SPECTRAL GAP ON RIEMANNIAN PATH SPACE OVER STATIC AND EVOLVING MANIFOLDS

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ABSTRACT. In this article, we continue the discussion of Fang-Wu (2015) to estimate the spectral gap of the Ornstein-Uhlenbeck operator on path space over a Riemannian manifold of pinched Ricci curvature. Along with explicit estimates we study the short-time asymptotics of the spectral gap. The results are then extended to the path space of Riemannian manifolds evolving under a geometric flow. Our paper is strongly motivated by Naber's recent work (2015) on characterizing bounded Ricci curvature through stochastic analysis on path space.

1. INTRODUCTION

Let (M,g) be a d-dimensional complete smooth Riemannian manifold with ∇ and Δ denoting respectively the Levi-Civita connection and the Laplacian on M. Given a C^1 vector field Z on M, we consider the Bakry-Emery curvature

$$\mathrm{Ric}^Z := \mathrm{Ric} - \nabla Z$$

for the so-called Witten Laplacian $L = \Delta + Z$ where Ric is the Ricci curvature tensor with respect to g. It is well known that the spectral gap of L can be estimated in terms of a lower curvature bound K, i.e.,

$$\mathrm{Ric}^Z \geq K$$

for some constant K, see e.g. [4, 5, 10]. These results reveal the close relationship between spectral gap, convergence to equilibrium and hypercontractivity of the corresponding semigroup. For example, Poincaré inequalities and log-Sobolev inequalities which can be used to characterize the convergence for the semigroup, imply certain lower bound for the spectral gap.

In this article, we extend this circle of ideas to the Riemannian path space over M and revisit the problem of estimating the spectral gap of the Ornstein-Uhlenbeck operator under the following general curvature condition: there exist constants k_1 and k_2 such that

$$k_1 \leq \mathrm{Ric}^Z \leq k_2.$$

Before moving on, let us briefly summarize some background results on stochastic analysis on path space over a Riemannian manifold. Stochastic analysis on path space attracted a lot of attention since 1992 when B.K. Driver proved quasi-invariance of the Wiener measure on the path space over a compact Riemannian manifold [11]. A milestone in the theory is the integration by parts formula

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(see e.g. [3, 15]) for the associated gradient operator induced by the quasi-invariant flow. This result is a main tool in proving functional inequalities for the corresponding Dirichlet form, for instance, the log-Sobolev inequality [1]; the constant in this inequality has been estimated in [19] in terms of curvature bounds.

Very recently, A. Naber [24] proved that certain log-Sobolev inequalities and L^p-inequalities on path space are equivalent to an upper bound for the norm of Ricci curvature on the base manifold M; R. Haslhofer and A. Naber [17] extended these results to characterize solutions of the Ricci flow, see also [18]. Inspired by this work, S. Fang and B. Wu [16] gave an estimate of the spectral gap under the curvature condition that

$$k_1 \le \mathrm{Ric}^Z \le k_2$$

for two constants k_1 and k_2 with $k_1+k_2\ge 0$. However, as far as the case "$k_1+k_2<0$" is concerned, the same argument may lead to a loss of information concerning k_2. We revisit this topic in this article. Our aim is to remove the restriction $k_1+k_2\ge 0$ in the curvature condition and to establish sharper short-time asymptotics for the spectral gap.

Our methods rely strongly on suitable extensions and generalizations of recent estimates on Riemannian path space, due to Naber [24], resp. Haslhofer and Naber [17, 18]. This work is crucial for our arguments, as it allows to characterize bounded Ricci curvature in terms of stochastic analysis on path space.

We start by briefly introducing the context. Let X_t^x be a diffusion process with generator L starting from $X_0^x=x$. We call X_t^x an L-diffusion process. We assume that X_t^x is non-explosive. Let $B_t=(B_t^1,\ldots,B_t^d)$ be a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega,\{\mathscr{F}_t\}_{t\ge0},\mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\ge0}$. It is well known that the L-diffusion process X_t^x starting from x solves the equation

$$\mathrm{d}X_t^x=\sqrt{2}\,u_t^x\circ \mathrm{d}B_t+Z(X_t^x)\,\mathrm{d}t,\quad X_0^x=x, \tag{1.1}$$

where u_t^x is the horizontal process of X_t^x taking values in the orthonormal frame bundle $\mathrm{O}(M)$ over M such that $\pi(u_0^x)=x$. Furthermore

$$//_{s,t}:=u_t^x\circ(u_s^x)^{-1}\colon T_{X_s^x}M\to T_{X_t^x}M,\quad s\le t,$$

defines parallel transport along the paths $r\mapsto X_r^x$. As usual, orthonormal frames $u\in\mathrm{O}(M)$ are identified with isometries $u\colon\mathbb{R}^d\to T_xM$ where $\pi(u)=x$.

For fixed $T>0$ define $W^T=C([0,T];M)$ and let

$$\mathscr{F}C_{0,T}^\infty=\left\{W^T\ni\gamma\mapsto f(\gamma_{t_1},\ldots,\gamma_{t_n})\colon\ n\ge1,\ 0<t_1<\ldots<t_n\le T,\ f\in C_0^\infty(M^n)\right\}$$

be the class of smooth cylindrical functions on W^T. Let $X_{[0,T]}=\{X_t\colon\ 0\le t\le T\}$ for fixed $T>0$. Then, for $F\in\mathscr{F}C_{0,T}^\infty$ with $F(\gamma)=f(\gamma_{t_1},\ldots,\gamma_{t_n})$, we define the intrinsic gradient as

$$D_tF(X_{[0,T]}^x)=\sum_{i=1}^n \mathbb{1}_{\{t\le t_i\}}\,//_{t,t_i}^{-1}\,\nabla_i f(X_{t_1}^x,\ldots,X_{t_n}^x),\quad t\in[0,T],$$

where ∇_i denotes the gradient with respect to the i-th component. The generator $\mathscr{L}$ associated to the Dirichlet form

$$\mathscr{E}(F,F)=\mathbb{E}\left[\int_0^T|D_tF|^2(X_{[0,T]})\,\mathrm{d}t\right]=\langle\mathscr{L}F,F\rangle$$

is called Ornstein-Uhlenbeck operator. Let gap($\mathscr{L}$) be the spectral gap of the Ornstein-Uhlenbeck operator $\mathscr{L}$.

In this article, we continue the topic of estimating gap($\mathscr{L}$) under general lower and upper bounds of the Ricci curvature. For the sake of conciseness, let us first introduce some notation: for constants

K_1 and K_2, define

$$C(T,K_1,K_2)=\begin{cases}1+K_2T+\dfrac{K_2^2T^2}{2}, & K_1=0;\\ (1+\beta)^2-\beta\sqrt{(2+\beta)\,(2+2\beta-\beta\,\mathrm{e}^{-K_1T})}\,\mathrm{e}^{-K_1T/2}, & K_1>0;\\ \dfrac{1}{2}+\dfrac{1}{2}\left(1+\beta(1-\mathrm{e}^{-K_1T})\right)^2, & K_1<0,\end{cases} \tag{1.2}$$

where $\beta=K_2/K_1$.

Theorem 1.1. *Let (M,g) be a complete manifold. Assume that*

$$k_1\le \mathrm{Ric}^Z\le k_2. \tag{1.3}$$

The following estimate holds:

$$\mathrm{gap}(\mathscr{L})^{-1}\le C(T,k_1,|k_1|\vee|k_2|)\wedge\left[C\left(T,k_1,\frac{k_2-k_1}{2}\right)\times C\left(T,\frac{k_1+k_2}{2},\frac{|k_1+k_2|}{2}\right)\right]. \tag{1.4}$$

Let us mention that the first bound in inequality (1.4), i.e.,

$$\mathrm{gap}(\mathscr{L})^{-1}\le C(T,k_1,|k_1|\vee|k_2|),$$

is due to Fang and Wu [16].

