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# ELASTOGRAPHY UNDER UNCERTAINTY.

**Jack S. Hale,[1] Patrick E. Farrell[2], Stéphane P.A. Bordas[1].**
**[1]Research Unit in Engineering Sciences, University of Luxembourg. [2]Mathematics Institute, University of Oxford.**

## ELASTOGRAPHY

*Elastography* is any method that can be used to extract quantitative or qualitative data about *elastic modulus distributions* from images of elastic solids (Parker, Doyley, and Rubens, 2011).

## WHY?

- Tumorous tissue is *significantly stiffer* than healthy tissue.
- If we can detect that change in stiffness we a useful extra imaging modality for cancer diagnosis.
- There is growing clinical evidence that elastography is useful (Parker, Doyley, and Rubens, 2011).

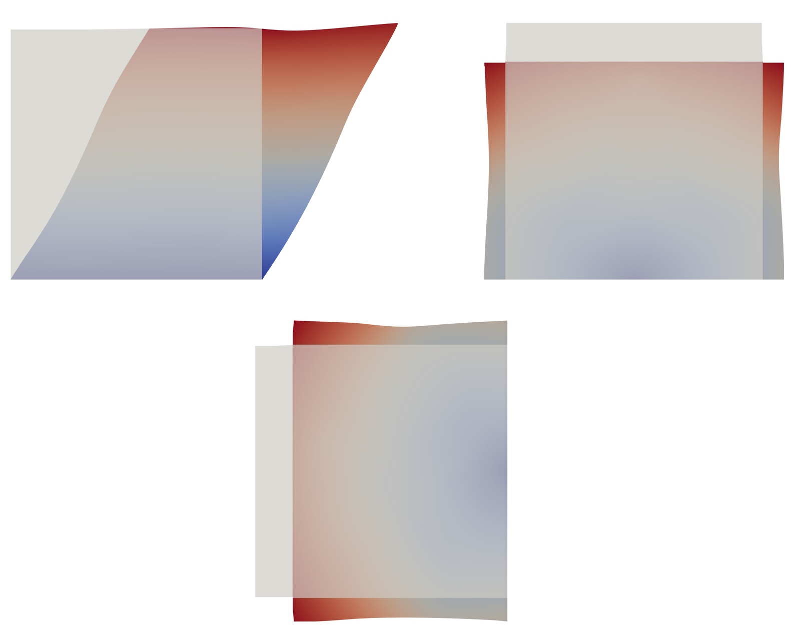
## QUESTIONS AND ISSUES

- Imaging modalities are *corrupted by noise*. How can we take this noise into account? How does it affect the results?
- If we only have *surface observations* of an object, how much do we really know about the parameters *inside*?
- The displacements of soft-tissues are related to the the stiffness parameters by a complex set of non-linear PDEs. Can we find the parameters in a *reasonable amount of time*?

## MODEL PROBLEM

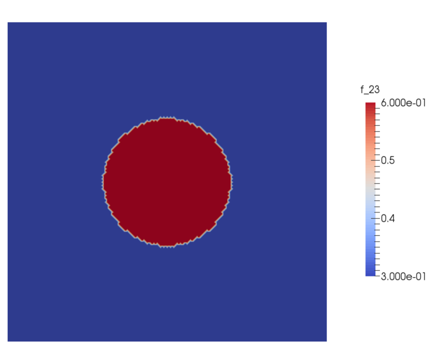
Given displacement observations on the surface of a block of soft tissue, possibly containing a stiff tumor, what can we infer about the material parameters of the tissue inside? How sure are we about what we infer?

## FIGURE 1



Left: Three virtual experimental results from applying three different loads to the same non-homogeneous block of soft tissue. We are only given the observations on the exterior surface, and they are corrupted by random white noise.

## FIGURE 2



Left: The true material parameter field used to generate the experimental data in Figure 1. A stiff circular tumour is surrounded by softer healthy tissue.

## REFERENCES

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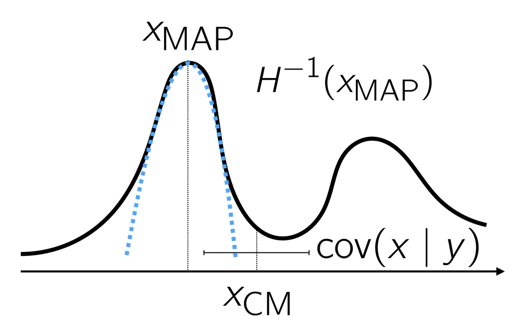
## METHODOLOGY

- We use the Bayesian framework for statistical inference (Stuart, 2010).
- Allows for rigorous statistical quantification of uncertainty arising from:
  - Partial observations.
  - Noisy instruments.
  - Model inadequacy.
- Soft tissue modelled by a fully non-linear hyperelastic PDE.
- Flexible Gaussian noise and prior modelling.
- We use derivatives of the finite element model to find the most likely material parameters and approximate the covariance structure.

