Strongly B-associative and preassociative functions

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Aggregation

A property of the arithmetic mean

$\mathbb{I} \equiv$ real interval

$$F(\mathbf{x}) := \sum_{i=1}^{n} \frac{x_i}{n}, \qquad \mathbf{x} \in \mathbb{I}^n, n \geq 1$$

We have

$$F(\mathbf{x}) \;=\; F(x_1, \ldots, x_\ell, y, \ldots, y, x_{\ell+k+1}, \ldots, x_n)$$

where $y = F(x_{\ell+1}, \ldots, x_{\ell+k})$.

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This property is called *B-associativity*.

A characterization of quasi-arithmetic means

Theorem (Kolmogorov - Nagumo) Let $F\colon \bigcup_{n\geq 1}\mathbb{I}^n \to \mathbb{I}$. The following conditions are equivalent.

(i) F is B-associative, and for every $n \geq 1$, F_n is

symmetric,

continuous,

strictly increasing in each argument,

reflexive, *i.e.*, $F(x,\ldots,x) = x$ for every x.

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(ii) There is a continuous and strictly monotonic $f\colon \mathbb{I} \to \mathbb{R}$ such that

$$F(\mathbf{x}) = f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big), \quad \mathbf{x} \in \mathbb{I}^n, n \geq 1.$$

These functions are called *quasi-arithmetic means*.

Notation

$X \equiv$ non-empty set

We regard n-tuples $\mathbf{x}$ in X^n as *n-strings* over X

0-string: ε
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$
$|\mathbf{x}| \;=\;$ length of $\mathbf{x}$
$\mathbf{x}^k = \mathbf{x} \cdots \mathbf{x}$ (k times)

$$X^* := \bigcup_{n \geq 0} X^n$$

We endow X^* with concatenation

Any $F : X^* \to Y$ is called a *variadic function*, and we set

$$F_n := F|_{X^n}.$$

B-associative functions

$F\colon X^* \to X \cup \{\varepsilon\}$

Definition. F is *B-associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \qquad \forall\, \mathbf{xyz} \in X^*$$

F is *B-associative* if its value on $\mathbf{u}$ does not change when replacing each letter of a substring $\mathbf{y}$ of consecutive letters of $\mathbf{u}$ by $F(\mathbf{y})$.

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Strongly B-associative functions

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For instance,

$$\begin{aligned} F(x_1x_2x_3x_4x_5) &= F(F(x_1x_3)x_2F(x_1x_3)x_4x_5), \\ &= F(F(x_1x_3)x_2F(x_1x_3)F(x_4x_5)F(x_4x_5)). \end{aligned}$$

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Example
F defined by $F(\varepsilon) = \varepsilon$ and $F(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ for $n \geq 1$ and $\mathbf{x} \in \mathbb{R}^n$ is strongly B-associative and symmetric.

Strongly B-associative functions

Fact.

$$\text{Strongly B-associative} \quad \left\{ \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \right\} \text{B-associative}$$

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F defined by $F(\varepsilon) = \varepsilon$ and

$$F(\mathbf{x}) = \sum_{i=1}^{n} \frac{2^{i-1}}{2^n - 1}\, x_i, \qquad n \geq 1, \mathbf{x} \in \mathbb{R}^n,$$

is B-associative but not strongly B-associative.

Strongly B-associative functions

Fact.

$$\text{B-associative} \;+\; \text{symmetric} \quad \left\{ \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \right\} \text{strongly B-associative}$$

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Example.
F defined by $F(\varepsilon) = \varepsilon$ and $F(\mathbf{x}) = x_1$ for every $n \geq 1$ and $\mathbf{x} \in \mathbb{R}^n$ is strongly B-associative but not symmetric.

Strongly B-associative functions

Proposition. Let $F\colon X^* \to X \cup \{\varepsilon\}$. The following condition are equivalent.