Remark 1.2. In explicit terms we may expand the upper bound as follows:

$$\begin{aligned}&C(T,k_1,|k_1|\vee|k_2|)\\ &\quad=\begin{cases}1+k_2T+\dfrac{k_2^2T^2}{2}, & k_1=0;\\ (\gamma+1)^2-\gamma\sqrt{(2+\gamma)\,(2\gamma+2-\gamma\mathrm{e}^{-k_1T})}\,\mathrm{e}^{-k_1T/2}, & k_1>0;\\ \dfrac{1}{2}+\dfrac{1}{2}\left(1+\gamma-\gamma\mathrm{e}^{-k_1T}\right)^2, & k_1+k_2\ge 0\ \text{ and }\ k_1<0;\\ \dfrac{1}{2}\left(1+\mathrm{e}^{-2k_1T}\right), & k_1+k_2<0,\end{cases}\end{aligned}$$

and

$$\begin{aligned}&C\left(T,k_1,\frac{k_2-k_1}{2}\right)\times C\left(T,\frac{k_1+k_2}{2},\frac{|k_1+k_2|}{2}\right)\\ &=\begin{cases}\left(1+\dfrac{k_2T}{2}+\dfrac{k_2^2T^2}{8}\right)\left(4-(12-3\,\mathrm{e}^{-\frac{k_2T}{2}})^{1/2}\,\mathrm{e}^{-\frac{k_2T}{4}}\right), & k_1=0;\\ \dfrac{1}{4}\left\{(\gamma+1)^2-(\gamma-1)\,(\gamma+3)^{1/2}\big(2\gamma+2-(\gamma-1)\,\mathrm{e}^{-k_1T}\big)^{1/2}\,\mathrm{e}^{-\frac{k_1T}{2}}\right\}\\ \qquad\times\left(4-(12-3\,\mathrm{e}^{-\frac{k_2T}{2}})^{1/2}\,\mathrm{e}^{-\frac{(k_1+k_2)T}{4}}\right), & k_1>0;\\ \dfrac{1}{2}\left\{1+\dfrac{1}{4}\left(\gamma+1-(\gamma-1)\,\mathrm{e}^{-k_1T}\right)^2\right\}\\ \qquad\times\left(4-(12-3\,\mathrm{e}^{-\frac{k_2T}{2}})^{1/2}\,\mathrm{e}^{-\frac{(k_1+k_2)T}{4}}\right), & k_1+k_2\ge 0\ \text{ and }\ k_1<0;\\ \frac{1}{4}\left\{1+\frac{1}{4}\left(\gamma+1-(\gamma-1)\,\mathrm{e}^{-k_1T}\right)^2\right\}\left(1+\mathrm{e}^{-(k_1+k_2)T}\right), & k_1+k_2<0,\end{cases}\end{aligned}$$

where $\gamma:=k_2/k_1$.

By means of Theorem 1.1 we are now in position to determine the asymptotic behavior of $\mathrm{gap}(\mathscr{L})$ as T tends to 0.

Theorem 1.3. *Assume $k_1 \le \mathrm{Ric}^Z \le k_2$. Then, as $T \to 0$, the following asymptotics hold:*

(i) *for $k_1 \ge 0$,*

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 + k_2 T + \frac{1}{2}\left(k_2^2 - \frac{(7k_1+k_2)(k_1+k_2)k_2}{6(3k_1+k_2)}\right)T^2 + \mathrm{o}(T^2);$$

(ii) *for $k_1 + k_2 \ge 0$ and $k_1 < 0$,*

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 + k_2 T + \frac{1}{2}\left(k_2^2 + \frac{2k_1^2 - k_2^2 - 5k_1k_2}{6}\right)T^2 + \mathrm{o}(T^2);$$

(iii) *for $k_1 + k_2 < 0$,*

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 - k_1 T + \frac{1}{2}\left(k_1^2 + \frac{3k_1^2 + k_2^2}{4}\right)T^2 + \mathrm{o}(T^2).$$

Remark 1.4. Note that as $T \to 0$, up to the first order, the two upper bounds in Theorem 1.1 have the same short-time behaviour, however when considered up to second order, our estimates provide sharper asymptotics (see the proof of Theorem 1.3). For instance, from [16, Proposition 3.6] we know that if $k_1 \to 0$, then

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 + k_2 T + \frac{1}{2}k_2^2 T^2 + \mathrm{o}(T^2). \tag{1.5}$$

In this case, from Theorem 1.3 we deduce that

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 + k_2 T + \frac{5}{12}k_2^2 T^2 + \mathrm{o}(T^2)$$

with a smaller coefficient of T^2 when compared to estimate (1.5).

In Section 3 below we shall extend these results to the path space of an evolving manifold (M, g_t). Stochastic analysis on evolving manifolds began with an appropriate notion of Brownian motion on (M, g_t) (called g_t-Brownian motion), see [2]. Since then there has been a lot of subsequent work, see for instance, [6, 7, 8, 21, 22, 23, 24]. Here, we deal with diffusions X_t generated by $L_t = \Delta_t + Z_t$ which are assumed to be non-explosive. The first-named author [7] developed a Malliavin calculus on the path space of X_t by means of an appropriate derivative formula and an integration by parts formula. Recently, Haslhofer and Naber [17] characterized solutions to the Ricci flow in terms of functional inequalities on path space. Inspired by this work, we consider in Section 3 an Ornstein-Uhlenbeck type operator on path space and derive a family of log-Sobolev inequalities and Poincaré inequalities on the path space to the L_t-diffusion under a generalized pinched curvature condition. This curvature condition encodes information on the time derivative of the metric as well. In the particular case of the Ricci flow the modified curvature tensor equals to zero.

The rest of the paper is organized as follows. In the next section we establish first a log-Sobolev inequality and a Poincaré inequality on Riemannian path space; these inequalities are the tools to establish our main results of Section 1. As already indicated, Section 3 is then devoted to the extension of the results to evolving manifolds under a geometric flow.

2. Proofs of main results

To prove the main results, we introduce a two-parameter family $\{Q_{r,t}\}_{0\le r<t}$ of multiplicative functionals as follows: the $Q_{r,t}$ are a random variable taking values in the linear automorphisms of $T_{X_r^x}M$ satisfying for fixed $r \ge 0$ the pathwise equation:

$$\frac{\mathrm{d}Q_{r,t}}{\mathrm{d}t} = -Q_{r,t}\,\mathrm{Ric}^Z_{//_{r,t}}, \quad Q_{r,r} = \mathrm{id}, \tag{2.1}$$

where $\mathrm{Ric}^Z_{//_{r,t}} = //_{r,t}^{-1} \circ \mathrm{Ric}^Z_{X_t^x} \circ //_{r,t}$, see [20] and [25, Theorem 4.1.1]. As usual, Ric^Z_x operates as a linear homomorphism on T_xM via $\mathrm{Ric}^Z_x v = \mathrm{Ric}^Z(\cdot, v)^\sharp$, $v \in T_xM$.

It is easy to see that if $\mathrm{Ric}^Z \geq K$ for some constant K, then for any $0 \leq r \leq t < T$,

$$\|Q_{r,t}\| \leq \mathrm{e}^{-K(t-r)}, \quad \text{a.s.},$$

where $\|\cdot\|$ denotes the operator norm. The functionals $Q_{r,t}$ (or the "damped parallel transport" defined as $//_{r,t} \circ Q_{r,t}$) are well-known ingredients in the stochastic representation of the heat flow on one-forms and for Bismut-type derivative formulas for the diffusion semigroup $\{P_t\}_{t\geq 0}$, see [3, 14].

On path space a canonical gradient operator is given in terms of $Q_{r,t}$. For any $F \in \mathscr{F}C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, the damped gradient $\tilde{D}_t F(X_{[0,T]}^x)$ is defined as

$$\tilde{D}_t F(X_{[0,T]}^x) = \sum_{i=1}^n \mathbb{1}_{\{t\leq t_i\}}\, Q_{t,t_i}\, //_{t,t_i}^{-1} \nabla_i f(X_{t_1}^x, \ldots, X_{t_n}^x), \quad t \in [0,T].$$

By estimating the damped gradient, a log-Sobolev inequality and a Poincaré inequality on path space can be obtained. Let us first introduce the following function: for any constants K_1, K_2 and c,

$$\Lambda^c(t,T,K_1,K_2) := \beta(t) + K_2 \int_0^t \beta(s)\, \mathrm{e}^{-(K_1+c)(t-s)}\, \mathrm{d}s,$$

where $\beta(t) = 1 + K_2 \int_t^T \mathrm{e}^{-(K_1-c)(s-t)}\, \mathrm{d}s$. Define

$$S(T,K_1,K_2) = \inf_{c\in\mathbb{R}} \sup_{t\in[0,T]} \Lambda^c(t,T,K_1,K_2).$$

Theorem 2.1. *Assume $k_1 \leq \mathrm{Ric}^Z \leq k_2$. Let*

$$H(T,k_1,k_2) := S(T,k_1,|k_1|\vee|k_2|) \wedge \left[S\left(T,k_1,\frac{k_2-k_1}{2}\right) S\left(T,\frac{k_2+k_1}{2},\frac{|k_2+k_1|}{2}\right)\right]. \tag{2.2}$$

Then for $F \in \mathscr{F}C_{0,T}^\infty$, we have

(i) $\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2H(T,k_1,k_2) \mathbb{E} \int_0^T |D_t F|^2\, \mathrm{d}t$;
(ii) $\mathbb{E}\big[F - \mathbb{E}[F]\big]^2 \leq H(T,k_1,k_2)\, \mathbb{E} \int_0^T |D_t F|^2\, \mathrm{d}t$.

