QUASI-POLYNOMIAL FUNCTIONS OVER BOUNDED DISTRIBUTIVE LATTICES

MIGUEL COUCEIRO AND JEAN-LUC MARICHAL

ABSTRACT. In [6] the authors introduced the notion of quasi-polynomial function as being a mapping $f\colon X^n \to X$ defined and valued on a bounded chain X and which can be factorized as $f(x_1, \ldots, x_n) = p(\varphi(x_1), \ldots, \varphi(x_n))$, where p is a polynomial function (i.e., a combination of variables and constants using the chain operations $\wedge$ and $\vee$) and φ is an order-preserving map. In the current paper we study this notion in the more general setting where the underlying domain and codomain sets are, possibly different, bounded distributive lattices, and where the inner function is not necessarily order-preserving. These functions appear naturally within the scope of decision making under uncertainty since, as shown in this paper, they subsume overall preference functionals associated with Sugeno integrals whose variables are transformed by a given utility function. To axiomatize the class of quasi-polynomial functions, we propose several generalizations of well-established properties in aggregation theory, as well as show that some of the characterizations given in [6] still hold in this general setting. Moreover, we investigate the so-called transformed polynomial functions (essentially, compositions of unary mappings with polynomial functions) and show that, under certain conditions, they reduce to quasi-polynomial functions.

1. INTRODUCTION

When we need to summarize, fuse, or merge a set of values into a single one, we usually make use of a so-called aggregation function, e.g., a mean or an averaging function. The need to aggregate values in a meaningful way has become more and more present in an increasing number of areas not only of mathematics or physics, but especially in applied fields such as engineering, computer science, and economical and social sciences. Various aggregation functions have been proposed in the literature, thus giving rise to the growing theory of aggregation which proposes, analyzes, and characterizes aggregation function classes. For recent references, see Beliakov et al. [2] and Grabisch et al. [13].

Among noteworthy aggregation functions is the (discrete) Sugeno integral, which was introduced by Sugeno [20, 21] as a way to compute the average of a function with respect to a nonadditive measure. Since its introduction, the Sugeno integral has been thoroughly investigated and is now considered as one of the most relevant aggregation functions in the qualitative setting of ordinal information (e.g., when the values to be aggregated are simply defined on a chain without further structure). For general background, see also the edited book [14].

As it was observed in [16], Sugeno integrals can be regarded as certain (lattice) polynomial functions, that is, functions which can be obtained as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. More precisely, given a bounded chain X, the Sugeno integrals are exactly those polynomial functions $f\colon X^n \to X$ which are idempotent, that is, satisfying $f(x,\ldots,x) = x$. This convenient description made it possible to naturally extend the definition of the Sugeno integrals to the case when X is a bounded distributive lattice (see [17]) and to derive several axiomatizations of this class (as idempotent polynomial functions); see [7, 9].

In many applications, the values to be aggregated are first to be transformed by an order-preserving unary function $\varphi\colon X \to Y$ so that the transformed values (which are usually real numbers) can be aggregated in a meaningful way by a function $g\colon Y^n \to Y$. The resulting composed function $f\colon X^n \to Y$ is then defined as $f = g \circ \varphi$, that is,

$$(1) \qquad\qquad f(x_1,\ldots,x_n) = g(\varphi(x_1),\ldots,\varphi(x_n)).$$

Such an aggregation model is used for instance in decision under uncertainty, where φ is called a utility function and f an overall preference functional. It is also used in multi-criteria decision making where the criteria are commensurate (i.e., expressed in a common scale). For a recent reference, see the edited book [3].

This aggregation model has also been investigated in a purely ordinal decision setting, where X and Y are bounded chains and $g\colon Y^n \to Y$ is a Sugeno integral or a polynomial function; see for instance [10, 11]. In the special case when $X = Y$, the corresponding compositions (1), which we call quasi-polynomial functions, were recently investigated and characterized by the authors as solutions of certain functional equations and in terms of necessary and sufficient conditions which have natural interpretations in decision making and aggregation theory; see [6].

In this paper, after recalling the basic concepts in lattice theory as well as few well-known results concerning polynomial functions (Sections 2 and 3), we investigate the quasi-polynomial functions when considered in the more general setting where the underlying domain and codomain sets are, possibly different, bounded distributive lattices, and where the inner unary functions are not necessarily order-preserving. Moreover, we show that some axiomatizations of the class of quasi-polynomial functions given in [6] still hold in this more general setting and, under certain assumptions, we propose further characterizations of this class by necessary and sufficient conditions given in terms of generalizations of well-established properties in aggregation theory. These results also lead to new characterizations of the class of polynomial functions (Section 4). Finally, we introduce the concept of transformed polynomial function and show that, under certain conditions, this notion is subsumed by that of quasi-polynomial function (Section 5).

2. Basic notions and terminology

Throughout this paper, let X be an arbitrary bounded distributive lattice with lattice operations $\wedge$ and $\vee$, and with least and greatest elements 0_X and 1_X, respectively, where the subscripts may be omitted when the underlying lattice is clear from the context. A *chain* is simply a lattice X such that, for every $a, b \in X$, we have $a \leqslant b$ or $b \leqslant a$. A subset S of a lattice X is said to be *convex* if, for every $a, b \in S$ and every $c \in X$ such that $a \leqslant c \leqslant b$, we have $c \in S$. For any subset

$S \subseteq X$, we denote by $\overline{S}$ the convex hull of S, that is, the smallest convex subset of X containing S. For any integer $n \geqslant 1$, we set $[n] = \{1, \ldots, n\}$.

