# Reduced order method combined with domain decomposition method

## IHP Workshop: Recent developments in numerical methods for model reduction

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# Outline



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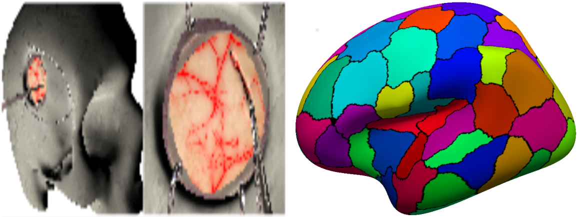
# Motivation

- Decrese the computational cost in solving computational mechanics problem in accurate mesh discretization



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Recent approach that combine reduced order modelling and spatial domain decomposition:

- Reduced basis element methd based on Lagrange multiplier (FETI) method
- Static condensation reduced basis element
- Reduced basis hybrid element method
- Substructuring and heterogenous domain decomposition method

Let $\Omega(\theta) = \Omega_1(\theta_1) \cup \Omega_2(\theta_2)$ and $\Gamma(\theta) = \partial\Omega_1(\theta_1) \cap \partial\Omega_2(\theta_2)$ the interface betwen the subdomains. The linear elasticity problem reads: find the displacement $u$ such that

$$\begin{aligned}
-div(\sigma(\mathbf{u}, \mu_i, \lambda_i)) = f_i(\beta, \rho) \quad & \text{in}\, \Omega_i(\theta_i) \\
\sigma(\mathbf{u}, \mu_1, \lambda_1) \cdot \mathbf{n} = \sigma(\mathbf{u}, \mu_2, \lambda_2) \cdot \mathbf{n} \quad & \text{on}\, \Gamma(\theta) \\
\mathbf{u}_1 = \mathbf{u}_2 \quad & \text{on}\, \Gamma(\theta) \\
\mathbf{u} = 0 \quad & \text{on}\, \partial\Omega_{D,i}(\theta_i) \\
\sigma(\mathbf{u}, \mu_i, \lambda_i) \cdot \mathbf{n} = \mathbf{g} \quad & \text{on}\, \partial\Omega_{N,i}(\theta_i)
\end{aligned}$$

Here the stress tensor $\sigma$ is related to the displacement by Hooke's law:

$$\sigma(\mathbf{u_i}) = 2\mu_i \epsilon(\mathbf{u}_i) + \lambda_i tr(\epsilon(\mathbf{u}_i))\mathbf{I} \quad \text{in}\, \Omega_i(\theta_i)$$

And source is defined as : $f_i(\beta, \rho) = (0.0, -\rho \cdot 9.8 \cdot e^{-c(\cdot(x-\beta_x)^2 + (y-\beta_y)^2}))$

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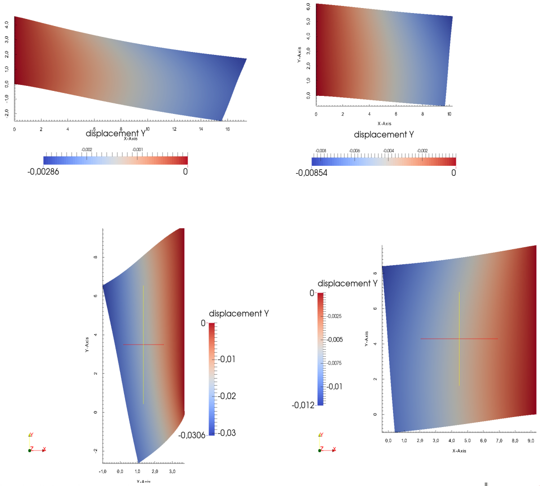
# Reference configuration



# Error estimation of RB-Greedy

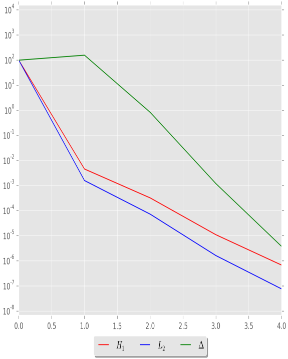



Figure: Reduced basis error with parametric manifold defined by the variation of the shape and position of the source

# FETI-Lagrange multiplier method

The algebraic system reads

$$\begin{bmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix} \tag{1}$$

where $B_i$ are signed boolean matrices. Cons:

- additional effort for each floating block and corner point
- mortar integration or interpolation for non-matching case

# Reduced basis with FETI solver

In no-floating case, the reduced FETI system could be rewriten in the following way:

$$(B_1^T Z_1 (A_1^r)^{-1} Z_1^T B_1 + B_2^T Z_2 (A_2^r)^{-1} Z_2^T B_2)\lambda = 0$$

$$A_1^r u_1^r + Z_1^T B_1 \lambda = Z_1^T f_1$$

$$A_2^r u_2^r + Z_2^T B_2 \lambda = Z_2^T f_2$$

where $Z_i$ matrix contains the reduced basis (column-wise)

