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BEREZIN-TOEPLITZ QUANTIZATION ON K3 SURFACES AND HYPERKÄHLER BEREZIN-TOEPLITZ QUANTIZATION

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Introduction

Abstract

Given a quantizable Kähler manifold, the Berezin-Toeplitz quantization scheme constructs a quantization in a canonical way. In their seminal paper Martin Bordemann, Eckhard Meinrenken and Martin Schlichenmaier proved that for a compact Kähler manifold such scheme is a well defined quantization which has the correct semiclassical limit.

However, there are some manifolds which admit more than one (non-equivalent) Kähler structure. The question arises then, whether the choice of a different Kähler structure gives rise to a completely different quantizations or the resulting quantizations are related.

An example of such objects are the so called K3 surfaces, which have some extra relations between the different Kähler structures. In this work, we consider the family of K3 surfaces which admit more than one quantizable Kähler structure and we use the relations between the different Kähler structures to study whether the corresponding quantizations are related or not. In particular, we prove that such K3 surfaces have always Picard number 20, which implies that their moduli space is discrete, and that the resulting quantum Hilbert spaces are always isomorphic, although not always in a canonical way. However, there exists an infinite subfamily of K3 surfaces for which the isomorphism is canonical.

We also define new quantization operators on the product of the different quantum Hilbert spaces and we call this process Hyperkähler quantization. We prove that these new operators have the semiclassical limit, as well as new properties inherited from the quaternionic numbers.

Motivation and Main Results

Classical physics studies the properties and behavior of planets, fluids, electromagnetism and other *macroscopic* structures. Its most remarkable characteristic is the fact that it is deterministic: a closed system's state in a future time t depends only on the system's state on a fixed time $t_0 < t$.

One of the main mathematical tools for studying classical systems are the so called *symplectic structures.* A *symplectic manifold* is a smooth manifold equipped with a closed non-degenerated two form ω, which is called *symplectic form.* Such a differential form identifies canonically the tangent space and the cotangent space of the manifold. Each classical system is described by a *Hamiltonian function* H, which is related to the energy of the system. The symplectic form allows one to transform such Hamiltonian function into a *Hamiltonian vector* field which describes the flow of the system. The Hamiltonian vector field X_H is characterized uniquely by:

$$\mathrm{d}H = \omega\left(X_H, \cdot\right).$$

The origin of symplectic structures can be found in [Lag08], where Lagrange applied his method of variation of the constants to study the motion of the Earth around the Sun. *Poisson structures* generalize symplectic structures and were first described by Siméon Denis Poisson ([Poi08]). It should be remarked however that even though symplectic and Poisson geometry have a physical origin, they are now independent mathematical concepts of great importance.

Quantum physics, on the other hand studies the behavior of *microscopic structures*, such as atoms and quarks. In the early 1890s, Max Planck was asked to design a light bulb that produced the maximum amount of light using the minimal amount of energy. His first predictions based on classical electromagnetism did not agree with experiments. In 1900, he formulated the idea that the energy is *quantized*.

Many problems in quantum physics have complex solutions. While they can always be expressed using real structures, many phenomena are naturally explained by the extra complex structure. For instance, quantum waves are modeled by the expression

$$\phi(x) = e^{ikx}$$

and the superposition of different waves corresponds to multiplication of their corresponding functions.

In the intersection of complex and symplectic geometry lies *Kähler geometry*, which enforces a *compatibility* condition between the two structures: a symplectic form ω and a complex structure I are called *compatible* if

$$g(\cdot,\cdot) := \omega\left(I\cdot,\cdot\right)$$

defines a *Riemannian metric*, which generalizes the notions of distance and angles to general manifolds. It is not surprising then that Kähler structures are a fundamental part of many mathematical models for quantum physics.

Quantization is a word used to describe any procedure of transforming *classical information* into *quantum information.* During the last two centuries, there have been different approaches to quantization, such as deformation quantization, path integral quantization, geometrical quantization and Berezin-Toeplitz quantization, each one carrying a different meaning to the word *information.* For instance Berezin-Toeplitz quantization, the object of study of this thesis, tranforms *functions* (which are used to measure physical properties of a classical system) into bounded operators acting on a Hilbert space (which measure physical properties of a quantum system).

A compact manifold is quantizable (in the sense of Berezin-Toeplitz quantization) if its Kähler form is integral. Given a quantizable simply-connected compact *Kähler manifold* one constructs a unique *prequantum line bundle* $L \to M$ in a canonical way: it is the only line bundle whose *curvature* is the *Kähler form.* This line bundle determines a non-empty finite-dimensional *Hilbert space* of *holomorphic sections* $H^0(M, L)$. Under appropriate assumptions, this line bundle is well-behaved and of interest. Berezin-Toeplitz quantization assigns to each function of the original space a family of operators acting on the family of Hilbert spaces

$$T : C^\infty(M) \longrightarrow End\left(H^0\left(M, L^m\right)\right) ,$$
$$f \longmapsto T_f^{(m)}$$

where $m \in \mathbb{N}$. In 1994, Martin Bordemann, Eckhard Meinrenken and Martin Schlichenmaier showed that for a compact Kähler manifold such scheme is a well defined quantization which has the correct *semiclassical limit*:

Theorem (Bordemann, Meinrenken, Schlichenmaier [BMS94])**.**

1. *For every* $f \in C^\infty(M)$*, there exists a* $C > 0$ *such that*

$$| f |_\infty - \frac{C}{m} \leq \left\| T_f^{(m)} \right\| \leq | f |_\infty .$$

In particular,

$$\lim_{m\to\infty} \left\| T_f^{(m)} \right\| = | f |_\infty .$$

2. *For every* $f, g \in C^\infty(M)$*,*

$$\left\| m\sqrt{-1}\left[T_f^{(m)}, T_g^{(m)}\right] - T_{\{f,g\}}^{(m)} \right\| = O\left(m^{-1}\right).$$

3. *For every* $f, g \in C^\infty(M)$*,*

$$\left\| T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)} \right\| = O\left(m^{-1}\right).$$

Hyperkähler manifolds are geometrical objects which admit three different Kähler structures (g, ω_{I_1}, I_1), (g, ω_{I_2}, I_2) and (g, ω_{I_3}, I_3) sharing the same Riemannian metric such that

$$I_1 I_2 = I_3.$$

They were first defined by Eugenio Calabi in 1978 and can be seen as a generalization of the quaternionic numbers. The simply-connected two-dimensional versions of these objects are called K3-surfaces. The existence of such manifolds creates a new interesting problem: given such a manifold, the construction of a Berezin-Toeplitz quantization is not canonical anymore, since one needs to choose a Kähler structure.

In this work, we study these different quantizations and we show that they share different properties. An important result is that the different quantum Hilbert spaces are isomorphic:

Theorem 5.16.
The spaces of holomorphic sections of the different quantum line bundles on a K3 surface have the same dimension. In particular, they are isomorphic as vector spaces.

However, in general there is no canonical choice of an isomorphism. However, Proposition 5.30 shows that an infinite subfamily allows such a canonical choice.

Another important result is the existence of K3 with two or more quantizations:

Theorem 5.28.
A K3 surface admits two or more quantizations if and only if there exists a fiber of its twistor space whose Kähler structure has Picard number 20.

This implies that the moduli space of such K3 surfaces is discrete. In particular, there is only an infinite countable number of them.

Later on we study some a generalization of the Berezin-Toeplitz operators. Fix $a = (a_1, a_2, a_3) \in \mathbb{N}^3_{\geq 0}$ and denote by $\mathbb{H}$ the quaternionic space. Denote by

$$\mathcal{H}_k = \mathcal{H}_k^a := \mathbb{H} \otimes_{\mathbb{R}} \left(H^0 \left(X, L_{I_1}^{\otimes k a_1} \right) \otimes_{\mathbb{C}} H^0 \left(X, L_{I_2}^{\otimes k a_2} \right) \otimes_{\mathbb{C}} H^0 \left(X, L_{I_3}^{\otimes k a_3} \right) \right)$$

the product Hilbert space. Since a is fixed, we will usually skip the superindex. We define a new kind of quantization operators called *Hyperkähler Berezin-Toeplitz operators* using the additive average of the original Berezin-Toeplitz operators with quaternionic coefficients

$$\tilde{\mathbb{T}}_f^{(k)} := \sum_{n=1}^{3} a_n i_n \tilde{\mathbb{T}}_f^{n,(k)},$$

where i_n are quaternionic $n = 1, 2, 3$. Denote by s the sum of the coefficients:

$$s := \sum_{n=1}^{3} a_n i_n.$$

The most important part of this thesis is the study of the properties of those operators and their behavior with respect to different products. Denote by i, j and k the three complex units of $\mathbb{H}$ such that $ij = k$. Consider following products:

1. The (Lorentzian) scalar product in which $\{1, i, j, k\}$ is an orthogonal basis with signature $(1, 3)$.

2. The cross product $i \times j = k$, $j \times i = -k$, $i \times i = 0$. Note that this product is only defined on the subspace of $\mathbb{H}$ generated by i, j and k.

3. The usual quaternionic product $i \star j = k$, $i \star i = j \star j = k \star k = -1$.

Note that the three spaces $H^0\left(X, L_{I_i}^{\otimes k a_i}\right)$ are vector spaces over the complex field $\mathbb{C}$.

One of the most important parts of this work is to study the behavior of the Hyperkähler Berezin-Toeplitz operators. In particular, we will show their behavior when multiplied with respect to the different products:

- First we will show that these new operators fulfill the semiclassical limit with respect to the scalar product $\cdot$:

  Theorem 6.9.
  Assume that i_1, i_2 and i_3 are quaternionic numbers which are orthogonal as vectors. The generalized Berezin-Toeplitz operators have:

  1. *For every $f \in C^\infty(M)$, there exists a $C > 0$ such that*
  $$| f |_\infty - \frac{C}{m} \leq \left\| s \cdot \tilde{\mathbb{T}}_f^{(m)} \right\| \leq | f |_\infty .$$
  In particular,
  $$\lim_{m \to \infty} \left\| \tilde{\mathbb{T}}_f^{(m)} \right\| = | f |_\infty .$$
  2. *For every $f, g \in C^\infty(M)$,*
  $$\left\| m\sqrt{-1} \left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^{\cdot} - s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)} \right\| = O\left(m^{-1}\right).$$
  3. *For every $f, g \in C^\infty(M)$,*
  $$\left\| \tilde{\mathbb{T}}_f^{(m)} \cdot \tilde{\mathbb{T}}_g^{(m)} - s \cdot \tilde{\mathbb{T}}_{fg}^{(m)} \right\| = O\left(m^{-1}\right).$$

  Moreover, the original theorem is shown as a particular case of this one: the case $a_2 = a_3 = 0$, $a_1 = 1$ recovers the original theorem.

- Then, we study their behavior with respect to the cross product. This gives new original properties of the Hyperkähler Berezin-Toeplitz operators which have no equivalent version in the case of the original operators and, in particular, show that Hyperkähler Berezin-Toeplitz operators inherit asymptotically different properties coming from the quaternionic numbers.

  Theorem 6.16.
  Assume that i_1, i_2 and i_3 are orthonormal. The generalized Berezin-Toeplitz operators have the following properties with respect to the cross product $\times$:

1. *For every* $f \in C^\infty(M)$

$$\left\| s \times \tilde{\mathbb{T}}_f^{(m)} \right\| \leq C \mid f \mid_\infty,$$

where

$$C = \left(\sum_{n=1}^{3} a_n \right)^2 - 1.$$

2. *For every* $f, g, h \in C^\infty(M)$,

$$\left\| \left(\tilde{\mathbb{T}}_f^{(m)} \times \tilde{\mathbb{T}}_g^{(m)} \right) \times \tilde{\mathbb{T}}_h^{(m)} + \left(\tilde{\mathbb{T}}_g^{(m)} \times \tilde{\mathbb{T}}_h^{(m)} \right) \times \tilde{\mathbb{T}}_f^{(m)} + \left(\tilde{\mathbb{T}}_h^{(m)} \times \tilde{\mathbb{T}}_f^{(m)} \right) \times \tilde{\mathbb{T}}_g^{(m)} \right\| = O\left(m^{-1}\right).$$

3. *For every* $f, g, h \in C^\infty(M)$,

$$\Bigg\| m\sqrt{-1} \left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^\times, \tilde{\mathbb{T}}_h^{(m)} \right]^\times - 4\tilde{\mathbb{T}}_f^{(m)} \left(s \cdot \tilde{\mathbb{T}}_{\{g,h\}}^{(m)} \right)$$
$$m\sqrt{-1} \left[\left[\tilde{\mathbb{T}}_g^{(m)}, \tilde{\mathbb{T}}_h^{(m)} \right]^\times, \tilde{\mathbb{T}}_f^{(m)} \right]^\times - 4\tilde{\mathbb{T}}_g^{(m)} \left(s \cdot \tilde{\mathbb{T}}_{\{h,f\}}^{(m)} \right)$$
$$m\sqrt{-1} \left[\left[\tilde{\mathbb{T}}_h^{(m)}, \tilde{\mathbb{T}}_f^{(m)} \right]^\times, \tilde{\mathbb{T}}_g^{(m)} \right]^\times - 4\tilde{\mathbb{T}}_h^{(m)} \left(s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)} \right) \Bigg\| = O\left(m^{-1}\right).$$

In fact, for the case $a_1 = 1$, $a_2 = a_3 = 0$, all the expressions of the previous theorem are identically zero.

- For quaternionic numbers without real part, the quaternionic product can be expressed as a combination of the scalar product and the cross product.

Theorem 6.24.
Assume that i_1, i_2 *and* i_3 *are orthogonal. Let* $f, g, h \in C^\infty(M)$. *The Hyperkähler Berezin-Toeplitz operators have the following properties with respect to the quaternionic product* $\star$:

1. *There exist a* $C > 0$ *such that:*

$$\mid f \mid_\infty - \frac{C}{m} \leq \left\| s \star \tilde{\mathbb{T}}_f^{(m)} \right\| \leq \left(\sum_{n=1}^{3} a_n \right)^2 \mid f \mid_\infty .$$

2. $$\left\| \tilde{\mathbb{T}}_f^{(m)} \star \tilde{\mathbb{T}}_g^{(m)} + \tilde{\mathbb{T}}_g^{(m)} \star \tilde{\mathbb{T}}_f^{(m)} - 2s \cdot \tilde{\mathbb{T}}_{fg}^{(m)} \right\| = O\left(m^{-1}\right).$$

3.

$$\left\| m\sqrt{-1}\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\star} - m\sqrt{-1}\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\times} - s\cdot\tilde{\mathbb{T}}_{\{f,g\}}^{(m)}\right\| = O\left(m^{-1}\right).$$

4.

$$\begin{aligned}&\Big\|\left(\tilde{\mathbb{T}}_f^{(m)}\star\tilde{\mathbb{T}}_g^{(m)}\right)\star\tilde{\mathbb{T}}_h^{(m)} - \left(s\cdot\tilde{\mathbb{T}}_{fg}^{(m)}\right)\tilde{\mathbb{T}}_h^{(m)}\\ &+\left(\tilde{\mathbb{T}}_g^{(m)}\star\tilde{\mathbb{T}}_h^{(m)}\right)\star\tilde{\mathbb{T}}_f^{(m)} - \left(s\cdot\tilde{\mathbb{T}}_{gh}^{(m)}\right)\tilde{\mathbb{T}}_f^{(m)}\\ &+\left(\tilde{\mathbb{T}}_h^{(m)}\star\tilde{\mathbb{T}}_f^{(m)}\right)\star\tilde{\mathbb{T}}_g^{(m)} - \left(s\cdot\tilde{\mathbb{T}}_{fh}^{(m)}\right)\tilde{\mathbb{T}}_g^{(m)}\Big\| = O\left(m^{-1}\right).\end{aligned}$$

5.

$$\begin{aligned}&\Big\|m\sqrt{-1}\left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\star},\tilde{\mathbb{T}}_h^{(m)}\right]^{\star} - 4\tilde{\mathbb{T}}_f^{(m)}\left(s\cdot\tilde{\mathbb{T}}_{\{g,h\}}^{(m)}\right)\\ &+m\sqrt{-1}\left[\left[\tilde{\mathbb{T}}_g^{(m)},\tilde{\mathbb{T}}_h^{(m)}\right]^{\star},\tilde{\mathbb{T}}_f^{(m)}\right]^{\star} - 4\tilde{\mathbb{T}}_g^{(m)}\left(s\cdot\tilde{\mathbb{T}}_{\{h,f\}}^{(m)}\right)\\ &+m\sqrt{-1}\left[\left[\tilde{\mathbb{T}}_h^{(m)},\tilde{\mathbb{T}}_f^{(m)}\right]^{\star},\tilde{\mathbb{T}}_g^{(m)}\right]^{\star} - 4\tilde{\mathbb{T}}_h^{(m)}\left(s\cdot\tilde{\mathbb{T}}_{\{f,g\}}^{(m)}\right)\Big\| = O\left(m^{-1}\right)\end{aligned}$$

This last theorem combines the previous two, but it is slightly weaker, in the sense that does not recover exactly the original properties. For instance, in the case $a_1 = 1$, $a_2 = a_3 = 0$, Theorem 6.24.3 results in a symmetrized version of Theorem 3.4.3:

$$\left\|T_f^{(m)}T_g^{(m)} + T_g^{(m)}T_f^{(m)} - 2T_{fg}^{(m)}\right\| = O\left(m^{-1}\right).$$

Chapter Summaries

In Chapters 1-4, we introduce the main notions and tools that will be used during this work. Most of the results found in these first chapters have been already proven and we are only collecting and adapting them to our case. The appropriate references to the original documents are provided in each chapter. On chapters 5 and 6 we show our original results.

Chapter 1 is dedicated to introduce to the reader the main concepts and properties of Kähler manifolds. We start explaining the basics notions of symplectic, Poisson and complex geometry that the reader will need to understand this work. We make a specific focus on holomorphic vector bundles since, as the reader will see, these play a critical role in this work. We also present a basic introduction to *singular homology* and different notions of *cohomologies* like *de Rham*, *Dolbeault* and *sheaf cohomology* that will be used along this work. We include different relations between them like Poincaré Duality (Theorem 1.8), de Rham Theorem (Theorem 1.9) and Dolbeault Theorem (Theorem 1.10). We also introduce the notion of intersection pairing of submanifolds and singular chains. Finally, we introduce Kähler manifolds. Hodge decomposition (Theorem 1.12) shows that cohomologies are *nicely behaved* in Kähler manifolds.

As we will see later on, any manifold which fulfills the necessary conditions for Berezin-Toeplitz quantization is also an algebraic variety. In Chapter 2 we introduce some useful notions and properties of algebraic varieties. In particular, we introduce Weil and Cartier divisors, which generalize the notion of hypersurface. If a manifold is a smooth projective variety (which is the case for quantizable manifolds), then both classes of divisors are equivalent. Moreover, there is a 1-1 relation between (classes of equivalent) divisors and (isomorphism classes of) line bundles, which will allow us to prove different properties of the quantum line bundle using divisors. We also define some classical invariants like the Picard number, the Chern class and the Euler characteristic.

Chapter 3 is dedicated to explaining the basics of Berezin-Toeplitz quantization. We start explaining the construction of the quantum line bundle and how its sections define an embedding into a projective space. Then we explain the construction of the Berezin-Toeplitz operators. As the reader will see, one is not only interested in the quantum line bundle but also in all its tensor powers, since the notion of semiclassical limit of a Berezin-Toeplitz operator refers to its *asymptotic* behavior over all the tensor powers of the quantum line. We explain the most important properties of such operators and we introduce the generalized Hardy space. Then we explain the relation between the functions on such space and the sections of the tensor powers of the quantum line bundle. The most important result of Chapter 3 is Theorem 3.4, which explains that the Berezin-Toeplitz operators have the correct *semiclassical limit*. Generalizing this theorem to Hyperkähler Berezin-Toeplitz operators is one of the most important results of this work.

In Chapter 4 we introduce K3 surfaces using Huybrechts notes ([Huy]). We first explain the different equivalent definitions and show some classical properties. In particular, we compute some of the invariants defined in Chapter 2 and show some peculiarities of the Neron-Severi group for K3 surfaces. We then define the *Twistor Space*, which is a complex manifold that encodes all the Kähler structures of a K3 surface. Then we do a basic introduction to lattices and we explain the construction of Hodge structures on lattices. The Global Torelli Theorem (Theorem 4.19) uses Hodge structures to classify K3 surfaces.

In Chapter 5 we start proving some relations between the different Kähler structures and the *holomorphic* tangent spaces. Using Poincaré duality and the relation between divisors and line bundles, we prove that the dimension of the quantum Hilbert space for a complex surface does not depend on the chosen Kähler structure but only on the Riemannian metric. Hence, for a

K3 surface one has that the different quantum Hilbert spaces are isomorphic (Theorem 5.16). However, in general there is no canonical choice for the isomorphism.

Then we use the Global Torelli Theorem to describe an infinite countable family of K3 surfaces for which there is a canonical choice (Proposition 5.30).

In Theorem 5.28 we also show that the Picard number of a K3 surface with two or more quantizable Kähler structures is 20. Conversely, any K3 surface with Picard number 20 admits two quantizations. This is again a consequence of the Global Torelli Theorem, which also allows us to show that the family of such K3 surfaces is infinite countable. Moreover, we show that an infinite subfamily of them are (resolution of) Kummer surfaces (Proposition 5.29).

We finally show that such a K3 surface admits not only three quantizations, but infinitely many of them, for which the previous results are still valid after rescaling the metric.

We start Chapter 6 introducing an approach done by Barron and Serajelahi to Hyperkähler quantization. They consider the tensor product of the quantum Hilbert spaces and Berezin-Toeplitz operators and prove a generalization of Theorem 3.4. Then we present a different approach, in which we furthermore tensor this higher quantum Hilbert space with the quaternionic space

$$\mathcal{H}_k := \mathbb{H} \otimes_{\mathbb{R}} \left(H^0\left(X, L_{I_1}^{\otimes k a_1}\right) \otimes_{\mathbb{C}} H^0\left(X, L_{I_2}^{\otimes k a_2}\right) \otimes_{\mathbb{C}} H^0\left(X, L_{I_3}^{\otimes k a_3}\right)\right)$$

and define the *Hyperkähler Berezin-Toeplitz operators*

$$\tilde{\mathbb{T}}_f^{(k)} := \sum_{n=1}^{3} a_n i_n \tilde{\mathbb{T}}_f^{n,(k)}$$

as the average of the original Berezin-Toeplitz operators with quaternionic coefficients.

To be able to work with these operators, we introduce a notion of asymptotic equivalence which greatly simplifies the notation during the proofs. We then show different properties of these operators with respect to the scalar product, cross product and quaternionic product.

First we use the scalar product to prove Theorem 6.9, which generalizes Theorem 3.4 and shows that generalized Berezin-Toeplitz operators have a semiclassical limit. Part of this theorem is proved using the triangle inequality and applying the original results for each component. However, for some parts the original result itself is not enough and one needs to study and reproduce the proof in this more general setting. This theorem can be considered as a generalization of the original theorem (see Theorem 3.4) since by considering only one Kähler structure the original result is recovered.

Moreover, the Hyperkähler Berezin-Toeplitz operators inherit asymptotically different properties from the quaternionic numbers, like relations between the cross product and the scalar product. In particular, we prove that Hyperkähler Berezin-Toeplitz operators fulfill the Jacobi Identity asymptotically with respect to the cross product. This is showed in Theorem 6.16, where we also compute the first order term. We should remark that these properties have no equivalent in the case of the original operators.

Finally Theorem 6.24 shows the properties of the Hyperkähler Berezin-Toeplitz operators with respect to the quaternionic product. The existence of a splitting of the quaternionic product up to "order" two

$$\left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\star, \tilde{\mathbb{T}}_h^{(m)}\right]^\star = \left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\cdot, \tilde{\mathbb{T}}_h^{(m)}\right] + \left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\times, \tilde{\mathbb{T}}_h^{(m)}\right]^\times$$

allows us to prove this last theorem by using Theorems 6.9 and 6.16.

1 Kähler Geometry

As the reader will see in Chapter 3, a *quantizable* manifold is a *Kähler manifold* whose *Kähler form* is *integral.* The first chapter of this thesis is devoted to introduce the reader to these concepts and explain some basic results on both Kähler geometry and algebraic topology that will be used along this work.

We will not enter into details in this chapter nor prove any result. A reader interested in a more detailed explanation will find references in each subsection.

1.1 Symplectic and Poisson Structures

As explained before, one of the main tools of classical physics are the symplectic structures. Such structures determine uniquely a Poisson structure in its algebra of functions.

When quantizing a manifold using Berezin-Toeplitz operators, one wants a map which transform functions into bounded operators on a certain Hilbert space and preserves this Poisson structure. Unfortunately, it is proven that the quantization procedure is not an algebra homomorphism ([Sch10, page 8]). However, Theorem 3.4 shows that the Poisson algebra is asymptotically recovered.