The Cartesian product X^n can be as well regarded as a bounded distributive lattice by defining $\wedge$ and $\vee$ componentwise, i.e.,

$$
\begin{aligned}
(a_1, \ldots, a_n) \wedge (b_1, \ldots, b_n) &= (a_1 \wedge b_1, \ldots, a_n \wedge b_n), \\
(a_1, \ldots, a_n) \vee (b_1, \ldots, b_n) &= (a_1 \vee b_1, \ldots, a_n \vee b_n).
\end{aligned}
$$

We denote the elements of X by lower case letters $a, b, c, \ldots$, and the elements of X^n by bold face letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$. We also use $\mathbf{0}$ and $\mathbf{1}$ to denote the least element and the greatest element, respectively, of X^n and we denote by $\mathbf{e}$ any n-tuple whose components are either 0 or 1, regardless of the underlying lattice. For $k \in [n]$, $c \in X$, and $\mathbf{x} \in X^n$, we use $\mathbf{x}_k^c$ to denote the n-tuple whose ith component is c, if $i = k$, and x_i, otherwise. We also let $\mathbf{x} \wedge c = (x_1 \wedge c, \ldots, x_n \wedge c)$ and $\mathbf{x} \vee c = (x_1 \vee c, \ldots, x_n \vee c)$, and denote by $[\mathbf{x}]_c$ (resp. $[\mathbf{x}]^c$) the n-tuple whose ith component is 0 (resp. 1), if $x_i \leqslant c$ (resp. $x_i \geqslant c$), and x_i, otherwise.

Let Y be an arbitrary bounded distributive lattice, possibly different from X. A mapping $\varphi \colon X \to Y$ is said to be a *lattice homomorphism* if it preserves the lattice operations, i.e.,

$$
\varphi(a \wedge_X b) = \varphi(a) \wedge_Y \varphi(b) \quad \text{and} \quad \varphi(a \vee_X b) = \varphi(a) \vee_Y \varphi(b).
$$

With no danger of ambiguity, we omit the subscripts X and Y. For further background on lattice theory, see, e.g., [5, 15, 19].

The *range* of a function $f \colon X^n \to Y$ is defined by $\mathcal{R}_f = \{f(\mathbf{x}) : \mathbf{x} \in X^n\}$. The *diagonal section* of f is the unary function $\delta_f \colon X \to Y$ defined by $\delta_f(x) = f(x, \ldots, x)$. A function $f \colon X^n \to Y$ is said to be *order-preserving* (resp. *order-reversing*) if, for every $\mathbf{a}, \mathbf{b} \in X^n$ such that $\mathbf{a} \leqslant \mathbf{b}$, we have $f(\mathbf{a}) \leqslant f(\mathbf{b})$ (resp. $f(\mathbf{a}) \geqslant f(\mathbf{b})$). By a *monotone* function we simply mean an order-preserving or order-reversing function. As a typical example, we have the ternary median function $\mathrm{med} \colon X^3 \to X$ which is given by

$$
\mathrm{med}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_3 \wedge x_1).
$$

For any integer $m \geqslant 1$, any vector $\mathbf{x} \in X^m$, and any function $f \colon X^n \to X$, we define $\langle \mathbf{x} \rangle_f \in X^m$ as the m-tuple $\langle \mathbf{x} \rangle_f = \mathrm{med}(f(\mathbf{0}), \mathbf{x}, f(\mathbf{1}))$, where the right-hand side median is taken componentwise. We then clearly have

$$
\tag{2} \langle \mathbf{x} \rangle_f = \langle \mathbf{x} \rangle_{\delta_f}.
$$

For $\varphi \colon X \to Y$ and $\mathbf{x} \in X^n$, we set $\varphi(\mathbf{x}) = (\varphi(x_1), \ldots, \varphi(x_n))$.

Given a function $f \colon X^n \to Y$, we define the function $\hat{f} \colon \{0,1\}^n \to Y$ as

$$
\tag{3} \hat{f}(\mathbf{e}) = \bigvee_{\substack{e_i' \in \{0,1\} \\ i \in [n] \,:\, e_i = 1}} \bigwedge_{\substack{e_i' \in \{0,1\} \\ i \in [n] \,:\, e_i = 0}} f(\mathbf{e}'), \qquad \mathbf{e} \in \{0,1\}^n.
$$

Note that, for each fixed $\mathbf{e} \in \{0,1\}^n$, from among the $2n$ operations $\vee$ and $\wedge$ only n are considered in (3), for if $e_i = 0$ (resp. $e_i = 1$) then the corresponding $\vee_{e_i' \in \{0,1\}}$ (resp. $\wedge_{e_i' \in \{0,1\}}$) is ignored and only $\wedge_{e_i' \in \{0,1\}}$ (resp. $\vee_{e_i' \in \{0,1\}}$) is considered. In particular, we have that

$$
\hat{f}(\mathbf{0}) = \bigwedge_{\mathbf{e}' \in \{0,1\}^n} f(\mathbf{e}') \quad \text{and} \quad \hat{f}(\mathbf{1}) = \bigvee_{\mathbf{e}' \in \{0,1\}^n} f(\mathbf{e}').
$$

For instance, if $n = 2$, we have

$$(4) \qquad \hat{f}(0,1) = \big(f(0,0) \wedge f(1,0)\big) \vee \big(f(0,1) \wedge f(1,1)\big).$$

If f is order-preserving then $\hat{f} = f|_{\{0,1\}^n}$. However, note that $\hat{f}$ is always order-preserving.

Remark 1. Note that there are $(2n)!$ ways of rearranging the $\vee$ and $\wedge$ in expression (3) and thus $(2n)!$ ways of constructing such an order-preserving function from a given $f\colon X^n \to Y$, each of which producing a possibly different order-preserving function. For instance, let $f\colon \{0,1\}^2 \to \{0,1\}$ be the Boolean addition, i.e., addition modulo 2. Then, as defined in (3), $\hat{f}$ is the binary conjunction $\wedge$. On the other hand, if $\hat{f}$ had been defined as

$$(5) \qquad \hat{f}(\mathbf{e}) = \bigwedge_{\substack{e_i' \in \{0,1\} \\ i \in [n]\,:\, e_i = 0}} \bigvee_{\substack{e_i' \in \{0,1\} \\ i \in [n]\,:\, e_i = 1}} f(\mathbf{e}'), \qquad \mathbf{e} \in \{0,1\}^n,$$

then $\hat{f}$ would have been the binary disjunction $\vee$. To avoid such ambiguities, we only refer to $\hat{f}$ as given in (3).

3. General background on polynomial functions

In this subsection we recall some important results on polynomial functions that will be needed hereinafter. For further background, we refer the reader to [5, 7, 8, 9, 12, 15, 19].

A (*lattice*) *polynomial function* on a lattice X is any map $p\colon X^n \to X$ which can be obtained as a composition of the lattice operations $\wedge$ and $\vee$, the projections $\mathbf{x} \mapsto x_i$, $i \in [n]$, and the constant functions $\mathbf{x} \mapsto c$, $c \in X$.