(i) F is strongly B-associative

(ii) F satisfies

$$F(\mathbf{xyz}) \;=\; F(F(\mathbf{xz})^{|\mathbf{x}|}\mathbf{y}F(\mathbf{xz})^{|\mathbf{z}|}) \qquad \forall\ \mathbf{xyz} \in X^*$$

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Proposition. If $F\colon X^* \to X \cup \{\varepsilon\}$ is strongly B-associative, then $\mathbf{y} \mapsto F(x\mathbf{y}z)$ is symmetric for every $xz \in X^2$.

Invariance by replication

$F\colon X^* \to Y$ is *invariant by replication* if $F(\mathbf{x}^k) = F(\mathbf{x})$ for all $\mathbf{x} \in X^*$ and $k \geq 1$.

Proposition. If $F\colon X^* \to X \cup \{\varepsilon\}$ is strongly B-associative, then the following conditions are equivalent.

(i) F is invariant by replication.

(ii) $\mathrm{ran}(F_n) \subseteq \mathrm{ran}(F_{kn})$ for every $n \geq 0$ and $k \geq 1$.

An alternative characterization of quasi-arithmetice means

Theorem (Kolmogorov - Nagumo) Let $F\colon \mathbb{I}^* \to \mathbb{I}$. The following conditions are equivalent.

(i) F is B-associative, and for every $n \geq 1$, F_n is

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- symmetric,
- continuous,
- strictly increasing in each argument,
- reflexive, *i.e.*, $F(x, \ldots, x) = x$ for every x.

(ii) F is a quasi-arithmetic mean.

Theorem. B-associativity and symmetry can be replaced by strong B-associativity. Moreover, reflexivity can be removed.

A composition-free version of strong B-associativity

Definition. $F\colon X^* \to Y$ is *strongly B-preassociative* if

$$\left.\begin{array}{c} |\mathbf{x}| = |\mathbf{x}'| \\ |\mathbf{z}| = |\mathbf{z}'| \\ F(\mathbf{x}\mathbf{z}) \;=\; F(\mathbf{x}'\mathbf{z}') \end{array}\right\} \implies F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}'\mathbf{y}\mathbf{z}').$$

Example. The length function $F\colon X^* \to \mathbb{R}\colon \mathbf{x} \mapsto |\mathbf{x}|$ is strongly B-preassociative.

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Example. The length function $F\colon X^* \to \mathbb{R}\colon \mathbf{x} \mapsto |\mathbf{x}|$ is strongly B-preassociative.

Proposition. Let $F\colon X^* \to X \cup \{\varepsilon\}$. The following conditions are equivalent.

(i) F is strongly B-associative.

(ii) F is strongly B-preassociative and satisfies $F(F(\mathbf{x})^{|\mathbf{x}|}) = F(\mathbf{x})$.

Factorization of strongly B-preassociative functions with strongly B-associative ones

$F\colon X^* \to X \cup \{\varepsilon\}$ is *ε-standard* if $F(\mathbf{x}) = \varepsilon \iff \mathbf{x} = \varepsilon$.

$$\delta_{F_n}(x) := F_n(x \cdots x)$$

Factorization of strongly B-preassociative functions with strongly B-associative ones

$F\colon X^* \to X \cup \{\varepsilon\}$ is *ε-standard* if $F(\mathbf{x}) = \varepsilon \iff \mathbf{x} = \varepsilon$.

$$\delta_{F_n}(x) := F_n(x \cdots x)$$

Theorem. (AC) Let $F\colon X^* \to Y$. The following conditions are equivalent.

(i) F is strongly B-preassociative & $\mathrm{ran}(F_n) = \mathrm{ran}(\delta_{F_n})$ for all n;

(ii) $F_n = f_n \circ H_n$ for every $n \geq 1$ where

- $H\colon X^* \to X \cup \{\varepsilon\}$ is ε-standard and strongly B-associative,
- $f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one for every $n \geq 1$.