Here we are only writing the basic definitions and a few important properties of both symplectic and Poisson structures, as well as some examples. We will skip many other interesting and important properties which are of no importance for this work, which includes a formal definition of the notion of foliation and leaf.

For a more complete, yet simple and fast introduction to symplectic geometry we recommend Ana Cannas da Silva *Lectures on symplectic geometry* ([CdS01]). Here we follow Sections 1 and 2 for the symplectic structures and Section 18 for the Poisson structures.

1.1.1 Symplectic Manifolds

Let ω be a de Rham 2-form on a manifold M. Remember that this means that, for each $p \in M$, the map

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

is bilinear skew-symmetric and it varies smoothly in p.

Definition 1.1. The 2-form ω is *symplectic* if:

- ω is *closed*:

$$\mathrm{d}\omega = 0.$$

- for each point $p \in M$, the linear form ω_p is *non-degenerate*:

$$\forall u \in T_pM\backslash 0 \quad \exists v \in T_pM \mid \omega_p(u,v) \neq 0.$$

A *symplectic manifold* is a pair (M, ω), where M is a manifold and ω a symplectic form.

Note that by non-degeneracy of the symplectic form, a symplectic manifold must be of even dimension.

Example 1.1. Let $M = \mathbb{R}^{2n}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$$

is symplectic. It is called the *standard symplectic* structure of $\mathbb{R}^{2n}$.

Example 1.2. Let $M = \mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. The form

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k$$

is symplectic. In fact, this is the same form as in the last example under the identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ given by

$$z_k = x_k + iy_k, \quad k = 1, \cdots, n.$$

Example 1.3 (Canonical symplectic structure of the cotangent bundle)**.** Let X be a n-dimensional manifold, $M = T^*X$ the cotangent space and $\pi : M \to X$ the projection. The *tautological 1-form* α is defined pointwise as

$$\alpha_p := (\mathrm{d}\pi_p)^* \xi \in T_p^* M,$$

where $\xi \in T_x^M$ and $p = (x, \xi)$.

The *canonical symplectic 2-form* ω on M is defined as

$$\omega := -\mathrm{d}\alpha.$$

Locally,

$$\alpha = \sum_{i=1}^{n} x_i \mathrm{d}\xi_i,$$

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}\xi_i.$$

As with all mathematical structures one wants to classify them up to "isomorphism". To his end, one needs a notion of morphism:

Definition 1.2. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A smooth map $f : M_1 \to M_2$ is called *symplectic* if

$$f^*\omega_2 = \omega_1.$$

If moreover f is a diffeomorphism, then it is called *symplectomorphism.*

Lemma 1.1. *[MS98, page 21] Let (M, ω) be a symplectic manifold of dimension $2n$. Then*

$$\Omega := \frac{1}{n!}\omega^n$$

is a volume form called Liouville form*. In particular, every symplectic manifold has a canonical orientation.*

Theorem 1.2. *(Darboux)[CdS01, Theorem 8.1]*
Let (M, ω) be a $2n$-dimensional symplectic manifold and let $p \in M$ be any point. Then there exists a coordinate chart (U, x_i, y_i) centered at p such that

$$\omega \mid_U = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i.$$

Such coordinates are called Darboux *or* symplectic coordinates*.*

The Darboux theorem implies that, locally, all symplectic manifolds of dimension $2n$ are indistinguishable and locally symplectomorphic to the standard symplectic structure of $\mathbb{R}^{2n}$ (see Example 1.1). As we will explain later in Section 1.2, complex manifolds use local holomorphic coordinates $z_1, \ldots, z_n$, where

$$z_j = x_j + iy_j.$$

As the reader can read at [Mor], in general there exists no "complex Darboux theorem", i.e., not all complex symplectic manifolds admit holomorphic coordinate systems which are symplectomorphic to Example 1.2.

1.1.2 Poisson Structures

The non-degeneracy of the symplectic form allows one to define a canonical isomorphism between the tangent and the cotangent spaces by contraction,

$$\begin{aligned} \widehat{\omega}_p : T_pM &\to T_p^*M \\ u_p &\mapsto \omega(u_p, \cdot), \end{aligned}$$

which extends to a isomorphism between vector fields and differential forms.

Remark. Some authors use instead the map

$$u_p \mapsto \omega(\cdot, u_p).$$

As ω is alternating, this only changes a sign.

Definition 1.3. Given a smooth function $f \in C^\infty(M)$ the *hamiltonian vector field associated to* f is

$$X_f := \widehat{\omega}^{-1}(\mathrm{d}f).$$

A vector field of such form is called *hamiltonian* and f is called the *hamiltonian function* of X_f.

Example 1.4. Consider the standard symplectic structure on $\mathbb{R}^{2n}$. Then

$$X_{x_i} = \frac{\partial}{\partial y_i}$$
$$X_{y_i} = -\frac{\partial}{\partial x_i}.$$

Proposition 1.3. *[CdS01, Proposition 8.1] The hamiltonian vector fields form a group under the sum. Moreover, this group is closed with respect to the Lie bracket. In particular, if X and Y are hamiltonian vector fields, then $[X, Y]$ is hamiltonian with hamiltonian function $\omega(Y, X)$.*

Proposition 1.4. *[CdS01, page 105] The symplectic form is invariant under the hamiltonian flow, i.e.,*

$$\mathcal{L}_{X_f}\omega = 0$$

for all $f \in C^\infty(M)$.

Definition 1.4. A *Poisson algebra* $(\mathcal{P}, \{\cdot, \cdot\})$ is a commutative associate algebra $\mathcal{P}$ with a *Lie bracket* $\{\cdot, \cdot\}$, i.e., a bilinear antisymmetric map

$$\mathcal{P} \times \mathcal{P} \to \mathcal{P}$$

which satisfies the *Jacobi identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad \forall f, g, h \in \mathcal{P}$$

and the Leibnitz rule

$$\{fg, h\} = \{f, h\}g + f\{g, h\}.$$

A *Poisson manifold* is a manifold M together with a Poisson structure for its algebra of functions $C^\infty(M)$.

Example 1.5. Consider (M, ω) a symplectic manifold. M has a canonical Poisson structure defined as follows:

$$\{f, g\} := \omega(X_f, X_g) = X_f \cdot g = \mathcal{L}_{X_f} g.$$

Proposition 1.5. *[CdS01, page 109] Consider a symplectic manifold* (M,ω) *together with its Lie algebra of vector fields* $(\chi(M),[\cdot,\cdot])$ *and the Poisson algebra of functions* $(C^\infty(M),\{\cdot,\cdot\})$*. The map*

$$\begin{aligned} C^\infty(M) &\to \chi(M) \\ H &\mapsto X_H \end{aligned}$$

is a Lie algebra anti-homomorphism.

Remark. Consider a symplectic manifold (M,ω) and coordinates $\{x_i, y_i\}$. It is easily proven that they are Darboux coordinates if and only if

$$\{x_i, x_j\} = 0 \qquad \{y_i, y_j\} = 0 \qquad \{x_i, y_j\} = \delta_i^j.$$

1.2 Complex Manifolds

Many of the complex structures constructions are parallel to the real ones (atlas, tangent space, etc.). However, complex manifolds are much more rigid than smooth manifolds. For instance, there are no holomorphic partitions of unit, which is an essential tool to construct many objects in smooth geometry. Similarly, on a compact complex manifold there are no non-constant holomorphic functions. As a consequence of this rigidity one usually considers both real and complex structures at the same time.

Sometimes one also needs to differentiate between complex and holomorphic structures. For instance, not all complex vector bundles have a holomorphic structure. However, the reader should be aware that frequently the two words are used to refer to the holomorphic case.

The following definitions and results are extracted mainly from Chapter 0 of Phillip Griffiths and Joseph Harris's *Principles of algebraic geometry* ([GH94]).

Definition 1.5. A *complex manifold* M is a differential manifold admitting an open cover $\{U_\alpha\}_{\alpha\in A}$ and coordinate maps

$$\varphi_\alpha : U_\alpha \to \mathbb{C}^n$$

such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all $\alpha, \beta \in A$.

Remark. As in the case of vector spaces, the real dimension of a complex manifold is always even, and its complex dimension is half its real one. As a general rule, whenever M is a complex manifold, M^n will denote its complex dimension.

As usual, a function or map will be called *holomorphic* if it is locally holomorphic when composed with the coordinate charts. *Submanifolds* are defined as in the real case, changing the word "smooth" by "holomorphic" in the definitions.

Example 1.6. $\mathbb{C}^n$ has a natural complex structure.

Example 1.7. Let $\mathbb{P}^n$ denote the quotient

$$\mathbb{P}^n := \frac{\left\{[z] \neq 0 \in \mathbb{C}^{n+1}\right\}}{[z] \sim [\lambda z]}.$$

$\mathbb{P}^n$ is a compact complex manifold called *complex projective space.* The "coordinates" $z = [z_0, \cdots, z_n]$ are called *homogeneous coordinates.* Note that $\mathbb{P}^n$ can be identified as the set of of lines through the origin in $\mathbb{C}^{n+1}$. The one dimensional projective space $\mathbb{P}^1$ is also known as the *Riemann sphere* and can be identified with $\mathbb{C} \cup \{\infty\}$.

Example 1.8. A one-dimensional complex manifold is called *Riemann surface.*

Example 1.9. Let $\Lambda \subset \mathbb{C}^n$ be a discrete lattice of maximal rank. The quotient $\mathbb{C}^n/\Lambda$ has the structure of a complex manifold induced by the projection map

$$\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$$

and it is called *complex torus.* A projective complex torus is also called *abelian variety.*

Different lattices give different complex structures. However, as smooth manifolds they are all diffeomorphic.

Let M be a complex manifold of complex dimension n, $p \in M$ any point and $z = (z_1, \cdots, z_n)$ a holomorphic coordinate system around p. Remember the definition of (real) tangent space at p:

Definition 1.6. The *real tangent space* $T_{\mathbb{R},p}(M)$ is the usual tangent space to M at p, where M is considered as a real manifold of dimension $2n$. It is defined as the space of $\mathbb{R}$-linear derivations on the ring of real-valued C^∞ functions in a neighborhood of p. If we write $z_j = x_j + iy_j$, then

$$T_{\mathbb{R},p}(M) = \mathbb{R}\left\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\}.$$

Then one also defines two other tangent spaces on M:

Definition 1.7. The *complexified tangent space* $T_{\mathbb{C},p}(M)$ to M at p is

$$T_{\mathbb{C},p}(M) := T_{\mathbb{R},p}(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

It can be realized as the space of $\mathbb{C}$-linear derivations in the ring of complex-valued C^∞ functions on M around p, i.e.,

$$T_{\mathbb{C},p}(M) = \mathbb{C}\left\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right\}. \tag{1.1}$$

Note that it can also be written as

$$T_{\mathbb{C},p}(M) = \mathbb{C}\left\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}\right\},$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Definition 1.8. The *holomorphic tangent space* $T'_p(M)$ to M at p is defined as

$$T'_p(M) := \mathbb{C}\left\{\frac{\partial}{\partial z_j}\right\} \subset T_{\mathbb{C},p}(M).$$

For a coordinate independent definition, the holomorphic tangent space can be realized as the subspace of $T_{\mathbb{C},p}(M)$ consisting of derivations that vanish on local antiholomorphic functions. The subspace

$$T''_p(M) := \mathbb{C}\left\{\frac{\partial}{\partial \overline{z}_j}\right\}$$

is called the *antiholomorphic tangent space*. Clearly

$$T_{\mathbb{C},p}(M) = T'_p(M) \oplus T''_p(M) \tag{1.2}$$

and

$$T''_p(M) = \overline{T'_p(M)}.$$

Remark. If M has (real) dimension $2n$, then $T_{\mathbb{C},p}(M)$ has dimension $4n$ and $T'_p(M)$ has dimension $2n$.

Remark. By duality the Equation 1.2 determines a decomposition

$$T^*_{\mathbb{C},p}(M) = T^{*\prime}_p(M) \oplus T^{*\prime\prime}_p(M) \tag{1.3}$$

and by linearity

$$\bigwedge^k T^*_{\mathbb{C},z}(M) = \bigoplus_{p+q=k}\left(\bigwedge^p T^{*\prime}_z(M) \otimes \bigwedge^q T^{*\prime\prime}_z(M)\right). \tag{1.4}$$

Definition 1.9. An *almost complex structure* on M is a smooth field of linear maps

$$x \mapsto I_x : T_xM \to T_xM \text{ linear, and } I_x^2 = -Id.$$

As explained in [CdS01], Section 13, a complex manifold determines uniquely an almost complex structure. In that case, I is called *integrable* and one says that I is a *complex structure*.

Remark. The involution I_z determines the holomorphic (resp. antiholomorphic) tangent space. In particular, given its $\mathbb{C}$ linear extension to $T_{\mathbb{C},z}M$, the linear map I_z has two eigenvalues: i and $-i$. The holomorphic (resp. antiholomorphic) tangent space is defined as its eigenspace of eigenvalue i (resp. $-i$).

Remark. The map

$$\begin{aligned} \pi : TM &\to T'M \\ v &\mapsto \frac{1}{2}(v \otimes 1 - Iv \otimes i) \end{aligned}$$

is a real vector bundle isomorphism.

1.3 Algebraic Topology

Invariants are a great way to study and classify spaces. There are many topological invariants, such as the *Betti numbers* and the *Euler characteristic*. In this section we introduce two of the most common invariants, the *singular homology groups* and *cohomology groups*, and we will explain the most important properties we will use in this work. Two classical references for algebraic topology are [Spa81] and [BT82].

1.3.1 Homology Groups

First we will start with the so called *homology groups*. Intuitively speaking, the n-homology group $H_n(X)$ of a topological space X counts the number of n-dimensional "holes" of X. In this section X will denote a non-empty topological space.

Consider $n+1$ affinely independent points $v_0, \dots, v_n \in \mathbb{R}^n$.

Definition 1.10. The *standard n-simplex* (or just *n-simplex*) is convex hull of $v_0, \dots, v_n$:

$$\Delta^n = \left\{ \sum_i \theta_i v_i \mid 0 \leq \theta_i, 0 \leq i \leq n, \sum_i \theta_i = 1 \right\}.$$

Remark. Note that, for a fixed n, all n-simplex are homeomorphic.

Example 1.10. 0. The 0-simplex is a point.

1. The 1-simplex is a segment.

2. The 2-simplex is a triangle.

3. The 3-simplex is a tetrahedron.

Definition 1.11. A *singular n-simplex* is a continuous map

$$\sigma_n : \Delta^n \to X.$$

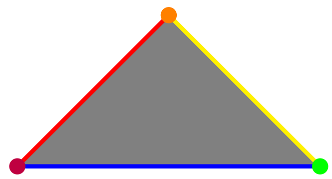
Remark. σ_n does not need to be injective.

Given a singular n-simplex σ_n, the simplex $-\sigma_n$ represents the same topological space but with the opposite orientation.

Definition 1.12. A *face* of a singular n-simplex σ_n is its restriction to the convex hull of a subset of $\{v_0, \ldots, v_n\}$ with the induced orientation.

Example 1.11. The faces of a singular 2-simplex are:

1. The singular 2-simplex itself.
2. Three singular 1-simplex or *sides*.
3. Three singular 0-simplex or *vertices*.
4. The empty set.



Remark. It is common to identify a singular simplex with its image.

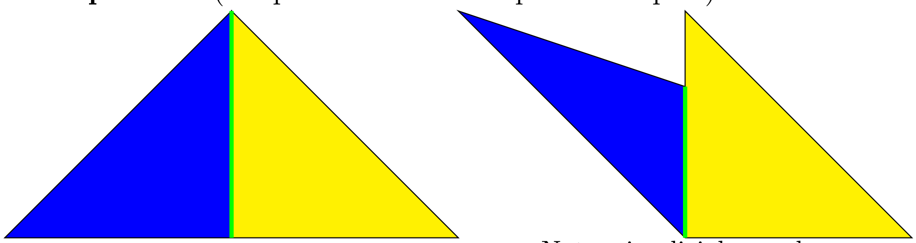
Definition 1.13. A *singular n-chain* is a finite formal sum of singular n-simplices.

Remark. The set of singular n-chains together with the sum form a group, which will be denoted by $\mathcal{C}_n(X)$.

Definition 1.14. A *simplicial complex* is a set of simplices $\mathcal{K}$ such that:

1. Any face of $\mathcal{K}$ is also in $\mathcal{K}$.
2. The intersection of two simplices is either a face of both or the empty set.

Example 1.12 (Simplicial and non-simplicial complex)**.**



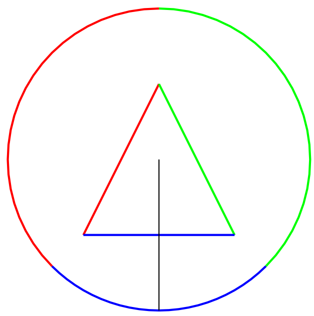
Simplicial complex

Not a simplicial complex
Intersection is not a face of Yellow

Definition 1.15. A *triangulation* of a topological space X is a simplicial complex $\mathcal{K}$, together with a homeomorphism

$$h : \mathcal{K} \xrightarrow{\simeq} X.$$

Example 1.13. A triangulation of a circle is given by the projection



Consider S^1 embedded into $\mathbb{R}^2$ and an empty triangle inside it. In other words, the triangle is a simplicial complex consistent of three points and three segments. From the center of the triangle, draw half line which intersects once the triangle and the circle. This defines a homeomorphism from the triangle to the circle and determines a triangulation.

Theorem 1.6. *[Whi57, Theorem 12A]*
Any smooth manifold admits a triangulation.

Corollary 1.7. *Any submanifold can be represented by a singular chain.*

Consider a n-simplex σ_n.

Definition 1.16. The *boundary* of σ_n, denoted by $\partial_n \sigma_n$, is the formal sum of the singular $n-1$-simplices represented by the restriction of σ_n to the faces of Δ_n, with alternating sign depending on the orientation.

This definition extends by linearity to the group of singular n-chains, giving rise to what is called the *boundary operator*

$$\partial_n : \mathcal{C}_n \to \mathcal{C}_{n-1}.$$

Note that $\partial_n \partial_{n+1} = 0$.

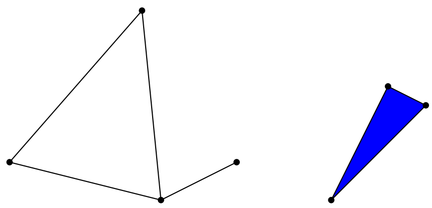
Definition 1.17. The *group of singular n-cycles* is the kernel of the boundary operator

$$\mathcal{Z}_n(X) := \ker(\delta_n).$$

Definition 1.18. The *n-homology group* of X is defined as the quotient group:

$$H_n(X) = \frac{\mathcal{Z}_n(X)}{\partial_{n+1}\mathcal{C}_{n+1}(X)}.$$

Example 1.14. The homology of the following complex is



$$H_k(X) = \begin{cases} \mathbb{Z}^2 \text{ for } k = 0, \\ \mathbb{Z} \text{ for } k = 1, \\ 0 \text{ for } k > 1. \end{cases}$$

An intuitive way to see this is to note that it has *two* connected components and *one* empty triangle.

Definition 1.19. The n-th Betti number $b_n(X)$ of a topological space X is the rank of its n-th cohomology group, i.e.,

$$b_n(X) := \text{rk}\left(H_n(X)\right).$$

Example 1.15. For the figure in Example 1.14,

$$h_k(X) = \begin{cases} 2 \text{ for } k = 0, \\ 1 \text{ for } k = 1, \\ 0 \text{ for } k > 1. \end{cases}$$

1.3.2 De Rham and Dolbeault Cohomology

As stated before, homology theory assigns some topological invariants to a topological space. Whenever this space has more refined algebraic structure (in a broad sense of the word), one uses the cohomology theory to define some algebraic invariants. There are many kinds of cohomologies. We will introduce the ones we will use in this work, starting with *de Rham* and *Dolbeault* cohomologies.

Consider a differential manifold M of dimension n and let $\mathcal{P}^p(M,\mathbb{R})$ be the space of smooth differential forms of degree p on M and

$$\mathcal{Z}^p(M,\mathbb{R}) \subset \mathcal{P}^p(M,\mathbb{R})$$

the subspace of closed p-forms. Since $\mathrm{d}^2 = 0$, then

$$\mathrm{d}\mathcal{P}^{p-1}(M,\mathbb{R}) \subset \mathcal{Z}^p(M,\mathbb{R})$$

Hence the following is well-defined:

Definition 1.20. The *de Rham cohomology groups* of M are defined as

$$H^p_{DR}(M,\mathbb{R}) := \frac{\mathcal{Z}^p(M,\mathbb{R})}{\mathrm{d}\mathcal{P}^{p-1}(M,\mathbb{R})}.$$

In the same way, denote by $\mathcal{P}^p(M)$ the space of $\mathbb{C}$-valued differential forms of degree p and $\mathcal{Z}^p(M) \subset \mathcal{P}^p(M)$ the subspace of closed p-forms.

Definition 1.21.

$$H^p_{DR}(M) := \frac{\mathcal{Z}^p(M,\mathbb{C})}{\mathrm{d}\mathcal{P}^{p-1}(M,\mathbb{C})} = H^p_{DR}(M,\mathbb{R}) \otimes \mathbb{C}.$$

Remark. $\mathcal{P}^p(M)$ are sections of $\bigwedge^k T^*_{\mathbb{C},z}(M)$. In particular, an element of $\mathcal{Z}^1(M,\mathbb{C})$ is a section of the complexified cotangent space, not the holomorphic one.

Remark. As for the homology groups, one denotes the dimension of a cohomology group by lower case letters. For instance,

$$h^p_{DR}(M,\mathbb{C}) = \dim\left(H^p_{DR}(M,\mathbb{C})\right).$$

Example 1.16. Consider $M = \mathbb{R}^n$.

$$H^p_{DR}(\mathbb{R}^n,\mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.17. Consider a manifold M. The 0-cohomology group counts the number of connected components of M, i.e.,

$$h^0_{DR}(M,\mathbb{R}) = \sharp\{\text{connected components}\}.$$

Remember that there is a decomposition

$$\bigwedge^k T^*_{\mathbb{C},z}(M) = \bigoplus_{p+q=k} \left(\bigwedge^p T^{*\prime}_z(M) \otimes \bigwedge^q T^{*\prime\prime}_z(M) \right).$$

Correspondingly one writes

$$\mathcal{P}^k(M) = \bigoplus_{p+q=k} \mathcal{P}^{p,q}(M),$$

where

$$\mathcal{P}^{p,q}(M) = \left\{ \varphi \in \mathcal{P}^k(M) \mid \ \varphi(z) \in \bigwedge^p T^{*\prime}_z(M) \otimes \bigwedge^q T^{*\prime\prime}_z(M) \ \forall z \in M \right\}.$$

A differential form $\varphi \in \mathcal{P}^{p,q}(M)$ is said to be of *type* (p,q). One denotes by $\pi^{(p,q)}$ the projection map

$$\mathcal{P}^*(M) \to \mathcal{P}^{p,q}(M)$$

and $\varphi^{(p,q)}$ for $\pi^{(p,q)}\varphi$.

Definition 1.22. The *Dolbeault operators*

$$\overline{\partial} = \mathcal{P}^{p,q}(M) \to \mathcal{P}^{p,q+1}(M)$$

$$\partial = \mathcal{P}^{p,q}(M) \to \mathcal{P}^{p+1,q}(M)$$

are defined by

$$\overline{\partial} := \pi^{(p,q+1)} \circ \mathrm{d}$$

$$\partial := \pi^{(p+1,q)} \circ \mathrm{d}.$$

Note that

$$\mathrm{d} = \partial + \overline{\partial}.$$

Definition 1.23. A form φ of type $(q,0)$ is called *holomorphic* if

$$\overline{\partial}\varphi = 0.$$

Remark. If $f : M \to N$ is a holomorphic map, then

$$\overline{\partial} f^* = f^* \overline{\partial},$$

where f^* is the usual pull-back of differential forms.

Let denote by $\mathcal{Z}^{p,q}_{\overline{\partial}}(M)$ the space of $\overline{\partial}$-closed forms of type (p,q). Since $\overline{\partial}^2 = 0$, one defines:

Definition 1.24. The *Dolbeault cohomology groups* are

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\mathcal{Z}^{p,q}_{\overline{\partial}}(M)}{\overline{\partial}(\mathcal{P}^{p,q-1}(M))}.$$

Sometimes one drops the subscript $\overline{\partial}$.

Remark. Any differential form can be expressed as a sum of p,q-forms. However, in general it is not true that

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M).$$

As we will explain later, this is true for Kähler manifolds (See Hodge decomposition, Theorem 1.12).