Fact 1. *Every polynomial function $p\colon X^n \to X$ is order-preserving and satisfies $\delta_p(c) = \langle c \rangle_p$ for every $c \in X$. In particular $\delta_p(c) = c$ for every $c \in \mathcal{R}_p$.*

Polynomial functions $p\colon X^n \to X$ satisfying $\mathcal{R}_p = X$ are referred to as (*discrete*) *Sugeno integrals*. By Fact 1, Sugeno integrals are exactly those polynomial functions $q\colon X^n \to X$ satisfying $q(\mathbf{0}) = 0$ and $q(\mathbf{1}) = 1$, and thus every polynomial function $p\colon X^n \to X$ is of the form $p = \langle q \rangle_p$ for a suitable Sugeno integral $q\colon X^n \to X$ (see [17]). The following result is due to Goodstein [12].

Proposition 2. *Every polynomial function is completely determined by its restriction to $\{0,1\}^n$. Moreover, a function $g\colon \{0,1\}^n \to X$ can be extended to a polynomial function $f\colon X^n \to X$ if and only if it is order-preserving, and in this case the extension is unique.*

As observed by Goodstein [12], polynomial functions on bounded distributive lattices have neat normal form representations. To this extent, for each $I \subseteq [n]$, let $\mathbf{e}_I$ be the element of X^n whose ith component is 1, if $i \in I$, and 0, otherwise. Let $\alpha_f\colon 2^{[n]} \to X$ and $\beta_f\colon 2^{[n]} \to X$ be the functions defined by $\alpha_f(I) = f(\mathbf{e}_I)$ and $\beta_f(I) = f(\mathbf{e}_{[n] \setminus I})$, respectively.

Proposition 3. *A function $f\colon X^n \to X$ is a polynomial function if and only if*

$$(6) \qquad f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(\alpha_f(I) \wedge \bigwedge_{i \in I} x_i \Big) \quad or \quad f(\mathbf{x}) = \bigwedge_{I \subseteq [n]} \Big(\beta_f(I) \vee \bigvee_{i \in I} x_i \Big).$$

The expressions given in (6) are usually referred to as the *disjunctive normal form* (DNF) representation and the *conjunctive normal form* (CNF) representation, respectively, of the polynomial function f.

In the sequel, we will make use of some characterizations of polynomial functions obtained in [7, 9]. For the sake of self-containment, we recall these results.

Let $S \subseteq X$. A function $f\colon X^n \to X$ is said to be

- *S-idempotent* if $\delta_f(c) = c$, for every $c \in S$.
- $\wedge_S$-*homogeneous* if $f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge c$ for all $\mathbf{x} \in X^n$ and $c \in S$.
- $\vee_S$-*homogeneous* if $f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee c$ for all $\mathbf{x} \in X^n$ and $c \in S$.
- *median decomposable* if $f(\mathbf{x}) = \mathrm{med}\left(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\right)$ for all $\mathbf{x} \in X^n$ and $k \in [n]$.

Theorem 4. *Let $f\colon X^n \to X$ be a function. The following conditions are equivalent:*

(i) *f is a polynomial function.*
(ii) *f is median decomposable.*
(iii) *f is order-preserving, and $\wedge_{\overline{\mathcal{R}}_f}$- and $\vee_{\overline{\mathcal{R}}_f}$-homogeneous.*

In the case when X is a chain, Theorem 4 can be drastically refined for the conditions provided only need to be verified on n-tuples of a certain prescribed type (see [8]). Moreover, further characterizations are available, and given in terms of conditions of somewhat different flavor, as the following theorem illustrates.

Let σ be a permutation on $[n]$. The *standard simplex* of X^n associated with σ is the subset $X_\sigma^n \subset X^n$ defined by $X_\sigma^n = \left\{ \mathbf{x} \in X^n \colon x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \cdots \leqslant x_{\sigma(n)} \right\}$.

Theorem 5 ([8]). *Let X be a bounded chain. A function $f\colon X^n \to X$ is a polynomial function if and only if it is $\overline{\mathcal{R}}_f$-idempotent, and comonotonic minitive and comonotonic maxitive, that is, for every permutation σ on $[n]$, and every $\mathbf{x}, \mathbf{x}' \in X_\sigma^n$,*

$$f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}') \quad \text{and} \quad f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}'), \quad \text{resp.}$$

4. Quasi-polynomial functions and quasi-Sugeno integrals

The notions of polynomial function and Sugeno integral can be naturally extended to functions defined on a bounded distributive lattice X and valued on a possibly different bounded distributive lattice Y via the concepts of quasi-polynomial function and quasi-Sugeno integral.

Definition 6. We say that a function $f\colon X^n \to Y$ is a *quasi-polynomial function* (resp. a *quasi-Sugeno integral*) if there exist a polynomial function (resp. a Sugeno integral) $p\colon Y^n \to Y$ and a unary function $\varphi\colon X \to Y$, satisfying $\varphi = \langle \varphi \rangle_\varphi$, such that $f = p \circ \varphi$, that is,

$$(7) \qquad\qquad f(x_1, \ldots, x_n) = p(\varphi(x_1), \ldots, \varphi(x_n)).$$

Remark 2. The condition $\varphi = \langle \varphi \rangle_\varphi$ in Definition 6 simply ensures that the values of φ are not too scattered with respect to $\varphi(0)$ and $\varphi(1)$. More precisely, the values of φ lie in the interval $[\varphi(0) \wedge \varphi(1), \varphi(0) \vee \varphi(1)]$. In the case when φ is monotone, this condition is satisfied since it translates into saying that $\varphi(0) \leqslant \varphi(x) \leqslant \varphi(1)$ or $\varphi(1) \leqslant \varphi(x) \leqslant \varphi(0)$.

It is easy to see that the functions p and φ in (7) need not be unique. For instance, if f is a constant $c \in Y$, then we could choose $p \equiv c$ and φ arbitrarily, or p idempotent and $\varphi \equiv c$. We make use of the following lemma to show that we can always choose δ_f for φ.

