventually defining the universal inner product by

$$(-|-)_{\text{UIP}} : \mathcal{F}(\mathcal{E}) \times \mathcal{F}(\mathcal{E}) \to \mathcal{F}(\mathcal{E}) \odot \mathcal{F}(\mathcal{E})/\!\sim\, =: \mathcal{Q}(\mathcal{F}(\mathcal{E})) \ .$$

Finally, the invariance conditions of a Courant algebroid require (in the generalized framework) that $\mathcal{F}(\mathcal{E})$ acts on the left and on the right on the value-space $\mathcal{Q}(\mathcal{F}(\mathcal{E}))$ of $(-|-)_{\text{UIP}}$. It is not very difficult to find such left and right actions μ^ℓ and μ^r, so that, in principle, the tuple

$$(\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{F}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{\text{UIP}})$$

is a rather canonical candidate for the free generalized Courant algebroid over $(\mathcal{E}, \lambda)$. The interesting point is that the actions μ^ℓ and μ^r are well-defined on the quotient $\mathcal{Q}(\mathcal{F}(\mathcal{E}))$ provided the bracket $[-,-]_{\text{ULB}}$ satisfies two new conditions.

3.2 Symmetric Leibniz algebroids

We now open a parentheses to discuss these new conditions. They are called symmetry conditions, can be written for any classical Leibniz algebroid, and the classical Leibniz algebroids that satisfy them are referred to as symmetric Leibniz algebroids.

Definition 2. *[10] A* **symmetric Leibniz algebroid** *is a classical Leibniz algebroid* $(\mathcal{E} \to M, [-,-], \lambda)$*, whose bracket satisfies, for any* $f \in C^\infty(M)$ *and* $X, Y, Z \in \mathcal{E}$*,*

$$X \circ fY - (fX) \circ Y = 0 \tag{1}$$

and

$$([fX, Y] - f[X, Y]) \circ Z + Y \circ ([fX, Z] - f[X, Z]) = 0 \ , \tag{2}$$

where $X \circ Y := [X, Y] + [Y, X]$ *.*

The name 'symmetric' is of course due to the involved symmetrized Leibniz bracket $\circ$. Furthermore, the new conditions are reminiscent (this is particularly obvious for the second requirement) of the differential operator conditions or locality conditions, which were already at the basis of a first subclass of classical Leibniz algebroids, Loday algebroids, which actually encode a differential operator condition for each argument of the corresponding Leibniz bracket – via a standard left and a generalized right anchor. We will think of the two preceding conditions as being some weak locality conditions.

The natural question is now how the two subclasses of classical Leibniz algebroids, Loday algebroids and symmetric Leibniz algebroids, are related. It turns out that they have quite a number of common elements, for instance, Leibniz algebra brackets, (twisted) Courant-Dorfman brackets, Grassmann-Dorfman brackets, Courant algebroid brackets, but, that no one of these classes in included in the other. Indeed, we already mentioned that a classical Leibniz algebroid associated to a Nambu-Poisson structure is Loday, but it can be checked that it is NOT symmetric Leibniz. On the other hand, we are able to construct the free symmetric Leibniz algebroid over any anchored module, which is of course symmetric Leibniz, but it can be shown that it is NOT Loday.

3.3 Free Courant algebroids (continuation)

According to what we said above, the actions μ^ℓ and μ^r, which are part of the candidate free generalized Courant algebroid

$$(\mathcal{F}(\mathcal{E}), [-,-]_{\mathrm{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{F}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{\mathrm{UIP}}) \ ,$$

are well-defined if the universal Leibniz bracket $[-,-]_{\mathrm{ULB}}$ is symmetric in the sense of Definition 2. 'Of course', this bracket does not satisfy the symmetry conditions (1) and (2), however, it is possible to 'symmetrize the situation' (what is denoted by $\mathcal{S}$ below) and to prove [10] the

Theorem 2. *The above-described tuple*

$$(\mathcal{SF}(\mathcal{E}), [-,-]_{ULB}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{SF}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{UIP})$$

is the free generalized Courant algebroid over $(\mathcal{E}, \lambda)$*.*

The reader will have observed that we did not yet define generalized Courant algebroids. This fact is due to 'pedagogical reasons': indeed, *now* this definition is natural, since it just mimics the preceding naturally constructed free generalized Courant algebroid.