1.3.3 Poincare Duality

Consider M a closed oriented n-dimensional manifold and fix $i_N : N \hookrightarrow M$ a k-dimensional closed oriented submanifold. Consider $\alpha \in H^k(M, \mathbb{R})$ a closed differential form on M. One *integrates* α over N as follows:

$$\int_N i_N^* \alpha$$

Since integration does not depend on sets of measure zero, this integration is well defined over simplicial complexes and, by linearity, over simplicial chains. The following result is of great importance in algebraic topology and geometry:

Theorem 1.8. *(Poincare Duality)[BT82, page 44]*
Let M be a closed oriented manifold and $[\sigma] \in H_k(M)$ a closed singular k-chain. Then there exists a unique $[\mu_\sigma] \in H^{n-k}(M, \mathbb{R})$ such that

$$\int_\sigma i_\sigma^* \alpha = \int_M \mu_\sigma \wedge \alpha \quad \forall [\alpha] \in H^k(M, \mathbb{R}).$$

In other words

$$H^k(M, \mathbb{R}) \cong H_{n-k}(M) \otimes \mathbb{R} \cong \Big(H^{n-k}(M, \mathbb{R})\Big)^*.$$

$[\mu_\sigma]$ is called the Poincare dual *of σ.*

Remark. Stokes' theorem (see [Lee13]) implied that the previous integrals are well-defined, i.e., they do not depend on the chosen element $\alpha \in [\alpha]$.

Definition 1.25. A closed k-form α is called *integral* (resp. *rational*) if

$$\int_N i_N^* \alpha \in \mathbb{Z} \text{ (resp. } \mathbb{Q}) \quad \forall k\text{-dimensional submanifold } N.$$

The subgroup of integral forms is denoted by $H^k(M, \mathbb{Z})$ (resp. $H^k(M, \mathbb{Q})$).

1.3.4 Intersection Theory

Intersection theory is a tool to generalize the concept of intersection to manifolds and homology theory.

Consider a smooth manifold M of dimension n, two smooth cycles $A \in H_k(M, \mathbb{Z})$ and $B \in H_{n-k}(M, \mathbb{Z})$ and a point $p \in A \cap B$ of transversal intersection of A and B. Let $\{v_1, \dots, v_k\}$ a basis of T_pA and $\{w_1, \dots, w_{n-k}\}$ a basis of T_pB.

Definition 1.26. The *intersection index* $i_p(A \cdot B)$ is defined as $+1$ if

$$\{v_1, \dots, v_k, w_1, \dots, w_{n-k}\}$$

is an oriented basis of T_pM and -1 otherwise.

If A and B intersect tranversely everywhere, the *intersection number* $(A \cdot B)$ is defined as

$$(A \cdot B) := \sum_{p \in A \cap B} i_p(A \cdot B).$$

It is possible, however, that the two cycles are not transverse everywhere (they may even be identical) or that one of them is not smooth. Fortunately, this definition depends only on the homology class and it is always possible to choose elements of $[A]$ and $[B]$ that are smooth transversal everywhere, giving a well defined map:

$$H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \to \mathbb{Z}.$$

For a more detailed explanation, see [GH94, Section 0.4].

Example 1.18. Any line L in $\mathbb{P}^2$ has self-intersection 1. This is easily seen by using an affine chart and moving the line to a parallel one. Only the intersection point at infinity remains.

Note that sometimes a curve can have negative self-intersection.

Example 1.19. Consider a blow-up

$$\pi : \tilde{S} \to S$$

of a smooth projective complex surface. The curve

$$E = \pi^{-1}(0)$$

has self-intersection -1. From an algebraic point of view, a negative self-intersection implies some kind of *rigidity.* A *divisor* (which we will define later) with negative self-intersection as no other linearly equivalent divisors.

1.3.5 Sheaf Cohomology, de Rham Theorem and Dolbeault Theorem

Definition 1.27. A *sheaf* $\mathcal{F}$ on a topological space X is a map which assigns to each open set $U \subset X$ a group $\mathcal{F}(U)$, called *sections* of $\mathcal{F}$ over U, and to each pair $U \subset V$ of open sets a map

$$r_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U),$$

called the *restriction map*, such that:

1. For any triple $U \subset V \subset W$ of open sets,

   $$r_{W,U} = r_{V,U} \circ r_{W,V}.$$

   We may write $\sigma \mid_U$ for $r_{V,U}(\sigma)$ without loosing any information.

2. (Locality) If $\sigma \in \mathcal{F}(U \cup V)$ and

   $$\sigma \mid_U = \sigma \mid_V = 0,$$

   then $\sigma = 0$.

3. (Gluing) For any pair $U, V \subset M$ of open sets and sections $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that

   $$\sigma \mid_{U \cap V} = \tau \mid_{U \cap V}$$

   there exists a section $\rho \in \mathcal{F}(U \cup V)$ such that

   $$\rho \mid_U = \sigma,$$

   $$\rho \mid_V = \tau.$$

Example 1.20. Consider M a smooth manifold. The sheaf of smooth functions C^∞ assigns to each open set U the set of smooth functions defined of U:

$$C^\infty : U \mapsto C^\infty(U).$$

Clearly smooth functions fulfill the three axioms.

Example 1.21. On a C^∞ manifold M, one defines the following sheaves:

- C^∞: smooth functions.
- $\mathcal{P}^p$: smooth p-forms.
- $\mathcal{Z}^p$: closed smooth p-forms.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$: locally constant functions with values in $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$.

Example 1.22. Consider a complex manifold M and a holomorphic line bundle $E \to M$ (defined below in Section 1.4). Then one defines the following sheaves in M:

- $\mathcal{O}$: holomorphic functions.
- $\mathcal{O}^*$: non-zero holomorphic functions.
- Ω^p: holomorphic p-forms.
- $\mathcal{P}^{p,q}$: smooth (p,q)-forms.
- $\mathcal{Z}^{p,q}_{\overline{\delta}}$: $\overline{\delta}$-closed smooth (p,q)-forms.
- $\mathcal{O}(E)$: holomorphic sections of E.

Consider now an arbitrary sheaf $\mathcal{F}$ on a real manifold M of dimension n. Next we describe how to construct the *cohomology* of $\mathcal{F}$.

Let $\underline{U} = \{U_\alpha\}$ be a locally finite open cover such that all finite intersections are diffeomorphic to $\mathbb{R}^n$. Such a cover is called a *good cover*. Define

$$C^j(\underline{U}, \mathcal{F}) := \prod_{\alpha_0 \neq \cdots \neq \alpha_j} \mathcal{F}(U_\alpha).$$

Remark. Every smooth manifold has such a cover. See [BT82, Theorem 5.1] for more details.

Definition 1.28. A *p-cochain* of $\mathcal{F}$ is an element

$$\sigma = \left\{ \sigma_I \in \mathcal{F}\left(\bigcap_{i \in I} U_i \right) \right\},$$

where $\#I = p + 1$.

Definition 1.29. The *coboundary operator* is the map

$$\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$$

defined by the formula

$$(\delta\sigma)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\ldots,\widehat{i_j},\ldots i_{p+1}} |_{U_{i_0} \cap \cdots \cap U_{p+1}} .$$

A p-cochain σ is called *cocyle* if $\delta\sigma = 0$ and *coboundary* if $\sigma = \delta\tau$. It is easy to see that $\delta^2 = 0$. Defining

$$\mathcal{Z}^p(\underline{U}, \mathcal{F}) := \{\sigma \in C^p(\underline{U}, \mathcal{F}) \mid \delta\sigma = 0\},$$

as the subgroup of cocycles, then the following is well defined:

$$H^p(\underline{U}, \mathcal{F}) := \frac{\mathcal{Z}^p(\underline{U}, \mathcal{F})}{\delta C^{p-1}(\underline{U}, \mathcal{F})}.$$

One proves then (see, for instance, [BT82, Section 5]) that this construction does not depend on the chosen good cover. One writes then

$$H^p(M, \mathcal{F}) = H^p(\underline{U}, \mathcal{F})$$

for any such cover. This is called the p-th *Čech cohomology group* of $\mathcal{F}$ on M.

Remark. Here I have defined the notion of *good cover* on the category of smooth manifolds. However, this definition is not valid for all categories and one must define good covers accordingly.

1.3.6 de Rham and Dolbeault Theorems

Theorem 1.9. *(de Rham Theorem) [GH94, page 44]*
There exists an isomorphism

$$H^*_{DR}(M, \mathbb{R}) \to H^*(M, \mathbb{R}).$$

Remark. In fact, the original de Rham Theorem says a bit less:

$$H^*_{DR}(M, \mathbb{R}) \to H^*_{sing}(M, \mathbb{R}).$$

The second term is proven to be isomorphic to the one we need. Since we will not use singular cohomology anywhere in this work, we have decided to skip its definition and rewrite the theorem accordingly.

Remark. The de Rham isomorphism restricts to integral forms. More precisely, the sheaf cohomology groups of the $\mathbb{Z}$-constant sheaf is isomorphic to the integral cohomology as defined in Definition 1.25. It also restricts to rational forms.

Theorem 1.10. *(Dolbeault Theorem) [GH94, page 45]*
If M is a complex manifold, then

$$H^q(M, \Omega^p) \cong H^{p,q}_{\bar{\delta}}(M).$$

Example 1.23. On a complex manifold, the sheaf of holomorphic functions is such that

$$H^q(M, \mathcal{O}) \cong H^{0,q}_{\bar{\partial}} \quad \forall q > n.$$

1.4 Complex and Holomorphic Vector Bundles

Definition 1.30. A *complex vector bundle* on a smooth manifold M is a vector bundle $E \to M$ such that the fibers are complex vector spaces.

Note that neither E nor M need to be complex manifolds, nor the projection needs to be a holomorphic map. An example of such bundle is the complexified tangent space $T_{\mathbb{C}}M$.

Definition 1.31. A *holomorphic vector bundle* on a complex manifold M is a complex vector bundle $E \to M$ together with a complex structure on E that admits biholomorphic trivialization maps, i.e., for any $x \in M$ there exists an open set $x \in U$ and a biholomorphic map

$$\varphi_U : E\mid_U \to U \times \mathbb{C}^k.$$

Example 1.24. The holomorphic tangent space $T'(M)$ is a holomorphic vector bundle.

Note that, as a direct consequence of the definition, the projection is an holomorphic map. Usual operation such as tensor product, dual bundle and wedge product are defined as usual.

One should note that there is no naturally defined exterior derivative d on the space of sections of a vector bundle. On a holomorphic vector bundle E however, there exists another operator

$$\overline{\partial} : A^{p,q}(E) \to A^{p,q+1}(E)$$

defined locally by

$$\overline{\partial}\sigma = \sum \overline{\partial}\omega_i \otimes e_i,$$

where

$$\sigma = \sum \omega_i \otimes e_i, \quad \omega_i \in A^{p,q}(U)$$

and $\{e_i\}$ is a local holomorphic frame.

Definition 1.32. A *hermitian metric* on a complex vector bundle $E \to M$ is a hermitian inner product on each fiber E_x of E, varying smoothly with $x \in E$.

A holomorphic vector bundle with such a hermitian metric is called a *hermitian vector bundle.*

Definition 1.33. A *connection* D on a complex vector bundle $E \to M$ is a map

$$D : \mathcal{P}^0(E) \to \mathcal{P}^1(E)$$

satisfying Leibnitz' rule

$$D(f \cdot \xi) = \mathrm{d}f \otimes \xi + f \cdot D(\xi)$$

for all sections $\xi \in \mathcal{P}^0(E)(U)$, $f \in C^\infty(U)$.

In general, there is no natural or canonical connection on a vector bundle. However, on a hermitian vector bundle there exists a canonical choice:

Lemma 1.11. *[GH94, page 73] If E is a hermitian vector bundle, there exists a unique connection D* compatible *with both the* complex structure *and the* metric*, i.e.:*

1. *Consider the splitting $D = D' + D''$, where*

$$D' : \mathcal{P}^0(E) \to \mathcal{P}^{1,0}(E)$$

$$D'' : \mathcal{P}^0(E) \to \mathcal{P}^{0,1}(E).$$

   D is called compatible with the complex structure *if*

$$D'' = \overline{\partial}.$$

2. *D is called* compatible with the metric *if, for any pair of sections sections $\xi, \eta \in \mathcal{P}^0(E)$,*

$$\mathrm{d}(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta).$$

1.5 Kähler Manifolds

Kähler geometry lies in the intersection of complex, riemannian and symplectic geometry. However, not every complex, riemannian and symplectic is Kähler; it requires an extra compatibility condition. This condition also allows one to recover one of the three structures from the other two. Moreover, by Hodge Theorem it is possible to express any de-Rham class of differential forms on Kähler manifolds as a linear combination of different $\overline{\delta}$ classes of (p, q)-forms. A proper introduction to symplectic and Kähler geometry can be found in Cannas da Silva notes ([CdS01]). For a complex and algebraic point of view, one can read Griffiths and Harris book ([GH94]).

Definition 1.34. A *Kähler manifold* is a quadruple (M, g, J, ω) where:

- (M, J) is a complex manifold.
- ω is a symplectic positive $(1, 1)$-form, known as *Kähler form.*
- g is a Riemannian metric.
- (g, J, ω) is a *compatible triple*, i.e.,

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot).$$

A manifold is called *Kählerian* if it admits any such structure.

Remark. Since it is possible to recover the third element of a compatible triple from the other two using the compatibility condition, sometimes a Kähler manifold is denoted only using two of then, e.g. (M, g, I) or (M, g, ω).

Later on we will consider Kähler manifolds which admits two or more Kähler structures sharing the same metric g. In this case, we will use subscripts to denote which complex structure is compatible with which symplectic form. In particular, we will write either the complex structure as a subscript of the symplectic form, e.g., (g, I, ω_I), or put a numerical subscript of both of then, e.g., (g, I_1, ω_1).

Example 1.25. $\mathbb{R}^{2n}$ with the standard symplectic and complex structures is a Kähler manifold.

Example 1.26. Any metric on a compact Riemann surface is Kähler.

Example 1.27. If Λ is a lattice in $\mathbb{C}^n$, the complex torus

$$T = \mathbb{C}^n/\Lambda$$

is Kähler with the hermitian metric

$$\mathrm{d}s^2 = \sum \mathrm{d}z_i \otimes \mathrm{d}\overline{z}_i.$$

Example 1.28. The projective space $\mathbb{P}^n$ is Kähler with the Fubini-Study metric and

$$\omega = \frac{i}{2}\partial\overline{\partial} \log \parallel Z \parallel^2,$$

where Z is a local lifting of the points of $U \subset \mathbb{P}^n$ to $\mathbb{C}^{n+1} - \{0\}$.

As explained before, on a complex manifold one defines (p, q)-forms. If M is moreover a Kähler manifold, one has the *Hodge decomposition*:

Theorem 1.12. *(Hodge decomposition)[GH94, page 116]*
Let M be a Kähler manifold. Then

$$H^{p,q}(M) = \overline{H^{q,p}(M)},$$
$$H^r(M, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(M).$$

Some consequences of the Hodge decomposition:

Corollary 1.13. *[GH94, pages 116-119] Consider M a compact complex manifold. Then*

1.
$$b_k = \dim(H^k) = \sum_{p+q=k} h^{p,q}.$$

2. *The odd Betti numbers $b_{2k+1}(M)$ are even.*

3.
$$H^{p,q}_{\overline{\partial}}(\mathbb{P}^n) = \begin{cases} 0, & \text{if } p \neq q, \\ \mathbb{C}, & \text{if } p = q. \end{cases}$$

2 Algebraic Varieties

A *complex (projective) algebraic variety* is defined to be the set of zeros of homogeneous polynomials with complex coefficients inside a projective space $\mathbb{P}^n$. If the variety is smooth, one also considers its induced complex structure as a submanifold, which in fact is independent of the embedding. As the reader will see in Chapter 3, a quantizable manifold is always projective. In this chapter, we will explain some results on algebraic geometry that we will use on the following chapters.

We start defining divisors, which generalize the concept of hypersurface and then we explain their relation with line bundles. Lefschetz Theorem will be used to characterize quantizable manifolds, as well as to prove an important result about K3 surfaces.

Anyone interested in a more algebraic view should check Hartshorne's *Algebraic Geometry* ([Har77]). We will just show the main properties we are interested in from a more differentiable point of view using Phillip Griffiths and Joseph Harris's *Principles of algebraic geometry* ([GH94]), Chapter 1.

2.1 Weil and Cartier Divisors

Divisors generalize the notion of hypersurface on algebraic varieties. Consider M a projective algebraic variety of dimension n.

Definition 2.1. An *analytic hypersurface* is an analytic subvariety of dimension $n-1$. In other words, it is defined locally as the zeros of a (local) holomorphic function. V is called *irreducible* if it cannot be expressed as a non-trivial union of hypersurfaces.

Definition 2.2. A *(Weil) divisor* D on M is a finite formal linear combination

$$D = \sum_i a_i V^i, \quad a_i \in \mathbb{Z},$$

where V_i are hypersurfaces. The group of divisors of M is denoted by $\mathrm{Div}(M)$.

Example 2.1. Consider $\mathbb{P}^n$ with projective coordinates $[z_0, \dots, z_n]$ and take D as the union of coordinate hypersurfaces:

$$D = \sum_{i=0}^{n} \{z_i = 0\}.$$

Remark. On a non-compact variety (in particular, non-projective), one defines divisors as locally finite sum of divisors. For instance, in $\mathbb{C}^n$ the following is a divisor:

$$D = \sum_{i \in \mathbb{Z}} \{z_1 - i = 0\}.$$

Definition 2.3. A divisor is called *effective* if

$$a_i \geq 0$$

for all i. In that case, one writes

$$D \geq 0.$$

Consider a (non-zero) local holomorphic function g defined around a point $p \in M$ and V an irreducible hypersurface defined by f around p. The *order* $\mathrm{ord}_{V,p}(g)$ of g along V is the largest integer a such that, in a neighborhood of p:

$$g = f^a h$$

for a locally holomorphic function h. Since the definition does not depend on the point p, one usually skips the subindex and writes $\mathrm{ord}_V(g)$. For meromorphic functions, one uses that locally they are of the form

$$f = \frac{g}{h}$$

with g and h locally holomorphic and relatively prime and defines

$$\mathrm{ord}_V(f) := \mathrm{ord}_V(g) - \mathrm{ord}_V(h), \quad f \in \mathcal{M}^*(M).$$

Definition 2.4. The *principal divisor* (f) of a meromorphic function $f \in \mathcal{M}^*(M)$ is defined as

$$(f) := \sum_V \mathrm{ord}_V(f) \cdot V,$$

where the sum is over all hypersurfaces of M.

Remark. Any meromorphic function on $\mathbb{P}^n$ (and other projective algebraic varieties) is *rational*, i.e., can be expressed locally as the quotient of homogeneous polynomials (see [GH94, page 168]).

Remark. Last definition is well defined since, for a non-zero meromorphic function, $\mathrm{ord}_V(f)$ is zero except for a finite number of hypersurfaces V.

Example 2.2. In $\mathbb{P}^2$,

$$\left(\frac{z_0^2}{z_1}\right) = 2\{z_0 = 0\} - \{z_1 = 0\}.$$

Definition 2.5. Two divisors D_1, D_2 are called *linearly equivalent* if

$$D_1 = D_2 + (f)$$

for a meromorphic function $f \in \mathcal{M}^*(M)$. One writes then

$$D_1 \sim D_2.$$

Definition 2.6. The *divisor class group* of an algebraic variety is defined as the quotient group of divisors by the subgroup of principal divisors:

$$Cl(M) := \frac{\mathrm{Div}(M)}{\{(f) \mid f \in \mathcal{M}^*(M)\}}.$$

There is a second notion of divisors:

Definition 2.7. A *Cartier divisor* is a pair

$$(\{U_\alpha\}, \{f_\alpha\})$$

where $\{U_\alpha\}$ is an open cover of M and $f_\alpha : U_\alpha \to \mathbb{C}$ are meromorphic function such that, in every non-empty intersection $U_\alpha \cap U_\beta \neq \emptyset$, the quotient f_α / f_β is a regular invertible function, i.e.,

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

Given a Cartier divisor, one constructs a Weil divisor by

$$D = \sum_V \mathrm{ord}_V(f_\alpha) \cdot V,$$

where, for each V, α is chosen in such a way that

$$V \cap U_\alpha \neq \emptyset.$$

Note that this definition is independent of the chosen α.

Remark. On a smooth projective variety, every Weil divisor can be constructed in this way (see [Har77, Chapter 2]). Therefore, from now on we will use the word *divisor* for both concepts.

In terms of sheaves, a divisor is a global section of the quotient sheaf

$$\mathcal{M}^*/\mathcal{O}^*,$$

where $\mathcal{M}^*$ is the sheaf of non-identically zero meromorphic functions and $\mathcal{O}^*$ the sheaf of nowhere-zero holomorphic functions.

Consider the short exact sequence of sheaves:

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0.$$

It gives rise to a long exact sequence:

$$\begin{array}{l} 0 \longrightarrow H^0(M, \mathcal{O}^*) \xrightarrow{\alpha} H^0(M, \mathcal{M}^*) \xrightarrow{\beta} H^0(M, \mathcal{M}^*/\mathcal{O}^*) \\ \xrightarrow{\gamma} H^1(M, \mathcal{O}^*) \longrightarrow H^1(M, \mathcal{M}^*) \longrightarrow \cdots \end{array} \tag{2.1}$$

- α is the natural inclusion of invertible (nowhere zero) holomorphic functions (non-zero constant maps for M compact) into the space of meromorphic ones.

- As stated before,

$$\mathrm{Div}(M) = H^0(M, \mathcal{M}^*/\mathcal{O}^*).$$

  The image of β is the subgroup of principal divisors.

- One shows that $H^1(M, \mathcal{M}^*) = 0$. Therefore,

$$Cl(M) = H^1(M, \mathcal{O}^*).$$

Example 2.3. Consider the Riemann sphere $\mathbb{P}^1$. It is well known that meromorphic functions are just rational functions in one variable, i.e., $\mathbb{C}(z)$. Moreover, given any two points p_1, $p_2 \in \mathbb{P}^1$, one has

$$\{p_1\} - \{p_2\} = \begin{cases} (z - p_1), & \text{if } p_2 = \infty, \\ \left(\frac{1}{z-p_2}\right), & \text{if } p_1 = \infty, \\ \left(\frac{z-p_1}{z-p_2}\right), & \text{otherwise.} \end{cases}$$

Therefore

$$Cl(M) = \mathbb{Z}\{p\}$$

where $p \in \mathbb{P}^1$ is an arbitrary point.

Theorem 2.1. *[Hir62]*
Linearly equivalent divisors give rise to homologically equivalent subvarieties.

Consider M an algebraic surface (i.e., an algebraic variety of complex dimension 2) and $D_i = \sum_j a_{i,j} V_j$, $i = 1, 2$ two (classes of) divisors. One then defines the intersection pairing of divisors by linearity:

$$(D_1, D_2) = \sum_{j,k} a_{1,j} a_{2,k} (V_j, V_k).$$

Theorem 2.2. *[SDTI13, page 8 and Chapter 3, Section 1.2]*
Every (quasi)-projective algebraic variety over $\mathbb{C}$ admits a triangulation.

Corollary 2.3. *Every divisor can be represented as a singular chain.*

2.2 Holomorphic Line Bundles

We have already defined holomorphic vector bundles before (see Section 1.4). Here we present some standard results for line bundles from an algebraic point of view.

Definition 2.8. The *Picard group* of M is the group of (isomorphism classes of) holomorphic line bundles of M, together with the tensor product as product and the dual bundle as inverse. It is denoted by $\mathrm{Pic}(M)$.

Consider a divisor $D = \{\{U_\alpha\}, \{f_\alpha\}\}$ and the quotients

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

It is immediate to check that

$$\{g_{\alpha\beta}\}_{\alpha\beta} \in H^1(M, \mathcal{O}^*),$$

therefore it defines a line bundle whose transition maps are precisely $\{g_{\alpha\beta}\}$. This line bundle is denoted by $[D]$ or $L(D)$.

The kernel of this map are the divisors such that

$$f_\alpha = f_\beta, \quad \forall\alpha,\ \beta,$$

i.e., the principal divisors.

Proposition 2.4. *[GH94, page 161] Any line bundle can be constructed in this way. Moreover, the map [ ] is a morphism, i.e.,*

$$[D + D'] = [D] \otimes [D'].$$

This defines an isomorphism

$$\mathrm{Pic}(M) \cong Cl(M).$$

In other words, there is a one-to-one correspondence between *divisor classes* and *isomorphism classes of line bundles*. Due to this relation, it is common to use the additive notation on line bundles, i.e., to write $L_1 + L_2$ instead of $L_1 \otimes L_2$ and $-L$ instead of L^*.

Remark. The map [ ] is the map γ found at the long exact sequence 2.1.