Lemma 7. *Every polynomial function $p \colon X^n \to X$ satisfies $p(\mathbf{x}) = p(\langle \mathbf{x} \rangle_p)$, $p(\mathbf{x} \vee c) = p(\mathbf{x}) \vee \langle c \rangle_p$, and $p(\mathbf{x} \wedge c) = p(\mathbf{x}) \wedge \langle c \rangle_p$ for every $\mathbf{x} \in X^n$ and every $c \in X$.*

Proof. Since p is $\vee_{\overline{\mathcal{R}}_p}$- and $\wedge_{\overline{\mathcal{R}}_p}$-homogeneous, we have $p(\mathbf{x}) = p(\langle \mathbf{x} \rangle_p)$. We then have $p(\mathbf{x}) \vee \langle c \rangle_p = p(\langle \mathbf{x} \rangle_p) \vee \langle c \rangle_p = p(\langle \mathbf{x} \vee c \rangle_p) = p(\mathbf{x} \vee c)$. The last identity can be proved dually. □

Proposition 8. *Let $p \colon Y^n \to Y$ be a polynomial function, $\varphi \colon X \to Y$ a unary function, and set $f = p \circ \varphi$. Then, we have $\delta_f = \langle \varphi \rangle_p$ and $f = p \circ \delta_f$. In particular, if p is a Sugeno integral, then we have $\varphi = \delta_f$.*

Proof. By Fact 1, we have $\delta_f = \delta_p \circ \varphi = \langle \varphi \rangle_p$ and hence, by Lemma 7, $f = p \circ \varphi = p \circ \langle \varphi \rangle_p = p \circ \delta_f$. The second part follows immediately. □

To explicitly describe all possible factorizations of quasi-polynomial functions into compositions of polynomial functions with unary maps, we shall make use of the following useful tool.

Lemma 9. *Let $p \colon Y^n \to Y$ be a polynomial function, $\varphi \colon X \to Y$ a unary function, and set $f = p \circ \varphi$. Then $\langle p(\mathbf{x}) \rangle_f = p(\langle \mathbf{x} \rangle_\varphi)$ for every $\mathbf{x} \in Y^n$. In particular, if $\varphi = \langle \varphi \rangle_\varphi$, then $f = \langle f \rangle_f$.*

Proof. For every $\mathbf{x} \in Y^n$, we have

$$
\begin{aligned}
\langle p(\mathbf{x}) \rangle_f &= \operatorname{med}\big(f(\mathbf{0}), p(\mathbf{x}), f(\mathbf{1})\big) \\
&= \big(f(\mathbf{0}) \wedge f(\mathbf{1})\big) \vee \big(p(\mathbf{x}) \wedge \big(f(\mathbf{0}) \vee f(\mathbf{1})\big)\big) \\
&= \big\langle \varphi(0) \wedge \varphi(1) \big\rangle_p \vee \big(p(\mathbf{x}) \wedge \big\langle \varphi(0) \vee \varphi(1) \big\rangle_p\big) \quad \text{(by Proposition 8)} \\
&= p\big(\big(\varphi(0) \wedge \varphi(1)\big) \vee \big(\mathbf{x} \wedge (\varphi(0) \vee \varphi(1))\big)\big) \quad \text{(by Lemma 7)} \\
&= p(\langle \mathbf{x} \rangle_\varphi).
\end{aligned}
$$

In particular, if $\varphi = \langle \varphi \rangle_\varphi$, then $\langle f \rangle_f = \langle p \circ \varphi \rangle_f = p \circ \langle \varphi \rangle_\varphi = p \circ \varphi = f$. □

The following proposition provides alternative factorizations of a quasi-polynomial function. Let $p_f \colon Y^n \to Y$ be the unique polynomial function extending $\hat{f}$ (see Proposition 2).

Proposition 10. *Let $f \colon X^n \to Y$ be a quasi-polynomial function, $p \colon Y^n \to Y$ a polynomial function, and $\varphi \colon X \to Y$ a unary function satisfying $\varphi = \langle \varphi \rangle_\varphi$. Then we have $f = p \circ \varphi$ if and only if $p_f = \langle p \rangle_f$ and $\delta_f = \langle \varphi \rangle_p$. In particular, we have $f = p_f \circ \delta_f$.*

Proof. We first establish the following claim.

Claim. *If $f = p \circ \varphi$, then $\delta_f = \langle \varphi \rangle_p$, $f = p \circ \delta_f$, and $\hat{f} = \langle p \rangle_f|_{\{0,1\}^n}$.*

Proof of the claim. The first two formulas follow immediately from Proposition 8. By Lemma 9, for every $\mathbf{e} \in \{0,1\}^n$, we have $\langle p(\mathbf{e}) \rangle_f = p(\langle \mathbf{e} \rangle_\varphi)$, where $\langle \mathbf{e} \rangle_\varphi$ is the n-tuple whose ith component is $\varphi(0) \vee \varphi(1)$, if $e_i = 1$, and $\varphi(0) \wedge \varphi(1)$, if $e_i = 0$. Since p preserves $\vee$ and $\wedge$ componentwise, when expanding $p(\langle \mathbf{e} \rangle_\varphi)$ in terms of $\vee$

and then in terms of $\wedge$, we obtain $\langle p(\mathbf{e}) \rangle_f = \hat{f}(\mathbf{e})$. To illustrate, if $\mathbf{e} = (0,1)$, we have

$$
\begin{aligned}
\langle p(0,1) \rangle_f &= p\big(\varphi(0) \wedge \varphi(1), \varphi(0) \vee \varphi(1)\big) \\
&= p\big(\varphi(0) \wedge \varphi(1), \varphi(0)\big) \vee p\big(\varphi(0) \wedge \varphi(1), \varphi(1)\big) \\
&= \big(p(\varphi(0), \varphi(0)) \wedge p(\varphi(1), \varphi(0))\big) \vee \big(p(\varphi(0), \varphi(1)) \wedge p(\varphi(1), \varphi(1))\big) \\
&= \big(f(0,0) \wedge f(1,0)\big) \vee \big(f(0,1) \wedge f(1,1)\big) \\
&= \hat{f}(0,1) \qquad \text{(by (4)).} \quad \square
\end{aligned}
$$

(Necessity) Suppose that $f = p \circ \varphi$. By the claim, we have $\delta_f = \langle \varphi \rangle_p$ and $p_f|_{\{0,1\}^n} = \hat{f} = \langle p \rangle_f|_{\{0,1\}^n}$. This completes the proof since two polynomial functions having the same values on $\{0,1\}^n$ coincide by Proposition 2.