Definition 3. *[10] A* **generalized Courant algebroid**

$$(\mathcal{E}_1, [-,-], \lambda, \mathcal{E}_2, \mu^\ell, \mu^r, (-|-))$$

is made of a symmetric Leibniz algebroid (or, maybe better, pseudo-algebra) $(\mathcal{E}_1, [-,-], \lambda)$*, a module* $\mathcal{E}_2$ *with a left and a right* $\mathcal{E}_1$*-action* μ^ℓ *and* μ^r*, and of an inner product* $(-|-)$ *on* $\mathcal{E}_1$ *valued in* $\mathcal{E}_2$*, such that, for any* $X, Y, Z \in \mathcal{E}_1$*, the invariance relations*

$$\mu^\ell(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z])$$

and

$$-\mu^r(X)(Y|Z) = ([Y,Z] + [Z,Y]|X) \ ,$$

as well as the compatibility condition

$$([X,Y]|Z) + (Y|[X,Z]) = ([Y,Z] + [Z,Y]|X) \ ,$$

are satisfied.

To understand the difference with standard Courant algebroids, it suffices to know that a possible non-degeneracy of the inner product $(-|-)$ implies the symmetry – in the sense of Definition 2 – of the Leibniz bracket $[-,-]$. Hence, we simply substituted the weaker symmetry conditions (1) and (2) to the usual non-degeneracy requirement, AND we replaced the standard module $C^\infty(M)$ with actions λ and $-\lambda$ by a more general module $(\mathcal{E}_2, \mu^\ell, \mu^r)$. These modifications then led to the broader category of *generalized* Courant algebroids, which admits free objects.

3.4 Application

As we met in this text already twice derived brackets, we recall their definition. If $(K, \{-,-\}, \Delta)$ is a differential graded Lie algebra (DGLA for short), then the new bracket

$$\{k', k''\}_\Delta := (-1)^{|k'|+1}\{\Delta k', k''\} \ ,$$

where $|-|$ denotes the degree in K, leads to a Leibniz algebra $(K, \{-,-\}_\Delta)$ and is referred to as the **derived bracket** implemented by the initial DGLA. Conversely, we may ask which Leibniz algebra or classical Leibniz algebroid brackets are, or can at least be represented by, derived brackets.

To understand the answer to this question, remember that above we built, for each anchored module $(\mathcal{E}, \lambda)$, the free generalized Courant algebroid

$$(\mathcal{SF}(\mathcal{E}), [-,-]_{\mathrm{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{SF}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{\mathrm{UIP}}) \ .$$

A similar construction allows to assign, to any symmetric Leibniz algebroid $(\mathcal{E}, [-,-], \lambda)$, its **associated generalized Courant algebroid**

$$(\mathcal{E}, [-,-], \lambda, \mathcal{Q}(\mathcal{E}), \mu^\ell, \mu^r, (-|-)) .$$

It can be shown [10] that the latter provides a DGLA $(K, \{-,-\}, \Delta)$, whose induced derived bracket algebra $(K, \{-,-\}_\Delta)$ is a *universal* derived bracket representation of $(\mathcal{E}, [-,-])$:

Theorem 3. *Symmetric Leibniz algebroid brackets admit universal derived bracket representations.*

3.5 Summary

We defined two subclasses of the class of classical Leibniz algebroids, Loday algebroids (with a standard left and a generalized right anchor, hence, with a locality condition on both arguments) and symmetric Leibniz algebroids (defined by two weak locality conditions). The two subclasses have a number of common elements, but no subclass is contained in the other one. Symmetric Leibniz algebroids are the basic ingredient of generalized Courant algebroids – a broader class that admits free objects over anchored modules. A construction, analogous to the free generalized Courant algebroid over an anchored module, associates a generalized Courant algebroid to any symmetric Leibniz algebroid. This generalized Courant algebroid allows to prove that any symmetric Leibniz algebroid bracket admits a universal derived bracket representation.

