

THE FARRELL–TATE AND BREDON HOMOLOGY FOR $\mathrm{PSL}_4(\mathbb{Z})$ VIA CELL SUBDIVISIONS

ANH TUAN BUI, ALEXANDER D. RAHM AND MATTHIAS WENDT

ABSTRACT. We provide some new computations of Farrell–Tate and Bredon (co)homology for arithmetic groups. For calculations of Farrell–Tate or Bredon homology, one needs cell complexes where cell stabilizers fix their cells pointwise. We provide two algorithms computing an efficient subdivision of a complex to achieve this rigidity property. Applying these algorithms to available cell complexes for $\mathrm{PSL}_4(\mathbb{Z})$ provides computations of Farrell–Tate cohomology for small primes as well as the Bredon homology for the classifying spaces of proper actions with coefficients in the complex representation ring.

1. INTRODUCTION

Understanding the structure of the cohomology of arithmetic groups is a very important problem with relations to number theory and various K-theoretic areas. Explicit cohomology computations usually proceed via the study of the actions of the arithmetic groups on their associated symmetric spaces, and recent years have seen several advances in algorithmic computation of equivariant cell structures for these actions. To approach computations of Farrell–Tate and Bredon (co)homology of arithmetic groups, one needs cell complexes having a *rigidity property*: Cell stabilizers must fix their cells pointwise. The known algorithms (using Voronoi decompositions and such techniques, cf. e.g. [7, 9]) do not provide complexes with this rigidity property, and both for the computation of Farrell–Tate cohomology (resp. the torsion at small prime numbers in group cohomology) of arithmetic groups as well as for the computation of Bredon homology, this lack of rigid cell complexes constitutes a significant bottleneck.

In theory, it is always possible to obtain this rigidity property via the barycentric subdivision. However, the barycentric subdivision of an n -dimensional cell complex can multiply the number of cells by $(n + 1)!$ and thus easily let the memory stack overflow. We provide two algorithms, called Rigid Facets Subdivision, cf. Section 2, and Virtually Simplicial Subdivision, cf. Section 3, as well as a combination of them (Hybrid Subdivision, cf. Section 4) which subdivide cell complexes for arithmetic groups such that stabilizers fix their cells pointwise, but only lead to a controlled increase (in terms of sizes of stabilizer groups) in the number of cells, avoiding an explosion of the data volume. An implementation of the algorithms, cf. [5], shows that cases like $\mathrm{PSL}_4(\mathbb{Z})$ can effectively be treated with it, using commonly available machine resources. For the sake of comparison, barycentric subdivision applied to the cell complex for $\mathrm{PSL}_4(\mathbb{Z})$ from [6] would produce 3540 times as many top-dimensional cells as Rigid Facets Subdivision does — see Table 1.

1.1. Computations of Farrell–Tate cohomology. Farrell–Tate cohomology is a modification of cohomology of arithmetic groups which is particularly suitable to investigate torsion related to finite subgroups (in particular, the torsion in cohomological degrees above the virtual cohomological dimension). While the known cell complexes for arithmetic groups can deal very well with the rational cohomology and torsion at primes which do not divide orders of finite subgroups, computations with these complexes run into serious trouble for small prime numbers because the differentials in the relevant spectral sequence are too complicated to evaluate. There is a suitable new technique called torsion subcomplex reduction, cf. [14], which produces significantly smaller cell complexes and therefore simplifies the equivariant spectral sequence calculations. To apply this simplification, however, one needs cell complexes with the abovementioned rigidity property. Applying Rigid Facets Subdivision to a cell complex for $\mathrm{PSL}_4(\mathbb{Z})$, we have computed the Farrell–Tate cohomology of $\mathrm{PSL}_4(\mathbb{Z})$, at the primes 3 and 5. These results can be found in Theorem 10 and Proposition 14. Since the

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TABLE 1. Numbers of cells in the individual dimensions of the studied non-rigid fundamental domain X for $\mathrm{PSL}_4(\mathbb{Z})$ from [6], its rigid facets subdivision $\mathrm{RFS}(X)$, its virtually simplicial subdivision $\mathrm{VSS}(X)$, its hybrid subdivision $\mathrm{HyS}(X)$ and its barycentric subdivision $\mathrm{BCS}(X)$.

Dimension	0	1	2	3	4	5	6
# Orbits in X	2	2	2	4	4	3	1
# Cells in X	304	1416	2040	1224	332	36	1
# Orbits in $\mathrm{RFS}(X)$	17	213	1234	3025	3103	1117	1
# Cells in $\mathrm{RFS}(X)$	4153	62592	268440	472272	370272	108096	96
# Orbits in $\mathrm{VSS}(X)$	17	219	1508	5082	8456	6720	2040
# Cells in $\mathrm{VSS}(X)$	4153	71952	390936	974304	1238688	783360	195840
# Orbits in $\mathrm{HyS}(X)$	17	213	1245	3095	3214	1180	12
# Cells in $\mathrm{HyS}(X)$	4153	62592	271416	483504	383520	114144	1152
# Cells in $\mathrm{BCS}(X)$	5353	110352	644136	1658304	2138688	1359360	339840

computation proceeds through a complete description of the reduced torsion subcomplex, we can compute the torsion above the virtual cohomological dimension in all degrees.

In the cases which are effectively of rank one (5-torsion in $\mathrm{PSL}_4(\mathbb{Z})$), we can check the results of the cohomology computation using torsion subcomplex reduction by comparing to a computation using Brown's formula.

1.2. Computations of Bredon homology. For any group G , Baum and Connes introduced a map from the equivariant K -homology of G to the K -theory of the reduced C^* -algebra of G , called the assembly map. For many classes of groups, it has been proven that the assembly map is an isomorphism; and the Baum–Connes conjecture claims that it is an isomorphism for all finitely presented groups G (counter-examples have been found only for stronger versions of the Baum–Connes conjecture). The assembly map is known to be injective for arithmetic groups. For an overview on the conjecture, see the monograph [13].

The geometric-topological side of Baum and Connes' assembly map, namely the equivariant K -homology, can be determined using an Atiyah–Hirzebruch spectral sequence with E_2 -page given by the Bredon homology $H_n^{\mathfrak{F}\mathrm{in}}(\underline{E}G; R_{\mathbb{C}})$ of the classifying space $\underline{E}G$ for proper actions with coefficients in the complex representation ring $R_{\mathbb{C}}$ and with respect to the system $\mathfrak{F}\mathrm{in}$ of finite subgroups of G . This Bredon homology can be computed explicitly, as described by Sanchez-Garcia [18, 19].

While for Coxeter groups with a small system of generators [19] and arithmetic groups of rank 2 [15], general formulae for the equivariant K -homology have been established, the only known higher-rank case to date is the example $\mathrm{SL}_3(\mathbb{Z})$ in [18]. Although there are by now considerably more arithmetic groups for which cell complexes have been worked out [6, 7, 9], no further computations of Bredon homology $H_n^{\mathfrak{F}\mathrm{in}}(\underline{E}G; R_{\mathbb{C}})$ have been done since 2008 because the relevant cell complexes fail to have the rigidity property required for Sanchez-Garcia's method. We discuss an explicit example, cf. Section 10, demonstrating that the rigidity property is essential for the computation of Bredon homology and cannot be circumvented by a different method.