Example 2.4. Consider $\mathbb{P}^1$. Let $p \in \mathbb{C}(z)$ be a meromorphic functions. Note that p has the same number of zeros than poles, when counted with multiplicity. For instance, $z(z-1)$ has two zeros at $\{z = 0\}$ and $\{z = 1\}$ and pole of order two at infinity. In particular,

$$\{p\} = \{\infty\} + (z - p)$$

and all points are equivalent as a divisor, hence

$$H^1(M, \mathcal{O}^*) = \mathbb{Z}\{p\}.$$

Note that this argument also applies to $\mathbb{P}^n$.

Remark. With the identification of last example, the canonical bundle corresponds to $-2\{p\}$.

Definition 2.9. Two line bundles L_1, $L_2 \in Pic(M)$. are *algebraically equivalent* if there is a connected scheme T, two closed points t_1, $t_2 \in T$ and a line bundle $L \to M \times T$ such that

$$L_{M\times\{t_1\}} \cong L_1,$$

$$L_{M\times\{t_2\}} \cong L_2.$$

Definition 2.10. $Pic^0(M)$ is the subgroup of $Pic(M)$ formed by the isomorphism classes of line bundles algebraically equivalent to zero.

Definition 2.11. The **Neron-Severi group** is the quotient group

$$NS(M) := \frac{Pic(M)}{Pic^0(M)}.$$

We have not defined the concept of *scheme* nor explained any property about them. However,as the reader will see in Proposition 4.4, for a K3 surface (the objects of our study) the subgroup $Pic^0(M)$ is trivial. Therefore we will not refer anymore to the concept of algebraically equivalent and we will use both concepts of Picard group and Neron-Severi group interchangeably. We will also use properties of one when working with objects of the other one. Nevertheless, all propositions will be stated using the correct group. A reader who is new to these concepts should keep always is mind that, for general algebraic varieties, they are different and have different properties.

Proposition 2.5. *[GH94, page 461] The Neron-Severi group* $NS(M)$ *is finitely generated.*

Definition 2.12. The rank of $NS(M)$ is called the *Picard number*:

$$\rho(M) := \mathrm{rk}(NS(M)).$$

Example 2.5. The Picard group of $\mathbb{P}^n$ is $\mathbb{Z}$.

Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

where the first map is the inclusion and the second map is given by the exponential. This sequence gives rise to a long exact sequence

$$\begin{array}{c} \cdots \longrightarrow H^1(M,\mathbb{Z}) \longrightarrow H^1(M,\mathcal{O}) \longrightarrow H^1(M,\mathcal{O}^*) \xrightarrow{\gamma} \\ H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathcal{O}^*) \longrightarrow \cdots \end{array} \tag{2.2}$$

Consider a line bundle $L = L(D) = [D] \to M$ associated to a divisor D. Denote by $\mathcal{L}(D)$ the space of meromorphic functions f on M such that $D + (f)$ is effective:

$$\mathcal{L}(D) := \{f \in \mathcal{M}^*(M) \mid D + (f) \geq 0\}.$$

Denote by $\mid D \mid \subset \mathrm{Div}(M)$ the set of all effective divisors linearly equivalent to D:

$$\mid D \mid := \left\{D' \in \mathrm{Div}(M) \mid D \sim D' \text{ and } D' \geq 0\right\}.$$

Let s_0 be a global holomorphic section of L with $(s_0) = D$. Then there exists an identification

$$\mathcal{L}(D) \xrightarrow{\otimes s_0} H^0(M, L).$$

Example 2.6. Consider $D = (2p)$. The line bundle $L = L(D)$ is the dual of the canonical bundle in $\mathbb{P}^1$. Then

$$\mathcal{L}(D) = \{f \in \mathbb{C}(z) \mid -2 \leq \mathrm{ord}_p(f) \leq 0\}.$$

Assume that $H^0(M, L) \neq \{0\}$ and take a basis $\{s_0, \ldots, s_N\}$ of $H^0(M, L)$ as a vector space. Denote by M_0 the set of common zeros of the elements of the basis, i.e.,

$$M_0 := \{z \in M \mid s_0(z) = \cdots = s_N(z) = 0\}.$$

One defines the map

$$i_L : M \backslash M_0 \to \mathbb{P}(H^0(M, L))$$

by

$$i_L(p) = [s_0(p), \ldots, s_N(p)]. \tag{2.3}$$

Definition 2.13. A line bundle L is called *positive* if its curvature is a positive form.

Definition 2.14. A line bundle L is called *very ample* if the map $i_L(p)$ defines an embedding. It is called *ample* if there exists a $k \in \mathbb{N}$ such that L^k is very ample.

Theorem 2.6. *(Kodaira Embedding Theorem)[GH94, page 181]*
Let M be a compact complex manifold and $L \to M$ a positive line bundle. Then, there exists a $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, the map

$$i_{L^k} : M \to \mathbb{P}^{N(k)}$$

is a well defined embedding, i.e., it is very ample.

2.3 Classical Invariants

In this subsection, we are gathering some classical definitions algebraic varieties and complex surfaces which we will use in later parts of this work. Let M be a compact complex manifold. The exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

gives a connecting morphism in cohomology

$$H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}).$$

Definition 2.15. The *(first) Chern class* of a line bundle $L \in \mathrm{Pic}(M)$ is defined as

$$\mathrm{c}_1(L) := \delta(L).$$

Remark. The operator c_1 is a morphism, i.e.,

$$\mathrm{c}_1(L_1 \otimes L_2) = \mathrm{c}_1(L_1) + \mathrm{c}_1(L_2),$$

$$\mathrm{c}_1(L_1^*) = -\mathrm{c}_1(L_1),$$

for any pair L_1, $L_2 \in \mathrm{Pic}(M)$.

Next proposition is often used as an alternate definition of the Chern class:

Proposition 2.7. *[GH94, page 141]*

1. *For any line bundle L with curvature form θ*

$$\mathrm{c}_1(L) = \left[\frac{i}{2\pi}\theta\right] \in H^2_{DR}(M).$$

2. *If $L = [D]$ for a divisor D, then*

$$\mathrm{c}_1(L) = \nu_D$$

where ν_D is the Poincare dual of D.

Theorem 2.8. *(Lefschetz Theorem)[GH94, page 163]*
Any two-form ω such that

$$\omega \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$$

is the first Chern class of a hermitian line bundle. Moreover, if M is simply-connected then L is unique.

Definition 2.16. Let L be a line bundle over a complex manifold M. The *Euler characteristic* of L is defined as:

$$\chi(M, L) := \sum (-1)^n h^n(M, L).$$

Assume now that M is a complex surface and let $L \to M$ be a line bundle.

Definition 2.17. The *intersection* of two line bundles L_1, L_2 over M is defined as the intersection of the divisors defining the line bundles, i.e.

$$(L_1, L_2) := (D_1, D_2)$$

where $L_i = L(D_i)$, $i = 1, 2$.

The Euler characteristic is sometimes used as an alternate definition of the intersection form of two line bundles:

Proposition 2.9. *[Huy, page 9] Given L_1, L_2 two line bundles over a surface M*

$$(L_1, L_2) = \chi(M, \mathcal{O}_X) - \chi(M, L_1^*) - \chi(M, L_2^*) + \chi(M, L_1 \otimes L_2).$$

Next we have the Riemann-Roch theorem, which plays an important role in the computation of dimension of the space of section of line bundles.

Theorem 2.10 (Riemann-Roch)**.** *[Huy, page 10]*
Given a line bundle $L \to M$ over a surface M,

$$\chi(M, L) = \frac{(L, L \otimes \omega_M^*)}{2} + \chi(M, \mathcal{O}_M),$$

where ω_M is the canonical bundle of M.

To finish this chapter, an important result on the cohomology of ample line bundles:

Theorem 2.11 (Kodaira-Ramanujam)**.** *[Huy, Theorem 2.1.8]*
Let M be a smooth projective surface. If L is a very ample line bundle, then

$$H^i(X, L \otimes \omega_M) = 0$$

for $i > 0$.

3 Berezin-Toeplitz Quantization

This chapter is important, not only for describing the results, but also for understanding the flow and direction of the work. Most of the hypothesis used later on appear due to some properties of quantizable manifolds, and the direction of the work reflects the interest of understanding the relation between the different quantizations of the K3 surfaces.

There are many notions of quantization, which are usually *asymptotically equivalent* when working with reasonable objects. We will restrict ourselves to the so called *Berezin-Toeplitz quantization.* But before starting, let us show a reason of the existence of different quantization schemes. To do so, consider the simple case of $M = \mathbb{R}^n$ and denote by $\hbar$ the Planck's constant.

Definition 3.1. A *full quantization* of M is a map

$$\mathcal{F} : f \mapsto \widehat{f}$$

taking classical observables f, i.e., continuous functions of $(q,p) \in T^*M$, to self adjoint operators $\hat{f}$ on a Hilbert space $\mathcal{H}$ such that:

1. $\mathcal{F}$ is linear:
$$\widehat{(f+g)} = \hat{f} + \hat{g},$$
$$\widehat{(\lambda f)} = \lambda \hat{f},$$
for $\lambda \in \mathbb{R}$.

2. $\mathcal{F}$ is a Lie-algebra morphism (up to a constant):
$$\widehat{\{f,g\}} = \frac{1}{\hbar}\left[\hat{f}, \hat{g}\right].$$

3. The constant function 1 maps to the identity:
$$\hat{1} = Id.$$

4. The coordinates $\hat{q}^i$ and $\hat{p}_j$ act irreducibly on $\mathcal{H} = L^2(M)$.

However, as stated in the Groenewold-van-Hove Theorem (see [VH51]), there exists no full quantization. To overcome this problem, two things are changed:

(i) One chooses a *polarization.* In our case, the operators will act only on the space of holomorphic sections of a certain line bundle.

(ii) One asks some of the properties to be fulfilled *asymptotically*, i.e., when the Planck's constant tends to zero
$$\hbar \to 0.$$

Different choices of a polarization and subsets of the previous properties give rise to different quantization schemes. In particular, Berezin-Toeplitz quantization fulfills properties 1 and 3; Theorem 3.4 shows precisely that 3 is true asymptotically. This chapter explains the construction of Berezin-Toeplitz operators, as well as shows their basic properties.

For a more complete review on Berezin-Toeplitz quantization, the reader should check [Sch10]. Any reader interested in a more detailed discussion about quantization in general can read *Foundation of Mechanics* [AM78].

3.1 Quantum Line Bundles

The object of study of geometric quantization are the operators on the Hilbert space of sections of a particular line bundle:

Definition 3.2. Let (M, ω) be a symplectic manifold. A *quantum line bundle* for M is a triple (L, h, ∇) where L is a complex line bundle, h is a Hermitian metric on L, and ∇ is a connection compatible with the metric h such that the *(pre)-quantum condition*

$$\mathrm{curv}_{L,\nabla}(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -i\omega(X, Y)$$

is fulfilled. A symplectic manifold M is called *quantizable* if there exists a quantum line bundle for it.

For Kähler manifolds, we require L additionally to be holomorphic, and that the connection is compatible with both the metric and the complex structure of the line bundle (see Lemma 1.11). One checks that with this requirement, ∇ is uniquely determined. In the frame of Berezin-Toeplitz quantization one studies the space of holomorphic sections of this vector bundle and its space of linear operators. Most of the results of BT-quantization are done in the case that this space is closed. This is the case, for instance, whenever M is compact. Since K3 surfaces, our main objects of study, are compact, we will assume from now on that M is compact.

Example 3.1. Consider $\mathbb{P}^n$ the projective space. The Fubini-Study form

$$\omega_{FS} := i\frac{\left(1+ \mid \omega \mid^2\right) \sum_{i=1}^n \mathrm{d}\omega_i \wedge \mathrm{d}\overline{\omega}_i - \sum_{i,j=1}^n \overline{\omega}_i \omega_j \mathrm{d}\omega_i \wedge \mathrm{d}\overline{\omega}_j}{\left(1+ \mid \omega \mid^2\right)^2}$$

defines a Kähler structure, where $\omega_i = \frac{z_i}{z_0}$ are affine coordinates. The quantum line bundle is the hyperplane section bundle.

3.2 Embedding into Projective Space

Assume that M is a quantizable compact Kähler manifold with a quantum line bundle L. The prequantum condition implies that L is positive. By

Kodaira's Embedding Theorem 2.6, there exists a m_0 such that, for every $m > m_0$, the map i_{L^m} (see Formula 2.3) defines an embedding into a projective space. In other words, L^{m_0} is a very ample line bundle and therefore L is an ample line bundle. In the following, we will assume that L is already very ample by rescaling the Kähler form to $m_0\omega$ and considering L^{m_0} as the starting line bundle. Note that the underlying complex manifold structure does not change by this operation. This embedding realizes M as a smooth projective variety.

Remark. The embedding i_L is a holomorphic map. However, it is not symplectic, i.e., in general

$$i_L^*(\omega_{FS}) \neq \omega.$$

3.3 Construction of Berezin-Toeplitz Operators

Berezin-Toeplitz quantization assigns to each (complex valued) differentiable function $f \in C^\infty(M)$ a quantum operator T_f which acts on the space $H^0(M, L)$ of holomorphic sections of the quantum line budnle L.

As explained before, the Berezin-Toeplitz operators acting on a Hilbert space of holomorphic sections of a fixed line bundle do not have all the desired properties of a full quantization. However, if one considers the whole family of line bundles

$$\left(L^m, h^{(m)}, \nabla^{(m)}\right),$$

then the resulting family of operators will have the correct *semiclassical limit*.

Take the Liouville form $\Omega = (1/n!)\omega^n$ as a volume form on M (see Lemma 1.1) and the scalar product and norm

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\Omega, \quad \|\varphi\| := \sqrt{\langle \varphi, \varphi \rangle}$$

on the space $\Gamma_\infty(M, L^m)$ of global C^∞-sections.

One then uses this norm to construct the L^2-completion of $H^0(M, L^m)$:

Definition 3.3. The $L^2(M, L^m)$ space is the vector space of *measurable* sections φ of L^m with finite norm:

$$\|\varphi\| = \int_M h^{(m)}(\varphi, \varphi)\Omega < \infty.$$

All the sections we use through this work are smooth (in particular measurable). Since we won't need any special property of measurable functions, we will skip its definition. Any interested reader is refered to Knapp's book *Basic real analysis* ([Kna05]). What we need is the following result:

Theorem 3.1. *[Kna05, Theorem 5.59] The space $L^2(M, L^m)$ is a Hilbert space.*

In particular, it is complete. Consider its finite-dimensional subspace of global holomorphic sections $H^0(M, L^m)$. Since it is a closed vector subspace, the orthogonal projection (with respect to the previous metric) is well-defined:

$$\Pi^{(m)} : L^2(M, L^m) \to H^0(M, L^m).$$

Definition 3.4. For $f \in C^\infty(M)$, the *Toeplitz operator* $T_f^{(m)}$ (of level m) is defined by

$$T_f^{(m)} := \Pi^{(m)}(f\cdot) : H^0(M, L^m) \to H^0(M, L^m).$$

As stated before, the *Toeplitz map*

$$T^{(m)} : f \mapsto T_f^{(m)}$$

is linear, but it is neither a Lie algebra homomorphism nor an associative algebra homomorphism. In general,

$$T_f^{(m)} T_g^{(m)} \neq T_{fg}^{(m)}.$$

3.4 The Generalized Hardy Space

In this section, we briefly introduce the generalized Hardy space and we show the relation between sections of the quantum Hilbert space and the functions on the Hardy space. This relation is used to proof Theorem 3.4.

Consider

$$(U, k) := (L^*, h^{-1})$$

the dual vector bundle (L, h) together with the dual metric. Consider the circle subbundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\}$$

and the disc bundle

$$\overline{D} := \{\lambda \in U \mid k(\lambda, \lambda) \leq 1\}.$$

Consider $\tau : U \to M$ the projection. Using the function

$$\widehat{k}(\lambda) := k(\lambda, \lambda)$$

define

$$\widehat{\alpha} := \frac{1}{2i}(\partial - \overline{\partial}) \log \widehat{k}$$

on $U \backslash 0$ and denote by α its restriction to Q.

Proposition 3.2. *[Sch10, page 20]*

$$\mu := \frac{1}{2\pi} \tau^* \Omega \wedge \alpha$$

is a volume form on Q.

Definition 3.5. The *generalized Hardy space* $\mathcal{H}$ is the closure of the space of those functions in $L^2(Q,\mu)$ which can be extended to holomorphic functions on the whole bundle $\overline{D}$.

The natural S^1-action on Q preserves $\mathcal{H}$. Moreover, it determines a decomposition

$$\mathcal{H} = \prod_{m=0}^{\infty} \mathcal{H}^{(m)},$$

where $c \in S^1$ acts on $\mathcal{H}^{(m)}$ as multiplication by c^m.

Proposition 3.3. *[Sch10, page 21] Sections of L^m can be identified with functions ϕ on Q which satisfy the equivariance condition*

$$\phi(c\lambda) = c^m \phi(\lambda), c \in S^1.$$

This identification is given by the map

$$\psi_m : L^2(M, L^m) \longrightarrow L^2(Q,\mu)$$

$$s \longmapsto \phi_s,$$

where

$$\phi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))).$$

Moreover, under this identification

$$\psi_m : H^0(M, L^m) \cong \mathcal{H}^{(m)}.$$

3.5 Basic Properties

As was shown by Bordemann, Meinrenken and Schlichenmaier in [BMS94], Berezin-Toeplitz operators have the *correct semiclassical limit.* Given a function $f \in C^\infty(M)$, denote by $\mid f \mid_\infty$ the supremum norm of f and by

$$\|T_f^{(m)}\| := \sup_{\substack{s \in H^0(M,L^m) \\ s \neq 0}} \frac{\|T_f^{(m)} s\|}{\|s\|}$$

the operator norm with respect to the hermitian metric.

Theorem 3.4 (Bordemann, Meinrenken, Schlichenmaier)**.** *[Sch10, Theorem 3.3]*

1. *For every $f \in C^\infty(M)$, there exists a $C > 0$ such that*

$$\mid f \mid_\infty - \frac{C}{m} \leq \|T_f^{(m)}\| \leq \mid f \mid_\infty .$$

In particular,

$$\lim_{m\to\infty} \|T_f^{(m)}\| = \mid f \mid_\infty .$$

2. *For every* $f, g \in C^\infty(M)$,

$$\left\| mi \left[T_f^{(m)}, T_g^{(m)} \right] - T_{\{f,g\}}^{(m)} \right\| = O\left(m^{-1}\right).$$

3. *For every* $f, g \in C^\infty(M)$,

$$\left\| T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)} \right\| = O\left(m^{-1}\right).$$

Proof. The complete proof of this theorem can be found in [BMS94], Theorems 4.1 and 4.2. Here, we are only making some remarks on the proof of the first inequality of (1), since later on we will adapt it to our needs.

Denote by

$$\pi : L \to M$$

the projection map and by k the hermitian metric on L. Consider $x_0 \in M$ the point were $|f|$ assumes its supremum and a point $\lambda_0 \in \pi^{-1}(x_0)$ such that $k(\lambda_0, \lambda_0) = 1$. By Proposition 3.3, there is a one to one correspondence between holomorphic sections $\phi^{(m)}$ of $L^{(m)}$ and equivariant holomorphic functions

$$\tilde{\phi}^{(m)} : Q^* \to \mathbb{C}.$$

Consider the sequence $\{\phi^{(m)}\}_m$ of sections corresponding to

$$\tilde{\phi}^{(m)}(\lambda) = k(\lambda_0, \lambda).$$

In the aforementioned paper it is proven that

$$\frac{\|T_f^{(m)}\phi^{(m)} - f(x_0)\phi^{(m)}\|}{\|\phi^{(m)}\|} = O(m^{-1}),$$

which implies the desired inequality. □

Corollary 3.5. *[Sch10, Proposition 3.3] Let* $f_1, f_2, \ldots, f_r \in C^\infty(M)$*, then*

$$\left\| T_{f_1}^{(m)} \cdots T_{f_r}^{(m)} - T_{f_1 \cdots f_r}^{(m)} \right\| = O\left(m^{-1}\right).$$

Corollary 3.6. *[Sch10, Proposition 3.4]*

$$\lim_{m \to \infty} \left\| \left[T_f^{(m)}, T_g^{(m)} \right] \right\| = 0.$$

Remark. If one reads the proof of last corollary ([BMS94]), one sees that, in fact

$$\left\| \left[T_f^{(m)}, T_g^{(m)} \right] \right\| = O\left(m^{-1}\right).$$

Since the dimension of $H^0(M, L^m)$ is finite for all $m \in \mathbb{N}$, the map $f \mapsto T_f^{(m)}$ is not injective. However,

Proposition 3.7. *[Sch10, page 10] If* $\|T_f^{(m)}\| \to 0$ *then* $f = 0$.

Proposition 3.8. *[Sch10, Proposition 3.6] The Toeplitz map is surjective.*

4 K3 Surfaces

In this chapter, we introduce some basic notions on K3 surfaces. By definition, these objects have three different Kähler structures (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K) sharing the same Riemannian metric such that $IJ = K$. Recall that a Kähler structure is a triple (g, I, ω_I), where g is a Riemannian metric, I a complex structure and ω_I a symplectic form, such that

$$g(\cdot, \cdot) := \omega_I(I\cdot, \cdot).$$

As the reader will see, such manifolds have in fact an infinite amount of Kähler structures compatible with g.

We will follow mostly Huybrechts' lecture notes ([Huy]). Most of the original results are written for arbitrary fields. However, for simplicity's sake, we will only deal with case of the field of complex numbers $\mathbb{C}$.

4.1 Basic Definition

Definition 4.1. A *(complex) K3 surface* is a compact connected complex manifold X of complex dimension two such that:

- $\Omega_X^2 \cong \mathcal{O}_X$.
- $H^1(X, \mathcal{O}_X) = 0$.

The first condition says that the *canonical bundle* is trivial since X compact. The second *almost* says that it is simply-connected. A priory, the fundamental group could have some torsion elements. However, as the next proposition says, K3 surfaces are, indeed, simply-connected:

Proposition 4.1. *[Huy, Page 12] The fundamental group $\pi_1(X)$ of a K3 surface X is trivial.*

Remark. An *(algebraic) K3 surface* is a complete separated non-singular variety X of complex dimension 2 such that $\Omega_X^2 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

An algebraic K3 surface is clearly also a complex K3 surface. However, there are non-projective complex K3 surfaces. We will mostly use the first definition, but since most of the work in K3 surfaces is done on an algebraic level, it is worth having both definitions in mind.

Definition 4.2. An *hyperkähler manifold* is a manifold which admits 3 different Kähler structures (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K) such that

$$IJ = K.$$

Note that the three Kähler structures share the same Riemann metric g.

Due to the next proposition, we will also refer to this definition when talking about K3 surfaces:

Proposition 4.2. *([Saf, page 1]) A K3 surface is a two dimensional hyperkähler manifold. Conversely, all simply-connected compact hyperkähler manifolds of dimension two are K3 surfaces.*

Definition 4.3. Consider an hyperkähler manifold and fix one of the Kähler structures, let's say (g, I, ω_I). Then according to the complex structure I, the 2-form

$$\Omega_I = \omega_J + i\omega_K \in \Omega_X^2$$

is a trivializing holomorphic symplectic form. Since Ω_I depends only on the chosen complex structure, it is called *the holomorphic symplectic form* of the hyperkähler manifold (X, g, I, ω_I).

Remark. It is a straightforward computation to see that Ω_I is indeed a holomorphic symplectic form. Any interested reader can find this computation as part of the proof of Proposition 4.2.

Example 4.1. Any non-singular degree 4 surface in $\mathbb{P}^3$ is a K3 surface. For instance, the set of zeroes of the polynomial

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0,$$

where $z = [z_0, \cdots, z_3]$ are homogeneous coordinates, is a K3 surface.

Example 4.2. Kummer surfaces are K3 surfaces (see Section 4.2).

