Monetary Policy, Factor Substitution, and the Speed of Convergence

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Introduction

We want to bring together three strands of discussion in macroeoconomic growth theory:

1. *(Optimal) Monetary growth theory*
2. *(Normalized) CES production functions*
3. *Speed of convergence analysis*

The most important benchmark papers are:

Turnovsky (JEDC 2002), Gokan (JEDC 2003), Klump/Saam (EL 2008)

Early ideas can already be found in Klump (1999):

„Keynes und die Neoklassiker: Verbindungen zwischen Keynesianischer Makroökonomik und Neoklassischer Wachstumstheorie"

The Intertemporal Monetary Growth Model

We let money enter the utility function (Sidrauski 1967) and work with a normalized CES production function (De LaGrandville 1989; Klump and De LaGrandville 2000)

$$u\left(c_t, m_t\right) = \frac{\left(c^{1-\theta} m^{\theta}\right)^{1-\eta} - 1}{1-\eta} \tag{1}$$

$$f(k) = C(\sigma)\left[\alpha(\sigma) k^{\psi} - \left(1-\alpha(\sigma)\right)\right]^{\frac{1}{\psi}} \text{ with } \alpha(\sigma) = \frac{k_0^{1-\psi}}{k_0^{1-\psi} + \mu_0},$$

$$C(\sigma) = y_0 \left[\frac{k_0^{1-\psi} + \mu_0}{k_0 + \mu_0}\right]^{\frac{1}{\psi}}, \quad \mu_0 = \frac{F_L}{F_k} = \frac{1-\alpha}{\alpha} k_0^{1-\psi} \quad \text{and} \quad \psi = \frac{\sigma-1}{\sigma} \tag{2}$$

$$y = y_0 [\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + \left(1-\pi_0\right)]^{\frac{1}{\psi}} .^{1} \tag{3}$$

$$\frac{dy}{d\sigma} = -\frac{1}{\sigma^2}\frac{1}{\psi^2}y\left\{\pi\ln\left(\frac{\pi_0}{\pi}\right)+(1-\pi)\ln\left(\frac{1-\pi_0}{1-\pi}\right)\right\} > 0 \quad with \quad \pi = \frac{f'(k)k}{y}. \tag{4}$$

$$\dot{v} = f(k)-(\delta+n)k+z-(\Pi+n)m-c \tag{5}$$

The Hamiltonian of the intertemporal optimization problem is:

$$H = u(c,m)+\psi\left[f(k)-(\delta+n)k+z-(\Pi+n)m-c\right] \tag{6}$$

We obtain a system of three differential equations:

$$\frac{\dot{c}}{c} = \frac{(f'(k)-(n+\delta+\rho))}{(\eta(1-\theta)+\theta)}+\frac{\theta(1-\eta)}{(\eta(1-\theta)+\theta)}(\phi+f'(k)-n-\delta-\frac{c}{m}\frac{\theta}{1-\theta}) \tag{7}$$

$$\frac{\dot{k}}{k} = \frac{f(k)}{k}-(\delta+n)-\frac{c}{k} \tag{8}$$

$$\frac{\dot{m}}{m} = \phi-n-\Pi = \phi+f'(k)-n-\delta-\frac{c}{m}\frac{\theta}{1-\theta}. \tag{9}$$

We obtain the following steady state conditions:

$$f'\left(k^*\right)=\delta+\rho+\mathrm{n} \tag{10}$$

$$c^* = f(k^*)-(\delta+n)k* \tag{11}$$

$$\phi-n = \Pi^* \tag{12}$$

$$\frac{c^*}{m^*}=\frac{1-\theta}{\theta}(\rho+\phi) \tag{13}$$

$$\frac{\partial\pi}{\partial\sigma}=\frac{1}{\sigma^2}(1-\pi)\pi\ln\left(\frac{k}{k_0}\right)>0 \;\; if \;\; k>k_0 \; . \tag{14}$$

$$\frac{\partial f'}{\partial\sigma}=\frac{y}{k}\frac{\partial\pi}{\partial\sigma}+\frac{\pi}{k}\frac{\partial y}{\partial\sigma}>0 \;\; if \;\; k>k_0. \tag{15}$$