Applying rigid facets subdivision to the cell complex for $\underline{\mathrm{E}}\mathrm{PSL}_4(\mathbb{Z})$ from [6], we obtain

$$H_n^{\mathfrak{F}\mathrm{in}}(\underline{\mathrm{E}}\mathrm{PSL}_4(\mathbb{Z}); R_{\mathbb{C}}) \cong \begin{cases} 0, & n \geq 4, \\ \mathbb{Z}, & n = 3, \\ 0, & n = 2, \\ \mathbb{Z}^4, & n = 1, \\ \mathbb{Z}^{25} \oplus \mathbb{Z}/2, & n = 0. \end{cases}$$

This is consistent with the rational homology of $\underline{\mathrm{B}}\mathrm{PSL}_4(\mathbb{Z})$ as inferred from the results of [6]. As further consistency checks, the authors have paid attention that the homology of the cell complex remains unchanged under our implementation of rigid facets subdivision, and that the equivariant Euler characteristic of $\underline{E}G$ vanishes before and after subdividing.

Organization of the paper. In Section 2, we provide the Rigid Facets Subdivision algorithm. In Section 3, we provide the Virtually Simplicial Subdivision algorithm. In Section 8, we apply the Rigid Facets Subdivision algorithm to a $\mathrm{PSL}_4(\mathbb{Z})$, and compare the result with a computation using Brown’s conjugacy classes cell complex. Finally in Section 10, we provide a counterexample in order to contradict the possibility to compute the Bredon homology from an arbitrary non-rigid cell complex.

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2. THE RIGID FACETS SUBDIVISION ALGORITHM

In this section, we discuss the rigid facets subdivision algorithm which rigidifies equivariant cell complexes. The core of the method is Algorithm 2, which is expected to run in reasonable time for input coming from cell complexes for arithmetic groups. The key fact which guarantees that rigid facets subdivision works, is Lemma 7 below.

Algorithm 1 Subdivide to get stabilizers which fix their cells pointwise

Input: An n -dimensional Γ -equivariant CW-complex X with finite cell stabilizers and a metric as in Remark 2.

Output: A rigidification of X (that is, an equivalent Γ -cell complex on which each stabilizer fixes its cell pointwise).