First, let us introduce some functional inequalities on path space under pinched curvature condition, which extend the estimates in [24]. For $F \in \mathscr{F}C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, we define a modified gradient as

$$\hat{D}_t F(X_{[0,T]}^x) = \sum_{i=1}^n \mathbb{1}_{\{t\leq t_i\}}\, \mathrm{e}^{-\frac{k_1+k_2}{2}(t_i-t)}\, //_{t,t_i}^{-1} \nabla_i f(X_{t_1}^x, \ldots, X_{t_n}^x), \quad t \in [0,T].$$

In what follows, if there is no ambiguity, we write briefly $D_t F$, $\tilde{D}_t F$ and $\hat{D}_t F$ instead of $D_t F(X_{[0,T]})$, $\tilde{D}_t F(X_{[0,T]})$ and $\hat{D}_t F(X_{[0,T]})$.

Proposition 2.2. *Let (M,g) be a complete Riemannian manifold. Let k_1, k_2 be two real constants such that $k_1 \leq k_2$. The following conditions are equivalent:*

(i) $k_1 \leq \mathrm{Ric}^Z \leq k_2$;
(ii) *for any $F \in \mathscr{F}C_{0,T}^\infty$,*

$$\left|\nabla_x \mathbb{E} F(X_{[0,T]}^x)\right| \leq \mathbb{E}|\hat{D}_0 F| + \frac{k_2-k_1}{2} \int_0^T \mathrm{e}^{-k_1 s}\, \mathbb{E}|\hat{D}_s F|\mathrm{d}s;$$

(iii) *for any $F \in \mathscr{F}C_{0,T}^\infty$ and constant c,*

$$\left|\nabla_x \mathbb{E} F(X_{[0,T]}^x)\right|^2 \leq \left(1 + \frac{k_2-k_1}{2} \int_0^T \mathrm{e}^{-(k_1-c)s}\, \mathrm{d}s\right) \left(\mathbb{E}|\hat{D}_0 F|^2 + \frac{k_2-k_1}{2} \int_0^T \mathrm{e}^{-(k_1+c)s}\, \mathbb{E}|\hat{D}_s F|^2 \mathrm{d}s\right);$$

(iv) *for any $F \in \mathscr{F}C^\infty_{0,T}$, constant c and $t_1 < t_2$ in $[0,T]$,*

$$\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\Big]-\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\Big]$$
$$\leq 2\int_{t_1}^{t_2}\left(1+\frac{k_2-k_1}{2}\int_t^T \mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{d}s\right)\left(\mathbb{E}|\hat{D}_tF|^2+\frac{k_2-k_1}{2}\int_t^T \mathrm{e}^{-(k_1+c)(s-t)}\,\mathbb{E}|\hat{D}_sF|^2\mathrm{d}s\right)\mathrm{d}t;$$

(v) *for any $F \in \mathscr{F}C^\infty_{0,T}$, constant c and $t_1 < t_2$ in $[0,T]$,*

$$\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_2}]^2\Big]-\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_1}]^2\Big]$$
$$\leq \int_{t_1}^{t_2}\left(1+\frac{k_2-k_1}{2}\int_t^T \mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{d}s\right)\left(\mathbb{E}|\hat{D}_tF|^2+\frac{k_2-k_1}{2}\int_t^T \mathrm{e}^{-(k_1+c)(s-t)}\,\mathbb{E}|\hat{D}_sF|^2\mathrm{d}s\right)\mathrm{d}t.$$

Proof. (a) The following inequalities are well known (see [13] and [25, Chapter 4]). For convenience of the reader we include them with precise statements.

1) for $F \in \mathscr{F}C^\infty_{0,T}$, one has

$$\nabla_x\mathbb{E}[F(X^x_{[0,T]})]=\mathbb{E}[\tilde{D}_0F(X^x_{[0,T]})];$$

2) for $F \in \mathscr{F}C^\infty_{0,T}$, one has

$$\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\Big]-\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\Big]$$
$$\leq 2\mathbb{E}\int_{t_1}^{t_2}|\tilde{D}_tF(X_{[0,T]})|^2\,\mathrm{d}t;$$

3) for $F \in \mathscr{F}C^\infty_{0,T}$, one has

$$\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_2}]^2\Big]-\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_1}]^2\Big]\leq\mathbb{E}\int_{t_1}^{t_2}|\tilde{D}_tF(X_{[0,T]})|^2\,\mathrm{d}t.$$

Hence it suffices to estimate $|\tilde{D}_tF(X_{[0,T]})|$. For the sake of brevity, let $k=\frac{k_1+k_2}{2}$ and $\tilde{k}=\frac{k_2-k_1}{2}$. It is easy to see that

$$\begin{aligned}\tilde{D}_tF&=\hat{D}_tF+\sum_{i=1}^N \mathbb{1}_{\{t\leq t_i\}}\left(\mathrm{e}^{k(t_i-t)}\,Q_{t,t_i}-\mathrm{id}\right)\mathrm{e}^{-k(t_i-t)}\,//_{t,t_i}^{-1}\nabla_iF\\&=\hat{D}_tF+\int_t^T \mathrm{e}^{-k(s-t)}\,\frac{\mathrm{d}\left(\mathrm{e}^{k(s-t)}\,Q_{t,s}\right)}{\mathrm{d}s}//_{t,s}^{-1}\hat{D}_sF\,\mathrm{d}s.\end{aligned}$$

As

$$\frac{\mathrm{d}\left(\mathrm{e}^{k(s-t)}\,Q_{t,s}\right)}{\mathrm{d}s}=-\,\mathrm{e}^{k(s-t)}\,Q_{t,s}\left(\mathrm{Ric}^Z_{//_{t,s}}-k\,\mathrm{id}\right),$$

we get

$$\begin{aligned}\left|\tilde{D}_tF\right|&\leq\left|\hat{D}_tF\right|+\int_t^T\|Q_{t,s}\|\cdot\|(\mathrm{Ric}^Z)^\sharp-k\,\mathrm{id}\|\cdot|\hat{D}_sF|\,\mathrm{d}s\\&\leq\left|\hat{D}_tF\right|+\tilde{k}\int_t^T\mathrm{e}^{-k_1(s-t)}\,|\hat{D}_sF|\,\mathrm{d}s.\end{aligned}$$

It follows that

$$\left|\tilde{D}_tF\right|^2\leq\mathrm{e}^{2ct}\left(\mathrm{e}^{-ct}\left|\hat{D}_tF\right|+\tilde{k}\int_t^T\mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{e}^{-cs}\,|\hat{D}_sF|\,\mathrm{d}s\right)^2.$$

Thus, by Cauchy's inequality, we obtain

$$\begin{aligned}\big|\tilde{D}_tF\big|^2 &\le e^{2ct}\left(1+\int_t^T \tilde{k}\,e^{-(k_1-c)(s-t)}\,ds\right)\left(e^{-2ct}\big|\hat{D}_tF\big|^2+\int_t^T \tilde{k}\,e^{-(k_1-c)(s-t)-2cs}\,|\hat{D}_sF|^2\,ds\right)\\ &=\left(1+\int_t^T \tilde{k}\,e^{-(k_1-c)(s-t)}\,ds\right)\left(\big|\hat{D}_tF\big|^2+\int_t^T \tilde{k}\,e^{-(k_1+c)(s-t)}\,|\hat{D}_sF|^2\,ds\right).\end{aligned}$$

This allows to complete the proof of (i) implies (ii)–(v).