# Nitsche formulation

Find $u \in H^1(\Omega)$ such that

$$\sum_i \int_{\Omega_i} \sigma(\mathbf{u}) : \nabla v - \int_\Gamma \langle \sigma(\mathbf{u}) \cdot n \rangle [\![ v ]\!]$$
$$- \int_\Gamma \langle \sigma(\mathbf{v}) \cdot n \rangle [\![ u ]\!] + \frac{\alpha}{h} \int_\Gamma [\![ u ]\!][\![ v ]\!] = \sum_i \int_i f_i v \, \forall v \in H_0^1(\Omega)$$

The implementation in FeNicS is based on recent "Multimesh, MultiMeshFunctionSpace" .

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# RB-Nitsche formualtion

Using an parametrization, the Nitsche formulation on the reference block could be writen as

$$
\begin{aligned}
\sum_i \int_{\Omega_i} G(\mu) \cdot \sigma(\mathbf{u},) : G(\mu) \cdot \nabla v \ \det(G^{-1}(\mu)) dV_i \\
- \int_{\Gamma} \langle \sigma(\mathbf{u}) \cdot n \rangle [\![ v ]\!] \| G(\mu) \cdot e_t \| dl \\
- \int_{\Gamma} \langle \sigma(\mathbf{v}) \cdot n \rangle [\![ u ]\!] \| G(\mu) \cdot e_t \| dl \\
+ \frac{\alpha}{h} \int_{\Gamma} [\![ u ]\!] [\![ v ]\!] \| G(\mu) \cdot e_t \| dl = \sum_i \int_i f_i v \, \forall v \in H^1(\Omega)
\end{aligned}
$$

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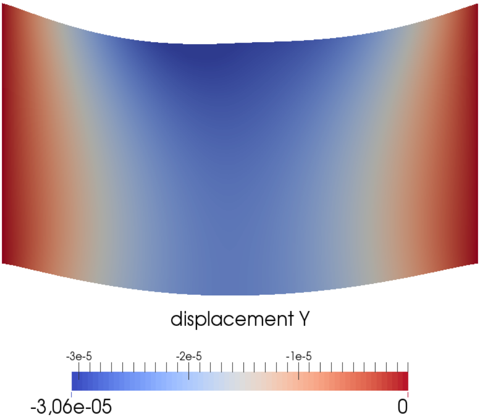
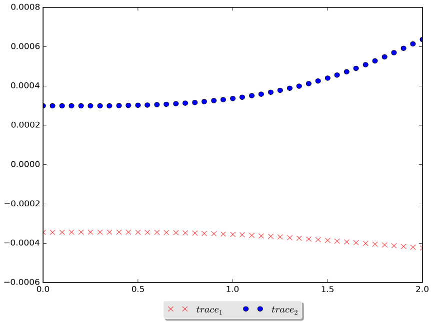
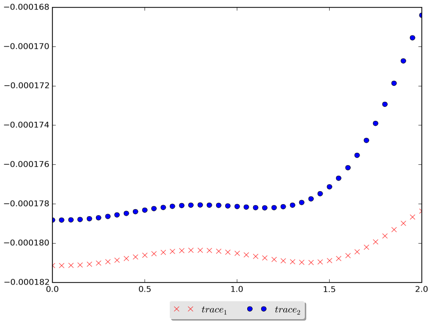
# Test case I



# Reduced basis pairs: (1,5), (2,5)

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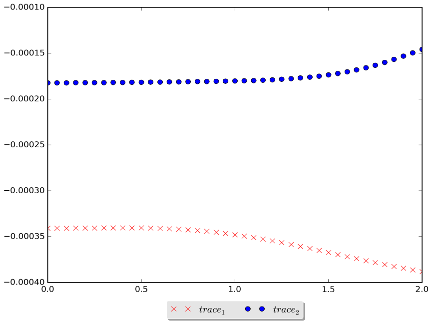
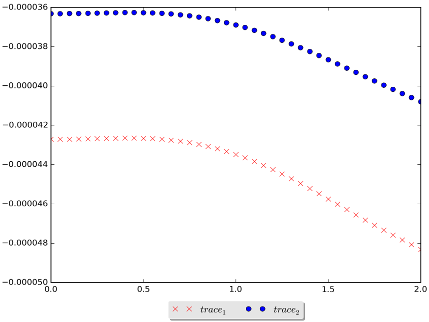
# Reduced basis pairs: (3,5), (4,5)