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for  $m$  running from 1 to  $n$  do
  for  $\sigma$  running through lifts of  $m$ -cells in  $\Gamma \backslash X^{(m)}$  do
    if  $\sigma$  is not rigid then
      Use Algorithm 2 or 3 to subdivide  $\sigma$  into a partition  $P$ , which is a union of rigid  $m$ -cells,
      disjoint up to boundaries, with a fundamental domain  $F$  for the  $\Gamma_\sigma$ -action on  $P$ .
      Run through all the  $(m + 1)$ -cells; if their boundaries contain  $\sigma$ ,
      then replace  $\sigma$  by its partition  $P$ .
      Replace the cell  $\sigma$  by  $F$  in  $\Gamma \backslash X^{(m)}$ .
    end if
  end for
end for

```

Definition 1. Following the notation in [4], we use the term Γ -equivariant CW-complex, or simply Γ -cell complex, to mean a CW-complex X on which a discrete group Γ acts cellularly, i.e., in such a way that the action induces a permutation of the cells of X . We say the cell complex is *rigid* if each element in the stabilizer of any cell fixes the cell pointwise.

Remark 2. The algorithm producing the subdivision of the Γ -equivariant CW-complex X only modifies combinatorial data, based on the barycentric subdivision of individual cells. We require X to come with a geometric realization, equipped with a metric such that each of the cells of X is convex, the restriction of the metric to each cell is $\mathrm{CAT}(0)$ and the cell stabilizers act by $\mathrm{CAT}(0)$ -isometries. We are not requiring that the metric is $\mathrm{CAT}(0)$ on the whole CW-complex. Note however that the examples we are most interested in, are those where Γ is an arithmetic group and the geometric realization of X is the associated symmetric space.

Definition 3. A *rigidification* \hat{X} of a Γ -cell complex X is a rigid Γ -cell complex \hat{X} with the same underlying topological space as X . The map passing through the underlying topological space is then a Γ -equivariant homeomorphism $\hat{X} \rightarrow X$ of Γ -spaces. Note that a Γ -equivariant homeomorphism of Γ -spaces does not need to preserve existing cell structure, so \hat{X} is allowed to have more cells than X .

The outer shell of the rigid facets subdivision is Algorithm 1, which subdivides (whenever necessary) representatives of cell orbits using Algorithm 2, respectively Algorithm 3.

Proposition 4. *Let Γ be a discrete group, and let X be a Γ -equivariant CW-complex having finitely many Γ -orbits and finite cell stabilizers. Assume furthermore that X is equipped with a metric as in Remark 2. Then Algorithm 1 finds a rigidification of X (with respect to the Γ -action). It terminates in finite time.*

Proof. The key step of the algorithm is proved by Lemma 7 below; the rest is a routine induction. Lemma 7 is the point where the convexity and isometry requirements are needed. By the finiteness assumptions for orbits and cell stabilizers, the loops are all deterministic over finite index sets. Each operation inside them takes finite time, cf. Corollary 8, whence the claim. \square

Observation 5. The outer shell Algorithm 1 can be used with any subdivision algorithm for the cells. In particular, replacing the use of Algorithm 2 by the barycentric subdivision in Algorithm 1, the claims of Proposition 4 still hold. However, as mentioned in the introduction, the reason for developing Algorithm 2 is to reduce the blow-up in the number of cells, so as to make the algorithm practically applicable to cell complexes for higher-rank arithmetic groups.

We now discuss the actual subdivision to rigidify cells, Algorithm 2.

Algorithm 2 — Rigid Facets Subdivision

Input: An m -cell σ with stabilizer group Γ_σ , with rigid faces, equipped with a metric as in Remark 2.

Output: A Γ_σ -equivariant set of rigid m -cells, disjoint up to boundaries, constituting a partition P of σ , together with a contractible fundamental domain F (a single m -cell) for the action of Γ_σ on P .

- Sort the $(m-1)$ -faces of σ into orbits $\{\{g_{jt}\sigma_j\}_t\}_j$ under the action of Γ_σ , where j is indexing the orbits and t is indexing the cells inside each orbit.
- Let T be the list containing the element $g_{11}\sigma_1$.

while $\#T < \#\{\text{orbits of } (m-1)\text{-faces of } \sigma\}$ **do**

Choose one cell $\tau \in \{g_{jt}\sigma_j\}$ in the next orbit $\{g_{jt}\sigma_j\}$ not yet represented in T ,

if the union of τ with the cells in T has vanishing naive Euler characteristic **then**

if $\{\tau\} \cup T$ is contractible (we can use Ellis' method [10] to check this) **then**

Construct the boundary S of $\{\tau\} \cup T$.

if S has the naive Euler characteristic of an $(m-2)$ -sphere **then**

if $\#T < \#\{\text{orbits of } (m-1)\text{-faces of } \sigma\} - 1$ **then**

Add the chosen cell τ to T .

else

if S is simply connected **then** Add the chosen cell τ to T .

end if

end if

end if

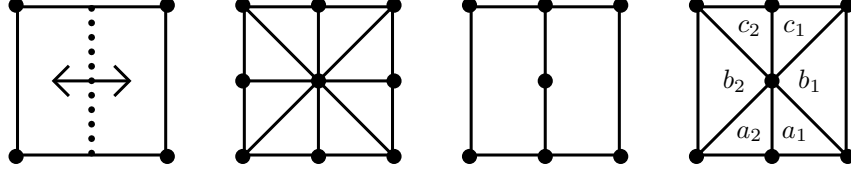
end if

end if

end while

- Use Algorithm 3 to construct the m -cell $F := |\bigcup_{\tau \in T} \text{convex envelope}(\tau, \text{barycenter}(\sigma))|$.
- Let Γ_σ^{pw} be the subgroup of Γ_σ which fixes the cell σ pointwise.
- Then $P := \bigcup_{1 \leq t \leq |\Gamma_\sigma / \Gamma_\sigma^{pw}|} g_{1t}F$ is the desired partition of σ .
- Return P and F .

FIGURE 1. From left to right: 1) A full square with natural cell structure and action by its stabilizer mirroring it onto itself. 2) Barycentric subdivision of the square. 3) Rigid facet subdivision of the square with respect to the mirroring action. 4) Virtually Simplicial Subdivision of the square with respect to the mirroring action; note that the fundamental domain F can be chosen arbitrarily by selecting one cell among each of the pairs $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$.



Remark 6. Essentially, Algorithm 2 produces a convex union of cells of the barycentric subdivision which is a fundamental domain for the Γ_σ -action. The slight complications arise from the fact that we do not actually want to compute the full barycentric subdivision, to gain computational feasibility.

Lemma 7 (Rigid Facets Lemma). *Let σ be a cell (with stabilizer Γ_σ), whose faces are all rigid and which is equipped with a metric as in Remark 2. Let Γ_σ^{pw} be the subgroup of Γ_σ which fixes the cell σ pointwise. Then there is a fundamental domain F for the action of $\Gamma_\sigma/\Gamma_\sigma^{pw}$ on σ such that σ is tessellated by $|\Gamma_\sigma/\Gamma_\sigma^{pw}|$ copies of F .*

Proof. First we have to check that the statement is well defined in the sense that $\Gamma_\sigma/\Gamma_\sigma^{pw}$ is a group. This is the case because for all g in Γ_σ , for all γ in Γ_σ^{pw} , for all x in σ we have $(g^{-1}\gamma g)x = g^{-1}(\gamma(gx)) = g^{-1}(gx) = x$. Therefore, as the kernel of the action of Γ_σ on σ , Γ_σ^{pw} is a normal subgroup; and there is a short exact sequence of groups,

$$1 \rightarrow \Gamma_\sigma^{pw} \rightarrow \Gamma_\sigma \rightarrow \Gamma_\sigma/\Gamma_\sigma^{pw} \rightarrow 1,$$

which makes our statement well defined.