In order to derive the speed of convergence we calculate the following Jacobian:

$$J = \begin{pmatrix} \dfrac{-\theta(1-\eta)(\rho+\phi)}{\theta+\eta(1-\theta)} & \dfrac{1+\theta(1-\eta)}{\theta+\eta(1-\theta)}k^*f''(k^*) & \dfrac{\theta(1-\eta)(\rho+\phi)}{\theta+\eta(1-\theta)} \\[3mm] -\dfrac{c^*}{k^*} & \rho & 0 \\[3mm] -(\rho+\phi) & k^*f''(k^*) & (\rho+\phi) \end{pmatrix} \tag{16}$$

$$\det J = \frac{(\phi+\rho)c^*f''(k^*)}{\theta+\eta(1-\theta)} < 0 \text{ if } f''(k^*) < 0 \text{ and trace } J > 0. \tag{17}$$

We can derive the one negative eigenvalue from the following characteristic equation:

$$\mu = -\varepsilon^3 + \varepsilon^2 \left(\rho + \frac{\eta(\rho+\phi)}{\theta+\eta(1-\theta)} \right) - \varepsilon \left(\frac{\eta(\rho+\phi)\rho}{\theta+\eta(1-\theta)} + c^*f''(k^*)\frac{(1+\theta(1-\eta))}{\theta+\eta(1-\theta)} \right)$$
$$+ c^*f''(k^*)\frac{(\rho+\phi)}{\theta+\eta(1-\theta)} \tag{18}$$

$$-\frac{\partial \mu}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \lambda}\frac{d\lambda}{d\phi} = \frac{\partial \mu}{\partial \phi} \tag{19}$$

with $\lambda = -\varepsilon$ where $\dfrac{d\varepsilon}{d\lambda} = -1$ and $\dfrac{d\mu}{d\varepsilon} < 0$ [2] so that $\dfrac{d\lambda}{d\phi} = \dfrac{d\mu}{d\phi}$.

$$\frac{\partial \mu}{\partial \phi}\left[\theta + \eta(1-\theta) - \frac{\eta(\rho+\phi)}{\varepsilon}\right] = c^* f''(k^*)\left(\frac{(1-\eta)^2}{\theta + \eta(1-\theta)}\right) < 0 \;[3] \tag{20}$$

We can analyze the effect of a change in the e. o. s. on the transitional effect of monetary policy:

$$-\frac{\partial \mu}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial \lambda}\frac{d\lambda}{d\phi d\sigma} = \frac{\partial \mu}{\partial \phi \partial \sigma}. \tag{21}$$

$$\frac{\partial \mu}{\partial \phi \partial \sigma}\left[\theta + \eta(1-\theta) - \frac{\eta(\rho+\phi)}{\varepsilon}\right] = \left(\frac{dc^* f''(k^*)}{d\sigma}\right)\left(\frac{(1-\eta)^2}{\theta + \eta(1-\theta)}\right). \tag{22}$$

$$\frac{dc^*}{d\sigma}f''(k^*) + \frac{\partial f''(k^*)}{\partial \sigma}c^* = \frac{dc^*}{d\sigma}f''(k^*) + (n+\delta+\rho)(1-\pi^*)\left(\frac{1}{\sigma^2}\right) \;[4]$$
$$+ \frac{\partial \pi^*}{\partial \sigma}(n+\delta+\rho)\frac{1}{\sigma} > 0 \quad \text{if } k^* > k_0 \tag{23}$$

Equation (23) together with (22) implies that $-\dfrac{d\lambda}{d\phi d\sigma} = \dfrac{\partial \mu}{\partial \phi \partial \sigma} > 0$ since $\lambda = -\varepsilon$

where $\dfrac{\partial \varepsilon}{\partial \lambda} = -1$ and $\dfrac{\partial \mu}{\partial \varepsilon} < 0$. This means that $\dfrac{d\lambda}{d\phi d\sigma}$ must be strictly negative.