(Sufficiency) Since f is a quasi-polynomial function, there exist a polynomial function $q \colon Y^n \to Y$ and a unary function $\psi \colon X \to Y$, satisfying $\psi = \langle \psi \rangle_\psi$, such that $f = q \circ \psi$. By the claim, we then have $\delta_f = \langle \psi \rangle_q$, $f = q \circ \delta_f$, and

$$
(8) \qquad \langle q \rangle_f|_{\{0,1\}^n} = \hat{f} = p_f|_{\{0,1\}^n} = \langle p \rangle_f|_{\{0,1\}^n},
$$

which, by Proposition 2 and (8), implies that

$$
(9) \qquad \langle q \rangle_f = \langle p \rangle_f.
$$

Therefore, we have

$$
\begin{aligned}
p \circ \varphi &= p \circ \langle \varphi \rangle_\varphi \\
&= \langle p \circ \varphi \rangle_{p \circ \varphi} \qquad \text{(by Lemma 9)} \\
&= \langle p \circ \varphi \rangle_{\delta_{p \circ \varphi}} \qquad \text{(by (2))} \\
&= \langle p \circ \varphi \rangle_{\langle \varphi \rangle_p} \qquad \text{(by Proposition 8)} \\
&= \langle p \circ \langle \varphi \rangle_p \rangle_{\langle \varphi \rangle_p} \qquad \text{(by Lemma 7)} \\
&= \langle p \circ \delta_f \rangle_{\delta_f} \\
&= \langle p \circ \delta_f \rangle_f \qquad \text{(by (2))} \\
&= \langle p \rangle_f \circ \delta_f \\
&= \langle q \rangle_f \circ \delta_f \qquad \text{(by (9))} \\
&= \langle q \circ \delta_f \rangle_f \\
&= \langle f \rangle_f \\
&= f \qquad \text{(by Lemma 9).} \quad \square
\end{aligned}
$$

Remark 3. As mentioned in Remark 1, there are several ways of constructing an order-preserving function from a given $f \colon X^n \to Y$, e.g., using an expression in DNF (as in (3)) or using an expression in CNF (as in (5)). Even though, in general, different constructions may lead to different order-preserving functions (as in the case of the Boolean sum), by the claim it follows that, if f is a quasi-polynomial function, then all possible rearrangements of $\vee$ and $\wedge$ produce the same order-preserving function $\hat{f}$. However, the converse statement does not hold, since there are order-preserving functions which do not constitute quasi-polynomial functions.

Recall that every polynomial function $p \colon X^n \to X$ can be represented as $\langle q \rangle_p$ for some Sugeno integral $q \colon X^n \to X$. Using this fact, we obtain the following result.

Proposition 11. *A function $f\colon X^n \to Y$ is a quasi-polynomial function if and only if it is a quasi-Sugeno integral.*

Proof. Since Sugeno integrals are polynomial functions, it follows that every quasi-Sugeno integral is a quasi-polynomial function. For the converse, let $f\colon X^n \to Y$ be a quasi-polynomial function as described in (7). Let $q\colon Y^n \to Y$ be a Sugeno integral such that $p = \langle q\rangle_p$. Since q is $\wedge_Y$- and $\vee_Y$-homogeneous (see Theorem 4), we have $q \circ \langle\varphi\rangle_p = \langle q \circ \varphi\rangle_p = \langle q\rangle_p \circ \varphi = p \circ \varphi = f$, which shows that f is a quasi-Sugeno integral. $\qquad\square$

Remark 4. Proposition 11 shows that monotone quasi-polynomial functions are of interest in the ordinal settings of decision making since they coincide with monotone quasi-Sugeno integrals, which were characterized as overall preference functionals for instance in decision under uncertainty; see [11].

We now consider a number of characterizations of the class of quasi-polynomial functions. These characterizations are inspired from those obtained by the authors [6] in the special case when $X = Y$ is a bounded chain.

We say that a function $f\colon X^n \to Y$ is *quasi-median decomposable* if, for every $\mathbf{x} \in X^n$ and every $k \in [n]$, we have

$$(10) \qquad\qquad f(\mathbf{x}) = \mathrm{med}\left(f(\mathbf{x}_k^0), \delta_f(x_k), f(\mathbf{x}_k^1)\right).$$

Note that every unary function $\varphi\colon X \to Y$ satisfying $\varphi = \langle\varphi\rangle_\varphi$ (in particular, every monotone unary function) is quasi-median decomposable. The following theorem provides a characterization for quasi-polynomial functions in terms of quasi-median decomposition.

Theorem 12. *A function $f\colon X^n \to Y$ is a quasi-polynomial function if and only if it is quasi-median decomposable.*

Proof. To verify that the condition is sufficient, just observe that applying (10) repeatedly to each variable of f we can straightforwardly obtain a representation of f as $f = p \circ \delta_f$ for some polynomial function p. Moreover, we have $\delta_f = \langle\delta_f\rangle_{\delta_f}$.

Conversely, suppose that $f\colon X^n \to Y$ is a quasi-polynomial function. By Proposition 8 and Lemma 9, we have $f = \langle f\rangle_f$ and there exists a polynomial function $p\colon Y^n \to Y$ such that $f = p \circ \delta_f$. By Theorem 4, for every $\mathbf{x} \in X^n$ and every $k \in [n]$, we have

$$f(\mathbf{x}) = \langle f(\mathbf{x})\rangle_f = \left\langle \mathrm{med}\left(p(\delta_f(\mathbf{x})_k^0), \delta_f(x_k), p(\delta_f(\mathbf{x})_k^1)\right)\right\rangle_f,$$

that is, since med and $\langle\cdot\rangle_f$ commute,

$$(11) \qquad f(\mathbf{x}) = \mathrm{med}\left(\langle p(\delta_f(\mathbf{x})_k^0)\rangle_f, \delta_f(x_k), \langle p(\delta_f(\mathbf{x})_k^1)\rangle_f\right).$$