Suppose that α is one of the facets of σ . We are going to prove that the size of the orbit of α , under the action of Γ_σ on the set of facets of σ , is $|\Gamma_\sigma/\Gamma_\sigma^{pw}|$. Let Γ_α be the stabilizer of α . We claim that $\Gamma_\alpha \cap \Gamma_\sigma = \Gamma_\sigma^{pw}$. The action of Γ_σ on the compact set $\text{closure}(\sigma)$ is by homeomorphisms; therefore, any element of Γ_σ fixing σ pointwise also fixes the boundary $\partial\sigma$ pointwise. Hence $\Gamma_\alpha \cap \Gamma_\sigma \supset \Gamma_\sigma^{pw}$. On the other hand, let $g \in \Gamma_\alpha \cap \Gamma_\sigma$. Then by assumption on the rigidity of the facets, g fixes the cell α pointwise. Since the cell σ is convex and the group acts by CAT(0)-isometries, the barycenter of σ preserves its distances to the boundary $\partial\sigma$ under the action of Γ_σ on σ , and hence remains fixed. As a further consequence of the CAT(0) isometry, the fixed point set of g extends, by preservation of the distances, from the convex envelope of α and the barycenter of σ to the whole cell σ . Hence, g is an element of Γ_σ^{pw} . Thus, we can conclude that $\Gamma_\alpha \cap \Gamma_\sigma = \Gamma_\sigma^{pw}$. Whence, the size of the orbit of α under the action of Γ_σ is $|\Gamma_\sigma/\Gamma_\sigma^{pw}|$.

Furthermore, from $\Gamma_\alpha \cap \Gamma_\sigma = \Gamma_\sigma^{pw}$, we see that $\Gamma_\sigma/\Gamma_\sigma^{pw}$ acts freely on the set of facets of σ . So, we can take one arbitrary representative α_k for each orbit of facets, to unite to a fundamental domain for $\Gamma_\sigma/\Gamma_\sigma^{pw}$ on the set of facets of σ . Taking the convex envelope e_k of α_k and the barycenter of σ , we get a fundamental domain $F := \bigcup_k e_k$ for $\Gamma_\sigma/\Gamma_\sigma^{pw}$ on σ . By the above orbit size calculation, it yields the desired tessellation. \square

Corollary 8. *Algorithm 2 terminates after finitely many steps and produces a rigid subdivision of the cell σ .*

Proof. The existence of the fundamental domain F for the action σ is guaranteed by Lemma 7. The contractibility of the fundamental domain T and the simply connectedness of its boundary S ensure that merging the union of cells

$$\bigcup_{\tau \in T} \text{convex envelope}(\tau, \text{barycenter}(\sigma))$$

into one cell F can be realized with the boundary construction $\partial F = \bigcup_{\tau \in T} \tau \cup \bigcup_{s \in S} e(s)$, where $e(s)$ is the convex envelope of s and the barycenter of σ . \square

Algorithm 3 Constructing the union of the convex envelopes

Input: A list T of j -cells, connected by adjacency,
and an m -cell σ such that all $\tau \in T$ are faces of σ .

Output: A $(j + 1)$ -cell F which has the same underlying topological space as

$$\bigcup_{\tau \in T} \text{convex envelope}(\tau, \text{barycenter}(\sigma)).$$

Record the barycenter of σ as a new vertex, with stabilizer Γ_σ .

Enumerate the finite set $S := \{\rho \text{ is } (j - 1)\text{-cell} \mid \exists! \tau \in T : \rho \in \partial\tau\}$.

Then S contains all the $(j - 1)$ -faces ρ of all $\tau \in T$ such that ρ is not a common face of any two j -cells $\tau_1, \tau_2 \in T$.

- For each $s \in S$, take the convex envelope $e(s)$ of s and the barycenter of σ .
- Record $e(s)$ as an oriented j -cell, with boundary

$$\partial e(s) = \{s\} \cup \bigcup_{\varepsilon \in \partial s} \{\text{convex envelope}(\varepsilon, \text{barycenter}(\sigma))\}.$$

For the stabilizers, we record $\Gamma_{e(s)} = \Gamma_s \cap \Gamma_\sigma$ and $\Gamma_{\text{convex envelope}(\varepsilon, \text{barycenter}(\sigma))} = \Gamma_\varepsilon \cap \Gamma_\sigma$.

The boundary of the $(j + 1)$ -cell F consists of all the j -faces $\tau \in T$ and $e(s)$ (for all $s \in S$).

Here, we have to take care of which of the newly constructed cells $e(s)$ are on the same Γ_σ -orbit.

In order to decide this, we make use of their common vertex, the barycenter of σ :

- Identify the orbits $e(s_1)$ and $e(s_2)$ if and only if $\exists \gamma \in \Gamma_\sigma : \gamma s_1 = s_2$.
- Attribute arbitrary orientations to Γ_σ -representatives of the new cells $e(s)$, and spread them on their Γ_σ -orbit using the above identifications.
- Return the $(j + 1)$ -cell F with boundary

$$\partial F = \bigcup_{\tau \in T} \tau \cup \bigcup_{s \in S} e(s)$$

and stabilizer $\Gamma_\sigma \cap \bigcap_{\tau \in T} \Gamma_\tau$, where the orbits are subject to the above specified identifications.

3. VIRTUALLY SIMPLICIAL SUBDIVISION

As a compromise between Rigid Facets Subdivision (RFS) and Barycentric Subdivision (BCS), we can make a simplification of RFS, which we shall call Virtually Simplicial Subdivision (VSS) and detail in Algorithm 3 below. For a cell complex X , $\text{VSS}(X)$ is a refinement of $\text{RFS}(X)$, and $\text{BCS}(X)$ is a further refinement of $\text{VSS}(X)$, as illustrated in Figure 1.

Algorithm 4 — Virtually Simplicial Subdivision

Input: An m -cell σ with stabilizer group Γ_σ , with rigid faces, equipped with a metric as in Remark 2.

Output: A Γ_σ -equivariant set of rigid m -cells, disjoint up to boundaries, constituting a partition P of σ , together with a fundamental domain F (a set of m -cells) for the action of Γ_σ on P .

- Sort the $(m - 1)$ -faces of σ into orbits $\{\{g_{jt}\sigma_j\}_t\}_j$ under the action of Γ_σ , where j is indexing the orbits and t is indexing the cells inside each orbit.
 - Use Algorithm 3 to construct the m -cells $F_j := \text{convex envelope}(g_{j1}\sigma_j, \text{barycenter}(\sigma))$, where j runs through all orbits.
 - Let Γ_σ^{pw} be the subgroup of Γ_σ which fixes the cell σ pointwise.
 - Then $P := \bigcup_j \bigcup_{1 \leq t \leq |\Gamma_\sigma / \Gamma_\sigma^{pw}|} g_{jt} F_j$ is the desired partition of σ .
 - Return P and the set $F := \{F_j\}_j$.
-

Corollary 9. *Algorithm 3 terminates after finitely many steps and produces a rigid subdivision of the cell σ .*

Proof. The existence of the fundamental domain F for the action σ is guaranteed by Lemma 7. We do not need any topological properties for F , because its cells already have the structure constructed with Algorithm 3. \square

TABLE 2. Time spent on a single processor for subdividing available cell complexes.

Arithmetic group	$SL_3(\mathbb{Z})$	Γ_1 in $SL_3(\mathbb{Z})$	Γ_2 in $SL_3(\mathbb{Z})$	$Sp_4(\mathbb{Z})$	$PSL_4(\mathbb{Z})$
Rigid Facets Subdivision	8s	17s	26s	24s	69 minutes
Virtually Simplicial Subdivision	9s	22s	33s	43s	111 minutes
Hybrid Subdivision	9s	20s	31s	33s	33 minutes
Barycentric Subdivision	13s	56s	104s	296s	?

4. HYBRID SUBDIVISION

The bulk of the processing time of Rigid Facets Subdivision (RFS) and Virtually Simplicial Subdivision (VSS) is being spent on the top-dimensional cells. We can save a considerable amount of processing time as compared to RFS if we do not merge the top-dimensional cells. And we can do that by applying RFS until we are in (top minus 1) dimensions, and then achieve the subdivision using VSS on the top-dimensional cells. We will call this process ‘‘Hybrid Subdivision (HyS)’’. It is faster than both RFS and VSS on large cell complexes, and it produces considerably less cells than VSS. We note however that the VSS process on the top-dimensional cells adds a few low-dimensional cells when connecting to the barycenter, so for instance in Table 5, the low-dimensional numbers of cells are slightly higher for HyS than for RFS, rather than being equal.

5. COMPARISON OF THE VARIOUS SUBDIVISION ALGORITHMS