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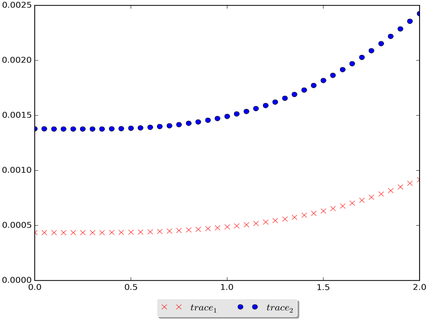
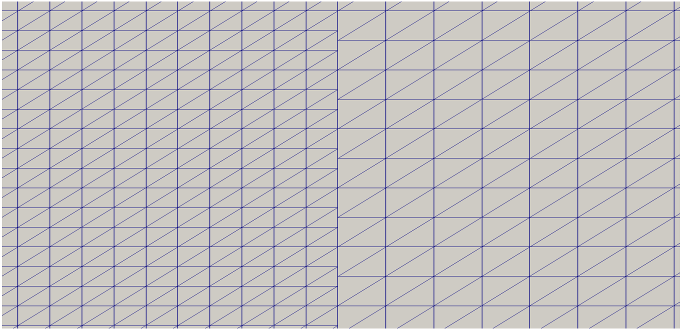
# Reduced basis pairs: (5,5)



# Test case II:Non-matching



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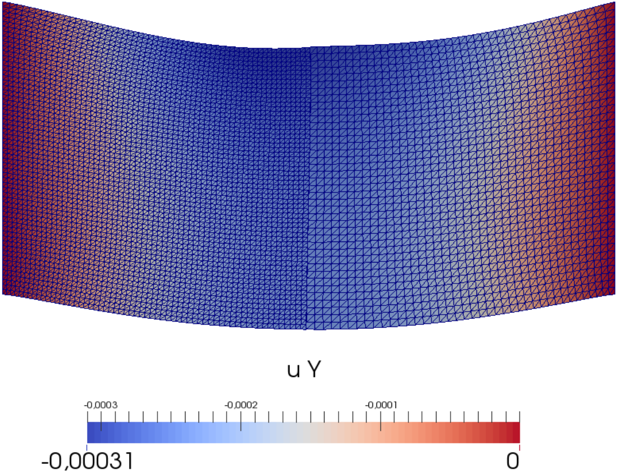
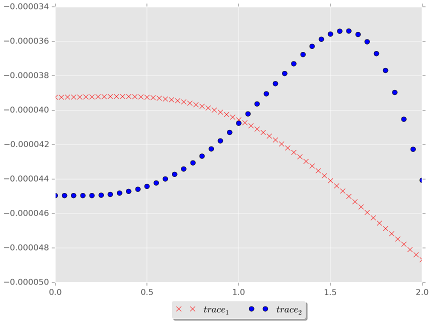
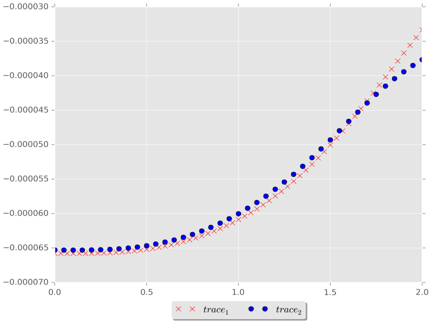

u Y
-0,0003
-0,0002
-0,0001
-0,00031
0

# Number of Reduced Basis:1-4 and 2-4

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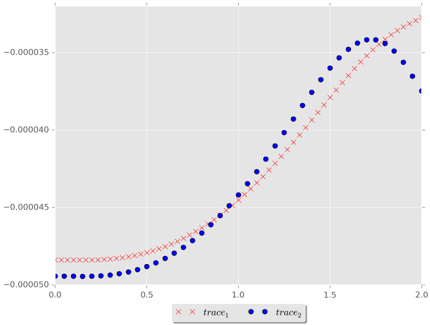
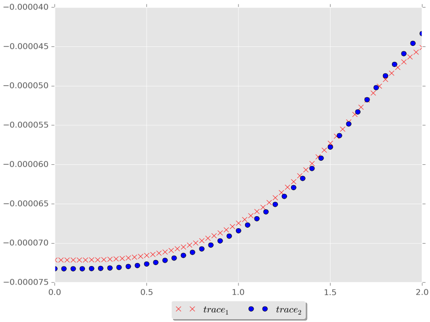
# Number of Reduced Basis:3-4 and 4-4

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# Conclusion

The preliminar numerical comparison evidences:

- more flexible approach in online gluing with Nitsche formulation
- combination of reduced basis provides a speed in solving the system

Working in progress:

- investigation on different reference lego block configuration
- integration of EIM tool
- extension to realistic configuration in biomechanics problem
- real-time cut-tracking
- release of a python module based on FeNicS, petsc4py and slepc4py for reduced order method approaches.

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# Acknowledgements

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European Research Council
Established by the European Commission

European Research Council Starting Independent Research Grant (ERC Stg grant agreement No. 279578) entitled "Towards real time multiscale simulation of cutting in non-linear materials with applications to surgical simulation and computer guided surgery" led by Stéphane P. A. Bordas

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## Thank you for your attention

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