(b) Conversely, to prove (ii)–(v) $\Rightarrow$ (i), by a similar argument as in [24, 26], it suffices to prove that (iii) implies (i). Following [24], we first take $F(X^x_{[0,T]}) = f(X^x_t)$ as test functional. In this case, (iii) reduces to

$$|\nabla P_t f|^2(x) \le \left[\left(1+\tilde{k}\int_0^t e^{-(k_1-c)r}\,dr\right)\left(1+\tilde{k}\int_0^t e^{-(k_1+c-k)r}\,dr\right)e^{-2kt}\right] P_t|\nabla f|^2(x). \tag{2.3}$$

By means of the formula from [25, Theorem 2.2.4]:

$$\mathrm{Ric}^Z(\nabla f,\nabla f)(x) = \lim_{t\to 0}\frac{P_t|\nabla f|^2(x)-|\nabla P_t f|^2(x)}{2t},\quad f\in C_0^\infty(M),$$

we obtain the inequality $\mathrm{Ric}^Z \ge k_1$. Taking however $F(X^x_{[0,T]}) = f(x)-\frac{1}{2}f(X^x_t)$ as test functional, then (iii) reduces to the inequality:

$$\begin{aligned}\left|\nabla f(x)-\frac{1}{2}\nabla P_t f(x)\right|^2 &\le \left(1+\frac{k_2-k_1}{2}\int_0^t e^{-(k_1-c)s}\,ds\right)\\ &\quad\times\left(\mathbb{E}\big|\nabla f(x)-\frac{1}{2}e^{-kt}\,//_{0,t}^{-1}\nabla f(X^x_t)\big|^2+\frac{k_2-k_1}{8}\left(\int_0^t e^{-(c-k_2)s}\,ds\right)e^{-2kt}\,P_t|\nabla f|^2(x)\right).\end{aligned}$$

Expanding the last inequality, we arrive at

$$\begin{aligned}&|\nabla P_t f(x)|^2-\left[\left(1+\tilde{k}\int_0^t e^{-(k_1-c)r}\,dr\right)\left(1+\tilde{k}\int_0^t e^{-(c-k_2)r}\,dr\right)e^{-2kt}\right]P_t|\nabla f|^2(x)\\ &\quad\le 2(k_2-k_1)\int_0^t e^{-(k_1-c)s}\,ds\,|\nabla f(x)|^2+4\langle\nabla f(x),\nabla P_t f(x)\rangle\\ &\qquad -4\left(1+\frac{k_2-k_1}{2}\int_0^t e^{-(k_1-c)s}\,ds\right)e^{-kt}\big\langle\nabla f(x),\mathbb{E}//_{0,t}^{-1}\nabla f(X^x_t)\big\rangle.\end{aligned}\tag{2.4}$$

Then by [9, Lemma 2.5] it is straightforward to derive the upper bound $\mathrm{Ric}^Z \le k_2$. □

Remark 2.3. In our paper [9] we use a direct method which does not need to use the advanced theory on path space, to prove the result that the pinched curvature condition is equivalent to the coupled conditions (2.3) and (2.4) when $c=(k_1+k_2)/2$.

Proof of Theorem 2.1. The following inequalities are well known (see [13] and [25, Chapter 4]). For convenience of the reader we include them here, as we have done in the proof of Proposition 2.2.

1) for $F\in\mathscr{F}C^\infty_{0,T}$, one has

$$\mathbb{E}[F^2\log F^2]-\mathbb{E}[F^2]\log\mathbb{E}[F^2]\le 2\mathbb{E}\int_0^T|\tilde{D}_tF(X_{[0,T]})|^2\,dt;$$

2) for $F\in\mathscr{F}C^\infty_{0,T}$, one has

$$\mathbb{E}\big[F-\mathbb{E}[F]\big]^2\le\mathbb{E}\int_0^T|\tilde{D}_tF(X_{[0,T]})|^2\,dt.$$

Hence, it suffices to estimate $\mathbb{E}\int_0^T|\tilde{D}_tF|^2\,dt$ where $\tilde{D}_tF=\tilde{D}_tF(X_{[0,T]})$. By [24], we know that

$$|\tilde{D}_tF|\le\big|D_tF\big|+(|k_1|\vee|k_2|)\int_t^T e^{-k_1(s-t)}\,|D_sF|\,ds.$$

It follows that for any constant c, we have

$$\left|\tilde{D}_t F\right|^2 \le \mathrm{e}^{2ct}\left(\mathrm{e}^{-ct}\left|D_t F\right| + (|k_1|\vee|k_2|)\int_t^T \mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{e}^{-cs}\,|D_s F|\,\mathrm{d}s\right)^2.$$

Thus, by Cauchy's inequality, we obtain

$$\left|\tilde{D}_t F\right|^2 \le \left(1+(|k_1|\vee|k_2|)\int_t^T \mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{d}s\right)\left(|D_t F|^2 + (|k_1|\vee|k_2|)\int_t^T \mathrm{e}^{-(k_1+c)(s-t)}\,|D_s F|^2\,\mathrm{d}s\right). \tag{2.5}$$

Let

$$\alpha_1(t) = 1+(|k_1|\vee|k_2|)\int_t^T \mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{d}s.$$

Then, integrating both sides of Eq. (2.5) from 0 to T yields

$$\begin{aligned}\int_0^T \left|\tilde{D}_t F\right|^2 \mathrm{d}t &\le \int_0^T \alpha_1(t)\left(|D_t F|^2 + (|k_1|\vee|k_2|)\int_t^T \mathrm{e}^{-(k_1+c)(s-t)}\,|D_s F|^2\,\mathrm{d}s\right)\mathrm{d}t\\ &= \int_0^T\left(\alpha_1(t) + (|k_1|\vee|k_2|)\int_0^t \alpha_1(s)\,\mathrm{e}^{-(k_1+c)(t-s)}\,\mathrm{d}s\right)|D_t F|^2\,\mathrm{d}t\\ &= \int_0^T \Lambda^c(t,T,k_1,|k_1|\vee|k_2|)|D_t F|^2\,\mathrm{d}t\\ &\le S(T,k_1,|k_1|\vee|k_2|)\int_0^T |D_t F|^2\,\mathrm{d}t.\end{aligned}$$

We are now going to prove

$$\mathbb{E}\int_0^T |\tilde{D}_t F(X_{[0,T]})|^2\,\mathrm{d}t \le S\left(T,k_1,\frac{k_2-k_1}{2}\right)S\left(T,\frac{k_1+k_2}{2},\frac{|k_2+k_1|}{2}\right)\int_0^T \mathbb{E}|D_t F(X_{[0,T]})|^2\,\mathrm{d}t.$$

Our first step is to show that

$$\mathbb{E}\int_0^T |\tilde{D}_t F(X_{[0,T]})|^2\,\mathrm{d}t \le S\left(T,k_1,\frac{k_2-k_1}{2}\right)\int_0^T \mathbb{E}|\hat{D}_t F(X_{[0,T]})|^2\,\mathrm{d}t.$$

Recall the notations introduced above

$$\hat{D}_t F(X_{[0,T]}) := \sum_{i=1}^n \mathbb{1}_{\{t\le t_i\}}\,\mathrm{e}^{-\frac{k_1+k_2}{2}(t_i-t)}\,//_{t,t_i}^{-1}\nabla_i f(X_{t_1},\ldots,X_{t_n})$$

and $k := \frac{k_1+k_2}{2}$, $\tilde{k} := \frac{k_2-k_1}{2}$. By Proposition 2.2, for any constant c, we have

$$\left|\tilde{D}_t F\right|^2 \le \left(1+\int_t^T \tilde{k}\,\mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{d}s\right)\left(\left|\hat{D}_t F\right|^2 + \int_t^T \tilde{k}\,\mathrm{e}^{-(k_1+c)(s-t)}\,|\hat{D}_s F|^2\,\mathrm{d}s\right).$$

Integrating both sides from 0 to T yields

$$\int_0^T \left|\tilde{D}_t F\right|^2 \mathrm{d}t \le \int_0^T\left(1+\int_t^T \tilde{k}\,\mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{d}s\right)\left(\left|\hat{D}_t F\right|^2 + \int_t^T \tilde{k}\,\mathrm{e}^{-(k_1+c)(s-t)}\,|\hat{D}_s F|^2\,\mathrm{d}s\right)\mathrm{d}t.$$

Let $\alpha_2(t) = 1+\int_t^T \tilde{k}\,\mathrm{e}^{-(k_1-c)(s-t)}\,\mathrm{d}s$. Then

$$\begin{aligned}\int_0^T \left|\tilde{D}_t F\right|^2 \mathrm{d}t &\le \int_0^T \alpha_2(t)\left(\left|\hat{D}_t F\right|^2 + \int_t^T \tilde{k}\,\mathrm{e}^{-(k_1+c)(s-t)}\,|\hat{D}_s F|^2\,\mathrm{d}s\right)\mathrm{d}t\\ &= \int_0^T \alpha_2(t)\left|\hat{D}_t F\right|^2\mathrm{d}t + \int_0^T \alpha_2(t)\int_t^T \tilde{k}\,\mathrm{e}^{-(k_1+c)(s-t)}\,|\hat{D}_s F|^2\,\mathrm{d}s\,\mathrm{d}t\\ &= \int_0^T\left(\alpha_2(t)+\tilde{k}\int_0^t \alpha_2(s)\,\mathrm{e}^{-(k_1+c)(t-s)}\,\mathrm{d}s\right)\left|\hat{D}_t F\right|^2\mathrm{d}t\\ &= \int_0^T \Lambda^c(t,T,k_1,\tilde{k})\left|\hat{D}_t F\right|^2\mathrm{d}t.\end{aligned}$$