Calibration of the Economy

For numerical simulations we calibrate the econony with parameters, if possible, close to the ones used by Turnovsky (2002) and Gokan (2003):

benchmark values:

point of normalization: $k_0 = y_0 = 1$; $\pi_0 = 1/3$

aggregate elasticity of substitution: $\sigma = 0.8$

key monetary policy parameter: $\phi = 0.03$

weight of money in utility: $\theta = 0.3$

elasticity of marginal utility: $\eta = 2.5$

other parameters:

rate of time preference: $\rho = 0.04$

rate of depreciation: $\delta = 0.04$

rate of population growth: $n = 0.01$.

Changes in the Parameters

• The speed of convergence will increase with the growth rate of the money supply

• This effect is the stronger the lower is the preference for money; however, the acceleration in the speed of convergence is stronger if the preference for money is higher.

• The speed of convergence effect of expansive monetary policy decreases if the e.o.s. increases

• Even with a high baseline profit share we are not able to replicate a 2% speed of convergence with values of the e.o.s. below one.

Table 1: Speed of convergence with **low** baseline profit share

Speed of convergence in percentage	Θ=0.8		Θ=0.3	
	λ	half time	λ	half time
	$k_0=y_0=1$; $\mu_0=2$	$\pi_0=1/3$		
$\sigma=1.2$	$\pi^*=0.43$	$c^*/y^*=0.76$		
$\Phi=0.01$	2.75	25.19	3.27	21.21
$\Phi=0.02$	2.85	24.36	3.29	21.01
$\Phi=0.03$	2.92	23.73	3.32	20.84
$\Phi=0.05$	3.03	22.88	3.36	20.61
$\sigma=1$	$\pi^*=1/3$	$c^*/y^*=0.81$		
$\Phi=0.01$	3.69	17.52	4.99	13.87
$\Phi=0.02$	4.13	16.79	5.05	13.73
$\Phi=0.03$	4.26	16.26	5.09	13.61
$\Phi=0.05$	4.47	15.52	5.16	13.44
$\sigma=0.8$	$\pi^*=0.26$	$c^*/y^*=0.86$		
$\Phi=0.01$	5.39	12.86	7.30	9.49
$\Phi=0.02$	5.67	12.24	7.38	9.39
$\Phi=0.03$	5.89	11.77	7.45	9.31
$\Phi=0.05$	6.23	11.12	7.55	9.18
$\sigma=0.6$	$\pi^*=0.19$	$c^*/y^*=0.89$		
$\Phi=0.01$	7.19	9.68	10.59	6.54
$\Phi=0.02$	7.59	9.13	10.69	6.48
$\Phi=0.03$	7.95	8.72	10.79	6.43
$\Phi=0.05$	8.50	8.15	10.93	6.34
$\sigma=0.2$	$\pi^*=0.12$	$c^*/y^*=0.94$		
$\Phi=0.01$	13.21	5.25	26.99	2.57
$\Phi=0.02$	14.39	4.81	27.17	2.55
$\Phi=0.03$	15.40	4.5	27.34	2.53
$\Phi=0.05$	17.03	4.07	27.64	2.51