Associative functions

Definition $F\colon X^* \to X^*$ is *associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}), \qquad \mathbf{xyz} \in X^*.$$

Examples.

- the sum $x_1 + \cdots + x_n$,
- the minimum $x_1 \wedge \ldots \wedge x_n$,
- variadic extensions of binary associative functions,

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Examples.

- the sum $x_1 + \cdots + x_n$,
- the minimum $x_1 \wedge \ldots \wedge x_n$,
- variadic extensions of binary associative functions,
- sorting in alphabetical order,
- letter removing, duplicate removing.

F_1 may differ from the identity map!

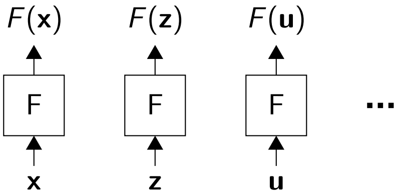
Associativity for string functions

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Examples. [. . .] duplicate removing

INPUT: $\mathbf{xzu}\cdots$ in blocks of unknown length given at unknown time intervals.

OUTPUT: $F(\mathbf{xzu}\cdots)$



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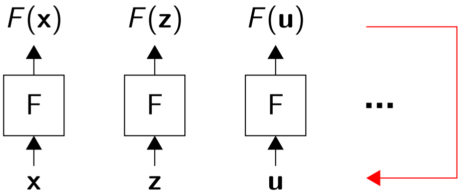

"Highly" distributed algorithms

Factorization of strongly B-preassociative functions with associative ones

$H\colon X^* \to X^*$ is *length-preserving* if $|H(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x} \in X^*$.

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$H\colon X^* \to X^*$ is *length-preserving* if $|H(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x} \in X^*$.

Theorem. (AC) Let $F\colon X^* \to Y$. The following conditions are equivalent.

(i) F is strongly B-preassociative.

(ii) $F_n = f_n \circ H_n$ for every $n \geq 1$ where

- $H\colon X^* \to X^*$ is associative, length-preserving and strongly B-preassociative,
- $f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one for every $n \geq 1$.

Quasi-arithmetic pre-mean functions

$\mathbb{I} \equiv$ non-trivial real interval.

Definition. $F\colon \mathbb{I}^* \to \mathbb{R}$ is a *quasi-arithmetic pre-mean function* if there are continuous and strictly increasing functions $f\colon \mathbb{I} \to \mathbb{R}$ and $f_n\colon \mathbb{R} \to \mathbb{R}$ $(n \geq 1)$ such that

$$F(\mathbf{x}) \; = \; f_n\Big(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big), \qquad n \geq 1, \mathbf{x} \in X^n.$$

Example. $F(\mathbf{x}) = \prod_{i=1}^{n} x_i$ is a quasi-arithmetic pre-mean function over $\mathbb{I} =]0, +\infty[$ (take $f_n(x) = \exp(nx)$ and $f(x) = \ln(x)$) which is not a quasi-arithmetic mean function.

Characterization of quasi-arithmetic pre-mean functions

Idea. Apply the factorization result to the quasi-arithmetic means.

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Theorem. Let $F\colon \mathbb{I}^* \to \mathbb{R}$. The following conditions are equivalent.

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Idea. Apply the factorization result to the quasi-arithmetic means.

Theorem. Let $F\colon \mathbb{I}^* \to \mathbb{R}$. The following conditions are equivalent.

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(ii) F is strongly B-preassociative, and for every $n \geq 1$, F_n is

- symmetric,
- continuous,
- strictly increasing in each argument.

Open problems

Characterization of the class of $F\colon X^* \to X^*$ which are associative, length-preserving and strongly B-preassociative?

Which of those B-associative functions that satisfy

$$F(xyz) = F(F(xz)yF(xz))$$

are strongly B-associative?

Reference. J.-L. Marichal and B. Teheux. Strongly barycentrically associative and preassociative functions. *Journal of Mathematical Analysis and Applications* 437, 181 – 193, 2016.