However, by using Lemma 9 twice, we obtain

$$\begin{aligned}
\langle p(\delta_f(\mathbf{x})_k^0)\rangle_f &= p(\langle\delta_f(\mathbf{x})_k^0\rangle_{\delta_f}) = p\left(\langle\delta_f(\mathbf{x})_k^{\delta_f(0)\wedge\delta_f(1)}\rangle_{\delta_f}\right) \\
&= \langle p\left(\delta_f(\mathbf{x})_k^{\delta_f(0)\wedge\delta_f(1)}\right)\rangle_f
\end{aligned}$$

and, since p preserves $\vee$ and $\wedge$ componentwise, we also have

$$\langle p(\delta_f(\mathbf{x})_k^0)\rangle_f = \langle f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1)\rangle_f = f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1).$$

Similarly, we have $\langle p(\delta_f(\mathbf{x})_k^1)\rangle_f = f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1)$. Combining this with (11), we see that f is quasi-median decomposable. $\qquad\square$

We say that a function $f\colon X^n \to Y$ is *quasi-$\vee$-homogeneous* (resp. *quasi-$\wedge$-homogeneous*) if for every $\mathbf{x} \in X^n$ and $c \in X$, we have

$$f(\mathbf{x} \vee c) = f(\mathbf{x}) \vee \delta_f(c) \qquad (\text{resp. } f(\mathbf{x} \wedge c) = f(\mathbf{x}) \wedge \delta_f(c)).$$

Note that, for every quasi-$\vee$-homogeneous (resp. quasi-$\wedge$-homogeneous) function $f\colon X^n \to Y$, the diagonal section δ_f preserves $\vee$ (resp. $\wedge$).

A function $f\colon X^n \to Y$ is said to be *horizontally $\wedge$-decomposable* (resp. *horizontally $\vee$-decomposable*) if for every $\mathbf{x} \in X^n$ and $c \in X$, we have

$$f(\mathbf{x}) = f(\mathbf{x} \vee c) \wedge f([\mathbf{x}]^c) \qquad (\text{resp. } f(\mathbf{x}) = f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c)).$$

By considering functions $f\colon X^n \to Y$ whose diagonal section δ_f is a lattice homomorphism (i.e., δ_f preserves both $\vee$ and $\wedge$), we can further extend some of the characterizations presented in [6] to the present setting. Note that any of these preservation conditions immediately ensures the order-preservation of δ_f, in which case, by Fact 1 and Proposition 8, if f is a quasi-polynomial function, then it is necessarily order-preserving.

Theorem 13. *Let $f\colon X^n \to Y$ be an order-preserving function whose diagonal section δ_f is a lattice homomorphism. The following are equivalent:*

(i) f *is a quasi-polynomial function.*
(ii) f *is quasi-$\wedge$-homogeneous and quasi-$\vee$-homogeneous.*
(iii) f *is quasi-$\wedge$-homogeneous and horizontally $\vee$-decomposable.*
(iv) f *is horizontally $\wedge$-decomposable and quasi-$\vee$-homogeneous.*

Proof. We show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. The equivalence $(i) \Leftrightarrow (iv)$ follows dually.

So suppose that $f\colon X^n \to Y$ is a quasi-polynomial function whose diagonal section δ_f is a lattice homomorphism. By Propositions 8 and 11, we may assume that $f = p \circ \delta_f$ for some Sugeno integral $p\colon Y^n \to Y$. Since p is $\wedge_Y$-homogeneous, we have for every $c \in X$,

$$\begin{aligned}
f(\mathbf{x} \wedge c) &= p(\delta_f(\mathbf{x} \wedge c)) = p(\delta_f(\mathbf{x}) \wedge \delta_f(c)) = p(\delta_f(\mathbf{x})) \wedge \delta_f(c) \\
&= f(\mathbf{x}) \wedge \delta_f(c),
\end{aligned}$$

which shows that f is quasi-$\wedge$-homogeneous. Dually, it can be shown that f is quasi-$\vee$-homogeneous.

To see that $(ii) \Rightarrow (iii)$, suppose that $f\colon X^n \to Y$ satisfies (ii). Since f is order-preserving, for every $c \in X$, we have

$$\begin{aligned}
f(\mathbf{x} \wedge c) \vee f([\mathbf{x}]_c) &= (f(\mathbf{x}) \wedge \delta_f(c)) \vee f([\mathbf{x}]_c) = f(\mathbf{x}) \wedge (\delta_f(c) \vee f([\mathbf{x}]_c)) \\
&= f(\mathbf{x}) \wedge f([\mathbf{x}]_c \vee c) = f(\mathbf{x}),
\end{aligned}$$

thus showing that (iii) holds.

To show that $(iii) \Rightarrow (i)$, by Theorem 12, it is enough to show that (iii) implies that f is quasi-median decomposable. Let $\mathbf{x} \in X^n$ and take $k \in [n]$. Since f is horizontally $\vee$-decomposable, we have

$$f(\mathbf{x}) = f(\mathbf{x} \wedge x_k) \vee f([\mathbf{x}]_{x_k}).$$

By quasi-$\wedge$-homogeneity, we have

$$f(\mathbf{x} \wedge x_k) = f(\mathbf{x}_k^1 \wedge x_k) = f(\mathbf{x}_k^1) \wedge \delta_f(x_k)$$

and by the definition of $[\mathbf{x}]_{x_k}$, we have $f([\mathbf{x}]_{x_k}) \leqslant f(\mathbf{x}_k^0)$. Thus,

$$
\begin{aligned}
f(\mathbf{x}) &= \left(f(\mathbf{x}_k^0) \vee f(\mathbf{x})\right) \wedge f(\mathbf{x}_k^1) = \left(f(\mathbf{x}_k^0) \vee (f(\mathbf{x}_k^1) \wedge \delta_f(x_k))\right) \wedge f(\mathbf{x}_k^1) \\
&= f(\mathbf{x}_k^0) \vee \left(f(\mathbf{x}_k^1) \wedge \delta_f(x_k)\right) = \operatorname{med}\left(f(\mathbf{x}_k^0), \delta_f(x_k), f(\mathbf{x}_k^1)\right),
\end{aligned}
$$

which completes the proof of the theorem. $\square$

Let Y be a bounded chain. We say that a function $f \colon X^n \to Y$ is *quasi-comonotonic minitive* (resp. *quasi-comonotonic maxitive*) if for every permutation σ on $[n]$ and every $\mathbf{x}, \mathbf{x}' \in X^n$ such that $\delta_f(\mathbf{x}), \delta_f(\mathbf{x}') \in Y_\sigma^n$, we have

$$
f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}') \quad (\text{resp. } f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}')).
$$