4 Infinity category of homotopy P-algebras

After the preceding extensive discussion of Leibniz algebroids, i.e., of the horizontal categorification of Leibniz algebras, we now address their vertical categorification, i.e., we report on homotopy or infinity Leibniz algebras, or, more generally, on homotopy algebras over an operad P.[1]

The afore-mentioned Ginzburg-Kapranov characterization

$$\mathrm{d} \in \mathrm{Der}_1(\mathcal{F}_{P^!}(sV^*)),\ \mathrm{d}^2 = 0$$

of a homotopy algebra structure over a quadratic Koszul operad P on a finite-dimensional graded vector space V, admits a coalgebraic variant, which does not require finite-dimensionality: a P_∞-algebra structure on a possibly infinite-dimensional graded vector space V is the same as a codifferential

$$\mathrm{d} \in \mathrm{CoDer}_1(\mathcal{F}_{P^{¡}}(s^{-1}V)),\ \mathrm{d}^2 = 0$$

[1]The result that homotopy P-algebras form an infinity category was proven in [13] for the operad $P = \mathsf{Lei}$ of Leibniz algebras. In fact, it holds in whole generality, with a similar explanation. The authors did not publish this extension. A possible reference is [4].

on the free Koszul P-dual coalgebra on the desuspended space $s^{-1}V$. Even better, there exist equivalences of categories

$$P_\infty\text{-}\mathtt{Alg} \simeq \mathtt{qfDGP^!A} \quad \text{and} \quad P_\infty\text{-}\mathtt{Alg} \simeq \mathtt{qfDGP^{¡}C} \tag{3}$$

between the category of P_∞-algebras and the category of quasi-free differential graded Koszul P-dual algebras or coalgebras. For the explicit construction of Leibniz infinity algebras and their morphisms, via the second categorical equivalence (3), we refer the interested reader to [1].

4.1 Concordances, gauge homotopies, Quillen homotopies

Although the objects and morphisms of $P_\infty\text{-}\mathtt{Alg}$ are well-understood in view of (3), the corresponding homotopies are not.

4.1.1 Concordances

A first concept of homotopy between two P_∞-morphisms between the same P_∞-algebras – due to Schlessinger and Stasheff [14] – is known under the name of concordances.

This notion of homotopy is, roughly, similar to homotopies between two smooth maps

$$p, q \in C^\infty(V, W)$$

between the same smooth manifolds V and W. Indeed, when considering the pullback chain maps

$$p^*, q^* \in \mathtt{Ch}(\Omega(W), \Omega(V))$$

between the de Rham complexes, a homotopy is a chain map

$$\eta^* \in \mathtt{Ch}(\Omega(W), \Omega_1 \otimes \Omega(V)) \;,$$

where the target is the complex obtained by left tensoring by the complex of differential forms of the topological 1-simplex Δ_1.

Analogously, if

$$p, q \in \mathrm{Hom}_{P_\infty\text{-}\mathtt{Alg}}(V, W)$$

are two P_∞-morphisms between the same P_∞-algebras V and W, the first categorical equivalence (3) provides two DGA-morphisms

$$p^*, q^* \in \mathrm{Hom}_{\mathtt{DGA}}(\mathcal{F}_{P^!}(W), \mathcal{F}_{P^!}(V)) \;,$$

so that a homotopy is a DGA-morphism

$$\eta^* \in \mathrm{Hom}_{\mathtt{DGA}}(\mathcal{F}_{P^!}(W), \Omega_1 \otimes \mathcal{F}_{P^!}(V)) \;,$$

where the target is the differential graded $P^!$-algebra obtained by tensoring by the differential graded commutative algebra Ω_1.

To deepen the understanding of the homotopies η^*, notice that, in both situations, this homotopy is a differential form

$$\eta^*_w(t, \mathrm{d}\, t) = \phi_w(t) + \mathrm{d}\, t\, \rho_w(t)$$

in Ω_1, parametrized by w in the source space, with coefficients in the second tensor factor of the target space. When translating, in whatever of the two considered cases, the chain map property of η^* in terms of ϕ and ρ, one finds

$$\mathrm{d}_t\, \phi = \mathrm{d}_V\ \rho(t) + \rho(t)\ \mathrm{d}_W\ , \tag{4}$$

where d_V and d_W are the differentials of the complexes with underlying space V and W, respectively.