Therefore, we have

$$\int_0^T \left|\tilde{D}_t F\right|^2 \mathrm{d}t \le \inf_{c\in\mathbb{R}} \sup_{t\in[0,T]} \Lambda^c(t,T,k_1,\tilde{k}) \int_0^T \left|\hat{D}_t F\right|^2 \mathrm{d}t.$$

Our second step is to prove

$$\int_0^T \mathbb{E}\left|\hat{D}_t F\right|^2 \mathrm{d}t \le S(T,k,|k|) \int_0^T \mathbb{E}|D_t F|^2 \mathrm{d}t.$$

To this end, we first observe that

$$\left|\hat{D}_t F\right| = \left|\sum_{i=1}^N \mathbb{1}_{\{t\le t_i\}} \mathrm{e}^{-k(t_i-t)} //_{t,t_i}^{-1} \nabla_i F\right| \le |D_t F| + |k| \int_t^T \mathrm{e}^{-k(s-t)} |D_s F| \,\mathrm{d}s.$$

Let $\alpha_3(t) = 1 + |k| \int_t^T \mathrm{e}^{-(k-c)(s-t)} \,\mathrm{d}s$ for some constant c. We have

$$\begin{aligned}\int_0^T \left|\hat{D}_t F\right|^2 \mathrm{d}t &\le \int_0^T \left(|D_t F| + |k| \int_t^T \mathrm{e}^{-k(s-t)} |D_s F| \,\mathrm{d}s\right)^2 \mathrm{d}t \\ &\le \int_0^T \left(\alpha_3(t) + |k| \int_0^t \alpha_3(s) \,\mathrm{e}^{-(k+c)(t-s)} \,\mathrm{d}s\right) |D_t F|^2 \,\mathrm{d}s.\end{aligned}$$

It is easy to see that

$$\Lambda^c(t,T,k,|k|) = \alpha_3(t) + |k| \int_0^t \alpha_3(s) \,\mathrm{e}^{-(k+c)(t-s)} \,\mathrm{d}s.$$

Hence, we arrive at

$$\int_0^T \mathbb{E}\left|\tilde{D}_t F\right|^2 \mathrm{d}t \le S(T,k_1,\tilde{k}) S(T,k,|k|) \int_0^T \mathbb{E}|D_t F|^2 \,\mathrm{d}t, \qquad \square$$

which completes the proof of Theorem 2.1.

In the proof of Theorem 1.1 the function $\Lambda := \Lambda^0$ will play an important role. More precisely, for constants K_1 and K_2, we have

$$\Lambda(t,T,K_1,K_2) = \begin{cases} (1+\beta)^2 - (\beta+\beta^2)\,\mathrm{e}^{-K_1 t} - \dfrac{2\beta+\beta^2}{2}\,\mathrm{e}^{-K_1(T-t)} + \dfrac{\beta^2}{2}\,\mathrm{e}^{-K_1(T+t)}, & \text{if } K_1 \ne 0, \\ 1 + K_2 T + \dfrac{K_2^2}{2}(2Tt - t^2), & \text{if } K_1 = 0 \end{cases}$$

where $\beta = K_2/K_1$. We choose here the value $c = 0$, which seems to give the best asymptotics as $T \to 0$.

Proposition 2.4. *Let K_1 and K_2 be two constants such that $K_2 \ge 0$. Then*

$$C(T,K_1,K_2) = \sup_{t\in[0,T]} \Lambda(t,T,K_1,K_2),$$

where $C(T,K_1,K_2)$ is defined as in (1.2).

Proof. For the case $K_1 + K_2 \ge 0$, the reader is referred to [16, Proposition 3.3]. It suffices to deal with the remaining case $K_1 + K_2 < 0$. The idea is similar to the proof of [16, Proposition 3.3].

When $K_1 + K_2 < 0$ and $K_2 \ge 0$, we must have $K_1 < 0$. Taking derivative of Λ with respect to t, we obtain

$$\Lambda'(t,T,K_1,K_2) = \frac{K_1}{2}\,\mathrm{e}^{-K_1 t}\left[2(\beta+\beta^2) - \beta^2\,\mathrm{e}^{-K_1 T} - (2\beta+\beta^2)\,\mathrm{e}^{-K_1 T}\,\mathrm{e}^{2K_1 t}\right],$$

where $\beta = K_2/K_1$. From this it is easy to see that there exists at most one point t such that

$$\Lambda'(t,T,K_1,K_2) = 0.$$

In addition, for the boundary values $t=0,T$, we have

$$\Lambda'(0,T,K_1,K_2)=\beta(K_1+K_2)\left(1-\mathrm{e}^{-K_1T}\right)<0;$$
$$\Lambda'(T,T,K_1,K_2)=-K_2\left(1-\mathrm{e}^{-K_1T}\right)-\frac{K_2^2}{2K_1}\left(1-\mathrm{e}^{-K_1T}\right)^2>0.$$

Thus, we obtain that the maximal value of Λ over the interval $[0,T]$ is reached either at $t=0$ or at $t=T$. Moreover, by inspection it is easy to see that $\Lambda(0,T,K_1,K_2)\le\frac{1}{2}+\frac{1}{2}\Lambda^2(0,T,K_1,K_2)=\Lambda(T,T,K_1,K_2)$. All this taken together, we may conclude that

$$\sup_{t\in[0,T]}\Lambda(t,T,K_1,K_2)=\Lambda(T,T,K_1,K_2). \qquad \square$$

Proof of Theorem 1.1. From Theorem 2.1 we conclude that

$$\mathrm{gap}(\mathscr{L})^{-1}\le H(T,k_1,k_2). \tag{2.6}$$

Moreover, it is easy to be observed that

$$S(T,K_1,K_2)\le\sup_{t\in[0,T]}\Lambda(t,T,K_1,K_2)=C(T,K_1,K_2),$$

which allows to complete the proof of Theorem 1.1. $\square$

Proof of Theorem 1.3. We check the short-time behavior of $C(T,K_1,K_2)$ for $K_2\ge 0$ first. If $K_1>0$, then

$$\begin{aligned}C(T,K_1,K_2)&=(1+\beta)^2-\beta\sqrt{(2+\beta)(2+2\beta-\beta\,\mathrm{e}^{-K_1T})}\,\mathrm{e}^{-K_1T/2}\\&=(1+\beta)^2-\beta(2+\beta)\,\mathrm{e}^{-K_1T/2}\sqrt{1+\frac{\beta}{2+\beta}(1-\mathrm{e}^{-K_1T})}.\end{aligned}$$

Note that

$$\begin{aligned}\sqrt{1+\frac{\beta}{2+\beta}(1-\mathrm{e}^{-K_1T})}&=1+\frac{\beta}{2(2+\beta)}(1-\mathrm{e}^{-K_1T})-\frac{\beta^2}{8(2+\beta)^2}(1-\mathrm{e}^{-K_1T})^2+\mathrm{o}(T^2)\\&=1+\frac{\beta}{2(2+\beta)}\left(K_1T-\frac{1}{2}(K_1T)^2\right)-\frac{\beta^2}{8(2+\beta)^2}(K_1T)^2+\mathrm{o}(T^2)\\&=1+\frac{\beta}{2(2+\beta)}K_1T-\frac{\beta(4+3\beta)}{8(2+\beta)^2}(K_1T)^2+\mathrm{o}(T^2).\end{aligned}$$

Thus,

$$\begin{aligned}C(T,K_1,K_2)&=(1+\beta)^2-\beta(2+\beta)\left(1-\frac{1}{2}K_1T+\frac{1}{8}(K_1T)^2+\mathrm{o}(T^2)\right)\\&\qquad\times\left(1+\frac{\beta}{2(2+\beta)}K_1T-\frac{\beta(4+3\beta)}{8(2+\beta)^2}(K_1T)^2+\mathrm{o}(T^2)\right)\\&=1+K_2T+\left(1-\frac{(K_1+K_2)K_1}{(2K_1+K_2)K_2}\right)\frac{K_2^2T^2}{2}+\mathrm{o}(T^2).\end{aligned}$$