The Riemann-Roch theorem (Theorem 2.10) allows one to compute the number of holomorphic sections of a very ample line bundle on a K3 surface:

Corollary 4.3. *For a K3 surface,*

$$\chi(X, L) = \frac{(L.L)}{2} + 2.$$

In particular, if L is a very ample line bundle,

$$h^0(X, L) = \frac{(L.L)}{2} + 2.$$

Proof. This is easily seen from the Riemann-Roch formula:

$$\chi(M, L) = \frac{(L, L \otimes \omega_M^*)}{2} + \chi(M, \mathcal{O}_M),$$

On one side,

$$\chi(M, L) = h^0(X, L) - h^1(X, L) + h^2(X, L).$$

Since L is very ample, by Theorem 2.11,

$$h^2(X, L) = h^1(M, L) = 0.$$

On the other side, by definition of a K3 surface, ω_M is trivial and

$$h^k(X, \mathcal{O}) = \begin{cases} 1 \text{ for } k = 0, \\ 0 \text{ for } k = 1, \\ 0 \text{ for } k > 2. \end{cases}$$

By Serre duality, $h^2(X, \mathcal{O}_M) = h^0(X, \mathcal{O}_M) = 1$ and therefore

$$\chi(M, \mathcal{O}_M) = 2.$$

□

Proposition 4.4. *[Huy, Section 2, Proposition 2.3] For a K3 surface,*

$$NS(X) \cong Pic(X).$$

Moreover, the intersection pairing is even, non-degenerate and of signature $(1, \rho(X) - 1)$*, where* $\rho(X) = rk(\mathrm{Pic}(X))$ *is the Picard number.*

Remember the Sequence 2.2:

$$\cdots \longrightarrow H^1(M, \mathbb{Z}) \longrightarrow H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{\ \gamma\ } H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathcal{O}^*) \longrightarrow \cdots$$

As explained in [Huy, Page 15] the map γ is injective, therefore this gives an embedding

$$Pic(X) \hookrightarrow H^2(X, \mathbb{Z}). \tag{4.1}$$

Proposition 4.5. *If L_1, L_2 are two line bundles such that $(L_1, L_1) > 0$, then*

$$(L_1, L_1)(L_2, L_2) \leq (L_1, L_2)^2.$$

Proof. Consider the line bundle

$$L := (L_1, L_1)L_2 - (L_1, L_2)L_1.$$

L is clearly is orthogonal to L_1. Last proposition says that the signature of $Pic(X)$ is $(1, \rho(X) - 1)$. Therefore, $(L, L) \leq 0$. However

$$(L, L) = (L_1, L_1)^2(L_2, L_2) - (L_1, L_1)(L_1, L_2)^2.$$

Since $(L_1, L_1) > 0$, that proves the result. □

Proposition 4.6. *[Huy, page 12] There are no non-trivial torsion line bundles on a K3 surface.*

Lemma 4.7. *[Huy, page 16] The Euler characteristic (Definition 2.16) of a K3 surface is*

$$e(X) = 24.$$

Corollary 4.8. *[Huy, page 16] The Betti numbers of a K3 surface are*

$$b_i = \begin{cases} 1 & \text{for } i = 0, 4; \\ 22 & \text{for } i = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since M is connected and compact, then

$$b_0 = b_4 = 1.$$

Since M is simply connected (and by Serre duality)

$$b_1 = b_3 = 0.$$

Recall that

$$e(X) = b_0 - b_1 + b_2 - b_3 + b_4.$$

Therefore $b_2 = 20$. □

4.2 Example: Kummer Surfaces

The original paper of Kummer can be found in [Kum75]. However, this section follows [Sco05]. Let $\mathbb{T}$ be the real 4-torus

$$\mathbb{T} = S^1 \times S^1 \times S^1 \times S^1,$$

where each S^1 is considered as the unit circle inside $\mathbb{C}$. Denote (z_1, z_2, z_3, z_4) the standard coordinates of $\mathbb{C}^4$. Consider the map

$$\sigma : \mathbb{T} \to \mathbb{T} \qquad \sigma(z_1, z_2, z_3, z_4) = (\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4)$$

given by complex-conjugation in each circle-factor. This involution has 16 fixed points and the quotient

$$\mathbb{T}/\sigma$$

has sixteen singular points. Denote by P the set of fixed points in $\mathbb{T}$.

Now, one could do a (complex) blow-up at those singular points to get a smooth manifold. However, to make it easier to relate the structures on the original torus $\mathbb{T}$ and the final manifold, it is more useful to do the blow-up in each of those fixed points *before* doing the quotient:

$$\pi : M \to \mathbb{T}.$$

In that way, one gets a manifold M where each point is replaced by a sphere of self-intersection -1. Note that the involution σ extends to M by fixing the spheres so the action of σ lifts to M. The *Kummer surface* associated to $\mathbb{T}$ is defined as the quotient

$$X := M/\sigma,$$

and the projection is denoted by τ:

$$\tau : M \to X.$$

Due to this construction, many properties of Kummer surfaces can be described in terms of the torus they come from.

To see that this surface is indeed a K3 surface, one considers a $\mathbb{Z}_2$-invariant holomorphic symplectic form on $\mathbb{T}$. Using a modified version of blow-ups called *symplectic cut* one sees that this lifts through π and then descends through τ into a holomorphic symplectic form on X.

Remark. The name of Kummer surface is sometimes used for the singular variety (before doing the blow-ups).

4.3 Twistor Space

We have defined a K3-surface as a surface with 3 Kähler structures. But in fact, as one can see at ([HKLR87]), it has an *infinite* number of them. Let (a, b, c) be a unit vector in $\mathbb{R}^3$.

Lemma 4.9. *Let I, J, K be three complex structures in X compatibles with a Riemannian metric g on a K3 surface such that $IJ = K$. Then $aI+bJ+cK$ is also a complex structure in X. Moreover, this new complex structure is compatible with the metric g, i.e., $g(aI + bJ + cK, \cdot)$ is a Kähler form.*

Proof. Let $p = (a, b, c) \in S^2$. Denote $\underline{I} = aI + bJ + cK$ the new almost complex structure. By direct computations

$$\underline{I}^2 = -Id.$$

Moreover, since $\underline{I}$ is covariantly constant, then it is integrable (see [Saf, page 2]).

Since a, b and c are constants and ω_I, ω_J and ω_K are symplectic (therefore closed), the two-form is closed:

$$\mathrm{d}(a\omega_I + b\omega_J + c\omega_K) = 0.$$

By bilinearity, it is clear they fulfill the compatibility condition

$$g(\cdot, \underline{I}\cdot) = a\omega_I + b\omega_J + c\omega_K,$$

and therefore the two-form $a\omega_I + b\omega_J + c\omega_K$ is non-degenerate, which implies that $(X, g, \underline{I}, a\omega_I + b\omega_J + c\omega_K)$ form a compatible triple.

□

Thus, we have a "sphere" of complex structures compatibles with the Riemannian metric of X. The idea of the twistor space is to incorporate all these structures into one complex structure of a larger manifold Z. This manifold is, from a smooth point of view, $X \times S^2$. However, from a complex point of view they are not isomorphic. The construction of the complex structure on Z goes as follows:

Consider the tangent space $TZ = TX \oplus TS^2$. Given a point $p = (m, \xi) \in Z$, consider the complex structure on T_pZ defined as

$$\underline{I}_p = (aI + bJ + cK, I_0)$$

where $(a, b, c) = \xi \in S^2 \subset \mathbb{R}^3$ and I_0 is the standard complex structure of $S^2 \cong \mathbb{P}^1$.

Theorem 4.10. *[HKLR87, page 554]*
Z is a complex manifold. Moreover, the projection

$$p : Z \to \mathbb{P}^1$$

is holomorphic.

Definition 4.4. Z is called the *twistor space* of X.

Definition 4.5. A *twistor line* is a fiber of the projection

$$p : Z \to \mathbb{P}^1.$$

Remark. A twistor line is homeomorphic to X.

Proposition 4.11. *The twistor lines are Kähler manifolds.*

Proof. Apply Lemma 4.9. □

For the sake of simplicity, when working with a particular complex structure I on a K3 surface X, we may denote it by X_I.

Note that X_I, X_J and X_K can be seen as immersed complex submanifolds of Z. In particular, they are isomorphic to the fibers with ξ equal to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Although not completely correct, we will often say that such a ξ is a point on the twistor space.

Remark. In general, the twistor space is not Kähler. Consider in Z the metric $g_Z = \pi^* g \pm p^* g_{\mathbb{C}P^1}$. This metric is *tamed* by the complex structure, i.e., $g(I\cdot, \cdot$ is a two-form. However, it is not closed.

Definition 4.6. The *quaternionic space* is the algebra $\mathbb{H}$ generated as a real vector space by $< 1, i, jj >$ such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Later on, it will be useful to consider the following definition:

Definition 4.7. Consider a complex structure $I_p = aI + bJ + cK$ on a K3 surface X, where $a, b, c \in \mathbb{R}$. The *associated quaternionic number* of I_p is defined as

$$i_p := ai + bj + ck.$$

Proposition 4.12. *Consider a quaternionic number* $q = a+bi+cj+dk \in \mathbb{H}$. *The following expression determines a norm:*

$$\|q\| := \sqrt{q\overline{q}} = \sqrt{(a + bi + cj + dk)(a - bi - cj - dk)}.$$

Definition 4.8. A quaternionic number $q = a + bi + cj + dk \in \mathbb{H}$ is called *real* if its *imaginary part* is zero, i.e.,

$$Im(q) := bi + cj + dk = 0.$$

Similarly, q is called *purely imaginary* if its *real part* is zero, i.e.,

$$Re(q) := a = 0.$$

Note that there is a 1-1 correspondence between complex (and Kähler) structures on X induced by the twistor space and purely imaginary quaternionic numbers of norm 1. Moreover, this correspondence is functorial with respect to the quaternionic product.

4.4 Lattices

This section contains some basic notions and results about lattices that will be useful for working with K3-surfaces.

Definition 4.9. A *lattice* Λ is a free $\mathbb{Z}$-module together with a symmetric non-degenerate bilinear form

$$(\quad . \quad) : \Lambda \times \Lambda \to \mathbb{Z}.$$

A lattice Λ is called *even* if $(x.x)$ is even for all $x \in \Lambda$, otherwise Λ is called *odd*. The determinant of the bilinear form is called the *discriminant* of the lattice and it is denoted by $disc(\Lambda)$.

Definition 4.10. Consider a basis $V = \{v_1, \cdots, v_m\} \subset \Lambda$ of Λ. The *intersection matrix* of $(\quad . \quad)$ with respect to V is defined as

$$M_V := \begin{pmatrix} (v_1.v_1) & \cdots & (v_1.v_m) \\ \cdots & \cdots & \cdots \\ (v_1.v_m) & \cdots & (v_m.v_m) \end{pmatrix}.$$

Remark. A lattice (with its bilinear form) its determined (up to isomorphism) by its intersection matrix.

Let $\Lambda^* := Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the dual lattice. Consider the injection

$$\begin{aligned} i_\Lambda : \Lambda &\hookrightarrow \Lambda^* \\ x &\mapsto (x.\ ). \end{aligned}$$

Then $i_\Lambda(\Lambda)$ has finite index in Λ^*.

Definition 4.11. A lattice Λ is called *unimodular* if i_Λ is an isomorphism.

Definition 4.12. Then the *signature* of Λ is defined as

$$sign(\Lambda) := (p, m - p).$$

where p is the number of positive eigenvalues (with multiplicity) of the bilinear form.

Remark. The signature is well defined since the intersection pairing is symmetric and non-degenerate.

Example 4.3. The *hyperbolic plane (lattice)* is the lattice U determined by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e., $U \cong \mathbb{Z}^2 = \mathbb{Z}e \oplus \mathbb{Z}f$ with the quadratic form $(e.e) = (f.f) = 0$ and $(e.f) = 1$. Clearly, $disc(U) = -1$.

Example 4.4. The E_8-lattice is given by the intersection matrix

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

and is even, unimodular, positive definite of rank eight with $disc(E_8) = 1$ and $sign(E_8) = (8, 0)$.

For any given lattice Λ the twist $\Lambda[m]$ is obtained by multiplying the intersection form $(\quad . \quad)$ of Λ by the integer m. In other words, Λ and $\Lambda[m]$ are equal as $\mathbb{Z}$-modules but

$$(\quad . \quad)_{\Lambda[m]} := m \cdot (\quad . \quad)_\Lambda.$$

Definition 4.13. The *K3 lattice*

$$\Lambda := E_8[-1]^{\oplus 2} \oplus U^{\oplus 3}$$

is an even, unimodular lattice of signature $(3, 19)$ and discriminant -1. The *extended K3 lattice* or *Mukai lattice* is

$$\tilde{\Lambda} := E_8[-1]^{\oplus 2} \oplus U^{\oplus 4},$$

which is even, unimodular of signature $(4, 20)$ and discriminant 1.

4.5 Hodge Structures

Let V be either a lattice or a finite-dimensional vector space over $\mathbb{Q}$. Let $V_{\mathbb{Q}}$, $V_{\mathbb{R}}$ and $V_{\mathbb{C}}$ denote the extension vector spaces (i.e., $V_K := V \otimes K$). Note that $V_{\mathbb{C}}$ comes with a well defined complex conjugation, which is a $\mathbb{R}$- linear isomorphism.

Definition 4.14. A *Hodge structure* on V of weight n is a decomposition of $V_{\mathbb{C}} = V \otimes \mathbb{C}$

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

Remark. When $n > 0$, often one assumes that $V^{p,q} = 0$ for $p < 0$. This is the case whenever V is given as the cohomology of a projective manifold.

Example 4.5. For Kähler manifolds, the lattices $H^k(X, \mathbb{Z})$ have a natural Hodge structure given by the Hodge decomposition into (p, q)-differential forms.

Definition 4.15. Two integral Hodge structures V and W are called *isogenous* if their rational extensions $V_{\mathbb{Q}}$ are isomorphic.

Example 4.6. $\mathbb{Z}$ and $2\mathbb{Z}$ with the trivial Hodge structures are isogenous.

Definition 4.16. The *Hodge classes* of V are the elements of $V \cap V^{k,k}$, where one uses the natural inclusion $V \subset V_{\mathbb{C}}$.

Definition 4.17. A *sub-Hodge structure* of a Hodge structure V of weight n is given by a $\mathbb{Z}$-submodule (resp. $\mathbb{Q}$-linear subspace) $W \subset V$ for which the V induces a Hodge structure on W, i.e., the following holds:

$$W_{\mathbb{C}} = \bigoplus (W_{\mathbb{C}} \cap V^{p,q}).$$

Any Hodge structure that does not contain any non-trivial sub-Hodge structure is called *irreducible*. A sub-Hodge structure is called *primitive* if V/W is torsion free.

Definition 4.18. The *transcendental lattice* T is the minimal primitive sub-Hodge structure such that $V^{2,0} = T^{2,0} \subset T_{\mathbb{C}}$.

Remark. Here *minimal* refers to the inclusion of sub-Hodge structures.

Definition 4.19. A *morphism of Hodge Structures of weight* k is a linear map

$$f : V \to W$$

such that

$$f(V^{p,q}) \subset W^{p+k,q+k}.$$

Many standard constructions from linear algebra generalize to Hodge structures. For instance, one defines the Hodge structure for the *direct sum* $V \oplus W$ of two Hodge structures V and W of the same weight n as follows:

$$(V \oplus W)^{p,q} = V^{p,q} \oplus W^{p,q}.$$

This is again a Hodge structure of weight n.

Similarly, one defines the tensor product $W \otimes V$, dual V^*, the space of morphism of weight k $Hom_k(V, W)$, the exterior product $\bigwedge^k V$ and the complex conjugate $\overline{V}$. For a detailed construction, see [Huy, page 39].

For any K3 surface, its lattice $H^2(X, \mathbb{Z})$ is abstractly isomorphic to the K3 lattice. However, in general this isomorphism do not extend to its Hodge Structures. In a similar way, the Mukai lattice is abstractly isomorphic to the total cohomology group $H^*(X, \mathbb{Z})$.

To any Hodge structure one associates the *Hodge filtration*

$$0 \subset F^n V_{\mathbb{C}} \subset F^{n-1} V_{\mathbb{C}} \subset \dots F^0 V_{\mathbb{C}} \subset V_{\mathbb{C}}$$

with $F^i V_{\mathbb{C}} = \bigoplus_{p \geq i} V^{p,q}$.

Consider the $\mathbb{Q}$-vector space $H^{2k}(X, \mathbb{Q})$ were X is an arbitrary algebraic variety.

Conjecture 4.13 (Hodge conjecture)**.** *On a smooth projective variety, the subspace of* $H^{2k}(X, \mathbb{Q})$ *spanned by algebraic classes (fundamental classes of subvarieties)* $[Z]$ *coincides with the space of Hodge classes, i.e.*

$$H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X) = \langle [Z] \rangle_{\mathbb{Q}}.$$

The Hodge conjecture is one of the most important unsolved problems in mathematics. It is know to be true for low dimensional manifolds. In case of complex surfaces, it suffices with the following theorem:

Theorem 4.14 (Lefschetz theorem on (1,1)-classes)**.** *[GH94, page 163] Any element of*

$$H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

is the cohomology class (in the sense of Poincaré Duality, Theorem 1.8) of a divisor on X*. In particular, the Hodge conjecture is true for* H^2*.*

Remark. This is the same theorem as Theorem 2.8 in an equivalent setting.

The notion of intersection pairing it is formalized by the notion of *polarization*. Denote by C the *Weil operator*, an operator which acts in $V^{p,q}$ by multiplication by i^{p-q}. Note that it preserves the real vector space $(V^{p,q} \oplus V^{q,p}) \cap V_{\mathbb{R}}$.

Definition 4.20. A *polarization* of weight n is a morphism

$$\psi : V \times V \to \mathbb{Q}(-n)$$

such that its $\mathbb{R}$-linear extension yields a positive definite symmetric forms

$$(u, v) \mapsto \psi(u, Cv)$$

on the real part $(V^{p,q} \oplus V^{q,p}) \cap V_{\mathbb{R}}$.

Remark. Note that $\psi(u, v) = 0$ unless $p_1 + p_2 = q_1 + q_2 = n$.

Definition 4.21. An isomorphism compatible with polarizations is called *Hodge isometry.*

There are some easy consequences of the definition of polarization:

Proposition 4.15. *[Huy, page 40-41]*

1. *If $n \equiv 0(2)$, then ψ is symmetric. Otherwise, alternating.*
2. *$\psi : V^{p,q} \times V^{q,p} \to \mathbb{C}$ is non-degenerate.*
3. *The restriction of a polarization to a sub-Hodge structure is a polarization.*
4. *Consider a polarization on a $\mathbb{Q}$-vector space and a sub-Hodge structure $V' \subset V$. Then this yields a direct sum*

$$V = V' \oplus V''$$

where V'' is the orthogonal complement. For polarizations on integral lattices, we only have $V' \oplus V'' \subset V$.

A polarization on $H^{2k}(X, \mathbb{Q})$ is constructed as follows:

Definition 4.22. Fix a rational (or integral) Kähler class $\omega \in H^2(X, \mathbb{Q})$. The *Hodge-Riemann* pairing on $H^n(X, \mathbb{Q})$, $n \leq d = dim_{\mathbb{C}} X$ is the pairing

$$(u, v) \mapsto (-1)^{n(n-1)/2} \int_X u \wedge v \wedge \omega^{d-n}.$$

Its *primitive part* is defined as

$$H^n(X, \mathbb{Q})_p := Ker\left(\wedge\omega^{d-n+1} : H^n(X, \mathbb{Q}) \to H^{2d-n+2}(X, \mathbb{Q})\right).$$

Lemma 4.16. *[Huy, page 41] The Hodge-Riemann pairing defines a polarization on the primitive part.*

Remark. If X is algebraic then $H^2(X,\mathbb{Z})$ is polarizable. However, the polarization comes from changing the sign of the intersection pairing on the Kähler form.

There exists a bijection between integral Hodge structures of weight one and complex tori (see [Huy, Page 43]). In the rest of the section, we will focus on Hodge structures of weight two, which describe K3 surfaces.

Definition 4.23. A Hodge structure V is of *K3 type* if it is of weight two with $V^{p,q} = 0$ for $|p-q| > 2$ and $dim_{\mathbb{C}}(V^{2,0}) = 1$.

Any Hodge structure of K3 type contains two natural sub-Hodge structures.

1. Hodge classes $V^{1,1} \cap V$.
2. The transcendental lattice T.

Recall that the Neron-Severi group can be considered has a sublattice of $H^2(X,\mathbb{Z})$ (Equation 4.1). Moreover,

$$V^{1,1} \cap V \cong NS(X) \cong Pic(X),$$

which realizes the Neron-Severi group as a sub-Hodge structure of $H^2(X,\mathbb{Z})$.

Lemma 4.17. *[Huy, Section 3, Lemma 2.7] The transcendental lattice of a polarizable K3 type structure is polarizable irreducible of K3 type.*

In general, computing directly the transcendental lattice may be complicated. However, the next lemma gives us a nice way to describe it:

Lemma 4.18. *[Huy, Section 3, Lemma 4.1] For the transcendental lattice of a complex K3 surface one has*

$$T(X) = NS(X)^{\perp}.$$

If X is projective, then $T(X)$ is an irreducible polarized Hodge structure.

The importance of the Hodge structures of K3 type becomes clear after reading the Global Torelli Theorem:

Theorem 4.19 (Global Torelli Theorem)**.** *[Huy, Section 3, Theorem 2.2] Given two complex K3 surfaces X and X', they are isomorphic if and only if there exists a Hodge isometry*

$$H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$$

respecting the intersection pairing. Moreover, for any Hodge isometry

$$\psi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$$

with $\psi(\mathcal{K}_X) \cap \mathcal{K}_{X'} \neq \emptyset$ *there exists a (unique) isomorphism*

$$f : X' \xrightarrow{\sim} X$$

with $f_* = \psi$*.*

Corollary 4.20. *[Huy, Section 14, Corollary 3.10] Let* X *and* X' *be arbitrary complex projective K3 surfaces. Then any Hodge isometry of the transcendental lattices*

$$\varphi : T(X) \xrightarrow{\sim} T(X')$$

can be extended to a Hodge isometry of Mukai lattices:

$$\widetilde{\varphi} : H^*(X, \mathbb{Z}) \xrightarrow{\sim} H^*(X', \mathbb{Z})$$

In fact, as explained in the next corollary, there is a nice way to enumerate K3 surfaces with Picard number 20:

Corollary 4.21. *[Huy, Section 14, Corollary 3.18] The map that associates to a complex K3 surface* X *with Picard number* $\rho(X) = 20$ *its transcendental lattice* $T(X)$ *describes a bijection*

$$\{X \mid \rho(X) = 20\} \longleftrightarrow \left\{ \begin{array}{l} \textit{positive definite, even,} \\ \textit{oriented lattices of rank } 2 \end{array} \right\},$$

where both sides have to be considered up to isomorphisms.

It is interesting to remark the explicit construction of such correspondence:

Consider a complex projective K3 surface (X, g, I, ω_I) of Picard number 20. Denote by $T(X)$ its transcendental lattice and consider a basis $\{\alpha, \beta\}$ of $T(X)$. The matrix T is defined by

$$T := \begin{pmatrix} (\alpha.\alpha) & (\alpha.\beta) \\ (\alpha.\beta) & (\beta.\beta) \end{pmatrix}.$$

Lemma 4.22. *[Huy, Section 14, Corollary 3.17.a] Let* X *be a complex projective K3 surface. Assume that* $\rho(X) = 20$*. Then* X *is a Kummer surface if and only if*

$$T(X) \cong T[2]$$

for some even lattice T*.*

5 Quantum Line Bundles on K3 Surfaces

In this chapter, we will show some original results on the relation of different Kähler structures and quantum line bundles on a K3 surface, as well as studying the existence of quantizable K3 surfaces.

Unless otherwise stated, all results in this section are original.

5.1 Relations between the Kähler Structures

First, we are presenting a list of relations on Kähler forms and the tangent spaces. Some of them are interesting per se and some will be used later to prove more important results.

Consider M a K3 surface and denote its three Kähler structures by (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K).

Lemma 5.1.

$$\omega_I(\cdot, -K\cdot) = \omega_J(\cdot, \cdot).$$

Proof. Use that for a compatible triple $g(\cdot, \cdot) = \omega_I(\cdot, I\cdot)$:

$$\omega_I(\cdot, -K\cdot) = g(\cdot, IK\cdot) = g(\cdot, -J\cdot) = \omega_J(\cdot, -J^2\cdot) = \omega_J(\cdot, \cdot).$$

□

Remark. The above relation can also be written as $\omega_I(-K\cdot, \cdot) = \omega_J(\cdot, \cdot)$ or $\omega_I(\cdot, J\cdot) = \omega_K(\cdot, \cdot)$.

The following corollaries are a direct application of the lemma above:

Corollary 5.2.

$$\omega_I(K\cdot, -K\cdot) = \omega_I(\cdot, \cdot).$$

Proof.

$$\omega_I(K\cdot, K\cdot) = -\omega_J(K\cdot, \cdot) = -\omega_I(\cdot, \cdot).$$

□

Corollary 5.3. *For any vector $v \neq 0$,*

$$\omega_I(Jv, Kv) > 0.$$

Proof.

$$\omega_I(Jv, Kv) = -\omega_J(Jv, v) = \omega_J(v, Jv) > 0.$$

□

Corollary 5.4. *For any vector v,*

$$\omega_I(Iv, Kv) = \omega_I(v, Jv) = 0.$$

Proof.

$$\omega_I(Iv, Kv) = -\omega_J(Iv, v) = \omega_K(v, v) = 0.$$

□

For the following lemma, denote the complex structures I, J and K by I_1, I_2 and I_3, respectively.