In the C^∞-case, the integration of Equation (4) from 0 to 1 shows that $h := \int_0^1 \mathrm{d}\, t\, \rho(t)$ is a chain homotopy between p^* and q^*, provided we assume that

$$\eta^*(0,0) = \phi(0) = p^* \quad \text{and} \quad \eta^*(1,0) = \phi(1) = q^*\ .$$

In the P_∞-case, we need not integrate, but have still to express the fact that η^* is an algebra morphism. It is straightforwardly seen that, in terms of ϕ and ρ, this property means that ϕ is a family

$$\phi(t) \in \mathrm{Hom}_{\mathtt{DGA}}(\mathcal{F}_{P^!}(W), \mathcal{F}_{P^!}(V)) \quad (t \in \Delta_1) \tag{5}$$

of DGA-morphisms and that ρ is a family

$$\rho(t) \in \phi\text{-}\,\mathrm{Der}(\mathcal{F}_{P^!}(W), \mathcal{F}_{P^!}(V)) \quad (t \in \Delta_1) \tag{6}$$

of ϕ-derivations.

Eventually, it is natural to define a homotopy, or, better, a **concordance** between two P_∞-morphisms p, q as families ϕ and ρ of the type (5) and (6), respectively, which satisfy (4), as well as $\phi(0) = p^*$ and $\phi(1) = q^*$.

Remark 2. *A priori one expects that homotopies or concordances can be composed horizontally and vertically and that these compositions are associative. However, whereas horizontal composition of concordances is quite obvious, vertical composition turns out to be problematic. This can be viewed as a first hint towards the fact that* P_∞-`Alg` *is not a 2-category, but possibly a higher one.*

4.1.2 Gauge and Quillen homotopies

Due to the second categorical equivalence (3), we have

$$\mathrm{Hom}_{P_\infty\text{-}\mathtt{Alg}}(V, W) \simeq \mathrm{Hom}_{\mathtt{DGC}}(\mathcal{F}_{P^¡}(V), \mathcal{F}_{P^¡}(W))\ . \tag{7}$$

Since $\mathcal{F}_{P^¡}(W)$ is a free coalgebra, a differential graded coalgebra (DGC for short) morphism is completely determined by its corestrictions $\mathcal{C} := \mathrm{Hom}_{\mathbb{R}}(\mathcal{F}_{P^¡}(V), W)$. It is known that $\mathcal{C}$

carries a Lie infinity structure and is referred to as the convolution Lie infinity algebra. One can thus consider its Maurer-Cartan elements $\mathrm{MC}(\mathcal{C})$ and it is rather easily seen that these elements are exactly the initially considered morphisms (7):

$$\mathrm{Hom}_{P_\infty\text{-}\mathtt{Alg}}(V,W) \simeq \mathrm{MC}(\mathcal{C}) \tag{8}$$

(we omitted suspension and dependencies of $\mathcal{C}$ on P, V and W). Therefore, looking for homotopies between P_∞-morphisms means defining homotopies between Maurer-Cartan elements. But: in the literature, one can find (even) several concepts of homotopy between Maurer-Cartan elements of a Lie infinity algebra, e.g., gauge homotopies and Quillen homotopies.

As concerns gauge homotopies, consider a Lie infinity algebra $(\mathcal{C},(\ell_i)_i)$ and fix any $r\in\mathcal{C}_0$. It can be shown that, if we restrict the map

$$V_r : \mathcal{C}_{-1} \ni \alpha \mapsto -\sum_i \frac{1}{i!}\ell_{i+1}(\alpha^{\otimes i}, r) \in \mathcal{C}_{-1}$$