Table 2: Speed of convergence with **high** baseline profit share

Speed of convergence in percentage	$\Theta=0.8$		$\Theta=0.3$	
	λ	half time	λ	half time
	$k_0=y_0=1$; $\mu_0=1/2$	$\pi_0=2/3$[1]		
$\sigma=1$	$\pi^{*}=2/3$	$c^{*}/y^{*}=0.63$		
$\Phi=0.01$	1.50	46.04	1.67	41.48
$\Phi=0.02$	1.54	44.99	1.68	41.16
$\Phi=0.03$	1.57	44.22	1.69	40.92
$\Phi=0.05$	1.6	43.17	1.71	40.59
$\sigma=0.8$	$\pi^{*}=0.45$	$c^{*}/y^{*}=0.75$		
$\Phi=0.01$	3.32	20.87	4.06	17.06
$\Phi=0.02$	3.45	20.09	4.10	16.89
$\Phi=0.03$	3.55	19.52	4.14	16.75
$\Phi=0.05$	3.70	18.72	4.19	16.55
$\sigma=0.6$	$\pi^{*}=0.29$	$c^{*}/y^{*}=0.83$		
$\Phi=0.01$	5.50	12.60	7.50	9.24
$\Phi=0.02$	5.79	11.98	7.58	9.15
$\Phi=0.03$	6.02	11.51	7.64	9.07
$\Phi=0.05$	6.37	10.87	7.75	8..96
$\sigma=0.2$	$\pi^{*}=0.13$	$c^{*}/y^{*}=0.92$		
$\Phi=0.01$	12.57	5.52	24.74	2.80
$\Phi=0.02$	13.66	5.08	24.91	2.78
$\Phi=0.03$	14.57	4.76	25.07	2.76
$\Phi=0.05$	16.06	4.32	25.35	2.73

Influence of the Normalization Point

- Klump and Saam (2008) have shown that the *choice of the normalization point* has a major impact on the speed of convergence:

- Xue and Yip (2011) have identified *two different effects* – the efficiency and the distribution effect

- The distribution effect can be elimated by *normalizing in the steady state*

Table 3: Speed of convergence with various baseline points

Speed of convergence in percentage	$\Theta=0.3$	
	λ	half time
	$k_0=2, y_0=1; \mu_0=2$	$\pi_0=1/2$
$\sigma=0.8$	$k^*=8.92$	$\pi^*=0.41$
$\Phi=0.01$	4.57	15.18
$\Phi=0.02$	4.61	15.03
$\Phi=0.03$	4.65	14.90
$\Phi=0.05$	4.70	14.71
	$k_0=5, y_0=1; \mu_0=2$	$\pi_0=5/7$
$\sigma=0.8$	$k^*=16.07$	$\pi^*=0.65$
$\Phi=0.01$	2.08	33.26
$\Phi=0.02$	2.10	32.98
$\Phi=0.03$	2.12	32.77
$\Phi=0.05$	2.14	32.26
	$k_0=1, y_0=0.5; \mu_0=2$	$\pi_0=1/3$
$\sigma=0.8$	$k^*=2.05$	$\pi^*=0.29$
$\Phi=0.01$	6.45	10.75
$\Phi=0.02$	6.51	10.64
$\Phi=0.03$	6.57	10.55
$\Phi=0.05$	6.67	10.41

The higher the baseline point the lower is the speed of convergence

Table 4: Speed of convergence with baseline point in the steady state

Speed of convergence in percentage	$\Theta=0.3$	
	λ	half time
baseline point in steady state of $\sigma=0.8$	$k_0=k_{\sigma=0.8}=4.409$, $y_0=1.547$	$\pi_0=0.26$
$\sigma=1.2$		
$\Phi=0.01$	5.77	12.02
$\Phi=0.02$	5.83	11.89
$\Phi=0.03$	5.88	11.79
$\Phi=0.05$	5.96	11.63
$\sigma=0.8$		
$\Phi=0.01$	7.30	9.49
$\Phi=0.02$	7.38	9.39
$\Phi=0.03$	7.45	9.31
$\Phi=0.05$	7.55	9.18
$\sigma=0.2$		
$\Phi=0.01$	15.64	4.43
$\Phi=0.02$	15.78	4.39
$\Phi=0.03$	15.89	4.36
$\Phi=0.05$	16.10	4.30

The impact of a change in the e.o.s. on the speed of convergence is largely due to the *distribution effect*

The impact of monetary policy is more channelled via the *efficiency effect*

Conclusion

- By using a *normalized CES production function* we are able to derive more realistic effects of monetary policy on the speed of convergence than Gokan (2002)

- Monetary policy is the more effective in speeding up convergence, the *lower ist the e.o.s.* and the *lower is the preference for money*

- The choice of the *normalization point* has a particular influence on the result as it determines the interplay between *distribution and efficiency effects* which can both be induced by changes in the e.o.s.