If $K_1<0$, then

$$\begin{aligned}C(T,K_1,K_2)&=\frac{1}{2}+\frac{1}{2}\left(1+\beta(1-\mathrm{e}^{-K_1T})\right)^2\\&=\frac{1}{2}+\frac{1}{2}\left(1+\beta K_1T-\beta\frac{(K_1T)^2}{2}+\mathrm{o}(T^2)\right)^2\\&=1+K_2T+\left(1-\frac{K_1}{K_2}\right)\frac{(K_2T)^2}{2}+\mathrm{o}(T^2).\end{aligned}$$

Hence, for $C(T,k_1,|k_1|\vee|k_2|)$, we obtain

$$C(T,k_1,|k_1|\vee|k_2|)=\begin{cases}1+k_2T+\dfrac{k_2^2}{2}T^2-\dfrac{k_1k_2(k_1+k_2)}{2(2k_1+k_2)}T^2+\mathrm{o}(T^2), & k_1\geq 0,\\ 1+k_2T+\dfrac{k_2^2}{2}T^2-\dfrac{k_1k_2}{2}T^2+\mathrm{o}(T^2), & k_1+k_2\geq 0 \text{ and } k_1<0,\\ 1-k_1T+\dfrac{k_1^2}{2}T^2+\dfrac{k_1^2}{2}T^2+\mathrm{o}(T^2), & k_1+k_2<0.\end{cases}$$

We now turn to estimate $C\big(T,k_1,\frac{k_2-k_1}{2}\big)C\big(T,\frac{k_1+k_2}{2},\frac{|k_2+k_1|}{2}\big)$.

(i) When $k_1+k_2<0$, we have

$$\begin{aligned}&C\left(T,k_1,\frac{k_2-k_1}{2}\right)C\left(T,\frac{k_2+k_1}{2},-\frac{k_2+k_1}{2}\right)\\&=1+k_2T+\frac{k_2^2}{2}T^2+\frac{3k_1^2+k_2^2}{8}T^2+\mathrm{o}(T^2);\end{aligned}$$

(ii) when $k_1+k_2\geq 0$ and $k_1\leq 0$,

$$\begin{aligned}&C\left(T,k_1,\frac{k_2-k_1}{2}\right)C\left(T,\frac{k_2+k_1}{2},\frac{k_2+k_1}{2}\right)\\&=1+k_2T+\frac{k_2^2}{2}T^2+\frac{2k_1^2-k_2^2-5k_1k_2}{12}T^2+\mathrm{o}(T^2);\end{aligned}$$

(iii) when $k_1>0$,

$$\begin{aligned}&C\left(T,k_1,\frac{k_2-k_1}{2}\right)C\left(T,\frac{k_2+k_1}{2},\frac{k_2+k_1}{2}\right)\\&=1-k_1T+\frac{k_1^2}{2}T^2-\frac{(7k_1k_2+k_2^2)(k_1+k_2)}{12(3k_1+k_2)}T^2+\mathrm{o}(T^2).\end{aligned}$$

Summarizing the estimates above, we conclude that as $T\to 0$,

$$C(T,k_1,|k_1|\vee|k_2|) \text{ and } C\left(T,k_1,\frac{k_2-k_1}{2}\right)C\left(T,\frac{k_1+k_2}{2},\frac{|k_2+k_1|}{2}\right)$$

have the same first order term, i.e. coefficient of T, and we only need to compare the coefficients of T^2.

(i) If $k_1\geq 0$, then

$$-\frac{(7k_1k_2+k_2^2)(k_1+k_2)}{12(3k_1+k_2)}+\frac{k_1k_2(k_1+k_2)}{2(2k_1+k_2)}=-\frac{(k_2^2-k_1^2)(4k_1+k_2)k_2}{12(3k_1+k_2)(2k_1+k_2)}\leq 0.$$

(ii) If $k_1+k_2\geq 0$ and $k_1<0$, then

$$\frac{2k_1^2-k_2^2-5k_1k_2}{12}+\frac{k_1k_2}{2}=\frac{(2k_1-k_2)(k_1+k_2)}{12}\leq 0.$$

(iii) If $k_1+k_2<0$, then

$$\frac{3k_1^2+k_2^2}{8}-\frac{k_1^2}{2}=\frac{k_2^2-k_1^2}{8}<0.$$

From this we conclude that

$$C\left(T,k_1,\frac{k_2-k_1}{2}\right)C\left(T,\frac{k_1+k_2}{2},\frac{|k_2+k_1|}{2}\right)$$

has a smaller coefficient in T^2. The proof is then completed by using Theorem 1.1. □

3. Extension to the path space of evolving manifolds

In this section, our base space is a differentiable manifold carrying a geometric flow of complete Riemannian metrics, more precisely, a d-dimensional differential manifold M equipped with a family of complete Riemannian metrics $(g_t)_{t\in[0,T_c)}$ for some $T_c \in (0,\infty]$, which is C^1 in t.

Let ∇^t and Δ_t be the Levi-Civita connection and the Laplace-Beltrami operator associated with the metric g_t, respectively. Let $(Z_t)_{t\in[0,T_c)}$ be a $C^{1,\infty}$-family of vector fields. Consider the diffusion process X_t^x generated by $L_t = \Delta_t + Z_t$ (called L_t-diffusion process) starting from x at time 0, which is assumed to be non-explosive before T_c (see [22] for sufficient conditions).

It is well known (e.g. [2, 12]) that X_t^x solves the equation

$$\mathrm{d}X_t^x = \sqrt{2}\, u_t^x \circ \mathrm{d}B_t + Z_t(X_t^x)\mathrm{d}t, \quad X_0^x = x = \pi(u_0^x),$$

where B_t is an $\mathbb{R}^d$-valued Brownian motion on a filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Here u_t^x is a horizontal process above X_t^x taking values in the frame bundle over M, constructed in such a way that the parallel transports

$$//_{s,t} := u_t^x \circ (u_s^x)^{-1} \colon (T_{X_s^x}M, g_s) \to (T_{X_t^x}M, g_t), \quad s \leq t,$$

along the paths of X are isometries, see [2] for the construction, as well as Section 3 in [9] for some details.

By Itô's formula, for any $f \in C_0^2(M)$ and $t \in [0, T_c)$, the process

$$f(X_t^x) - f(x) - \int_0^t (L_r f)(X_r^x)\,\mathrm{d}r = \sqrt{2}\int_0^t \left\langle //_{0,r}^{-1} \nabla^r f(X_r^x), u_0^x\,\mathrm{d}B_r \right\rangle_0$$

is a martingale up to T_c, where $\langle\cdot,\cdot\rangle_0$ is the inner product on T_xM given by the initial metric g_0. In other words, X_t^x is a diffusion generated by L_t.

For the sake of brevity, we introduce the following notation: for $X, Y \in TM$ such that $\pi(X) = \pi(Y)$ let

$$\mathscr{R}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y \right\rangle_t - \frac{1}{2}\partial_t g_t(X,Y)$$

where Ric_t is the Ricci curvature tensor with respect to the metric g_t and $\langle\cdot,\cdot\rangle_t = g_t(\cdot,\cdot)$. In what follows, given functions ϕ, ψ on $[0,T_c)\times M$, we write $\psi \leq \mathscr{R}_t^Z \leq \phi$ if

$$\psi|X|_t^2 \leq \mathscr{R}_t^Z(X,X) \leq \phi|X|_t^2$$

holds for all $X \in TM$, where $|X|_t := \sqrt{g_t(X,X)}$.