to the Maurer-Cartan quadric $\mathrm{MC}(\mathcal{C})$ inside the vector space $\mathcal{C}_{-1}$, we get a vector field $V_r|_{\mathrm{MC}(\mathcal{C})}$ of $\mathrm{MC}(\mathcal{C})$. Now, two Maurer-Cartan elements $\alpha,\beta\in\mathrm{MC}(\mathcal{C})$ are **gauge homotopic**, if they are connected by an integral curve of $V_r|_{\mathrm{MC}(\mathcal{C})}$ for some $r\in\mathcal{C}_0$. On the other hand, two Maurer-Cartan elements $\alpha,\beta\in\mathrm{MC}(\mathcal{C})$ are **Quillen homotopic**, if there exists a Maurer-Cartan element $\gamma\in\mathrm{MC}(\mathcal{C}\otimes\Omega_1)$ of the Lie infinity algebra obtained by tensoring $\mathcal{C}$ with the differential graded commutative algebra Ω_1 of differential forms of the 1-simplex Δ_1, i.e., if there is a Maurer-Cartan element

$$\gamma(t,\mathrm{d}\,t) = \gamma_1(t) + \mathrm{d}\,t\,\gamma_2(t) \quad (\gamma_1(t)\in\mathcal{C}_{-1})\ ,$$

such that $\gamma(0,0)=\gamma_1(0)=\alpha$ and $\gamma(1,0)=\gamma_1(1)=\beta$.

4.2 Infinity category of infinity P-algebras

As explained above, we have at least three concepts of homotopy between P_∞-morphisms at our disposal: concordances, gauge homotopies and Quillen homotopies. We proved [5] that these notions are all equivalent (we will briefly come back to this fact later on). In the sequel, we prefer Quillen homotopies, i.e., for any fixed $V,W\in P_\infty$-`Alg`, the homotopies or 2-morphisms are

$$\infty\text{-}2\text{-}\mathrm{Mor} = \mathrm{MC}(\mathcal{C}\otimes\Omega_1)\ ,$$

and, clearly, see (8), the 1-morphisms are

$$\infty\text{-}1\text{-}\mathrm{Mor} = \mathrm{MC}(\mathcal{C}\otimes\Omega_0)\ ,$$

where Ω_0 are the differential forms of the 0-simplex. Hence, it is natural to define ∞-n-Mor as

$$\infty\text{-}n\text{-}\mathrm{Mor} := \mathrm{MC}(\mathcal{C}\otimes\Omega_{n-1}) \quad (n\geq 1)\ .$$

The merging simplicial set

$$\mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)\in\mathtt{SSet}$$

is actually well-known. Indeed, when integrating nilpotent Lie infinity algebras $\mathcal{C}$, Getzler found [6]

$$\int \mathcal{C} \xrightarrow{\sim} \mathrm{MC}(\mathcal{C} \otimes \Omega_\bullet)$$

and he proved that this simplicial set is in fact a Kan complex. However, it is known that, in the presence of an ∞-**category**, the morphisms and higher morphisms form a **Kan complex**, for any two fixed objects. Hence, the preceding results and definitions allow to realize that [13]

Theorem 4. *The category P_∞-`Alg` of homotopy algebras over a quadratic Koszul operad P is an ∞-category.*

Let us mention that we consider the category `SSet` of simplicial sets together with its standard cofibrantly generated model structure. This means that a simplicial map is a weak equivalence if its geometric realization is a weak equivalence for the Quillen model structure of the category `Top` of topological spaces, i.e., if this realization is a weak homotopy equivalence; further, a simplicial map is a cofibration if it is a monomorphism; and finally, a simplicial map is a fibration if it has the right lifting property (RLP for short) with respect to the generating trivial cofibrations, i.e., with respect to all canonical inclusions $\Lambda^i[n] \hookrightarrow \Delta[n]$ of an (n, i)-horn into the corresponding simplicial n-simplex ($n \in \mathbb{N}$ and $i \in \{0, \ldots, n\}$).