1 Appendix

Reformulation of the normalized CES production function

$$f(k) = C(\sigma)\left[\alpha(\sigma)k^{\psi} - (1-\alpha(\sigma))\right]^{\frac{1}{\psi}}$$

$$= y_0 \left[\frac{k_0^{1-\psi} + \mu_0}{k_0 + \mu_0}\right]^{\frac{1}{\psi}} \left[\frac{k_0^{1-\psi}}{k_0^{1-\psi} + \mu_0}k^{\psi} - \left(1 - \frac{k_0^{1-\psi}}{k_0^{1-\psi} + \mu_0}\right)\right]^{\frac{1}{\psi}}$$

$$= y_0 \left[\frac{k_0^{1-\psi}}{k_0 + \mu_0}k^{\psi} - \frac{k_0^{1-\psi} + \mu_0}{k_0 + \mu_0} + \frac{k_0^{1-\psi}}{k_0 + \mu_0}\right]^{\frac{1}{\psi}}$$

$$= y_0 \left[\frac{k_0}{k_0 + \mu_0}\frac{k^{\psi}}{k_0^{\psi}} - \frac{\mu_0}{k_0 + \mu_0}\right]^{\frac{1}{\psi}}$$

$$= y_0 \left[\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + (1-\pi_0)\right]^{\frac{1}{\psi}}$$

Proof of Equation (4):

$$y = y_0 \left[\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + (1-\pi_0)\right]^{\frac{1}{\psi}}$$

$$\ln y = \ln y_0 + \frac{1}{\psi}\ln\left[\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + (1-\pi_0)\right]$$

$$\frac{1}{y}\frac{dy}{d\sigma} = -\frac{1}{\psi^2}\frac{1}{\sigma^2}\ln\left[\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + (1-\pi_0)\right] + \frac{1}{\psi}\frac{1}{\sigma^2}\frac{\pi_0 \left(\frac{k}{k_0}\right)^{\psi}\ln\left(\frac{k}{k_0}\right)}{\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + (1-\pi_0)}$$

$$\frac{dy}{d\sigma} = -\frac{1}{\psi^2}\frac{1}{\sigma^2}y\left\{\ln\left[\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + (1-\pi_0)\right] - \frac{\psi\pi_0 \left(\frac{k}{k_0}\right)^{\psi}\ln\left(\frac{k}{k_0}\right)}{\pi_0 \left(\frac{k}{k_0}\right)^{\psi} + (1-\pi_0)}\right\}$$

applying: $\left(\frac{k}{k_0}\right)^{\psi} = \frac{\pi(1-\pi_0)}{\pi_0(1-\pi)}$

$$\frac{dy}{d\sigma} = -\frac{1}{\psi^2}\frac{1}{\sigma^2}y\left\{\pi\ln\left(\frac{\pi_0}{\pi}\right) + \ln\left(\frac{1-\pi_0}{1-\pi}\right) - \pi\ln\left(\frac{1-\pi_0}{1-\pi}\right)\right\}$$

$$\frac{dy}{d\sigma} = -\frac{1}{\psi^2}\frac{1}{\sigma^2}y\left\{\pi\ln\left(\frac{\pi_0}{\pi}\right) + (1-\pi)\ln\left(\frac{1-\pi_0}{1-\pi}\right)\right\}$$

The characteristic equation is clearly negative for any negative root as can be seen from the plot below.