Lemma 5.5. *Let M be a K3 surface and consider the holomorphic tangent spaces $T^{(1,0)}_{I_n}M$, $n = 1, 2, 3$. Then the maps*

$$\pi_n : T^{(1,0)}_{I_{n-1}}M \to T^{(1,0)}_{I_n}M \quad n \in \mathbb{Z}/3\mathbb{Z}$$

defined by $\pi_n(v) = \frac{1}{2}(v - iI_n v)$ define a canonical isomorphism of real vector bundles.

Proof. First we need to check that the maps are well defined. Remember that, in the construction of the holomorphic tangent space, we use a similar map

$$\pi_n : TM \to T^{(1,0)}_{I_n}M \quad n \in \mathbb{Z}/3\mathbb{Z}.$$

This map is extended by linearity to $TM \otimes \mathbb{C}$

$$\pi_n(iv) = i\pi(v),$$

thus it defines a well defined map on $T^{(1,0)}_{I_{n-1}}M$. Consider now the composition

$$\pi := \pi_{n+3} \circ \pi_{n+2} \circ \pi_{n+1} : T^{(1,0)}_{I_n}M \to T^{(1,0)}_{I_n}M.$$

We are going to compute $\pi(v)$ for $v \in T^{(1,0)}_{I_n}M$:

$$\begin{aligned}
\pi(v) &= \pi_n \circ \pi_{n+2} \circ \pi_{n+1}(v) \\
&= \frac{1}{2}\pi_n \circ \pi_{n+2}(v - iI_{n+1}v) \\
&= \frac{1}{4}\pi_n((v - iI_{n+1}v) - iI_{n+2}(v - iI_{n+1}v)) \\
&= \frac{1}{4}\pi_n(v - iI_{n+1}v - iI_{n+2}v + i^2 I_{n+2}I_{n+1}v) \\
&= \frac{1}{4}\pi_n(v - iI_{n+1}v - iI_{n+2}v + I_n v) \\
&= \frac{1}{8}\left((v - iI_{n+1}v - iI_{n+2}v + I_n v)\right. \\
&\left. -iI_n(v - iI_{n+1}v - iI_{n+2}v + I_n v)\right) \\
&= \frac{1}{8}\left((v - iI_{n+1}v - iI_{n+2}v + I_n v)\right. \\
&\left. -iI_n v + i^2 I_n I_{n+1}v + i^2 I_n I_{n+2}v - iI_n I_n v)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left((v - iI_{n+1}v - iI_{n+2}v + I_n v)\right.\\
&\left.-iI_n v + I_{n+1}I_n v + I_{n+2}I_n v + iv)\right)\\
&= \frac{1}{8}\left((v - iI_{n+1}v - iI_{n+2}v + iv)\right.\\
&\left.v + iI_{n+1}v + iI_{n+2}v + iv)\right)\\
&= \frac{1}{4}(1+i)v.
\end{aligned}
$$

Hence π is an isomorphism and so, all three π_n are also isomorphisms. □

Remark. In general, is not true that

$$\pi_n^* H^0(M, T_{I_n}^{(1,0)}M) = H^0(M, T_{I_{n-1}}^{(1,0)}M).$$

Remark. The map from previous lemma determines a dual map on the cotangent spaces, which is also a real isomorphism. By linearity, it extends to a real isomorphism of the space of smooth $(k, 0)$-forms, $k \geq 0$.

Lemma 5.6. *The intersection of two different holomorphic spaces is the zero section:*

$$T_I^{(1,0)}M \cap T_J^{(1,0)}M = \{0\}$$

Proof. Remember that the holomorphic tangent space $T_I^{(1,0)}M$ is defined as the eigenspace of

$$I : T_{\mathbb{C}}M \to T_{\mathbb{C}}M$$

of eigenvalue i. Therefore, a non-zero element

$$v \in T_I^{(1,0)}M \cap T_J^{(1,0)}M$$

would fulfill $Iv = Jv = iv$. Hence $Kv = IJv = -v$ and $K^2 v = v$, which is false. □

Lemma 5.7. *Consider $\Omega_K = \omega_I + i\omega_J$ the holomorphic symplectic form with respect to K (see Definition 4.3). Then*

$$\omega_I(\cdot, (1 - iK)\cdot) = \Omega_K.$$

Proof. Apply linearity and Lemma 5.1. □

Lemma 5.8.

$$\pi_K^* \omega_I = i\pi_K^* \omega_J = \Omega_K,$$

where $\Omega_K = \omega_I + i\omega_J$ is the trivializing holomorphic symplectic form.

Proof.

-
$$
\begin{aligned}
2\pi_K^*\omega_I(\cdot,\cdot) &= 2\omega_I(\pi_K\cdot,\pi_K\cdot)\\
&= \omega_I((1-iK)\cdot,(1-iK)\cdot)\\
&= \omega_I(\cdot,\cdot)+i\omega_I(\cdot,-K\cdot)+i\omega_I(-K\cdot,\cdot)-i^2\omega_I(K\cdot,-K\cdot)\\
&= \omega_I(\cdot,\cdot)+i\omega_J(\cdot,\cdot)+i\omega_J(\cdot,\cdot)+\omega_I(\cdot,\cdot)\\
&= 2\omega_I(\cdot,\cdot)+2i\omega_J(\cdot,\cdot)\\
&= 2\Omega_K(\cdot,\cdot).
\end{aligned}
$$

-
$$
\begin{aligned}
2\pi_K^*\omega_J(\cdot,\cdot) &= 2\omega_J(\pi_K\cdot,\pi_K\cdot)\\
&= \omega_J((1-iK)\cdot,(1-iK)\cdot)\\
&= \omega_J(\cdot,\cdot)-i\omega_J(\cdot,K\cdot)-i\omega_J(K\cdot,\cdot)-i^2\omega_J(K\cdot,-K\cdot)\\
&= \omega_J(\cdot,\cdot)-i\omega_I(\cdot,\cdot)-i\omega_I(\cdot,\cdot)+\omega_J(\cdot,\cdot)\\
&= 2\omega_J(\cdot,\cdot)-2i\omega_I(\cdot,\cdot)\\
&= -2i\Omega_K(\cdot,\cdot).
\end{aligned}
$$

□

Corollary 5.9.
$$\pi_K^*\Omega_J = i\Omega_K.$$

Proof.

$$
\begin{aligned}
\pi_K^*\Omega_J &= \pi_K^*\omega_K + i\pi_K^*\omega_I\\
&= 0 + i\Omega_K.
\end{aligned}
$$

where we use:

- By definition, for any $v, w \in T_{\mathbb{C}}M$,
$$\pi_K^*\omega(u,w) = \omega(\pi(u),\pi(v)).$$
By construction, $\pi(u),\pi(v) \in T^(1,0)_K M$. Since ω_K is a $(1,1)$-form, then
$$\pi_K^*\omega_K = 0.$$
- By Lemma 5.8,
$$\pi_K^*\omega_I.$$

□

Lemma 5.10.
$$\omega_K \wedge \omega_J = 0.$$

Proof. Consider $\Omega_I = \omega_J + i\omega_K$. Then

$$\begin{aligned}\Omega_I \wedge \Omega_I =& \omega_J \wedge \omega_J - \omega_K \wedge \omega_K + 2i\omega_J \wedge \omega_K \\ \overset{(1)}{=}& n!V(g) - n!V(g) + 2i\omega_J \wedge \omega_K \\ =& 2i\omega_J \wedge \omega_K.\end{aligned}$$

where at (1) we use that on a Kähler structure, $\omega^2 = n!V(g)$.

However, since there are no non-zero $4, 0$ holomorphic forms, then

$$\Omega_I \wedge \Omega_I = 0.$$

The desired result follows. □

5.2 The Dimension of the Quantum Hilbert Spaces

In this section, we will show that the dimension of the space of holomorphic sections of the quantum line bundle does not depend on the chosen Kähler structure, but only on the Riemannian metric. For now, let M denote a two-dimensional compact Kähler manifold.

Definition 5.1. Let $\alpha \in H^k(M, \mathbb{Q})$, $\beta \in H^{k'}(M, \mathbb{Q})$. The *cup product* $\alpha \cup \beta$ of α and β is defined as the pullback

$$\alpha \cup \beta := \Delta^*(\alpha \otimes \beta)$$

via the diagonal map $\Delta : M \to M \times M$ of the class $\alpha \otimes \beta$ on $M \times M$ defined by

$$\alpha \otimes \beta(\sigma \times \tau) = \alpha(\sigma) \cdot \beta(\tau).$$

Lemma 5.11. *[Mun84, Section 48] Let ψ, φ be closed forms in M representing α and β. Then $\psi \wedge \varphi$ represents $\alpha \cup \beta$.*

Lemma 5.12. *[GH94, Page 142] Let $L(D) \to X$ be a very ample line bundle of curvature ω. Then the Poincaré dual of $[\sqrt{-1}/2\pi\omega]$ is (D).*

Remark. In the previous lemma, the divisor D is considered as a singular chain. This can be done by Theorem 2.2.

Lemma 5.13. *[GH94, Page 470] Let D, D' be divisors and L, L' their corresponding line bundles. The Poincaré dual of the intersection pairing is the cup product. In particular*

$$(D \cdot D') = (c_1(L) \cup c_1(L'))[M].$$

Proposition 5.14. *Let $L(D) \to M$ be a very ample quantum line bundle over a 2 dimensional compact Kähler manifold (M, g, I, ω). Then the self-intersection $(L.L)$ depends only on the metric g. In particular*

$$(D \cdot D) = -\frac{\pi^2}{2} V(g).$$

Proof. Theorem 2.2 says that divisors can be expressed as a singular chain. In particular, we can apply Poincare Duality (Theorem 1.8) to divisors.

By Lemma 5.12, the Poincaré Dual of (D) is its Chern class

$$\mu_D = c_1(L) = -i\frac{\pi}{2}\omega.$$

Therefore, using 5.13:

$$\begin{aligned}(D \cdot D) =& (c_1(L) \cup c_1(L'))[M] \\ =& -\frac{\pi^2}{4}(\omega \cup \omega)[M].\end{aligned}$$

By Lemma 5.11,

$$(\omega \cup \omega)[M] = \int_M \omega \wedge \omega.$$

Finally, we use that for a Kähler structure $\omega^n = n!V(g)$ where $V(g)$ is the Riemannian volume form. In particular, for a surface $n = 2$ and we have

$$\begin{aligned}(D \cdot D) =& -\frac{\pi^2}{4} 2 \int_M \omega^2 \\ =& -\frac{\pi^2}{2} \int_M V(g),\end{aligned}$$

which is independent of the Kähler form. □

Corollary 5.15. *On a K3 surface, all very ample quantum line bundles have the same self-intersection.*

Theorem 5.16.
The spaces of holomorphic sections of the different quantum line bundles on a K3 surface have the same dimension. In particular, they are isomorphic as vector spaces.

Proof. According to Corollary 4.3, for a very ample line bundle L, the Riemann-Roch formula reduces to

$$h^0(X, L) = \frac{(L \cdot L)}{2} + 2.$$

As explained at Definition 2.17, this is equivalent to

$$h^0(X, L) = \frac{(D \cdot D)}{2} + 2$$

where $L = L(D)$. Apply this formula to the three quantum line bundles and use Corollary 5.15. □

Remark. One should note that the last theorem does not give a canonical choice of such isomorphism.

Remark. Remember that the quantum line bundle is the one corresponding to $m = 1$. For other values of m, one must multiply the metric by m. Therefore, the previous theorem is actually valid for any (fixed) level.

Remark. This last theorem applies not only to the three Kähler structures (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K), but also to any other quantizable Kähler structure corresponding to a point $(a_1, a_2, a_3) \in S^1$ of the twistor space.

Now the question is how many integral Kähler forms exists on a K3 surface. To answer this question, we need to introduce a couple of results from number theory:

Definition 5.2. A *Pythagorean quadruple* is a tuple of integers a, b, c and e, such that $e > 0$ and $a^2+b^2+c^2 = e^2$. It is called *primitive* if $gcd(a, b, c) = 1$, where *gcd* denotes the greatest common divisor.

Lemma 5.17. *[Spi62] The set of all primitive Pythagorean quadruples, is parametrized by,*

$$\begin{aligned} a =& m^2 + n^2 - p^2 - q^2, \\ b =& 2(mq + np), \\ c =& 2(nq - mp), \\ e =& m^2 + n^2 + p^2 + q^2. \end{aligned}$$

where m, n, p, q are non-negative integers, $gcd(m, n, p, q) = 1$ and $m + n + p + q \equiv 1(mod 2)$

Remark. If $n = q = 0$, we recover the solutions for the Pythagorean triple.

Lemma 5.18. *[Sch08] Points with rational coordinates are dense on the unit sphere S^2.*

Note that there is a surjective correspondence (but not injective) between Pythagorean quadruples and points with rational coordinates in the unit sphere S^2, given by

$$(a, b, c, e) \mapsto (a/e, b/e, c/e).$$

Assume now that the three Kähler structures (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K) are rational.. If (a, b, c) is a rational point on the sphere S^2, then

$$a\omega_I + b\omega_J + c\omega_K$$

is rational. Also, by construction it is Kähler and therefore positive. After multiplying by an integer, we can assume that the Kähler form is integral, thus ample. After multiplying again by another integer, we can assume it is very ample. We have thus the next result:

Corollary 5.19. *Consider X a K3 surface. Given any two fibers of the twistor space, corresponding to two rational points $p_1, p_2 \in S^2$, consider their rational Kähler structures ω_1 and ω_2. Then there exists an $m \in \mathbb{Z}$ such that the quantum line bundles associated to $m\omega_1, m\omega_2$, that we will denote by $L_{m\omega_1}$ and $L_{m\omega_1}$, are well defined and*

$$h(X, L^n_{m\omega_1}) = h(X, L^n_{m\omega_2})$$

for all $n \in \mathbb{N}$.

Corollary 5.20. *Consider a K3 surface with Kähler structures (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K).*

- *If the three Kähler forms are rational, then the set of quantizable Kähler structures is countable dense on the twistor space.*
- *If exactly two of the Kähler forms are rational, then the set of quantizable Kähler structures is countable dense on a S^1 inside the twistor space.*

Remark. Note that, while the set of rational points in the sphere is dense, its complement is also dense and, in fact, much bigger (since the Lebesgue measure of $\mathbb{Q}$ in $\mathbb{R}$ is zero).

Remark. In the previous two corollaries, the metric is only fixed up to multiplication. For a fixed m, there are only a finite number of integral Kähler structures compatible with mg.

5.3 Existence of Quantizable K3 Surfaces

In the previous section, we have assumed the existence of a K3 Surface with 2 or more quantizable Kähler structures. In this section we show that there exist an infinite countable amount of them.

Consider a K3 surface X. Recall the definition of the Hodge-Riemann pairing:

Definition 4.22. *Fix a rational (or integral) Kähler class $\omega \in H^2(X, \mathbb{Q})$. The* Hodge-Riemann *pairing on $H^n(X, \mathbb{Q})$, $n \leq d = dim_{\mathbb{C}} X$ is the pairing*

$$(v, w) \mapsto (-1)^{n(n-1)/2} \int_X v \wedge w \wedge \omega^{d-n}.$$

Since $n = d = 2$, on a K3 surface the Hodge-Riemann pairing is independent of the chosen Kähler form. This is an important fact, since it means that for a fixed K3 surface, its K3 lattice (including the bilinear form, Definition 4.9) does not depend on the chosen Kähler structure. However one should keep in mind that the Hodge decompositions of the lattices (given by the usual decomposition into (p, q) forms) are obviously different.

Remember that the *transcendental lattice* T of a Hodge structure on a lattice V is the minimal primitive sub-Hodge structure such that $V^{2,0} = T^{2,0} \subset T_{\mathbb{C}}$ (see Definition 4.18)

Since we consider different complex and Kähler structures on a fixed K3 surface X, we will mark which one we are referring to by using a subindex. For instance, we will write either $NS_I(X)$, NS_I or $NS(X_I)$ to denote the Neron-Severi group of X when considered as the Kähler manifold (X, g, I, ω_I) and either $T_I(X)$, T_I or $T(X_I)$ for its corresponding transcendental lattice.

Lemma 5.21. *If ω_J and ω_K are integral Kähler forms, then*

$$T_I \otimes \mathbb{Q} = \langle \omega_K, \omega_J \rangle_{\mathbb{Q}} .$$

In particular,

$$rk(T_I) = 2.$$

Proof. Denote by $H_I = H^2(X, \mathbb{Z})$ the lattice of integral forms. Recall that $\Omega_I = \omega_J + i\omega_K$ is a nowhere zero $(2,0)$-form. Since on a K3 surface $h^{2,0} = 1$, then we have

$$H_I^{2,0}(X, \mathbb{C}) = \langle \Omega_I \rangle_{\mathbb{C}} .$$

Therefore, $\langle \omega_K, \omega_J \rangle$ is a sub-Hodge structure of H_I. Minimality it is trivial. □

By standard linear algebra, we have:

Lemma 5.22. *Consider $V \times V \to \mathbb{R}$ a non-degenerated pairing and an element $v \in V$ such that $(v.v) \neq 0$, then*

$$rk(v^{\perp}) = rk(V) - 1.$$

Proposition 5.23. *Assume that ω_J and ω_K are rational. Then the Picard numbers (i.e., the ranks of the Neron Severi groups, see Definition 2.12) are*

$$\rho(X_J) = \rho(X_K)$$

and

$$\rho(X_I) = 20.$$

Proof. Consider first the case of X_I. From Lemma 4.18,

$$T(X)^{\perp} = NS(X) \cong Pic(X),$$

where they are considered as sublattices of $H^2(X, \mathbb{Z})$ (see K3 lattice for the intersection pairing, Definition 4.13). We know (Corollary 4.8) that

$$b_2 = rk(H^2(X, \mathbb{Z})) = 22.$$

Recall (Lemma 5.21) that

$$T_I \otimes \mathbb{Q} = \langle \omega_K, \omega_J \rangle_{\mathbb{Q}}\,.$$

In particular, we can describe the Neron-Severi group as the orthogonal of those two elements:

$$NS_I(X) \otimes \mathbb{Q} = \langle \omega_K, \omega_J \rangle^{\perp} \subset H^2(X, \mathbb{Q}).$$

Then apply Lemma 5.22 twice using those generators to get that

$$rk(NS_I(X) \otimes \mathbb{Q}) = 20.$$

Sine tensoring by $\mathbb{Q}$ does not change the rank, this proves the desired property.

For the other two, the proof is similar. The main difference is in the description of the transcendental lattice. Consider ω_I as a linear combination of elements of $NS_I(X)$. Let V be the minimal sublattice of $H^2(X, \mathbb{Z})$ such that $\omega_I \in V_{\mathbb{C}}$. Then, as in Lemma 5.21,

$$T_J(X) \otimes \mathbb{Q} = (V \oplus < \omega_K >) \otimes \mathbb{Q},$$

$$T_K(X) \otimes \mathbb{Q} = (V \oplus < \omega_J >) \otimes \mathbb{Q}.$$

The rest of the proof is analogous: one can describe the Neron-Severi groups for J and K as the orthogonal of the transcendental lattices. Since they both have the same number of generators, Lemma 5.21 ensures that the Neron-Severi groups will have the same rank. □

Corollary 5.24. *Assume the three Kähler structures are integral. Then the Picard numbers are*

$$\rho(X_I) = \rho(X_J) = \rho(X_K) = 20.$$

Corollary 5.25. *Assume the three Kähler structures are integral. Then* $(T_I \otimes \mathbb{Q}) \cap (T_J \otimes \mathbb{Q}) =< \omega_K >_{\mathbb{Q}}$.

Proof. Clearly since $\omega_K \in T_I \cap T_J$ and $\omega_I \notin T_I \otimes \mathbb{Q}$. □

By Corollary 4.21, there is a bijection between (isomorphism classes) of K3 surfaces with Picard number 20 and positive definite, even, oriented lattices of rank two.

$$\{X \mid \rho(X) = 20\} \longleftrightarrow \left\{ \begin{array}{l} \text{positive definite, even,} \\ \text{oriented lattices of rank 2} \end{array} \right\},$$

Lemma 5.26. *Consider* $V = span_{\mathbb{R}}(\omega_I, \omega_J, \omega_K)$ *the vector space generated by the three Kähler forms. Then any* $\omega \in V$, $\omega \neq 0$, *determines a Kähler structure.*

Proof. It is exactly as in Lemma 4.9, but allowing any non-zero triple (a, b, c). □

Proposition 5.27. *Assume that a K3 surface X admits two (or more) quantizable Kähler structures denoted by (g, I_1, ω_{I_1}) and (g, I_2, ω_{I_2}). Then there exists an integer m, and complex structures I, J such that (mg, I, ω_I) and (mg, J, ω_J) are orthogonal quantizable Kähler structures.*

Proof. Since both Kähler structures share the metric, then

$$\omega_{I_1} \neq \omega_{I_2}.$$

Consider

$$\omega_0 := \omega_2 - \frac{(\omega_1, \omega_2)}{(\omega_1, \omega_1)}\omega_1.$$

By construction ω_0 is rational. Consider the following objects:

- $m \in \mathbb{N}$ such that
$$m\omega_0 \in H^2(M, \mathbb{Z}).$$
-
$$I := I_1.$$
- J the complex structure corresponding to ω_0 (which exist because Lemma 5.26).
-
$$\omega_I := m\omega_{I_1}.$$
-
$$\omega_J := m\omega_0.$$

By construction (mg, I, ω_I) and (mg, J, ω_J) are quantizable and orthogonal. □

Theorem 5.28.
A K3 surface admits two or more quantizations if and only if there exists a fiber of its twistor space whose Kähler structure has Picard number 20.

Proof.

- Consider a K3 surface with Kähler structures (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K). Assume that $\rho_K(X) = 20$. In particular the rank of the transcendental lattice is two
$$rk(T_K(X)) = 2.$$

By construction of the transcendental lattice, $\omega_I, \omega_J \in T_K(X) \otimes \mathbb{C}$ and therefore

$$span_{\mathbb{C}}(\omega_I, \omega_J) = T_K(X).$$

As a consequence, any element in $T_K(X)$ is Kähler and determines a quantizable Kähler structure.

- Denote by (g, I_1, ω_{I_1}) and (g, I_2, ω_{I_2}) the two quantizable Kähler structures. Without loss of generality we can choose fibers of the twistor space with complex structures I, J, K such that

$$\omega_{I_1}, \omega_{I_2} \in span_{\mathbb{C}}(\omega_I, \omega_J).$$

By Proposition 5.27, we can assume them to be orthogonal, i.e., them to be

$$\omega_{I_1} = \omega_I,$$

$$\omega_{I_2} = \omega_J.$$

Therefore, as in Proposition 5.23,

$$\rho_K(X) = 20.$$

□

Corollary 4.21 gives a 1-1 correspondence between (isomorphism classes of) K3 surfaces with Picard number 20 and an infinite countable family of matrices. This implies that there exists an infinite countable number of K3 surfaces with two or more quantizable Kähler structures.

Remark. The proof of the Global Torelli theorem actually constructs the moduli space of K3 surfaces. In particular, it shows that the moduli space of (marked) K3 surfaces with Picard number ρ is a smooth non-compact manifold of dimension $20 - \rho$.

Proposition 5.29. *There exist an infinite number of Kummer surfaces with at least two integral Kähler forms. Furthermore, the set of K3 surfaces with such property which are not Kummer surfaces is also infinite.*

Proof. Apply the construction in Corollary 4.21 to the families

$$K_n := \begin{pmatrix} 4n & 0 \\ 0 & 4n \end{pmatrix}$$

and

$$T_n := \begin{pmatrix} 4n & 1 \\ 1 & 4n \end{pmatrix}$$

for $n \geq 1$ By Lemma 4.22 the first ones correspond to a family of Kummer surfaces and the second one does not. □

5.3.1 The Canonical Isomorphism

In this last section, we will show a subfamily of K3 surfaces admitting two quantizations whose quantum Hilbert spaces admit a canonical isomorphism. By Theorem 5.28, such an X as Picard number 20. Without loss of generality, assume that ω_I and ω_J are integral and $\rho_K(X) = 20$. By Corollary 4.21 X determines a matrix :

$$T = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

Proposition 5.30. *Assume that $a = c$. Then there exists two quantizable Kähler structures whose quantum Hilbert spaces are canonically isomorphic.*

Proof. Consider the 2-forms ω_1, ω_2 corresponding to the lattice basis.

Note that those forms correspond to Kähler forms and determine quantizable Kähler structures. Now apply Corollary 4.20 and Theorem 4.19 to the linear map determined by

$$\omega_1 \mapsto \omega_2$$

$$\omega_2 \mapsto -\omega_1$$

and denote by f the corresponding endomorphism

$$f : X_K \to X_K,$$

which is a K-holomorphic map. Note that, by construction,

$$f^*(g, I_1, \omega_1) = (g, I_2, \omega_2).$$

In particular,

$$f^* H^0(X, L_{I_1}) = H^0(X, L_{I_2}).$$

□

Remark. In terms of Corollary 4.21, the construction of the last proposition corresponds to conjugating T by the element

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Remark. In particular, if $b = 0$, f determines a canonical isomorphism between $H^0(X, L_I)$ and $H^0(X, L_J)$.