Hence, a simplicial set $S \in$ `SSet` is fibrant if the map $S \to *$ from S to the terminal simplicial set $*$ is a fibration, i.e., has the RLP with respect to the inclusions $\Lambda^i[n] \hookrightarrow \Delta[n]$, i.e., if any simplicial map $\Lambda^i[n] \to S$ extends to a simplicial map $\Delta[n] \to S$, or, still, if *any horn* in $S = (S_0, S_1, \ldots)$ *has a filler*. It is this property that we translate saying that S is a **Kan complex**. There are three other, similar properties: *any inner horn* in S *has a unique filler*, *any horn* in S *has a unique filler*, and *any inner horn* in $S = (S_0, S_1, \ldots)$ *has a filler*. The first of these three properties means that S is the **nerve of some category**, the second means that S is the **nerve of some groupoid** (it is easy to understand that fillers of outer horns ($i = 0$ and $i = n$) correspond to inverse maps), and the last encodes correctly the idea that an ∞-category is made of objects S_0, morphisms S_1, and higher morphisms S_i ($i \geq 2$), such that compositions of i-morphisms are well-defined and associative only up to higher morphisms $> i$. Hence, the last one of the three properties means that S is an ∞-**category**.

4.3 Application

If P is the Leibniz operad `Lei`, Theorem 4 allows to conclude that the category

$$\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$$

of Leibniz infinity algebras is an ∞-**category**. On the other hand, Baez and Crans initiated [2] the study of the category $2\mathtt{Lie}_\infty$-`Alg` of Lie infinity algebras, whose underlying vector space V has only two terms V_0 and V_1. Their results can be generalized [13] to the category $2\mathtt{Lei}_\infty$-`Alg` of 2-term Leibniz infinity algebras and they show that this category carries actually a strict 2-categorical structure. The latter is merely the pullback of the God-given 2-categorical structure on the equivalent category

$$\texttt{Lei2Alg} \simeq 2\mathsf{Lei}_\infty\texttt{-Alg} \tag{9}$$

of Leibniz 2-algebras, i.e., of categorified Leibniz algebras. However, this **pullback strict 2-categorical structure** is rather mysterious in the homotopy algebra setting $2\mathsf{Lei}_\infty$- $\texttt{Alg}$.

The point is that it can be proven [13] that this 'artificial' 2-categorical structure on $2\mathsf{Lei}_\infty\texttt{-Alg}$ is nothing but the shadow of the above-constructed quite natural ∞-categorical structure on $\mathsf{Lei}_\infty\texttt{-Alg}$:

Theorem 5. *The ∞-categorical structure of $\mathsf{Lei}_\infty\texttt{-Alg}$ projects onto the strict 2-categorical structure of $2\mathsf{Lei}_\infty\texttt{-Alg}$, which is obtained via transfer of the canonical strict 2-categorical structure of $\texttt{Lei2Alg}$.*

This insight answers questions by Baez-Crans and Schreiber-Stasheff [15].

Remark 3. *A generalization of the correspondence (9) can be found in [12]: Lie 3-algebras are defined and it is proven that these are in 1-to-1 correspondence with the 3-term Lie infinity algebras, whose bilinear and trilinear maps vanish in degree $(1,1)$ and in total degree 1, respectively.*

We now further describe the preceding idea of projection. We start recalling the main aspect of Getzler's proof that $\mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)$ is a Kan complex. Remember first the maps

$$B_n^i : \mathrm{MC}(\mathcal{C}\otimes\Omega_n) \to \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}^i(\mathcal{C}\otimes\Omega_n) \subset \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}(\mathcal{C}\otimes\Omega_n) \quad (n \geq 0, 0 \leq i \leq n)\ ,$$

which send a higher morphism to an ordinary morphism and another component. Let us mention, for the sake of completeness, that

$$\mathrm{mc}^i(\mathcal{C}\otimes\Omega_n) = \{(\delta\otimes\mathrm{id} + \mathrm{id}\otimes\mathrm{d})\varepsilon, \varepsilon \in (\mathcal{C}\otimes\Omega_n)^0, \varepsilon(\mathrm{e}_{i+1}) = 0\}\ ,$$

where δ (resp., d) is the differential of $\mathcal{C}$ (resp., of Ω_n) and where $(e_i)_i$ is the standard basis of $\mathbb{R}^{n+1}$, i.e., where the $(e_i)_i$ are the vertices of the standard topological n-simplex Δ_n. The main insight is that these maps B_n^i admit inverse maps $\mathcal{B}_n^i$, which allow to prove the Kan property for $\mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)$. Indeed, using the B_n^i and $\mathcal{B}_n^i$,