As illustrated in Figure 1, all of the cells constructed by Rigid Facets Subdivision are also constructed by barycentric subdivision; and usually, Rigid Facets Subdivision is coarser than barycentric subdivision. However, the worst-case complexity of Rigid Facets Subdivision is not better than that of barycentric subdivision: If X is a single n -simplex with the natural permutation action of the symmetric group Σ_{n+1} acting on the vertices, then any rigidification will need to produce at least $(n + 1)! = \#\Sigma_{n+1}$ top-dimensional cells for X . However, the point is that the average cell complex for interesting arithmetic groups has many of its cells already almost rigid and, only very few with maximally possible stabilizer. Therefore, the complexity for the cases of interest is significantly better than that of the barycentric subdivision, as evidenced by Table 1. As we see from Figure 1, Rigid Facets Subdivision is a minimal rigidification only for the top-dimensional cells; in lower dimensions it creates some superfluous cells at the barycenters of facets, and cells connecting those barycenters with the constructed fundamental domain for the boundary of the subdivided cell.

In Table 1, note that we do not have the numbers of orbits for the barycentric subdivision, because the latter was not constructed; only the numbers of cells have been calculated. In such a construction, the boundaries of the 4-cells would span a 1658304×2138688 -matrix, and even though that matrix is quite sparse, the authors have not tried to store that amount of information on a machine. In retrospective however, considering that the amount of cells that have been constructed with Virtually Simplicial Subdivision comes close to the amount of cells expected for barycentric subdivision, it turns out that this issue would not prevent applying barycentric subdivision (which in the end was not done, because Table 2 was interpreted by the authors as an exponential growth of the runtime).

In contrast, for $SL_3(\mathbb{Z})$, the barycentric subdivision is almost reached (see Table 3) : In dimensions 0 and 1, the same cell structure is obtained with both subdivision methods, the only difference being that the four tetrahedra which represent the top-dimensional cells, are merged into one polyhedron when passing to the rigid facets subdivision, along three triangles of trivial stabilizers. Note also that Soulé did in [22] carry out the barycentric subdivision by hand. This means for the 2-torsion subcomplex of $SL_3(\mathbb{Z})$, which we will discuss in Section 6, that we can extract it equivalently by hand from [22], or by machine after rigid facets subdivision.

In order to benchmark the run-time, in Table 2, we denote the subgroup of $SL_3(\mathbb{Z})$ consisting of

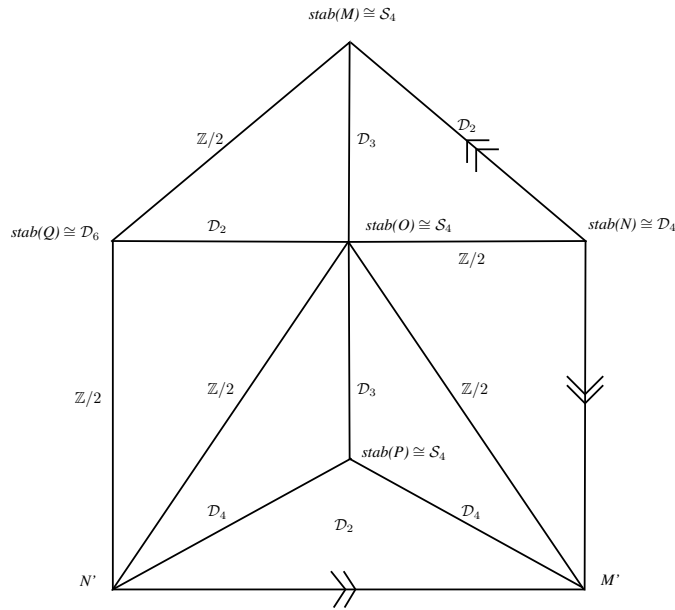
- (1) all matrices whose first column agrees with the first standard basis vector modulo 2 by Γ_1 ;
- (2) all matrices which are upper triangular modulo 2 by Γ_2 .

TABLE 3. Numbers of cells in the individual dimensions, of the studied non-rigid fundamental domain X for $SL_3(\mathbb{Z})$, its rigid facets subdivision $RFS(X)$, its virtually simplicial subdivision $VSS(X)$, its hybrid subdivision $HyS(X)$ and its barycentric subdivision $BCS(X)$ (the latter three coincide for $SL_3(\mathbb{Z})$).

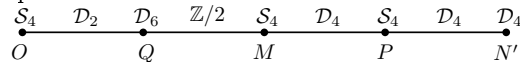
Dimension	0	1	2	3
# Orbits in X	1	1	2	1
# Cells in X	16	24	10	1
# Orbits in $RFS(X)$	5	11	8	1
# Cells in $RFS(X)$	51	194	168	24
# Orbits in $VSS(X)$, $HyS(X)$ and $BCS(X)$	5	11	11	4
# Cells in $VSS(X)$, $HyS(X)$ and $BCS(X)$	51	194	240	96

6. EXAMPLE: FARRELL–TATE COHOMOLOGY OF $SL_3(\mathbb{Z})$

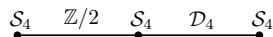
From the non-rigid cell complex describing the action of $SL_3(\mathbb{Z})$ on its symmetric space, provided by Mathieu Dutour Sikirić in [8], we obtain, after applying Rigid Facets Subdivision, the following 2-torsion subcomplex, in accordance with Soulé’s subdivision [22].



Here, the three edges NM , NM' and $N'M'$ have to be identified as indicated by the arrows. All of the seven triangles belong with their interior to the 2-torsion subcomplex, each with stabilizer $\mathbb{Z}/2$, except for the one which is marked to have stabilizer \mathcal{D}_2 . Using torsion subcomplex reduction (cf. [14]), we reduce this subcomplex to



and then to



which is the geometric realization of Soulé’s diagram of cell stabilizers. This yields the mod 2 Farrell cohomology as specified in [22].

Also for $PSL_4(\mathbb{Z})$, Rigid Facets Subdivision allows us to obtain the 2-torsion subcomplex, but because of the complexity the latter (see Table 4), the current implementation of Torsion Subcomplex Reduction does not get it down to manageable size. In upcoming joint work, the authors are going to

TABLE 4. Numbers of cells in the 2-torsion subcomplex for $\mathrm{PSL}_4(\mathbb{Z})$ before reduction, obtained after rigid facets subdivision of the studied cell complex, sorted into isomorphism types of their stabilizers. Here, G_1 and G_2 are non-trivial group extensions $1 \rightarrow \mathcal{A}_4 \times \mathcal{A}_4 \rightarrow G_1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$ and $1 \rightarrow G_0 \rightarrow G_2 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$ for $1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^4 \rightarrow G_0 \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow 1$

Stabilizer type	\mathcal{A}_4	G_1	$(\mathbb{Z}/2\mathbb{Z})^2$	\mathcal{S}_4	$\mathbb{Z}/2\mathbb{Z}$	\mathcal{D}_8	$(\mathbb{Z}/2\mathbb{Z})^3$	G_2	$\mathbb{Z}/4\mathbb{Z}$
Vertices	2	1	5	4	1	2	1	1	0
Edges	2	0	24	2	101	5	1	0	5
2-cells	0	0	27	0	326	0	0	0	4
3-cells	0	0	8	0	340	0	0	0	0
4-cells	0	0	0	0	116	0	0	0	0

TABLE 5. Numbers of cells in the individual dimensions, of the non-rigid fundamental domain X for $\mathrm{Sp}_4(\mathbb{Z})$ described in [12] and implemented by Dutour Sikirić [8], its rigid facets subdivision $\mathrm{RFS}(X)$, its virtually simplicial subdivision $\mathrm{VSS}(X)$, its hybrid subdivision $\mathrm{HyS}(X)$ and its barycentric subdivision $\mathrm{BCS}(X)$.

Dimension	0	1	2	3	4
In X : # Orbits # Cells	2 76	2 216	3 180	3 40	1 1
In $\mathrm{RFS}(X)$: # Orbits # Cells	8 185	31 800	54 1048	30 448	1 16
In $\mathrm{VSS}(X)$: # Orbits # Cells	8 185	39 1152	106 2536	122 2352	49 784
In $\mathrm{HyS}(X)$: # Orbits # Cells	8 185	35 864	72 1336	53 816	10 160
In $\mathrm{BCS}(X)$: # Orbits # Cells	11 513	90 3488	295 7904	368 7392	154 2464

present a new implementation of Torsion Subcomplex Reduction, which will overcome this problem using Discrete Morse Theory.

7. EXAMPLE: FARRELL-TATE COHOMOLOGY OF $\mathrm{Sp}_4(\mathbb{Z})$ AT ODD PRIMES

We shortly discuss the results of applying rigid facets subdivision and torsion subcomplex reduction (for odd primes) to the non-rigid cell complex describing the action of $\mathrm{Sp}_4(\mathbb{Z})$ on the associated symmetric space $\mathrm{Sp}_4(\mathbb{R})/\mathrm{U}(2)$, described in [12] and implemented by Mathieu Dutour Sikirić in [8].

The 3-torsion subcomplex for $\mathrm{Sp}_4(\mathbb{Z})$ is a single vertex whose stabilizer group is the group [72, 30] in the GAP `SmallGroup` library. This group is of the form $C_3 \times ((C_6 \times C_2) \rtimes C_2)$ and its mod 3 cohomology is isomorphic to the mod 3 cohomology of $C_3 \times S_3$. In particular, we get isomorphisms

$$H^i(\mathrm{Sp}_4(\mathbb{Z}), \mathbb{F}_3) \cong H^i(C_3 \times S_3, \mathbb{F}_3), \quad i > \mathrm{vcd}(\mathrm{Sp}_4(\mathbb{Z})).$$