From Corollary 4.20 one also gets the following: if $T_K(X)$ does not admit a symmetry, then there exists no isomorphism between the ω_I and ω_J Kähler structures of X.

Remark. Assume that $a = c$ for all three transcendental lattices. Applying Lemma 5.30 to each one, one gets that there exists a line bundle L and automorphisms $f_i : X_{I_i} \to X_{I_i}$ such that all quantum line bundles will be of the form

$$\bigotimes_{i=1}^{3} f_i^* L^{\otimes a_i}$$

for different choices of $a_i \in \mathbb{Z}$, $i = 1, 2, 3$.

6 Hyperkähler Quantization

In the last chapter of this thesis, we will define some generalizations of Berezin-Toeplitz operators for the product of quantum Hilbert spaces.

Consider (g, ω_1, I_1), (g, ω_2, I_2) and (g, ω_3, I_3) three integral Kähler structures on a K3 surface such that $I_1 I_2 = I_3$. Consider $p = (a_1, a_2, a_3) \in \mathbb{Z}^3$ and the Kähler structure associated to the Kähler form

$$\omega_p = \sum_{i=1}^{3} a_i \omega_i .$$

Consider the corresponding quantum line bundle L_p and quantum Hilbert space $H^0(M, L_p)$.

Note that such a family of ω_p does not cover all possible quantum line bundles and their tensor powers: one would need $\{\omega_1, \omega_2, \omega_3\}$ to be a basis for the lattice of integral Kähler forms. However, it is enough to consider this case since we are only interested in the asymptotic behavior and, for any integral form ω, there exists a $k \in \mathbb{N}$ such that

$$k\omega \in < \omega_1, \omega_2, \omega_3 > .$$

Since we are already multiplying the Kähler form ω to ensure that its corresponding quantum line bundle is very ample, multiplying again will not change the results.

Remark. The case where not all the three original Kähler structures are integral can be studied by declaring some of the $a_i = 0$. For instance, whenever one has two integral Kähler forms, one takes $a_3 = 0$ and uses the convention

$$L^0 = \mathbb{C}$$

for any line bundle L. All computations below work without problem.

Consider also the original three quantum line bundles L_i and quantum Hilbert spaces $H^0(X, L_i)$, $i \in \{1, 2, 3\}$. Note that

$$\operatorname{curv}(L_p) = \omega_p = \sum_{i=1}^{3} a_i \omega_i = \operatorname{curv}\left(\bigotimes_{i=1}^{3} L_i^{\otimes a_i} \right).$$

Since K3 surfaces are simply connected, Lefschetz Theorem implies that

$$L_p = \bigotimes_{i=1}^{3} L_i^{\otimes a_i}$$

and determines a map

$$F : \bigotimes_{i=1}^{3} \Gamma\left(X, L_i^{\otimes a_i}\right) \to \Gamma\left(X, L_p\right). \qquad (6.1)$$

Consider the product Hilbert space

$$\mathcal{H} := H^0(X, L_{I_1}^{\otimes a_1}) \otimes H^0(X, L_{I_2}^{\otimes a_2}) \otimes H^0(X, L_{I_3}^{\otimes a_3}).$$

One then has a map

$$\phi = \pi \circ F : \mathcal{H} \to H^0(X, L_p),$$

where π is the same map that is used to construct the Berezin-Toeplitz operators (Definition 3.4).

It is natural then, to ask whether $\mathcal{H}$ and $H^0(X, L_p)$ are related. In the next sections, we will propose and study some "quantizations" in the space $\mathcal{H}$ defined above. First we will explain and adapt Barron and Serajelahi's work, where they study the product of the Berezin-Toeplitz operators. Later we will introduce a new operator constructed by doing an additive average of the Berezin-Toeplitz operators with coefficients in the quaternionic space. In future works, it would be interesting to study the map ϕ relating the two quantizations.

6.1 Multiplicative Hyperkähler Quantization

In [BS14], Barron and Serajelahi study quantization of Hilbert Spaces of the form

$$H^0(X, L_{I_1}) \otimes H^0(X, L_{I_2}) \otimes H^0(X, L_{I_3}),$$

where (g, ω_1, I_1), (g, ω_2, I_2) and (g, ω_3, I_3) are three Kähler structures on a K3 surface X.

Many of the results found there are parallel to those showed by Bordemann et al in [BMS94] (see Section 3.5 of this thesis for more details). We will generalized them to tensor products of the form

$$H^0(X, L_{I_1}^{\otimes a}) \otimes H^0(X, L_{I_2}^{\otimes b}) \otimes H^0(X, L_{I_3}^{\otimes c}).$$

All the results and proofs in this section are parallel to those of the aforementioned paper, where they have been proven for $a = b = c = 1$. We only give the proofs of those who require extra work and the reader is refered to the original paper ([BS14]) for the rest of them.

Denote

$$\mathcal{H}_k := H^0(X, L_{I_1}^{\otimes ka}) \otimes H^0(X, L_{I_2}^{\otimes kb}) \otimes H^0(X, L_{I_3}^{\otimes kc})$$

and

$$\mathbb{T}_f^{(k)} := T_{f,I_1}^{(ka)} \otimes T_{f,I_2}^{(kb)} \otimes T_{f,I_3}^{(kc)}.$$

Lemma 6.1. *[BS14, Lemma 4.9] If M_j, N_j are linear operators on a (finite dimensional) Hilbert space V_j $(j = 1, 2, 3)$, then*

$$\begin{aligned}\|M_1 \otimes M_2 \otimes M_3 - N_1 \otimes N_2 \otimes N_3\| \leq & \|M_1 - N_1\|\|M_2 - N_2\|\|M_3 - N_3\| \\ & + \|M_1 - N_1\|\|M_2\|\|N_3\| \\ & + \|M_1\|\|N_2\|\|M_3 - N_3\| \\ & + \|N_1\|\|M_2 - N_2\|\|M_3\|.\end{aligned}$$

Lemma 6.2. *[BS14, Lemma 4.10]*

$$\begin{aligned}[M_1 \otimes M_2 \otimes M_3, N_1 \otimes N_2 \otimes N_3] = & [M_1, N_1] \otimes [M_2, N_2] \otimes [M_3, N_3] \\ & + [M_1, N_1] \otimes N_2 M_2 \otimes M_3 N_3 \\ & + M_1 N_1 \otimes [M_2, N_2] \otimes N_3 M_3 \\ & + N_1 M_1 \otimes M_2 N_2 \otimes [M_3, N_3].\end{aligned}$$

Theorem 6.3.

1. *For $f \in C^\infty(M)$, there exist a constant $C > 0$ such that, as $k \to \infty$,*

$$\left(|\, f\,|_\infty - \frac{C}{k} \right)^3 \leq \|\mathbb{T}_f^{(k)}\| \leq \|f\|_\infty^3.$$

In particular,

$$\lim_{k \to \infty} \|\mathbb{T}^{(k)}\| = \|f\|_\infty^3.$$

2. *For $f, g \in C^\infty(M)$, $a, b, c \in \mathbb{N} \cup \{0\}$ (not all simultaneously zero)*

$$\begin{aligned}&\|abc(ik)^3 [T_{f;I_1}^{(ka)}, T_{g;I_1}^{(ka)}] \otimes [T_{f;I_2}^{(kb)}, T_{g;I_2}^{(kb)}] \otimes [T_{f;I_3}^{(kc)}, T_{g;I_3}^{(kc)}] \\ &- T_{\{f,g\}_{I_1};I_1}^{ka} \otimes T_{\{f,g\}_{I_2};I_2}^{kb} \otimes T_{\{f,g\}_{I_3};I_3}^{kc}\| = O(k^{-1}).\end{aligned}$$

3. *For $f_1, \cdots, f_p \in C^\infty(M)$,*

$$\|\mathbb{T}_{f_1}^{(k)} \dots \mathbb{T}_{f_p}^{(k)} - \mathbb{T}_{f_1 \dots f_p}^{(k)}\| = O(k^{-1}).$$

Proof. For (1) and (3), see the original paper ([BS14, Page 18]). (2) needs to be adapted:

$$
\begin{aligned}
&\|abc(ik)^3[T^{(ka)}_{f;I_1}, T^{(ka)}_{g;I_1}] \otimes [T^{(kb)}_{f;I_2}, T^{(kb)}_{g;I_2}] \otimes [T^{(kc)}_{f;I_3}, T^{(kc)}_{g;I_3}] \\
&\qquad - T^{ka}_{\{f,g\}_{I_1};I_1} \otimes T^{kb}_{\{f,g\}_{I_2};I_2} \otimes T^{kc}_{\{f,g\}_{I_3};I_3}\| \\
&\overset{(1)}{=} \|aik[T^{(ka)}_{f;I_1}, T^{(ka)}_{g;I_1}] - T^{ka}_{\{f,g\}_{I_1};I_1}\|\|bik[T^{(kb)}_{f;I_2}, T^{(kb)}_{g;I_2}] - T^{kb}_{\{f,g\}_{I_2};I_2}\| \\
&\qquad \|cik[T^{(kc)}_{f;I_3}, T^{(kc)}_{g;I_3}] - T^{kc}_{\{f,g\}_{I_3};I_3}\| \\
&\qquad + \|aik[T^{(ka)}_{f;I_1}, T^{(ka)}_{g;I_1}] - T^{ka}_{\{f,g\}_{I_1};I_1}\|\|bik[T^{(kb)}_{f;I_2}, T^{(kb)}_{g;I_2}]\|\|T^{kc}_{\{f,g\}_{I_3};I_3}\| \\
&\qquad + \|bik[T^{(kb)}_{f;I_2}, T^{(kb)}_{g;I_2}]\|\|T^{kb}_{\{f,g\}_{I_2};I_2}\|\|cik[T^{(kc)}_{f;I_3}, T^{(kc)}_{g;I_3}] - T^{kc}_{\{f,g\}_{I_3};I_3}\| \\
&\qquad + \|T^{ka}_{\{f,g\}_{I_1};I_1}\|\|bik[T^{(kb)}_{f;I_2}, T^{(kb)}_{g;I_2}] - T^{kb}_{\{f,g\}_{I_2};I_2}\|\|cik[T^{(kc)}_{f;I_3}, T^{(kc)}_{g;I_3}]\| \\
&\overset{(2)}{\leq} O(\frac{1}{k^3abc}) + O(\frac{1}{ka})O(1) \mid \{f,g\}_{I_3} \mid_\infty \\
&\qquad + O(1) \mid \{f,g\}_{I_2} \mid_\infty O(\frac{1}{kc}) + \mid \{f,g\}_{I_1} \mid_\infty O(\frac{1}{kb})O(1) \\
&\overset{(3)}{=} O(k^{-1}),
\end{aligned}
$$

where we use:

1. Lemma 6.1.
2. Theorem 3.4 and Corollary 3.5.
3. That a, b and c are constants.

□

Remark.

1. Unlike the original Berezin-Toeplitz map, the assignment $f \to \mathbb{T}^{(m)}_f$ is not linear.
2. Note that the formula in (2) does not involve the commutator of the product of operators, but the product of the commutators.

6.2 Additive Hyperkähler Quantization

In the last section, we explained the behavior of the product of Berezin-Toeplitz Operators. This operator was originally studied by Barron and Serajelahi and we have presented the most important results and adapted them in this more general setting.

Now we present an original operator defined using the additive average of the Berezin-Toeplitz operators. By tensoring with the quaternionic space, we have defined some new operators that have the desired asymptotic properties (see Theorem 6.9), as well as different other interesting properties.

All results in this section are original.

Fix $a = (a_1, a_2, a_3) \in \mathbb{Q}^3_{\geq 0}$. Consider three different Hilbert spaces of holomorphic sections

$$H^0(X, L_{I_n}^{\otimes k a_n}), \quad n = 1, 2, 3$$

where $k \in \mathbb{N}$ such that $ka_n \in \mathbb{N}$ for $n = 1, 2, 3$. Consider the product Hilbert space

$$\mathcal{H}_k = \mathcal{H}_k^a := \mathbb{H} \otimes_{\mathbb{R}} \left(H^0(X, L_{I_1}^{\otimes k a_1}) \otimes_{\mathbb{C}} H^0(X, L_{I_2}^{\otimes k a_2}) \otimes_{\mathbb{C}} H^0(X, L_{I_3}^{\otimes k a_3}) \right).$$

A priori the vector bundles L_{I_i} are not necessary the three vector bundles L_I, L_J and L_K, nor X needs to be a K3 surface.

Since a is fixed, we will usually skip the superindex.

Remark. The three imaginary units of $\mathbb{H}$ will be denoted i, j and k as usual. Note that the three spaces $H^0\left(X, L_{I_i}^{\otimes k a_i}\right)$ are vector spaces over the complex field $\mathbb{C}$. To differentiate the action of the complex unit on these vector spaces from the quaternionic numbers in $\mathbb{H}$, we will denote it by t.

Denote by δ_i^j the Kronecker delta, i.e.,

$$\delta_i^j = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

and consider the following operator

$$\tilde{\mathbb{T}}_f^{n,(k)} := T^{(ka_1)}_{1+\delta_n^1(f-1), I_1} \otimes T^{(ka_2)}_{1+\delta_n^2(f-1), I_2} \otimes T^{(ka_3)}_{1+\delta_n^3(f-1), I_3}.$$

For instance, for $n = 1$ this denotes the operator which applies the corresponding Berezin-Toeplitz operator on the first factor and the identity operator on the other two:

$$\tilde{\mathbb{T}}_f^{1,(k)} = T^{(ka_1)}_{f, I_1} \otimes T^{(ka_2)}_{1, I_2} \otimes T^{(ka_3)}_{1, I_3} = T^{(ka_1)}_{f, I_1} \otimes Id \otimes Id.$$

Consider three purely imaginary quaternonic numbers $i_1, i_2, i_3 \in \mathbb{H}$. As stated before, we want to consider the additive average of the three Berezin-Toeplitz operators, i.e., we want to study the following operator

Definition 6.1. The *Hyperkähler Berezin-Toeplitz operator* $\tilde{\mathbb{T}}_f^{(k)}$ of *level* k, *structure* (i_1, i_2, i_3) and *weight* (a_1, a_2, a_3) is defined as

$$\tilde{\mathbb{T}}_f^{(k)} := \sum_{n=1}^{3} a_n i_n \tilde{\mathbb{T}}_f^{n,(k)}.$$

Remark. In general, the choice of i_n, $n = 1, 2, 3$ is arbitrary. However, for a K3 surface or an hyperkähler manifold one chooses its associated quaternionic numbers (see Definition 4.7).

Example 6.1. (Classical Berezin-Toeplitz operators) Consider $a_1 = 1$, $a_2 = a_3 = 0$. Denote $L = L_{I_1}$. Then

$$\mathcal{H}_m = \mathbb{H} \otimes_{\mathbb{R}} H^0(X, L^{\otimes m})$$

and

$$\tilde{\mathbb{T}}_f^{(m)} = i_1 T_f^{(m)}.$$

The reader may remark that one should also divide by

$$s := \sum_{j=1}^{3} a_j i_j.$$

However, as the reader will see soon, we plan to study the properties of the operators using different products: scalar product, cross product and quaternionic product. For some of those products and some choices of i_n, $i = 1, 2, 3$, the quaternionic number s may be a zero divisor. The main consequence of that is the fact that the constact function 1 does not map to the identity, but a multiple of it:

$$\tilde{\mathbb{T}}_1^{(m)} = sId \neq Id.$$

However, as the reader will see, this will have no further consequences. For instance, Theorem 6.9.2 is a generalization of Theorem 3.4.2.

$$\left\| mt \left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^{\cdot} - s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)} \right\| = O\left(m^{-1}\right).$$

By choosing $a_2 = a_3 = 0$, $a_1 = 1$ in the former one recovers the original formulation

$$\left\| mt \left[T_f^{(m)}, T_g^{(m)} \right] - T_{\{f,g\}}^{(m)} \right\| = O\left(m^{-1}\right).$$

Remark. Before starting to show the results and their proofs, we would like to state some simplifications and other important remarks:

- All the results and proofs below work for general s. However, since we are intered in the case of K3 surfaces, we will asume that $\|s\| = 1$, so the resulting space corresponds to a Kähler structure on the twistor space. In the general case, the norm may appear in some expressions. However, we will show some simple examples with $a_1 = a_2 = 1$, $a_3 = 0$. While this does not correspond the the case that $\|s\| = 1$, it allows us to show the reader the meaning of some results by using simpler expressions that have less coefficients.

- We will assume that the a_i are positive. For negative a_i, one takes the Kähler structure $(g, -I_i, -\omega_i)$ together with the dual quantum line bundle and $a'_i = -a_i$.

- As the reader will see, we are only interested in the case where i_n, $n = 1, 2, 3$ are orthogonal. Although the general case is not conceptually more involved, the length of the computations increases unnecessarily. Therefore, most of the results shown here assume that i_n, $n = 1, 2, 3$ are orthogonal. We have computed the general case of some results so that the reader can understand how to proceed in the general case.

As stated above, we will consider three different products induced by the scalar, cross and quaternionic products:

1. The usual quaternionic product $i \star j = k$, $i^2 = j^2 = k^2 = -1$.
2. The cross product: $i \times j = k$, $j \times i = -k$, $i \times i = 0$. Note that this is not defined for real numbers.
3. The (Lorentzian) scalar product in which $\{1, i, j, k\}$ is an orthogonal basis with signature $(1, 3)$. We will denote it by $i \cdot j$. When necessary, compose it with the natural inclusion $\mathbb{R} \hookrightarrow \mathbb{H}$ and consider it as a map

$$\mathbb{H} \times \mathbb{H} \to \mathbb{R} \hookrightarrow \mathbb{H}.$$

Note that with this notation $a \cdot b = Re(a \star b)$. Moreover, if a and b are purely imaginary, then $a \times b = Im(a \star b)$. In particular, for purely imaginary numbers

$$a \star b = a \times b + a \cdot b.$$

When considered as a vector space over $\mathbb{R}$, one has the scalar multiplication. This coincides with the quaternionic multiplication ($\star$). Given a real number x and a quaternionic one q, the reader will see both notations rq and $r \star q$.

When using "Lie brackets", we will write a superscript to denote the product used:

$$[\tilde{\mathbb{T}}_f^{i,(k)}, \tilde{\mathbb{T}}_g^{j,(k)}]^\star, [\tilde{\mathbb{T}}_f^{i,(k)}, \tilde{\mathbb{T}}_g^{j,(k)}]^\times, [\tilde{\mathbb{T}}_f^{i,(k)}, \tilde{\mathbb{T}}_g^{j,(k)}]^\cdot.$$

The first result of the section is independent of the product:

Lemma 6.4. *The Hyperkähler Berezin-Toeplitz operators are bounded operators:*

$$\|\tilde{\mathbb{T}}_f^{(k)}\| \leq (a_1 + a_2 + a_3) \mid f \mid_\infty .$$

Proof.

$$\begin{aligned}\|\tilde{\mathbb{T}}_f^{(k)}\| &= \| \sum_{n=1}^{3} a_n i_n \tilde{\mathbb{T}}_f^{n,(k)} \| \\ &\leq \sum_{n=1}^{3} a_n \|\tilde{\mathbb{T}}_f^{n,(k)}\| \\ &\overset{(1)}{\leq} \left(\sum_{n=1}^{3} a_n \right) \mid f \mid_\infty ,\end{aligned}$$

where at (1) we use Theorem 3.4.1. □

We will now proceed to show the main properties of the Hyperkähler Berezin-Toeplitz operators with respect to the different products.

6.2.1 Scalar product

The following results concern the behavior of the Hyperkähler Berezin-Toeplitz operators with respect to the scalar product. The first result is a generalization of Theorem 3.4.1:

Proposition 6.5. *Assume that i_n are orthogonal. For $f \in C^\infty(M)$, there exist a constant $C > 0$ such that, as $k \to \infty$,*

$$\mid f \mid_\infty - \frac{C}{k} \leq \|s \cdot \tilde{\mathbb{T}}_f^{(k)}\| \leq \mid f \mid_\infty .$$

Proof.

- For the first equality, we are using the proof of the original theorem. We have recalled some steps of it in Theorem 3.4.1. The complete proof can be found in [BMS94]. As the reader can see, during the proof it is constructed a sequence of sections

$$\phi^{(m)} \in H^0(X, L^m)$$

of L^m such that

$$\frac{\|T_f^{(m)} \phi^{(m)} - f(x_0)\phi^{(m)}\|}{\|\phi^{(m)}\|} = O(m^{-1}),$$

where x_0 is a maximum of f. For this proof, one considers 3 sequences $\phi_n^{(m)}$ for each I_n, $i = 1, 2, 3$ constructed in the same way. Denote by $\phi^{(m)}$ their product

$$\phi^{(m)} = \bigotimes_{n=1}^{3} \phi_n^{(m)}.$$

Then we have

$$\begin{aligned}\frac{\|s\cdot\tilde{\mathbb{T}}_f^{(m)}\phi^{(m)}-f(x_0)\phi^{(m)}\|}{\|\phi^{(m)}\|} &= \frac{\|-\sum_{n=1}^{3}a_n^2\tilde{\mathbb{T}}_f^{n,(m)}\phi^{(m)}+s\cdot sf(x_0)\phi^{(m)}\|}{\|\phi^{(m)}\|}\\ &\leq\sum_{n=1}^{3}\frac{\|-a_n^2\tilde{\mathbb{T}}_f^{n,(m)}\phi^{(m)}+a_n^2f(x_0)\phi^{(m)}\|}{\|\phi^{(m)}\|}\\ &\overset{(1)}{\leq}\sum_{n=1}^{3}\frac{\|-(a_n^2T_{f,I_n}^{(ma_n)}\phi_n^{(m)}-a_n^2f(x_0)\phi_n^{(m)})\|}{\|\phi_n^{(m)}\|}\\ &=O(m^{-1}),\end{aligned}$$

where at (1) we use Lemma 6.1. From this point on, one uses applies the original argument to each of the summands.

- For the second inequality:

$$\begin{aligned}\|s\cdot\tilde{\mathbb{T}}_f^{(k)}\| &= \|\sum_{n=1}^{3}a_n^2\tilde{\mathbb{T}}_f^{n,(k)}\|\\ &\leq\sum_{n=1}^{3}a_n^2\|\tilde{\mathbb{T}}_f^{n,(k)}\|\\ &\overset{(2)}{\leq}\sum_{n=1}^{3}a_n^2\mid f\mid_\infty\\ &=\mid f\mid_\infty .\end{aligned}$$

where at (2) we use Theorem 3.4.1.

□

Lemma 6.6. *The Lie bracket is bilinear with respect to the scalar product:*

$$[i_n\tilde{\mathbb{T}}_f^{n,(k)},i_m\tilde{\mathbb{T}}_g^{m,(k)}]^{\cdot}=i_n\cdot i_m[\tilde{\mathbb{T}}_f^{n,(k)},\tilde{\mathbb{T}}_g^{m,(k)}].$$

In particular, if $n\neq m$*,*

$$[i_n\tilde{\mathbb{T}}_f^{n,(k)},i_m\tilde{\mathbb{T}}_g^{m,(k)}]^{\cdot}=0.$$

Proof.

$$\begin{aligned}[i_n\tilde{\mathbb{T}}_f^{n,(k)},i_m\tilde{\mathbb{T}}_g^{m,(k)}]^{\cdot} &= i_n\tilde{\mathbb{T}}_f^{n,(k)}\cdot i_m\tilde{\mathbb{T}}_g^{m,(k)}-i_m\tilde{\mathbb{T}}_g^{m,(k)}\cdot i_n\tilde{\mathbb{T}}_f^{n,(k)}\\ &= i_n\cdot i_m\tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{m,(k)}-i_m\cdot i_n\tilde{\mathbb{T}}_g^{m,(k)}\tilde{\mathbb{T}}_f^{n,(k)}\\ &= i_n\cdot i_m\tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{m,(k)}-i_n\cdot i_m\tilde{\mathbb{T}}_g^{m,(k)}\tilde{\mathbb{T}}_f^{n,(k)}\\ &= i_n\cdot i_m[\tilde{\mathbb{T}}_f^{n,(k)},\tilde{\mathbb{T}}_g^{m,(k)}].\end{aligned}$$

Clearly for $n \neq m$,

$$[\tilde{\mathbb{T}}_f^{n,(k)}, \tilde{\mathbb{T}}_g^{m,(k)}] = 0,$$

since each operator acts on a different space. □

The following proposition generalizes Theorem 3.4.2 for the scalar product:

Proposition 6.7. *For every $f, g \in C^\infty(M)$,*

$$\left\| mt \left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^{\cdot} - s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)} \right\| \leq O(m^{-1}) + \left\| \sum_{\substack{u,v=1 \\ u \neq v}}^{3} i_u \cdot i_v \tilde{\mathbb{T}}_{\{f,g\}}^{v,(m)} \right\|.$$

Proof. Remember that

$$[\tilde{\mathbb{T}}_f^{i,(k)}, \tilde{\mathbb{T}}_g^{j,(k)}]^{\cdot} = 0$$

for $i \neq j$. Therefore,

$$\begin{aligned}
&\left\| mt \left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^{\cdot} - s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)} \right\| \\
&\overset{(1)}{=} \left\| \sum_{v=1}^{3} \left(mta_v^2 i_v \cdot i_v [\tilde{\mathbb{T}}_f^{v,(m)}, \tilde{\mathbb{T}}_g^{v,(m)}] - a_v s \cdot i_v \tilde{\mathbb{T}}_{\{f,g\}}^{v,(m)} \right) \right\| \\
&= \left\| -a^2 \sum_{v=1}^{3} \left(mt_v [\tilde{\mathbb{T}}_f^{v,(m)}, \tilde{\mathbb{T}}_g^{v,(m)}] - \tilde{\mathbb{T}}_{\{f,g\}}^{v,(m)} \right) + \sum_{\substack{u,v=1 \\ u \neq v}}^{3} i_u \cdot i_v \tilde{\mathbb{T}}_{\{f,g\}}^{v,(m)} \right\| \\
&\leq \sum_{v=1}^{3} a^2 \left\| mt [\tilde{\mathbb{T}}_f^{v,(m)}, \tilde{\mathbb{T}}_g^{v,(m)}] - \tilde{\mathbb{T}}_{\{f,g\}}^{v,(m)} \right\| + \left\| \sum_{\substack{u,v=1 \\ u \neq v}}^{3} i_u \cdot i_v \tilde{\mathbb{T}}_{\{f,g\}}^{v,(m)} \right\| \\
&\overset{(2)}{=} (a_1^2 + a_2^2 + a_3^2) O(m^{-1}) \\
&= O(m^{-1}),
\end{aligned}$$

□

where

(1) uses Lemma 6.6.