$$\begin{array}{ccc}
\texttt{SSet}(\Lambda^i[n], \mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)) & \dashrightarrow & \mathrm{MC}(\mathcal{C}\otimes\Omega_n) \\
\downarrow & & \uparrow \\
\texttt{SSet}(\Lambda^i[n], \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}(\mathcal{C}\otimes\Omega_\bullet)) & \longrightarrow & \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}^i(\mathcal{C}\otimes\Omega_n)
\end{array}\ ,$$

we obtain the dotted map

$$\texttt{SSet}(\Lambda^i[n], \mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)) \longrightarrow \texttt{SSet}(\Delta[n], \mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet))\ ,$$

so that any horn of $\mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)$ has actually a filler and $\mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)$ is Kan.

We are now prepared for the announced more detailed description of the projection in Theorem 5. Just as we referred to higher morphisms of the ∞-category $\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$ as ∞-n- Mor ($n \in \{1, 2, 3, \ldots\}$), we will refer to the morphisms and homotopies of the 2-category $2\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$ as 2-n- Mor ($n \in \{1, 2\}$). It is clear that any 2-1-morphism between two Leibniz infinity algebras having just 2 terms, is a morphism between these two Leibniz infinity algebras, hence an ∞-1-morphism between fixed algebras, or, still, an element of $\mathrm{MC}(\mathcal{C})$.

Let now α, β be two 2-1-morphisms between the same two 2-term Leibniz infinity algebras, i.e., two elements of $\mathrm{MC}(\mathcal{C})$, and let $\gamma \in \mathrm{MC}(\mathcal{C}\otimes\Omega_1)$ be a homotopy or ∞-2-morphism between them. Using the recalled correspondence

$$B_1^0 : \mathrm{MC}(\mathcal{C} \otimes \Omega_1) \rightleftarrows \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}^0(\mathcal{C} \otimes \Omega_1) : \mathcal{B}_1^0 \;,$$

we obtain

$$\gamma = \mathcal{B}_1^0 B_1^0 \gamma = \alpha + \mathcal{E}(\alpha, \varepsilon)$$

and

$$\beta = \gamma(1) = \alpha + \mathcal{E}(\alpha, \varepsilon(1)) \tag{10}$$

(if we view ε as defined, not on the interval $[0, 1]$, but on the standard topological 1-simplex, then $\varepsilon(1)$ means $\varepsilon(e_2)$; observe also that $\varepsilon(0) = \varepsilon(e_1) = 0$). In [13], we revisited the construction of the $\mathcal{B}_n^i$, adopting our own approach, what allowed us to compute the expression $\mathcal{E}(\alpha, \varepsilon(1))$ very explicitly. It turned out that Equation (10) exactly means that $\varepsilon(1)$ is a 2-2-morphism between the 2-1-morphisms α, β, i.e., a homotopy between α, β in the sense of the 2-category $2\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$. In other words, for any 2-1-morphisms α, β between the same algebras, we associated to every homotopy $\gamma \in \infty\text{-2- Mor}(\alpha, \beta)$ a homotopy $\varepsilon(1) \in \text{2-2- Mor}(\alpha, \beta)$. It is possible to show [13] that this assignment

$$\pi(\alpha, \beta) : \infty\text{-2- Mor}(\alpha, \beta) \ni \gamma \mapsto \varepsilon(1) \in \text{2-2- Mor}(\alpha, \beta)$$

is not only well-defined but also surjective (instead of $B_1^0, \mathcal{B}_1^0$, we could have used just as well $B_1^1, \mathcal{B}_1^1$).

Hence, for any vertically composable $\varepsilon(1), \varepsilon'(1)$, we can choose preimages γ, γ', set $\gamma'' := \gamma' \circ \gamma$, and project γ'' to $\varepsilon''(1)$. The point is that, despite the ill-definedness of the preimages γ, γ' and the ill-definedness of their composite γ'', the resulting $\varepsilon''(1)$ is well-defined with respect to $\varepsilon(1), \varepsilon'(1)$, so that we can set $\varepsilon''(1) := \varepsilon'(1) \circ \varepsilon(1)$. Exactly as for concordances (see Remark 2), the composition of horizontally composable $\varepsilon(1), \varepsilon'(1)$ is not problematic. Eventually, the projection of the ∞-categorical structure on $\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$ defines on $2\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$ a strict 2-categorical structure – which turns out to be exactly the a bit obscure 2-categorical structure obtained on this category via the pullback of the 2-category structure on $\mathtt{Lei2Alg}$.