## FIGURE 3

$$\pi_{\text{posterior}} \sim \mathcal{N}(x_{\text{MAP}}, \mathbf{H}^{-1})$$

$$\pi_{\text{posterior}}^{\text{approx}} \sim \mathcal{N}(x_{\text{MAP}}, \mathbf{H}^{-1}(x_{\text{MAP}}))$$




Left: The Bayesian posterior encodes the probability of the all possible parameters given our experimental observations. We find the *maximum a posteriori* point through gradient-driven optimisation. We construct a *Gaussian approximation* of the covariance structure at this point.

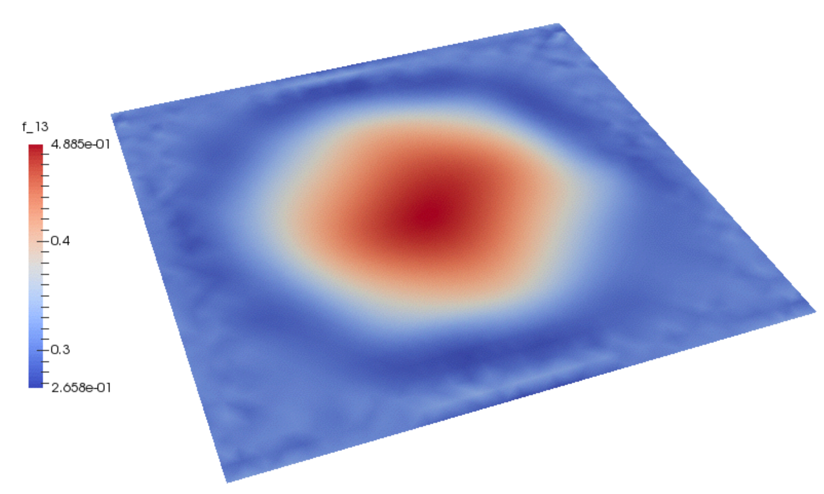
## COMPUTATIONAL TECHNIQUES

- Automatic construction of forward and adjoint models with `dolfin-adjoint` (Farrell et al., 2013). *Easy to change physical model.*
- Efficient algebraic multigrid preconditioning of forward and adjoint models. *Forward runs dominate overall cost, reduce as much as possible.*
- Gauss-Newton Conjugate-Gradient method to find maximum a posteriori point. *Scales well on mesh refinement.*
- Matrix-free Krylov-Schur algorithm for principal component analysis of prior pre-and-post-conditioned Hessian of likelihood. *Fixed cost for given observations/model.*
- Optimal low-rank update from prior to posterior covariance (Spantini et al., 2014). *Reduces Hessian actions/forward model runs.*

## RESULTS

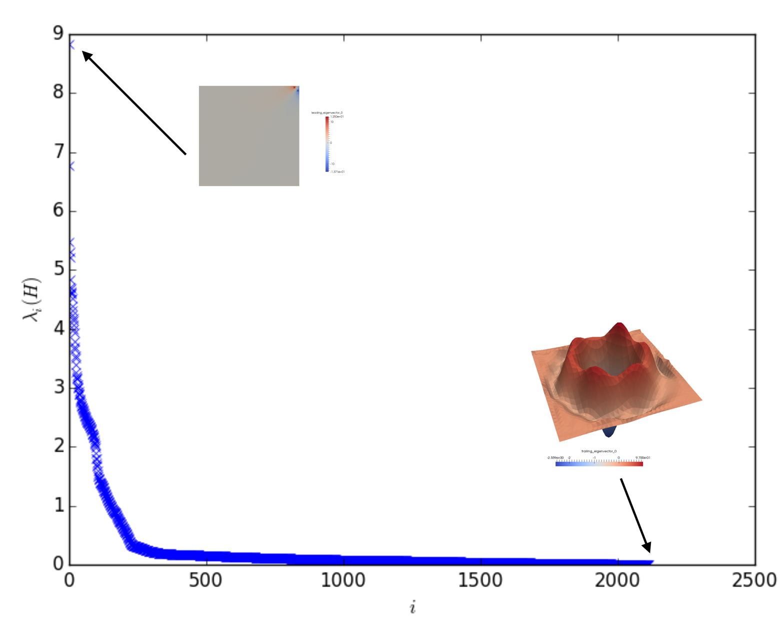
We can recover 10000s of parameters and quantify uncertainty in minutes on a laptop. *Practical.*

## FIGURE 4



Left: Recovered MAP point, cf. Figure 1. We *can* detect the stiff inclusion inside the object just from the noisy surface observations.

## FIGURE 5




Left: Low-rank structure of spectrum of posterior covariance. Data is only informative on low-rank subspace of original parameter space. Top left eigenvector points towards direction in parameter space most-constrained by the observations, bottom right towards least-constrained.