The Hilbert–Poincaré series for the mod 3 cohomology of $C_3 \times S_3$ is given by

$$\mathrm{HP}_{C_3 \times S_3}(T; 3) = \frac{(1+T)(1+T^3)}{(1-T^2)(1-T^4)}.$$

In [3, (6.7)], the Hilbert–Poincaré-series for $\mathrm{Sp}_4(\mathbb{Z})$ has been computed. In the notation of loc.cit., the difference to the Hilbert–Poincaré-series for $C_3 \times S_3$ above is given by

$$P(\Gamma_0^6) - P(B' \rtimes \mathbb{Z}/2) = \frac{1+T^3+T^4+T^5}{1-T^4} - \frac{1+T+T^3+T^4}{1-T^4} = \frac{T^5-T}{1-T^4} = -T.$$

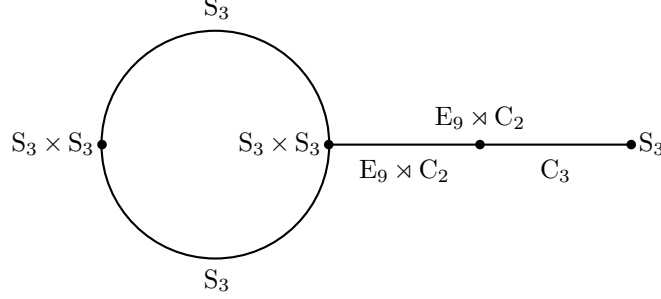
In particular, the Hilbert–Poincaré series for $C_3 \times S_3$ and $\mathrm{Sp}_4(\mathbb{Z})$ agree above the virtual cohomological dimension of $\mathrm{Sp}_4(\mathbb{Z})$ and thus the computation via rigid facets subdivision and torsion subcomplex reduction agrees with the computation of Brownstein and Lee in [3].

The 5-torsion subcomplex for $\mathrm{Sp}_4(\mathbb{Z})$ is a single vertex whose stabilizer group is C_{10} , whose mod 5 cohomology is isomorphic to the mod 5 cohomology of C_5 . This agrees directly with [3, Corollary 6.3] in degrees above the virtual cohomological dimension.

Table 5 provides an overview of the numbers of cells resp. orbits in the complexes subdivided by rigid facets subdivision and barycentric subdivision, respectively.

8. EXAMPLE: FARRELL–TATE COHOMOLOGY OF $\mathrm{PSL}_4(\mathbb{Z})$ AT THE PRIME 3

Applying the rigid facets subdivision algorithm to the $\mathrm{PSL}_4(\mathbb{Z})$ -equivariant cell complex from [6], extracting the 3-torsion subcomplex, and reducing it using the methods of [14], we get the following graph of groups \mathcal{T} , decorated with the groups stabilizing the cells that are the pre-images of the projection to the quotient space.



Here, $E_9 \rtimes C_2$ denotes the semi-direct product of the elementary abelian group of order 9 with C_2 , where C_2 acts by inversion (corresponding to the group [18,4] in GAP's `SmallGroups` library). The machine computation provided the following system of morphisms among the above cell stabilizers. The $E_9 \rtimes C_2$ edge stabilizer admits an isomorphism of groups ϕ (not the identity, though) to the $E_9 \rtimes C_2$ vertex stabilizer and an inclusion into the $S_3 \times S_3$ vertex stabilizer. Of the two S_3 edge stabilizers, one has maps $\mathrm{diag}(1, 1)$ and $\mathrm{diag}(1, 0)$ to the two $S_3 \times S_3$ vertex stabilizers, and the other one has maps $\mathrm{diag}(1, -1)$ and $\mathrm{diag}(-1, -1)$ to the two $S_3 \times S_3$ vertex stabilizers. The C_3 edge stabilizer admits an inclusion into the $E_9 \rtimes C_2$ vertex stabilizer, and an inclusion into the S_3 vertex stabilizer.

By the properties of torsion subcomplex reduction, the $\mathrm{PSL}_4(\mathbb{Z})$ -equivariant cohomology of the 3-torsion subcomplex is isomorphic to the $\mathrm{PSL}_4(\mathbb{Z})$ -equivariant cohomology of the above graph of groups \mathcal{T} . Similarly, the Farrell–Tate cohomology of $\mathrm{PSL}_4(\mathbb{Z})$ at the prime 3 is isomorphic to the Farrell–Tate cohomology of the above graph of groups. In the following, we evaluate the isotropy spectral sequence

$$E_1^{p,q} = \bigoplus_{\sigma \in \mathcal{T}_p} H^q(\mathrm{Stab}(\sigma); \mathbb{F}_3) \Rightarrow \widehat{H}^{p+q}(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{F}_3)$$

converging to Farrell–Tate cohomology. As we only consider a graph, the spectral sequence is concentrated in the two columns $p = 0, 1$. The differential d_1 is induced from the inclusions of subgroups, up to the sign coming from the choice of orientation of the graph. Since the spectral sequence is only concentrated in the first two columns, we will have $E_2 = E_\infty$. Since we are interested in field coefficients, there are no extension problems to solve at the E_∞ -page.

The relevant cohomology groups of the finite groups are:

- $H^\bullet(C_3; \mathbb{F}_3) \cong \mathbb{F}_3[x](a)$ with $\deg a = 1$ and $\deg x = 2$.
- $H^\bullet(S_3; \mathbb{F}_3) \cong \mathbb{F}_3[y](b)$ with $\deg b = 3$ and $\deg y = 4$.
- By the Künneth formula, $H^\bullet(S_3 \times S_3; \mathbb{F}_3) \cong H^\bullet(S_3; \mathbb{F}_3)^{\otimes 2}$.
- By the Hochschild–Serre spectral sequence, $H^\bullet(E_9 \rtimes C_2; \mathbb{F}_3) \cong \mathbb{F}_3[x_1, x_2](a_1, a_2)^{C_2}$ where $\deg x_i = 2$, $\deg a_i = 1$ and C_2 acts by multiplication with -1 on all the generators.

To describe the d_1 -differential, it is enough to note that the restriction map associated to the inclusion $C_3 \hookrightarrow S_3$ is the inclusion of C_2 -invariants.

Now, for the evaluation of the spectral sequence, we first deal with the edges attached to the loop.

- (1) The restriction map

$$\phi \oplus (\mathrm{Res}_{C_3}^{S_3} \circ \mathrm{pr}_2^*) : H^\bullet(E_9 \rtimes C_2; \mathbb{F}_3) \rightarrow H^\bullet(E_9 \rtimes C_2; \mathbb{F}_3) \oplus H^\bullet(C_3; \mathbb{F}_3)$$

is injective, and the cokernel is isomorphic to $H^\bullet(C_3; \mathbb{F}_3)$. Here ϕ denotes the isomorphism $E_9 \rtimes C_2 \cong E_9 \times C_2$ appearing as stabilizer inclusion in the reduced torsion subcomplex.

- (2) The inclusion of the dihedral vertex group into the cyclic edge group is an injection

$$\mathrm{Res}_{C_3}^{S_3} : H^\bullet(S_3; \mathbb{F}_3) \hookrightarrow H^\bullet(C_3; \mathbb{F}_3)$$

given by the inclusion of the invariant elements for the C_2 -action by -1 . Therefore, the cokernel is concentrated in degrees $1, 2 \pmod 4$ (except for the degree 0).

Therefore, we can reduce the E_1 -page of the spectral sequence as follows: from (1), we find that we can remove the two summands for $E_9 \rtimes C_2$ from the columns $p = 0$ and $p = 1$, respectively; but in turn, we have to replace the restriction map for $E_9 \rtimes C_2 \hookrightarrow S_3 \times S_3$ by a map from the cohomology of $S_3 \times S_3$ to the cokernel of the restriction map in (2). However, since the latter is concentrated in degrees 1 and 2, the induced restriction map is trivial in the generating degrees 3 and 4, hence it is trivial. Therefore, the E_1 -page decomposes: one contribution comes from the cokernel of the restriction map $H^\bullet(S_3; \mathbb{F}_3) \hookrightarrow H^\bullet(C_3; \mathbb{F}_3)$, the other contribution comes from the loop connecting the two copies of $S_3 \times S_3$ via the S_3 -edges.

The cohomology of the loop is computed as follows:

(3) The restriction maps $\text{diag}(1, 1), \text{diag}(1, 0), \text{diag}(1, -1)$ and $\text{diag}(-1, -1)$:

$$\mathbb{F}_3[x_4, y_4](a_3, b_3) \cong H^\bullet(S_3 \times S_3; \mathbb{F}_3) \rightarrow H^\bullet(S_3; \mathbb{F}_3) \cong \mathbb{F}_3[z_4](c_3)$$

are surjective. On the kernel of $\text{diag}(1, 1)$, the restriction of $\text{diag}(1, -1)$ is still surjective. On the kernel of $\text{diag}(-1, -1)$, the restriction of $\text{diag}(1, 0)$ is still surjective.

Therefore, the cohomology of one edge of the loop is killed already by the restriction map from any one of the vertex groups. Number the $S_3 \times S_3$ -vertices by 1 and 2, and the S_3 -edges by a and b . The restriction from the vertex group 1 to the edge a is surjective. Removing this part from the spectral sequence, the restriction from the vertex group 2 to the edge a is trivial, but we still have the restriction to the edge b . This kills the edge cohomology b , showing that the differential d_1 is surjective in the loop part of the E_1 -page. The kernel of the differential consists then exactly of two copies of the kernel of a restriction map $\text{Res}_{S_3}^{S_3 \times S_3}$.