5 A tale of five homotopies

We mentioned at the beginning of Subsection 4.2, that there exist at least three concepts of homotopy for P_∞-morphisms: concordances, gauge homotopies and Quillen homotopies. In

[5], we describe a fourth and a fifth notion, cylinder homotopies and operadic homotopies, and prove the

Theorem 6. *The concepts of concordance, gauge homotopy, Quillen homotopy, cylinder homotopy, and operadic homotopy are equivalent.*

Remark 4. *In fact, the notion of operadic homotopy is homotopically equivalent to the others. To our knowledge, we give in [5] the first explicit recipe to write a definition of operadic homotopy. This receipt is far from being simple. It involves nested trees in homotopy transfer formulas. Indeed, the prime (non-trivial) tool used to establish the equivalences of the different concepts is the homotopy transfer theorem for homotopy cooperads – which proves (of course) that, if a differential graded* $\mathbb{S}$*-module is a homotopy retract of a differential graded* $\mathbb{S}$*-module that carries a homotopy cooperad structure, then it is possible to transfer this homotopy cooperad structure to the retract.*

References

[1] M. Ammar and N. Poncin, *Coalgebraic approach to the Loday infinity category, stem differential for 2n-ary graded and homotopy algebras*, Ann. Inst. Fourier, 60(1), 2010, 355–387

[2] J.C. Baez and A.S. Crans, *Higher-dimensional algebra VI: Lie 2-algebras*, Theo. Appl. Cat., 12, 2004, 492–538

[3] G. Bonavolontà and N. Poncin, *On the category of Lie n-algebroids*, J. Geo. Phys., 73, 2013, 70–90

[4] V. Dolgushev, A. Hoffnung, and C. Rogers, What do homotopy algebras form?, `ArXiv:1406.1751`

[5] V. Dotsenko and N. Poncin, *A tale of three homotopies*, Appl. Cat. Struct., 24(6), 2016, 845–873

[6] E. Getzler, *Lie theory for nilpotent* L_∞*-algebras*, Ann. of Math. (2), 170(1), 2009, 271–301

[7] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J., 76, 1994, 203–272

[8] J. Grabowski and G. Marmo, *Binary operations in classical and quantum mechanics*, in "Classical and Quantum Integrability", Banach Center Publ., 59, 2003, 163–172

[9] R. Ibanez, M. de Leon, J.C. Marrero, and E. Padron, *Leibniz algebroid associated with a Nambu-Poisson structure*, J. Phys. A, Math. Gen., 32, 1999, 8129–8144

[10] B. Jubin, N. Poncin, and Kyosuke Uchino, *Free Courant and derived Leibniz pseudoalgebras*, J. Geo. Mech., 8(1), 2016, 71–97

[11] D. Khudaverdian, J. Grabowski, and N. Poncin, *The supergeometry of Loday algebroids*, J. Geo. Mech., 5(2), 2013, 185–213

[12] D. Khudaverdian, A. Mandal, and N. Poncin, *Higher categorified algebras versus bounded homotopy algebras*, Theo. Appl. Cat., 25(10), 2011, 251–275

[13] D. Khudaverdian, N. Poncin, and J. Qiu, *Infinity category of homotopy Leibniz algebras*, Theo. Appl. Cat., 29(12), 2014, 332–370, and ArXiv:1308.2583

[14] M. Schlessinger and J.D. Stasheff, *Deformation theory and rational homotopy type*, University of North Carolina preprint, ArXiv preprint 1211.1647, 1979

[15] U. Schreiber and J.D. Stasheff, *Structure of Lie n-algebras*, http://www.math.uni-hamburg.de/home/schreiber/struc.pdf