$\dfrac{\partial \mu}{\partial \varepsilon} < 0$ if $\varepsilon < 0$
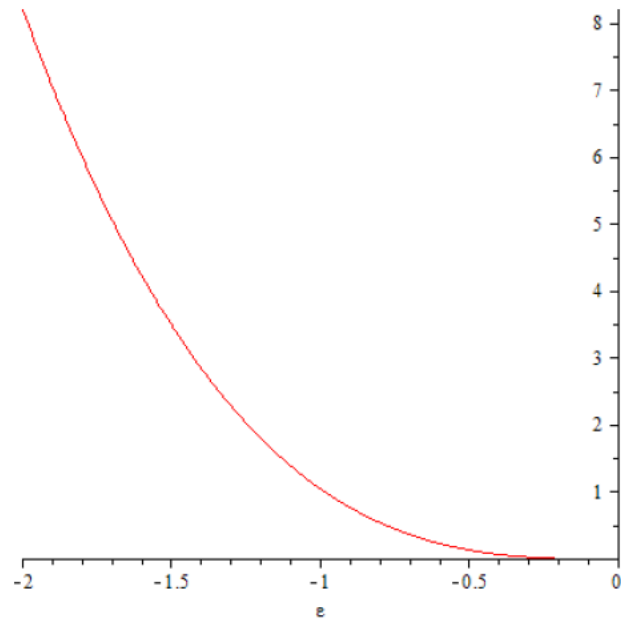
Derivation of equation (22)

$$f''\left(k^*\right) = -f'\left(k^*\right)(1-\psi)\left(\dfrac{\pi_0\left(\dfrac{k^*}{k_0}\right)^{\psi}}{\pi_0\left(\dfrac{k^*}{k_0}\right)^{\psi} + (1-\pi_0)} - 1\right)\dfrac{1}{k}$$

$$= -f'\left(k^*\right)(1-\psi)(1-\pi) = -\left(n+\delta+\rho\right)(1-\pi)\dfrac{1}{\sigma}$$

$$f'(k^*) = (n + \delta + \rho)$$

$$\frac{dk^*}{d\sigma} f''(k^*) = -\frac{\partial f'(k^*)}{\partial \sigma}$$

$$\frac{dk^*}{d\sigma} = -\frac{\partial f'(k^*)}{\partial \sigma} \frac{1}{f''(k^*)} > 0 \quad since \quad f''(k^*) < 0 \quad and \quad \frac{\partial f'(k^*)}{\partial \sigma} > 0$$

$$c^* = f(k^*) - (n + \delta)k^*$$

$$\frac{dc^*}{d\sigma} = \frac{\partial f(k^*)}{\partial \sigma} + f'(k^*)\frac{dk^*}{d\sigma} - (n + \delta)\frac{dk^*}{d\sigma}$$

$$= \frac{\partial f(k^*)}{\partial \sigma} + \frac{dk^*}{d\sigma}(f'(k^*) - (n + \delta))$$

$$= \frac{\partial f(k^*)}{\partial \sigma} + \frac{dk^*}{d\sigma}\rho > 0$$

$$\pi = \frac{f'(k)k}{f(k)} = \pi_0 \left(\frac{k}{k_0}\right)^\psi [\pi_0 \left(\frac{k}{k_0}\right)^\psi + (1 - \pi_0)]^{-1}$$

$$\frac{\partial \pi}{\partial \sigma} = \frac{1}{\sigma^2}(1 - \pi)\pi \ln\left(\frac{k}{k_0}\right) > 0 \quad if \quad k > k_0$$

$$\frac{dc^*}{d\sigma} f''(k^*) + \frac{\partial f''(k^*)}{\partial \sigma} c^* = \frac{dc^*}{d\sigma} f''(k^*) + (n + \delta + \rho)(1 - \pi^*)\left(\frac{1}{\sigma^2}\right)$$

$$+ \frac{\partial \pi^*}{\partial \sigma}(n + \delta + \rho)\frac{1}{\sigma} > 0 \quad if \quad k^* > k_0$$

Influence of elasticity of substitution on the speed of convergence

$$\frac{\partial \mu}{\partial \sigma} = -\varepsilon \frac{(1 + \theta(1 - \eta))}{\theta + \eta(1 - \theta)} * \frac{\partial(c^* f''(k^*))}{\partial \sigma} + \frac{\partial(c^* f''(k^*))}{\partial \sigma} \frac{(\rho + \phi)}{\theta + \eta(1 - \theta)} < 0 \quad if \varepsilon < 0$$

$$\frac{\partial(c^* f''(k^*))}{\partial \sigma} = \frac{dc^*}{d\sigma} f''(k^*) + \frac{\partial f''(k^*)}{\partial \sigma} c^* > 0$$