Theorem 10. *The Farrell–Tate cohomology of the group $\text{PSL}_4(\mathbb{Z})$ (with coefficients in \mathbb{F}_3) in degrees ≥ 2 is given as follows:*

$$\widehat{H}^\bullet(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_3) \cong \left(\ker \text{Res}_{S_3}^{S_3 \times S_3} \right)^{\oplus 2} \oplus \text{coker} \left(H^{\bullet-1}(S_3; \mathbb{F}_3) \rightarrow H^{\bullet-1}(C_3; \mathbb{F}_3) \right).$$

This, in particular also computes the 3-torsion group cohomology of $\text{PSL}_4(\mathbb{Z})$ above the virtual cohomological dimension.

The cokernel of the d_1 -differential in degree 0 and hence the first cohomology $\widehat{H}^1(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_3)$ is of \mathbb{F}_3 -rank 1, coming from the loop of the 3-torsion graph.

Remark 11. The kernel comes from the $p = 0$ column of the $E_2 = E_\infty$ -page. The cokernel comes from the $p = 1$ column and consequently has a shift. The cup product on the kernel is the one induced from cohomology of $S_3 \times S_3$, and cup-product with classes in the cokernel is always 0.

Inspired by Grunewald, we consider the Hilbert–Poincaré series of the Farrell–Tate cohomology of $\text{PSL}_4(\mathbb{Z})$ with coefficients in \mathbb{F}_ℓ :

$$\text{HP}_{\text{PSL}_4(\mathbb{Z})}(T; \ell) := \sum_{q=1}^{\infty} \dim \widehat{H}^q(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_\ell) \cdot T^q.$$

Corollary 12. *The Hilbert–Poincaré series of the 3-torsion Farrell–Tate cohomology of $\text{PSL}_4(\mathbb{Z})$ (for degrees ≥ 1) is then*

$$\text{HP}_{\text{PSL}_4(\mathbb{Z})}(T; 3) = T + \frac{2(T^3 + T^4 + T^6 + T^7)}{(1 - T^4)^2} + \frac{T^2 + T^3}{1 - T^4}.$$

Remark 13. Note that the above calculation describes the Farrell–Tate cohomology in all degrees, not just some small ones. Essentially, the computer calculation produces the reduced torsion subcomplex (which encodes the cohomology for all degrees). The spectral sequence is evaluated using the cup-product structure. Note that the finiteness results for group homology imply that the cup-product structure for both group and Farrell–Tate cohomology is finitely generated. Using suitable commutative algebra packages, such computations of the ring structure (and therefore additive computations for all cohomological degrees) could probably also be automated.

We make the following consideration on the compatibility of our result for Farrell–Tate cohomology with the result of Dutour–Ellis–Schürmann [6] for group homology in low degrees. The isomorphism types computed in the latter article are to correspond as follows to the evaluation of our above Hilbert–Poincaré series in those degrees.

$$H_q(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{Z}) \cong \begin{cases} 0, & q = 1, \\ (\mathbb{Z}/2)^3, & q = 2, \\ \mathbb{Z} \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/5, & q = 3, \\ (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/5, & q = 4, \\ (\mathbb{Z}/2)^{13}, & q = 5, \end{cases} \quad \text{---} \quad \dim \widehat{H}^q(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{F}_3) = \begin{cases} 1, & q = 1, \\ 1, & q = 2, \\ 3, & q = 3, \\ 2, & q = 4, \\ 0, & q = 5. \end{cases}$$

For this to be consistent, the Farrell–Tate cohomology groups in degrees 1 and 2 need to vanish in group homology; so, these should be annihilated by differentials from the orbit space. We have evidence for this in degree 1, since the loop in the graph becomes contractible in the orbit space of the full locally symmetric space. In degree 3, one of the summands in $H_3(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{Z})$ is rationally non-trivial and must come from the orbit space. This means that only the submodule $(\mathbb{Z}/3)^2$ can come from Farrell–Tate cohomology, and the third dimension that we observe in degree 3 Farrell–Tate cohomology must belong to the degree 2 stabilizer cohomology class that is annihilated by the above mentioned differentials from the orbit space.

From Theorem 10, we deduce that the degree 2 Farrell–Tate class can only come from

$$\mathrm{coker}(\mathrm{H}^{\bullet-1}(\mathrm{S}_3; \mathbb{F}_3) \rightarrow \mathrm{H}^{\bullet-1}(\mathrm{C}_3; \mathbb{F}_3)).$$

Then, this class and its group homology counterpart sit at position $p = 1, q = 1$ in the respective equivariant spectral sequence, and hence the annihilating differential, emanating from the orbit space homology module $\mathbb{Z} \subset H_3(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{Z})$ sitting at position $p = 3, q = 0$, must be of second degree.

In degrees 4 and 5, the dimensions already agree via the Universal Coefficient Theorem, so here we infer that the submodule $(\mathbb{Z}/3)^2$ in degree 3 should actually come from Farrell–Tate cohomology, so it should be stabilizer cohomology that is not hit by higher degree differentials.

9. HOMOLOGICAL 5-TORSION IN $\mathrm{PSL}_4(\mathbb{Z})$

We applied the rigid facets subdivision algorithm to the $\mathrm{PSL}_4(\mathbb{Z})$ -equivariant cell complex of [6], extracted the 5-torsion subcomplex, and reduced it using the methods of [14] to the following graph



The d^1 -differential of the equivariant spectral sequence on \mathcal{T} is zero, because the isomorphisms at edge end and edge origin cancel each other. Then the $E_1 = E_\infty$ page is concentrated in the columns $p = 0$ and 1, with dimensions over \mathbb{F}_5 being 1 in rows q congruent to 3 or 4 mod 4, and zero otherwise. This yields

Proposition 14. *We observe on Farrell cohomology: $\dim_{\mathbb{F}_5} \widehat{H}^{p+q}(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{F}_5) = \begin{cases} 1, & p+q \equiv 1 \pmod{4}, \\ 0, & p+q \equiv 2 \pmod{4}, \\ 1, & p+q \equiv 3 \pmod{4}, \\ 2, & p+q \equiv 4 \pmod{4}. \end{cases}$*

We check this result with a computation of $\widehat{H}^{\bullet}(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{F}_5)$ using Brown’s complex [2, last chapters]. In this case, it is standard that the set $\{1, \zeta_5, \zeta_5^3, \zeta_5^4\}$ is an integral basis of $\mathcal{O}_{\mathbb{Q}(\zeta_5)}$ and in particular $\mathbb{Z}[\zeta_5] = \mathcal{O}_{\mathbb{Q}(\zeta_5)}$ is a Dedekind ring.

We can therefore use Reiner’s result [17] to determine conjugacy classes of C_5 -subgroups in $\mathrm{GL}_4(\mathbb{Z})$. Since both \mathbb{Z} and $\mathbb{Z}[\zeta_5]$ have trivial class group, there is only one isomorphism class of $\mathbb{Z}[\zeta_5]$ -module with nontrivial action and \mathbb{Z} -rank 4. Hence, there is a unique conjugacy class of cyclic order 5 subgroup in $\mathrm{GL}_4(\mathbb{Z})$. Since the center of $\mathrm{GL}_4(\mathbb{Z})$ is of order 2, the same is true for $\mathrm{PGL}_4(\mathbb{Z})$.