By Theorem 5, it follows that every quasi-polynomial function whose diagonal section δ_f is a lattice homomorphism is quasi-comonotonic minitive and maxitive. Moreover, it is easy to verify that if a function is quasi-comonotonic minitive (resp. quasi-comonotonic maxitive), then it is quasi-$\wedge$-homogeneous (resp. quasi-$\vee$-homogeneous). Thus, using Theorem 13 (*ii*), we obtain the following characterization of quasi-polynomial functions valued on bounded chains.

Theorem 14. *Let X be a bounded distributive lattice, Y a bounded chain, and $f \colon X^n \to Y$ an order-preserving function whose diagonal section δ_f is a lattice homomorphism. Then f is a quasi-polynomial function if and only if it is quasi-comonotonic minitive and quasi-comonotonic maxitive.*

Remark 5. Theorems 12, 13, and 14 extend to the present setting some results obtained in [6] when $X = Y$ is a bounded chain. Order-preserving quasi-polynomial functions are also characterized in the latter case by order-preservation and horizontally $\wedge$- and $\vee$-decompositions (see [6, Theorem 11]). It remains open whether this result still holds in the present general setting.

We now use Theorem 13 to derive further characterizations of the class of polynomial functions. We first recall the following lemma given in [9].

Lemma 15. *A unary function $f \colon X \to X$ is a polynomial function if and only if f is a solution of the equation $f \circ f = f$, is a lattice homomorphism, and has a convex range.*

The following proposition provides conditions under which a quasi-polynomial function is a polynomial function.

Proposition 16. *Let $f \colon X^n \to X$ be a quasi-polynomial function. Then f is a polynomial function if and only if it is $\mathcal{R}_f$-idempotent and δ_f is a lattice homomorphism with a convex range.*

Proof. The necessity is straightforward (see [9]). Let us prove the sufficiency. Since f is $\mathcal{R}_f$-idempotent, we have $\delta_f \circ \delta_f = \delta_f$ and, by Lemma 15, δ_f is a polynomial function. Since f is a quasi-polynomial function, by Proposition 8, f is a polynomial function. $\square$

By combining Theorem 13 with Proposition 16, we obtain the following characterizations of the class of polynomial functions.

Corollary 17. *Let $f \colon X^n \to X$ be an order-preserving and $\mathcal{R}_f$-idempotent function whose diagonal section δ_f is a lattice homomorphism with a convex range. The following are equivalent:*

> (i) f is a polynomial function.
> (ii) f is quasi-$\wedge$-homogeneous and quasi-$\vee$-homogeneous.
> (iii) f is quasi-$\wedge$-homogeneous and horizontally $\vee$-decomposable.
> (iv) f is horizontally $\wedge$-decomposable and quasi-$\vee$-homogeneous.

5. Transformed polynomial functions and Sugeno integrals

We have defined quasi-polynomial functions by considering polynomial functions whose variables are first transformed by a certain unary function. Instead of transforming the variables, we could transform the polynomial function itself. This leads to the following definition.

Definition 18. We say that a function $f\colon X^n \to Y$ is a *transformed polynomial function* (resp. *a transformed Sugeno integral*) if there exist a polynomial function (resp. a Sugeno integral) $p\colon X^n \to X$ and a function $\psi\colon X \to Y$ such that $f = \psi \circ p$, that is,

$$(12) \qquad f(x_1, \ldots, x_n) = \psi(p(x_1, \ldots, x_n)).$$

The functions ψ and p in (12) need not be unique. For instance, if f is a constant $c \in Y$, then we could choose $\psi \equiv c$ and p arbitrarily. However, if $c \in X$, then we could as well choose $p \equiv c$ and ψ arbitrarily except $\psi(c) = c$. The following result shows that we can always choose δ_f for ψ.

Proposition 19. *Let $f\colon X^n \to Y$ be a transformed polynomial function as described in (12). Then, we have $f = \delta_f \circ p$. In particular, if p is a Sugeno integral, then we have $\psi = \delta_f$.*

Proof. Since any polynomial function $p\colon X^n \to X$ is $\mathcal{R}_p$-idempotent, we have $\delta_f \circ p = \psi \circ \delta_p \circ p = \psi \circ p = f$. The second part follows immediately since then $\delta_p = \mathrm{id}_X$ is the identity function on X. $\qquad\square$

We also have the following result, which is the counterpart of Proposition 11.

Proposition 20. *A function $f\colon X^n \to Y$ is a transformed polynomial function if and only if it is a transformed Sugeno integral.*

Proof. Clearly, any transformed Sugeno integral is a transformed polynomial function. Conversely, let $f\colon X^n \to Y$ be a transformed polynomial function as described in (12). Let $q\colon X^n \to X$ be a Sugeno integral such that $p = \langle q \rangle_p = \delta_p \circ q$ (cf. Fact 1). It follows that $f = \psi \circ p = \psi \circ \delta_p \circ q = \delta_f \circ q$ is a transformed Sugeno integral. $\quad\square$

Denote the domain of a function f by $\mathcal{D}_f$. Recall that a function h is a *right-inverse* [1, p. 25] of a function f if $\mathcal{D}_h = \mathcal{R}_f$, $\mathcal{R}_h \subseteq \mathcal{D}_f$, and $f \circ h = \mathrm{id}_{\mathcal{R}_f}$. It can be shown that the statement "every function has at least one right-inverse" is equivalent to the axiom of choice. Even though, in general, we have to appeal to the axiom of choice in order to ensure the existence of right-inverses, this requirement is not necessary in many concrete situations, for instance, when dealing with monotone functions over the real numbers.

We say that f is *quasi-idempotent* if $\mathcal{R}_{\delta_f} = \mathcal{R}_f$. The terminology "quasi-idempotent" is justified by the following result (see [18] for the real case).