Similarly to Eq. (2.1) we define a two-parameter family of multiplicative functionals $\{Q_{r,t}\}_{r\leq t}$ as solution to the following equation: for $0 \leq r \leq t < T_c$ let

$$\frac{\mathrm{d}Q_{r,t}}{\mathrm{d}t} = -Q_{r,t}\,\mathscr{R}^Z_{//_{r,t}}, \quad Q_{r,r} = \mathrm{id}, \tag{3.1}$$

where by definition

$$\mathscr{R}^Z_{//_{r,t}} := //_{r,t}^{-1} \circ \mathscr{R}_t^Z(X_t^x) \circ //_{r,t}.$$

For fixed $T \in (0,T_c)$, recall that W^T denotes the path space of M and

$$\mathscr{F}C_{0,T}^\infty = \left\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}),\ n \geq 1, 0 < t_1 < \ldots < t_n \leq T, f \in C_0^\infty(M^n) \right\}$$

the space of smooth cylindrical functions on W^T. For $F \in \mathscr{F}C_{0,T}^\infty$ we consider again different types of gradients:

(i) *intrinsic gradient*:

$$D_t F(X_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t\leq t_i\}}\, //_{t,t_i}^{-1}\, \nabla_i^{t_i} f(X_{t_1}, \ldots, X_{t_n}), \quad t \in [0,T];$$

(ii) *damped gradient*:

$$\tilde{D}_t F(X_{[0,T]}) = \sum_{i=1}^{n} \mathbb{1}_{\{t\le t_i\}}\, Q_{t,t_i}\, //_{t,t_i}^{-1}\, \nabla_i^{t_i} f(X_{t_1},\dots,X_{t_n}), \quad t\in[0,T];$$

(iii) *modified gradient*:

$$\hat{D}_t F(X_{[0,T]}) = \sum_{i=1}^{n} \mathbb{1}_{\{t\le t_i\}}\, \mathrm{e}^{-\frac{1}{2}\int_t^{t_i}(k_1+k_2)(r)\mathrm{d}r}\, //_{t,t_i}^{-1}\, \nabla_i^{t_i} f(X_{t_1},\dots,X_{t_n}), \quad t\in[0,T].$$

We again write briefly D_tF, $\tilde{D}_tF$ and $\hat{D}_tF$ instead of $D_tF(X_{[0,T]})$, $\tilde{D}_tF(X_{[0,T]})$ and $\hat{D}_tF(X_{[0,T]})$ if there is no ambiguity.

In terms of the intrinsic gradient D_t, the Ornstein-Uhlenbeck operator is defined as

$$\langle \mathscr{L}F,F\rangle = \mathbb{E}\int_0^T |D_sF|_s^2\,\mathrm{d}s.$$

Our aim is to give an estimate for the spectral gap of $\mathscr{L}$, denoted by $\mathrm{gap}(\mathscr{L})$. To this end, we use the Poincaré inequality and log-Sobolev inequality of the next theorem. For the precise statement some notation is required. Given three functions K_1, K_2 and c in $C([0,T];\mathbb{R})$, we define

$$\tilde{\Lambda}^c(t,T,K_1,K_2) = \alpha(t) + K_2(t)\int_0^t \alpha(s)\,\mathrm{e}^{-\int_s^t (K_1+c)(r)\mathrm{d}r}\,\mathrm{d}s$$

where

$$\alpha(t) = 1 + \int_t^T K_2(s)\,\mathrm{e}^{-\int_t^s (K_1-c)(r)\mathrm{d}r}\,\mathrm{d}s.$$

Furthermore let

$$\tilde{S}(T,K_1,K_2) = \inf_{c\in C([0,T])}\sup_{t\in[0,T]} \tilde{\Lambda}^c(t,T,K_1,K_2).$$

Note that if K_1,K_2,c are constants then

$$\tilde{\Lambda}^c(t,T,K_1,K_2) = \Lambda^c(t,T,K_1,K_2).$$

Analogously to Theorem 2.1 recall the following two inequalities.

Theorem 3.1. *Assume that there exist continuous functions k_1,k_2 such that for every vector field X,*

$$k_1(t)\,|X|_t^2 \le \mathscr{R}_t^Z(X,X) \le k_2(t)\,|X|_t^2, \quad t\in[0,T]. \tag{3.2}$$

Then,

(i) *for every cylindrical function $F\in\mathscr{F}C_{0,T}^\infty$,*

$$\mathbb{E}[F^2\log F^2] - \mathbb{E}[F^2]\log\mathbb{E}[F^2] \le 2\tilde{H}(T,k_1,k_2)\int_0^T \mathbb{E}|D_sF|_s^2\,\mathrm{d}s,$$

where

$$\tilde{H}(T,k_1,k_2) = \tilde{S}(T,k_1,|k_1|\vee|k_2|)\wedge\left[\tilde{S}\left(T,k_1,\frac{k_2-k_1}{2}\right)\tilde{S}\left(T,\frac{k_1+k_2}{2},\frac{|k_1+k_2|}{2}\right)\right];$$

(ii) *for every cylindrical function $F\in\mathscr{F}C_{0,T}^\infty$,*

$$\mathbb{E}\big[F-\mathbb{E}[F]\big]^2 \le \tilde{H}(T,k_1,k_2)\int_0^T \mathbb{E}|D_sF|_s^2\,\mathrm{d}s.$$

Similarly to Section 2, we need the characterizations of modified pinched curvature condition on path space to prove Theorem 3.1. In the following, we will use the notation:

$$\mathbb{E}^{(x,t)}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|\mathscr{F}_t,\,X_t=x].$$

Proposition 3.2. *Let $(M,g_t)_{t\in[0,T_c)}$ be a smooth manifold carrying a family of complete metrics g_t. Let k_1,k_2 be two continuous functions in $C([0,T_c);\mathbb{R})$ such that $k_1\le k_2$. For any $T\in(0,T_c)$, the following conditions are equivalent:*

(i) *for any* $t\in[0,T]$,
$$k_1(t)\le \mathscr{R}_t^Z\le k_2(t);$$
(ii) *for any* $F\in\mathscr{F}C_{0,T}^\infty$,
$$\left|\nabla_x^t\,\mathbb{E}^{(x,t)}(F(X_{[0,T]}))\right|_t\le \mathbb{E}^{(x,t)}(|\hat{D}_tF|_t)+\int_t^T\tilde{k}(s)\,\mathrm{e}^{-\int_t^s k_1(r)\mathrm{d}r}\,\mathbb{E}^{(x,t)}(|\hat{D}_sF|_s)\mathrm{d}s$$
where $\tilde{k}=(k_2-k_1)/2$;

(iii) *for any* $F\in\mathscr{F}C_{0,T}^\infty$ *and any continuous function* c *on* $[0,T]$,
$$\begin{aligned}\left|\nabla_x^t\,\mathbb{E}^{(x,t)}(F(X_{[0,T]}))\right|_t^2\le&\left(1+\int_t^T\tilde{k}(s)\,\mathrm{e}^{-\int_t^s(k_1(r)-c(r))\mathrm{d}r}\,\mathrm{d}s\right)\\&\times\left(\mathbb{E}^{(x,t)}(|\hat{D}_tF|_t^2)+\int_t^T\tilde{k}(s)\,\mathrm{e}^{-\int_t^s(k_1(r)+c(r))\mathrm{d}r}\,\mathbb{E}^{(x,t)}(|\hat{D}_sF|_s^2)\,\mathrm{d}s\right);\end{aligned}$$
(iv) *for any* $F\in\mathscr{F}C_{0,T}^\infty$, *any continuous function* c *on* $[0,T]$, *and any* $t_1<t_2$ *in* $[0,T]$,
$$\begin{aligned}&\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\Big]-\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\Big]\\&\quad\le 2\int_{t_1}^{t_2}\left(1+\int_t^T\tilde{k}(s)\,\mathrm{e}^{-\int_t^s(k_1(r)-c(r))\mathrm{d}r}\,\mathrm{d}s\right)\\&\qquad\times\left(\mathbb{E}|\hat{D}_tF|_t^2+\int_t^T\tilde{k}(s)\,\mathrm{e}^{-\int_t^s(k_1(r)+c(r))\mathrm{d}r}\,\mathbb{E}|\hat{D}_sF|_s^2\mathrm{d}s\right)\mathrm{d}t;\end{aligned}$$
(v) *for any* $F\in\mathscr{F}C_{0,T}^\infty$, *any continuous function* c *on* $[0,T]$, *and any* $t_1<t_2$ *in* $[0,T]$,
$$\begin{aligned}&\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_2}]^2\Big]-\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_1}]^2\Big]\\&\quad\le\int_{t_1}^{t_2}\left(1+\int_t^T\tilde{k}(s)\,\mathrm{e}^{-\int_t^s(k_1(r)-c(r))\mathrm{d}r}\,\mathrm{d}s\right)\\&\qquad\times\left(\mathbb{E}|\hat{D}_tF|_t^2+\int_t^T\tilde{k}(s)\,\mathrm{e}^{-\int_t^s(k_1(r)+c(r))\mathrm{d}r}\,\mathbb{E}|\hat{D}_sF|_s^2\mathrm{d}s\right)\mathrm{d}t.\end{aligned}$$

Remark 3.3. In case where $Z_t=0$, and thus $L_t=\Delta_t$, it has been proved in [17] that the inequalities (ii)–(v) in Proposition 3.2 with $c\equiv 0$ characterize solutions of the Ricci flow. More precisely, the condition $\mathscr{R}_t^0=0$, i.e.,
$$\partial_t g_t=2\mathrm{Ric}_t,\tag{3.3}$$
is equivalent to the inequalities of (ii)–(v) for $k_1=k_2=c=0$ and $\hat{D}_sF=D_sF$. Note that (3.3) describes backward Ricci flow which corresponds to forward Ricci flow if one passes to the new family of metrics $g_t':=g_{T-t}$ where time is running backwards, as is done in [17].