Now there is a necessary modification to deal with the case $\mathrm{SL}_4(\mathbb{Z})$, along the lines of the discussion in [16]. While conjugacy classes of C_5 -subgroups in $\mathrm{GL}_4(\mathbb{Z})$ correspond to isomorphism classes of $\mathbb{Z}[\zeta_5]$ -modules, the conjugacy classes of C_5 -subgroups in $\mathrm{SL}_4(\mathbb{Z})$ correspond to such modules *equipped with an additional orientation*, i.e., a choice of isomorphism $\det M \cong \bigwedge_{\mathbb{Z}}^4 M \cong \mathbb{Z}$. The conjugacy class in $\mathrm{GL}_4(\mathbb{Z})$ lifts to $\mathrm{SL}_4(\mathbb{Z})$, and the corresponding module has two different choices of orientation. The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ acts on the set of oriented modules. The action exchanges the orientations. Therefore, there is one conjugacy class of C_5 -subgroup in $\mathrm{SL}_4(\mathbb{Z})$ stabilized by $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$.

The centralizer of this C_5 -subgroup is the group of norm-1 units of $\mathbb{Z}[\zeta_5]$, which by Dirichlet’s unit theorem is isomorphic to

$$\ker(\mathbb{Z}[\zeta_5]^\times \rightarrow \mathbb{Z}^\times) \cong \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}.$$

As in [16, Section 5], the normalizer is an extension of the centralizer by an action of the stabilizer of the corresponding oriented module in the Galois group. We noted above that the Galois group $\mathbb{Z}/4\mathbb{Z}$ exchanges the two orientations of the trivial module, hence the stabilizer is the subgroup $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$. The normalizer therefore is of the form $(\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$. The action of $\mathbb{Z}/2\mathbb{Z}$ on

$\mathbb{Z}/10\mathbb{Z}$ is by multiplication with -1 because the action of the Galois group is via the identification $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/5\mathbb{Z}^\times$. The action of $\mathbb{Z}/2\mathbb{Z}$ on \mathbb{Z} is trivial: the full Galois group acts on \mathbb{Z} via a surjective homomorphism $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}^\times \cong \mathbb{Z}/2\mathbb{Z}$. The stabilizer of the oriented module in the Galois group lies in the kernel of the above action, as claimed. Therefore, the normalizer is in fact of the form $D_{10} \times \mathbb{Z}$.

Applying the formulas from [16, Section 3], the Farrell–Tate cohomology of the normalizer is of the form $\mathbb{F}_5[a_2^{\pm 2}](b_1^3)^{\oplus 2} \oplus \mathbb{F}_5[a_2^{\pm 2}](b_1^3)_{-1}^{\oplus 2}$ where the lower subscript -1 indicates a degree shift by -1 .

The Hilbert–Poincaré series for the positive degrees is $\frac{T^3+2T^4+T^5}{1-T^4} = \frac{T^3(1+T)^2}{1-T^4}$.

The computations in [6] show that the 5-torsion in integral homology of $\mathrm{PSL}_4(\mathbb{Z})$ of dimension 1 in degrees $0, 3 \pmod 4$ and trivial otherwise. By the universal coefficient theorem, this agrees with the above computation.

10. NECESSITY OF RIGIDITY FOR BREDON HOMOLOGY

From a non-rigid cell complex, i.e., a cell complex where cell stabilizers do not necessarily fix the corresponding cell pointwise, one can compute classical group homology via the equivariant spectral sequence with coefficients in the orientation module. Such an orientation module, where elements of the stabilizer group act by multiplication with 1 or -1 , depending on whether they preserve or reverse the orientation of the cell, cannot exist for Bredon homology. We make this precise in the following statement:

Proposition 15. *There is no module-wise variation of the Bredon module with coefficients in the complex representation ring and with respect to the system of finite subgroups such that Bredon homology can be computed from a non-rigid cell complex.*

Proof. We provide a counterexample in order to contradict the possibility to compute the Bredon homology from an arbitrary non-rigid cell complex.

Consider the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$. A model for $\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z})$ is given by the modular tree [21]. There is a rigid cell complex structure T_1 on it, given as follows. By [21], the modular tree admits a strict fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$, of the shape

$$\mathbb{Z}/3\mathbb{Z} \bullet \text{---} \bullet \mathbb{Z}/2\mathbb{Z}$$

with vertex stabilizers as indicated and trivial edge stabilizer. We obtain the cell complex T_1 by tessellating the modular tree with the $\mathrm{PSL}_2(\mathbb{Z})$ -images of this fundamental domain. Obviously, T_1 is rigid, and it yields the Bredon chain complex

$$0 \rightarrow R_{\mathbb{C}}(\langle 1 \rangle) \xrightarrow{d} R_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z}) \oplus R_{\mathbb{C}}(\mathbb{Z}/3\mathbb{Z}) \rightarrow 0.$$

The map d in the above Bredon chain complex is injective, and as $R_{\mathbb{C}}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}^n$, we read off

$$H_1^{\mathfrak{F}^{\mathrm{in}}}(\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}); R_{\mathbb{C}}) = 0, \quad H_0^{\mathfrak{F}^{\mathrm{in}}}(\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}); R_{\mathbb{C}}) \cong \mathbb{Z}^4.$$

Now we equip the modular tree with an alternative equivariant cell structure T_2 , induced by the non-strict fundamental domain

$$\mathbb{Z}/3\mathbb{Z} \bullet \text{---} \text{---} \bullet \mathbb{Z}/3\mathbb{Z}$$

where the (set-wise) edge stabilizer is $\mathbb{Z}/2\mathbb{Z}$, flipping the edge onto itself. It can be seen as a ramified double cover of the fundamental domain for T_1 discussed above. A system of representative cells for T_2 is given by the edge of double length, and one vertex of stabilizer type $\mathbb{Z}/3\mathbb{Z}$. This yields a chain complex

$$0 \rightarrow R_{\mathbb{C}}(\widetilde{\mathbb{Z}/2\mathbb{Z}}) \rightarrow R_{\mathbb{C}}(\mathbb{Z}/3\mathbb{Z}) \rightarrow 0,$$

where the tilde could be any construction which takes the non-trivial $\mathbb{Z}/2\mathbb{Z}$ -action on the edge of double length into account (similar to the coefficients in the orientation module for group homology computed from non-rigid cell complexes). But no matter how this construction is done, from $R_{\mathbb{C}}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}^3$, we can never reach $H_0^{\mathfrak{F}^{\mathrm{in}}}(\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z}); R_{\mathbb{C}}) \cong \mathbb{Z}^4$. Hence T_2 is our desired counterexample. \square

Remark 16. We could of course drop the condition “module-wise” in the above proposition, and investigate whether there is a reasonable construction which maps the representation ring to a complex of modules and yields a quasi-isomorphism from the total complex to the Bredon complex for the subdivided tree. But with such a construction, one would only superficially hide the fact that one needs to know how to subdivide in order to get the constructed complexes right. This means that it will not be practicable to compute Bredon homology with respect to the system of finite

subgroups and coefficients in the complex representation ring without subdividing the cell complex under consideration to make it rigid.

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ANH TUAN BUI, UNIVERSITY OF SCIENCE - HO CHI MINH CITY, FACULTY OF MATH & COMPUTER SCIENCE, 227 NGUYEN VAN CU ST., DISTRICT 5, HO CHI MINH CITY, VIETNAM BATUAN@HCMUS.EDU.VN

ALEXANDER D. RAHM, MATHEMATICS RESEARCH UNIT OF UNIVERSITÉ DU LUXEMBOURG, MAISON DU NOMBRE, 6, AVENUE DE LA FONTE, L-4364 ESCH-SUR-ALZETTE ALEXANDER.RAHM@UNI.LU

MATTHIAS WENDT, MATHEMATISCHES INSTITUT, ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG, ERNST-ZERMELO-STR. 1, 79104 FREIBURG IM BREISGAU, GERMANY M.WENDT.C@GMAIL.COM