Proposition 21. *Let $f\colon X^n \to Y$ be a function. Under the axiom of choice, f is quasi-idempotent if and only if there is an idempotent function $g\colon X^n \to X$ and a function $\psi\colon \mathcal{R}_g \to Y$ such that $f = \psi \circ g$. In this case, $\psi = \delta_f$.*

Proof. Sufficiency is straightforward. We have $\mathcal{R}_{\delta_f} = \mathcal{R}_\psi = \mathcal{R}_{\psi \circ g} = \mathcal{R}_f$. For the necessity we observe that, if $h \colon \mathcal{R}_{\delta_f} \to X$ is a right-inverse of δ_f, then the function $g \colon X^n \to X$, defined by $g(\mathbf{x}) = x_1$ if $x_1 = \cdots = x_n$ and $g(\mathbf{x}) = (h \circ f)(\mathbf{x})$ otherwise, is idempotent and satisfies $f = \delta_f \circ g$. $\qquad\square$

The following theorem yields a characterization of transformed polynomial functions whose diagonal sections are lattice homomorphisms. These functions are clearly order-preserving by Fact 1 and Proposition 19.

Theorem 22. *Let $f \colon X^n \to Y$ be a function and assume that δ_f is a lattice homomorphism. Then, under the axiom of choice, f is a transformed polynomial function if and only if it is a quasi-idempotent quasi-polynomial function.*

Proof. Suppose $f \colon X^n \to Y$ is a quasi-idempotent quasi-polynomial function. Let $h \colon \mathcal{R}_{\delta_f} \to X$ be a right-inverse of δ_f, i.e., $\delta_f \circ h = \mathrm{id}_{\mathcal{R}_{\delta_f}}$, and define $\psi = \delta_f$ and $p = p_{h \circ f}$, where $p_{h \circ f}$ is the unique polynomial function extending $h \circ f|_{\{0,1\}^n}$ (see Proposition 2), that is,

$$p_{h \circ f}(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big((h \circ f)(\mathbf{e}_I) \wedge \bigwedge_{i \in I} x_i \Big).$$

Since δ_f preserves $\vee$ and $\wedge$, we have $\psi \circ p = \delta_f \circ p_{h \circ f} = p_f \circ \delta_f = f$, which shows that f is a transformed polynomial function.

Conversely, suppose $f \colon X^n \to Y$ is a transformed polynomial function. By Propositions 20 and 21, f is quasi-idempotent. Moreover, by Proposition 19, we have $f = \delta_f \circ p = p_{\delta_f \circ p} \circ \delta_f = p_f \circ \delta_f$, which shows that f is a quasi-polynomial function. $\qquad\square$

Remark 6.   (i) We have seen in the proof of Theorem 22 that a transformed polynomial function $f \colon X^n \to Y$ whose δ_f is a lattice homomorphism satisfies the equations $f = p_f \circ \delta_f = \delta_f \circ p_{h \circ f}$ for any right-inverse h of δ_f. In a sense, the transformation and polynomial function commute since $p_{h \circ f}$ has just the same $\vee$-$\wedge$ form as p_f.

  (ii) Quasi-idempotency is necessary in Theorem 22. Indeed, the real quasi-polynomial function $f \colon [0,1]^2 \to [0,1]$, defined by $f = p \circ \varphi$, where $\varphi(x) = 1$ if $x \in [\frac{1}{2}, 1]$ and 0 otherwise, and $p(x_1, x_2) = \mathrm{med}(x_1 \wedge x_2, \frac{1}{2}, x_1 \vee x_2)$, is not quasi-idempotent since $f(1,0) = f(0,1) = \frac{1}{2} \notin \mathcal{R}_{\delta_f}$.

  (iii) By combining Theorem 22 with existing characterizations of quasi-polynomial functions (such as those presented in Section 4), we immediately generate characterizations of transformed polynomial functions.

By combining Proposition 16 and Theorem 22, we obtain the following result which yields conditions under which a transformed polynomial function is a polynomial function. Observe that, in Theorem 22, the appeal to the axiom of choice is used only to show that the conditions are sufficient.

Proposition 23. *Let $f \colon X^n \to X$ be a transformed polynomial function. Then f is a polynomial function if and only if it is $\mathcal{R}_f$-idempotent and δ_f is a lattice homomorphism with a convex range.*

6. Concluding remarks and future work

In this paper we considered quasi-polynomial functions as mappings defined and valued on, possibly different, bounded distributive lattices, and not necessarily order-preserving. The relevance of this concept in fields such as decision making was made apparent by showing that monotone quasi-polynomial functions coincide exactly with those overall preference functionals which can be factorized into Sugeno integrals applied to a utility function. We provided several axiomatizations for this class of quasi-polynomial functions, which subsume those presented in [6], and explicitly described all possible factorizations of a given quasi-polynomial function as a composition of a polynomial function with a unary map. Moreover, we introduced the notion of transformed polynomial function as a natural counterpart of quasi-polynomial function, and characterized the class of transformed polynomial functions accordingly. As it turned out, under the axiom of choice, those transformed polynomial functions whose diagonal section is a lattice homomorphism constitute a proper subclass of quasi-polynomial functions.

Looking at natural extensions to this framework, we are inevitably drawn to consider the multi-sorted setting. More precisely, we are interested in mappings $f\colon X_1 \times \cdots \times X_n \to Y$ which can be factorized as a composition

$$f(x_1, \ldots, x_n) = p(\varphi_1(x_1), \ldots, \varphi_n(x_n))$$

where $p\colon Y^n \to Y$ is a polynomial function, and each $\varphi_i\colon X_i \to Y$ is a unary map defined and valued on, possibly different, bounded distributive lattices X_i and Y. These functions appear naturally within the scope of multicriteria decision making (see for instance Bouyssou et al. [4]), and their axiomatization constitutes a topic of future research.

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Mathematics Research Unit, University of Luxembourg, 162A, avenue de la Faïencerie, L-1511 Luxembourg, Luxembourg

E-mail address: `miguel.couceiro[at]uni.lu`

Mathematics Research Unit, University of Luxembourg, 162A, avenue de la Faïencerie, L-1511 Luxembourg, Luxembourg

E-mail address: `jean-luc.marichal[at]uni.lu`