Proof of Propostion 3.2. By [7, Theorem 4.3] we know that for $F\in\mathscr{F}C_{0,T}^\infty$, one has
$$\left|\nabla_x^t\,\mathbb{E}^{(x,t)}(F(X_{[0,T]}))\right|_t\le\mathbb{E}^{(x,t)}|\tilde{D}_tF|_t,$$
and
$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\right]\\&\quad-\mathbb{E}\left[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\log\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\right]\le 2\int_{t_1}^{t_2}\mathbb{E}|\tilde{D}_sF|_s^2\,\mathrm{d}s.\end{aligned}$$
Analogously, by a similar discussion as in the proof of [7, Theorem 4.3], we have
$$\mathbb{E}\left[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_2}]^2\right]-\mathbb{E}\left[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_1}]^2\right]\le\int_{t_1}^{t_2}\mathbb{E}|\tilde{D}_sF|_s^2\,\mathrm{d}s.$$
Hence it suffices again to estimate $|\tilde{D}_tF|_t$.

Defining

$$\bar{k} = \frac{k_1 + k_2}{2} \quad \text{and} \quad \tilde{k} = \frac{k_2 - k_1}{2},$$

recall

$$\hat{D}_t F(X_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t \le t_i\}} \, \mathrm{e}^{-\int_t^{t_i} \bar{k}(r)\mathrm{d}r} \, //_{t,t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, \ldots, X_{t_n}).$$

Then, we have

$$\begin{aligned}
\tilde{D}_t F &= \hat{D}_t F + \sum_{i=1}^n \mathbb{1}_{\{t \le t_i\}} \left(\tilde{Q}_{t,t_i} - \mathrm{id}\right) \mathrm{e}^{-\int_t^{t_i} \bar{k}(r)\mathrm{d}r} \, //_{t,t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, \ldots, X_{t_n}) \\
&= \hat{D}_t F - \int_t^T Q_{t,s} \left(\mathscr{R}^Z_{//_{t,s}} - \bar{k}(s)\mathrm{id}\right) //_{t,s}^{-1} \hat{D}_s F \, \mathrm{d}s
\end{aligned} \tag{3.4}$$

where $\tilde{Q}_{t,s} = \mathrm{e}^{\int_t^s \bar{k}(r)\mathrm{d}r} Q_{t,s}$. Using similar arguments as in the proof of Proposition 2.2, we obtain "(i) $\Rightarrow$ (ii)–(v)".

Conversely, to prove "(ii)–(v) $\Rightarrow$ (i)", the essential part is to prove (iii) $\Rightarrow$ (i). The trick is again to use the test functionals $F(X_{[0,T]}) = f(X_t)$ and $F(X_{[0,T]}) = f(X_s) - \frac{1}{2} f(X_t)$. We refer the reader to [24, 9] for detailed calculations. □

Proof of Theorem 3.1. For convenience of the reader, we first recall that for $F \in \mathscr{F}C^\infty_{0,T}$, one has

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \le 2 \int_0^T \mathbb{E}|\tilde{D}_s F|_s^2 \, \mathrm{d}s$$

and

$$\mathbb{E}\big[F - \mathbb{E}[F]\big]^2 \le \int_0^T \mathbb{E}|\tilde{D}_s F|_s^2 \, \mathrm{d}s.$$

Hence it suffices to estimate $\int_0^T \mathbb{E}|\tilde{D}_s F|_s^2 \, \mathrm{d}s$. Under condition (3.2), we obtain the bounds

$$\left|\mathscr{R}^Z_t(X,X)\right| \le \big(|k_1| \vee |k_2|\big)(t) \, |X|_t^2$$

and

$$\mathscr{R}^Z_t(X,X) \ge k_1(t) |X|_t^2$$

for all $X \in TM$. Then

$$\begin{aligned}
\tilde{D}_t F &= D_t F + \sum_{i=1}^n \left(\int_t^{t_i} \frac{\mathrm{d}Q_{t,s}}{\mathrm{d}s} \mathrm{d}s\right) //_{t,t_i}^{-1} \nabla_i^{t_i} f(X_{t_1}, \ldots, X_{t_n}) \\
&= D_t F - \int_t^T Q_{t,s} \mathscr{R}^Z_{//_{t,s}} //_{t,s}^{-1} D_s F \, \mathrm{d}s
\end{aligned}$$

which implies that

$$|\tilde{D}_t F|_t \le |D_t F|_t + \int_t^T (|k_1| \vee |k_2|)(s) \, \mathrm{e}^{-\int_t^s k_1(r)\mathrm{d}r} |D_s F|_s \, \mathrm{d}s.$$

Using a similar argument as in the proof of Theorem 2.1, we arrive at

$$\int_0^T |\tilde{D}_t F|_t^2 \, \mathrm{d}t \le \int_0^T \tilde{\Lambda}^c(t, T, k_1, |k_1| \vee |k_2|) |D_t F|_t^2 \, \mathrm{d}t. \tag{3.5}$$

On the other hand, by Proposition 3.2, we have

$$|\tilde{D}_t F|_t^2 \le \left(1 + \int_t^T \tilde{k}(s) \, \mathrm{e}^{-\int_t^s (k_1 - c)(r)\mathrm{d}r} \, \mathrm{d}s\right) \left(|\hat{D}_t F|_t^2 + \int_t^T \tilde{k}(s) \, \mathrm{e}^{-\int_t^s (k_1 + c)(r)\mathrm{d}r} |\hat{D}_s F|_s^2 \, \mathrm{d}s\right). \tag{3.6}$$

Moreover, for $|\hat{D}_t F|_t$, it is easy to see that

$$|\hat{D}_t F|_t = \left| D_t F + \sum_{i=1}^n \mathbb{1}_{\{t \le t_i\}} \left(\mathrm{e}^{-\int_t^{t_i} \bar{k}(r)\mathrm{d}r} - 1 \right) //_{t,t_i}^{-1} \nabla_i^{t_i} f \right|_t$$
$$\le |D_t F|_t + \int_t^T |\bar{k}(s)| \, \mathrm{e}^{-\int_t^s \bar{k}(r)\mathrm{d}r} |D_s F|_s \, \mathrm{d}s. \tag{3.7}$$

Combining this with Eq. (3.6), and using similar arguments as in the proof of Theorem 2.1, we obtain

$$\int_0^T |\tilde{D}_t F|_t^2 \, \mathrm{d}t \le \tilde{S}\left(T, k_1, \frac{k_2 - k_1}{2}\right) \int_0^T \tilde{\Lambda}^c \left(t, T, \frac{k_2 + k_1}{2}, \frac{|k_2 + k_1|}{2}\right) |D_t F|_t^2 \, \mathrm{d}t.$$

From this and by means of Eq. (3.5), the proof is directly completed. □

The following result is a direct consequence of Theorem 3.1.

Theorem 3.4. *Assume that there exist two continuous functions k_1 and k_2 such that*

$$k_1(t) \, |X|_t^2 \le \mathscr{R}_t^Z(X, X) \le k_2(t) \, |X|_t^2, \quad t \in [0, T]$$

for any vector field X on M. Then

$$\mathrm{gap}(\mathscr{L})^{-1} \le \tilde{H}(T, k_1, k_2).$$

For the special case that k_1 and k_2 are constants, the following asymptotics hold as $T \to 0$:

(i) *for $k_1 \ge 0$,*

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 + k_2 T + \frac{1}{2}\left(k_2^2 - \frac{(7k_1 + k_2)(k_1 + k_2)k_2}{6(3k_1 + k_2)}\right) T^2 + \mathrm{o}(T^2);$$

(ii) *for $k_1 + k_2 \ge 0$ and $k_1 < 0$,*

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 + k_2 T + \frac{1}{2}\left(k_2^2 + \frac{2k_1^2 - k_2^2 - 5k_1 k_2}{6}\right) T^2 + \mathrm{o}(T^2);$$

(iii) *for $k_1 + k_2 < 0$,*

$$\mathrm{gap}(\mathscr{L})^{-1} \le 1 - k_1 T + \frac{1}{2}\left(k_1^2 + \frac{3k_1^2 + k_2^2}{4}\right) T^2 + \mathrm{o}(T^2).$$

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