(2) uses Theorem 3.4.2 on each operator.

The next proposition generalizes Theorem 3.4.3:

Proposition 6.8.

$$\left\|\tilde{\mathbb{T}}_f^{(k)}\cdot\tilde{\mathbb{T}}_g^{(k)} - s\cdot\tilde{\mathbb{T}}_{fg}^{(k)}\right\| \leq O\left(k^{-1}\right)$$
$$+\left\|\sum_{\substack{n\neq m\\ n,m=1}}^{3} a_n a_m i_n\cdot i_m \tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{m,(k)} - \sum_{\substack{n\neq m\\ n,m=1}}^{3} a_n a_m i_n\cdot i_m \tilde{\mathbb{T}}_{fg}^{m,(k)}\right\|.$$

Proof.

$$\tilde{\mathbb{T}}_f^{(k)}\cdot\tilde{\mathbb{T}}_g^{(k)} - s\tilde{\mathbb{T}}_{fg}^{(k)} = \sum_{n=1}^{3} a_n i_n \tilde{\mathbb{T}}_f^{n,(k)} \sum_{m=1}^{3} a_m i_m \tilde{\mathbb{T}}_g^{m,(k)} \tag{6.2}$$
$$-\sum_{n=1}^{3} a_n i_n \sum_{m=1}^{3} a_m i_m \tilde{\mathbb{T}}_{fg}^{m,(k)} \tag{6.3}$$
$$=\sum_{n,m=1}^{3} a_n a_m i_n\cdot i_m \tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{m,(k)} \tag{6.4}$$
$$-\sum_{n,m=1}^{3} a_n a_m i_n\cdot i_m \tilde{\mathbb{T}}_{fg}^{m,(k)} \tag{6.5}$$
$$=\sum_{n=1}^{3} a_n^2 i_n\cdot i_n \tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{n,(k)} \tag{6.6}$$
$$-\sum_{n=1}^{3} a_n^2 i_n\cdot i_n \tilde{\mathbb{T}}_{fg}^{n,(k)} \tag{6.7}$$
$$+\sum_{\substack{n\neq m\\ n,m=1}}^{3} a_n a_m i_n\cdot i_m \tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{m,(k)} \tag{6.8}$$
$$-\sum_{\substack{n\neq m\\ n,m=1}}^{3} a_n a_m i_n\cdot i_m \tilde{\mathbb{T}}_{fg}^{m,(k)}. \tag{6.9}$$

From expressions 6.6 and 6.7, and using Theorem 3.4.3 one gets

$$\left\|\sum_{n=1}^{3} a_n^2 i_n\cdot i_n \tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{n,(k)} - \sum_{n=1}^{3} a_n^2 i_n\cdot i_n \tilde{\mathbb{T}}_{fg}^{n,(k)}\right\|$$
$$=\left\|\sum_{n=1}^{3} a_n^2 i_n\cdot i_n \left(\tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{n,(k)} - \tilde{\mathbb{T}}_{fg}^{n,(k)}\right)\right\|$$
$$\leq \sum_{n=1}^{3} a_n^2 \|i_n\cdot i_n\| \left\|\tilde{\mathbb{T}}_f^{n,(k)}\tilde{\mathbb{T}}_g^{n,(k)} - \tilde{\mathbb{T}}_{fg}^{n,(k)}\right\|$$

$$
\begin{aligned}
&= \sum_{n=1}^{3} a_n^2 \|i_n \cdot i_n\| O(k^{-1}). \\
&= O(k^{-1}).
\end{aligned}
$$

□

Theorem 6.9.
Assume that i_1, i_2 and i_3 are orthogonal. The Hyperkähler Berezin-Toeplitz operators have the following properties:

1. *For every $f \in C^\infty(M)$, there exists a $C > 0$ such that*

$$
(| f |_\infty - \frac{C}{m}) \leq \|s \cdot \tilde{\mathbb{T}}_f^{(m)}\| \leq | f |_\infty .
$$

   In particular,

$$
\lim_{m \to \infty} \|\tilde{\mathbb{T}}_f^{(m)}\| = | f |_\infty .
$$

2. *For every $f, g \in C^\infty(M)$,*

$$
\left\| mt \left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^{\cdot} - s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)} \right\| = O\left(m^{-1}\right).
$$

3. *For every $f, g \in C^\infty(M)$,*

$$
\left\| \tilde{\mathbb{T}}_f^{(m)} \cdot \tilde{\mathbb{T}}_g^{(m)} - s \cdot \tilde{\mathbb{T}}_{fg}^{(m)} \right\| = O\left(m^{-1}\right).
$$

Proof. Part (1) is just Proposition 6.5. Parts (2) and (3) are Propositions 6.7 and 6.8 for an orthogonal basis. □

Corollary 6.10. *Theorem 6.9 holds for a K3 surface and the three quantum line bundles L_I, L_J and L_K.*

Proof. The associated structure (i_1, i_2, i_3) is (i, j, k), whose elements are orthogonal. □

The next example shows that one recovers the properties of original Berezin-Toeplitz operators as a particular case of the Hyperkähler Berezin-Toeplitz operators.

Example 6.2. (Berezin-Toeplitz operators) Consider the case where $a_1 = 1$, $a_2 = a_3 = 0$. Then Theorem 6.9 reads as follows:

1. For every $f \in C^\infty(M)$, there exists a $C > 0$ such that

$$
(| f |_\infty - \frac{C}{m}) \leq \| - T_f^{(m)}\| \leq | f |_\infty .
$$

   In particular,

$$
\lim_{m \to \infty} \|T_f^{(m)}\| = | f |_\infty .
$$

2. For every $f, g \in C^\infty(M)$,

$$\left\| mt\left[T_f^{(m)}, T_g^{(m)}\right] - T_{\{f,g\}}^{(m)} \right\| = O\left(m^{-1}\right).$$

3. For every $f, g \in C^\infty(M)$,

$$\left\| T_f^{(m)} T_g^{(m)} - T_{fg}^{(m)} \right\| = O\left(m^{-1}\right).$$

Theorem 6.9 generalizes the properties of Berezin-Toeplitz operators to the Hyperkähler Berezin-Toeplitz operators and allows one to recover the original properties. Roughly speaking, the above proofs use the fact that the scalar product handles terms of the form

$$T_{f,I_1}^{(m)} T_{g,I_1}^{(m)} \otimes Id \otimes Id$$

and eliminates terms of the form

$$T_{f,I_1}^{(m)} \otimes T_{g,I_2}^{(m)} \otimes Id.$$

Example 6.3. Consider a K3 surface with Kähler structures (g, I, ω_I), (g, J, ω_J) and (g, K, ω_K). Consider the case $a_1 = a_2 = 1$, $a_3 = 0$. Then

$$\tilde{\mathbb{T}}_f^{(k)} \cdot \tilde{\mathbb{T}}_g^{(k)} = T_{f,I}^{(k)} T_{g,I}^{(k)} \otimes Id + Id \otimes T_{f,J}^{(k)} T_{g,J}^{(k)}.$$

6.2.2 Cross product

Next, we will study the properties of the Hyperkähler Berezin-Toeplitz operators with respect to the cross product. As the reader will see, the properties of Hyperkähler Berezin-Toeplitz operators with respect to the cross product are not a direct generalization of the original properties. Before doing that, however, we need to introduce a equivalence relation which will greatly simplify the proofs:

Definition 6.2. Let H^m be a sequence of Hilbert spaces and $\tilde{\mathbb{T}}_1^{(m)}$ and $\tilde{\mathbb{T}}_2^{(m)}$ two family of operators acting on H^m. $\tilde{\mathbb{T}}_1^{(m)}$ and $\tilde{\mathbb{T}}_2^{(m)}$ are called *asymptotically equivalent* (denoted by $\tilde{\mathbb{T}}_1^{(m)} \sim \tilde{\mathbb{T}}_2^{(m)}$) if

$$\left\| \tilde{\mathbb{T}}_1^{(m)} - \tilde{\mathbb{T}}_2^{(m)} \right\| = O(m^{-1}).$$

Remark. One should note that the norm depends on the Hilbert space H^m and one should write

$$\left\| \tilde{\mathbb{T}}_1^{(m)} - \tilde{\mathbb{T}}_2^{(m)} \right\|_{H^m}.$$

However, on most of cases the norm is clear and writing it only creates a more inconvenient notation. Therefore we will not write it.

Example 6.4. For $n \neq m$,

$$\tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{m,(k)} = \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{n,(k)}.$$

In particular,

$$\tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \sim \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{n,(k)}.$$

Example 6.5. In general is not true that

$$\tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{n,(k)} = \tilde{\mathbb{T}}_g^{n,(k)} \tilde{\mathbb{T}}_f^{n,(k)}.$$

However, by Corollary 3.5

$$\tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{n,(k)} \sim \tilde{\mathbb{T}}_g^{n,(k)} \tilde{\mathbb{T}}_f^{n,(k)}.$$

Lemma 6.11.

$$\left[i_n \tilde{\mathbb{T}}_f^{n,(k)}, i_m \tilde{\mathbb{T}}_g^{m,(k)} \right]^{\times} = 2 i_n \times i_m \tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{m,(k)}.$$

In particular, if $n = m$,

$$\left[i_n \tilde{\mathbb{T}}_f^{n,(k)}, i_n \tilde{\mathbb{T}}_g^{n,(k)} \right]^{\times} = 0.$$

Proof.

$$\begin{aligned} \left[i_n \tilde{\mathbb{T}}_f^{n,(k)}, i_m \tilde{\mathbb{T}}_g^{m,(k)} \right]^{\times} &= i_n \tilde{\mathbb{T}}_f^{n,(k)} \times i_m \tilde{\mathbb{T}}_g^{m,(k)} - i_m \tilde{\mathbb{T}}_g^{m,(k)} \times i_n \tilde{\mathbb{T}}_f^{n,(k)} \\ &= i_n \times i_m \tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{m,(k)} - i_m \times i_n \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{n,(k)} \\ &= i_n \times i_m \tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{m,(k)} + i_n \times i_m \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{n,(k)}. \end{aligned}$$

If $m = n$, last expression is identically zero since $i \times i = 0$ for any imaginary quaternionic number. Otherwise, $\tilde{\mathbb{T}}_f^{n,(k)}$ and $\tilde{\mathbb{T}}_g^{m,(k)}$ commute, which gives the desired result. ☐

Corollary 6.12. *For any two functions $f, g \in C^\infty(M)$*

$$\left[\tilde{\mathbb{T}}_f^{(k)}, \tilde{\mathbb{T}}_g^{(k)} \right]^{\times} = 2 \sum_{\substack{n \neq m \\ n,m=1}} a_n a_m i_n \times i_m \tilde{\mathbb{T}}_f^{n,(k)} \tilde{\mathbb{T}}_g^{m,(k)} = 2 \tilde{\mathbb{T}}_f^{(k)} \times \tilde{\mathbb{T}}_g^{(k)}.$$

In particular, for $s = a_1 i + a_2 j + a_3 k$:

$$\begin{aligned} \frac{1}{2} \left[\tilde{\mathbb{T}}_1^{(k)}, \tilde{\mathbb{T}}_f^{(k)} \right]^{\times} =& a_2 a_3 i \left(\tilde{\mathbb{T}}_f^{3,(k)} - \tilde{\mathbb{T}}_f^{2,(k)} \right) \\ &+ a_1 a_3 j \left(\tilde{\mathbb{T}}_f^{1,(k)} - \tilde{\mathbb{T}}_f^{3,(k)} \right) \\ &+ a_2 a_1 k \left(\tilde{\mathbb{T}}_f^{2,(k)} - \tilde{\mathbb{T}}_f^{1,(k)} \right). \end{aligned}$$

In other words, the operator

$$\left[\tilde{\mathbb{T}}_1^{(k)}, \tilde{\mathbb{T}}_f^{(k)}\right]^{\times}$$

measures how different the original Berezin-Toeplitz operators of f act on the original Hilbert spaces.

Example 6.6. Consider a K3 surface with Kähler structures (g, ω_I, I), (g, ω_J, J) and (g, ω_K, K). Consider the case $a_1 = a_2 = 1$, $a_3 = 0$. Then

$$-i \cdot \left[\tilde{\mathbb{T}}_1^{(k)}, \tilde{\mathbb{T}}_f^{(k)}\right]^{\times} = T_{f,I}^{(k)} \otimes Id - Id \otimes T_{f,J}^{(k)}$$

compares the Berezin-Toeplitz operators of f for the two Kähler structures (g, I, ω_I) and (g, J, ω_J).

Remark. Consider again the case $a_1 = a_2 = 1$ and $a_3 = 0$. Consider the sequence

$$\phi^{(m)} = \phi_1^{(m)} \otimes \phi_2^{(m)}$$

as in the proof of Proposition 6.5. Assume that $f \neq 0$ and let x_0 be a maximum of f. Without loss of generality, assume that $\|\phi^{(m)}\| = 1$.

$$\begin{aligned}
\left\|(s \times \tilde{\mathbb{T}}_f^{(m)})\phi^{(m)}\right\| =& \left\|i_3(T_{f,I}^{(m)} \otimes Id - Id \otimes T_{f,J}^{(m)})\phi^{(m)} - (i_3 - i_3)f(x_0)\phi^{(m)}\right\| \\
\leq& \|i_3(T_{f,I}^{(m)} \otimes Id)\phi_1^{(m)} \otimes \phi_2^{(m)} - i_3 f(x_0)\phi_1^{(m)} \otimes \phi_2^{(m)}\| \\
&+ \|i_3(Id \otimes T_{f,J}^{(m)})\phi_1^{(m)} \otimes \phi_2^{(m)} - i_3 f(x_0)\phi_1^{(m)} \otimes \phi_2^{(m)}\| \\
=& O(m^{-1}).
\end{aligned}$$

In particular

$$\lim_{m\to\infty}(Id \otimes T_{f,J}^{(m)})\phi_1^{(m)} \otimes \phi_2^{(m)} = \lim_{m\to\infty}(T_{f,I}^{(m)} \otimes Id)\phi_1^{(m)} \otimes \phi_2^{(m)} \neq 0.$$

Proposition 6.13. *Assume that i_n, $n = 1, 2, 3$ are orthonormal. For any $f \in C^\infty(M)$*

$$\left\|s \times \tilde{\mathbb{T}}_f^{(m)}\right\| \leq C \mid f \mid_\infty,$$

where

$$C = \left(\sum_{n=1}^{3} a_n\right)^2 - 1.$$

Proof.

$$\begin{aligned}
\left\| s \times \tilde{\mathbb{T}}_f^{(m)} \right\| &= \left\| \sum_{n=1}^{3} a_n i_n \times \sum_{m=1}^{3} a_m i_m \tilde{\mathbb{T}}_f^{m,(k)} \right\| \\
&\overset{(1)}{=} \left\| \sum_{\substack{n,m=1 \\ n\neq m}}^{3} a_n a_m i_n \times i_m \tilde{\mathbb{T}}_f^{m,(k)} \right\| \\
&\leq \sum_{\substack{n,m=1 \\ n\neq m}}^{3} a_n a_m \left\| \tilde{\mathbb{T}}_f^{m,(k)} \right\| \\
&\overset{(2)}{\leq} \sum_{\substack{n,m=1 \\ n\neq m}}^{3} a_n a_m \mid f \mid_\infty \\
&= \left(\left(\sum_{n=1}^{3} a_n \right)^2 - \left(\sum_{n=1}^{3} a_n^2 \right) \right) \mid f \mid_\infty \\
&= \left(\left(\sum_{n=1}^{3} a_n \right)^2 - 1 \right) \mid f \mid_\infty,
\end{aligned}$$

where we use

1. The triangle inequality and the fact that i_n, $n = 1, 2, 3$ are orthonormal.

2. Theorem 3.4.1.

□

Proposition 6.14. *The bracket* $[\quad,\quad]^\times$ *fulfills the Jacobi identity asymptotically, i.e.*

$$\left\| \left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^\times, \tilde{\mathbb{T}}_h^{(m)} \right]^\times + \left[\left[\tilde{\mathbb{T}}_g^{(m)}, \tilde{\mathbb{T}}_h^{(m)} \right]^\times, \tilde{\mathbb{T}}_f^{(m)} \right]^\times + \left[\left[\tilde{\mathbb{T}}_h^{(m)}, \tilde{\mathbb{T}}_f^{(m)} \right]^\times, \tilde{\mathbb{T}}_g^{(m)} \right]^\times \right\| = O\left(m^{-1}\right).$$

Proof. A straightforward computation like in Lemma 6.11 shows that

$$\left[\left[\tilde{\mathbb{T}}_f^{(k)}, \tilde{\mathbb{T}}_g^{(k)} \right]^\times, \tilde{\mathbb{T}}_h^{(k)} \right]^\times \sim 4 \sum_{l,m,n} a_l a_m a_n i_l \times (i_m \times i_n) \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)}.$$

Moreover,

$$\left(i_l \times (i_m \times i_n) + i_m \times (i_n \times i_l) + i_n \times (i_l \times i_m)\right) \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} = 0$$

because the cross product fulfills the Jacobi identity. □

Remark. By Corollary 6.12, last proposition can be written in the following way:

$$\left\| \left(\tilde{\mathbb{T}}_f^{(m)} \times \tilde{\mathbb{T}}_g^{(m)}\right) \times \tilde{\mathbb{T}}_h^{(m)} + \left(\tilde{\mathbb{T}}_g^{(m)} \times \tilde{\mathbb{T}}_h^{(m)}\right) \times \tilde{\mathbb{T}}_f^{(m)} + \left(\tilde{\mathbb{T}}_h^{(m)} \times \tilde{\mathbb{T}}_f^{(m)}\right) \times \tilde{\mathbb{T}}_g^{(m)} \right\| = O\left(m^{-1}\right).$$

Proposition 6.15. *Assume that i_n, $i = 1, 2, 3$ are orthonormal. Then*

$$\left\| mt \left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^\times, \tilde{\mathbb{T}}_h^{(m)} \right]^\times - 4\tilde{\mathbb{T}}_f^{(m)} \left(s \cdot \tilde{\mathbb{T}}_{\{g,h\}}^{(m)} \right) \right.$$
$$mt \left[\left[\tilde{\mathbb{T}}_g^{(m)}, \tilde{\mathbb{T}}_h^{(m)} \right]^\times, \tilde{\mathbb{T}}_f^{(m)} \right]^\times - 4\tilde{\mathbb{T}}_g^{(m)} \left(s \cdot \tilde{\mathbb{T}}_{\{h,f\}}^{(m)} \right)$$
$$\left. mt \left[\left[\tilde{\mathbb{T}}_h^{(m)}, \tilde{\mathbb{T}}_f^{(m)} \right]^\times, \tilde{\mathbb{T}}_g^{(m)} \right]^\times - 4\tilde{\mathbb{T}}_h^{(m)} \left(s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)} \right) \right\| = O\left(m^{-1}\right).$$

Proof.

$$\begin{aligned}
& m \left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)} \right]^\times, \tilde{\mathbb{T}}_h^{(m)} \right]^\times + m \left[\left[\tilde{\mathbb{T}}_g^{(m)}, \tilde{\mathbb{T}}_h^{(m)} \right]^\times, \tilde{\mathbb{T}}_f^{(m)} \right]^\times \\
&\quad + m \left[\left[\tilde{\mathbb{T}}_h^{(m)}, \tilde{\mathbb{T}}_f^{(m)} \right]^\times, \tilde{\mathbb{T}}_g^{(m)} \right]^\times \\
&= \left[\left[\sum_{r=1}^{3} a_r i_r \tilde{\mathbb{T}}_f^{r,(m)}, \sum_{s=1}^{3} a_s i_s \tilde{\mathbb{T}}_g^{s,(m)} \right]^\times, \sum_{t=1}^{3} a_t i_t \tilde{\mathbb{T}}_h^{t,(m)} \right]^\times \\
&\quad + \left[\left[\sum_{s=1}^{3} a_s i_s \tilde{\mathbb{T}}_g^{s,(m)}, \sum_{t=1}^{3} a_t i_t \tilde{\mathbb{T}}_h^{t,(m)} \right]^\times, \sum_{r=1}^{3} a_r i_r \tilde{\mathbb{T}}_f^{r,(m)} \right]^\times \\
&\quad + \left[\left[\sum_{t=1}^{3} a_t i_t \tilde{\mathbb{T}}_h^{t,(m)}, \sum_{r=1}^{3} a_r i_r \tilde{\mathbb{T}}_f^{r,(m)} \right]^\times, \sum_{s=1}^{3} a_s i_s \tilde{\mathbb{T}}_g^{s,(m)} \right]^\times \\
&= \sum_{i,j,k=1}^{3} m a_r a_s a_t \left[\left[i_r \tilde{\mathbb{T}}_f^{r,(m)}, i_s \tilde{\mathbb{T}}_g^{s,(m)} \right]^\times, i_t \tilde{\mathbb{T}}_h^{t,(m)} \right]^\times \\
&\quad + \sum_{i,j,k=1}^{3} m a_r a_s a_t \left[\left[i_s \tilde{\mathbb{T}}_g^{s,(m)}, i_t \tilde{\mathbb{T}}_h^{t,(m)} \right]^\times, i_r \tilde{\mathbb{T}}_f^{r,(m)} \right]^\times
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i,j,k=1}^{3} m a_r a_s a_t \left[\left[i_t \tilde{\mathbb{T}}_h^{t,(m)}, i_r \tilde{\mathbb{T}}_f^{r,(m)} \right]^{\times}, i_s \tilde{\mathbb{T}}_g^{s,(m)} \right]^{\times} \\
&\overset{(1)}{=} 4 \sum_{i,j,k=1}^{3} m a_r a_s a_t \Big((i_r \times i_s) \times i_t \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{s,(m)} \tilde{\mathbb{T}}_h^{t,(m)} \\
&+ (i_s \times i_t) \times i_r \tilde{\mathbb{T}}_g^{s,(m)} \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} + (i_t \times i_r) \times i_s \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{s,(m)} \Big),
\end{aligned}
$$

where at (1) we use Lemma 6.11. Now there are three possibilities:

- If all three subindices are different, then the operators commute and

$$
\begin{aligned}
&(i_r \times i_s) \times i_t \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{s,(m)} \tilde{\mathbb{T}}_h^{t,(m)} + (i_s \times i_t) \times i_r \tilde{\mathbb{T}}_g^{s,(m)} \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \\
&\quad + (i_t \times i_r) \times i_s \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{s,(m)} \\
&= \left((i_r \times i_s) \times i_t + (i_s \times i_t) \times i_r + (i_t \times i_r) \times i_s \right) \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{s,(m)} \tilde{\mathbb{T}}_h^{t,(m)} \\
&= 0
\end{aligned}
$$

  by Jacobi identity.

- If all the three indices are the same, then the term is identically zero because $(i_t \times i_t) \times i_t$ is always zero.

- Only two subindices are equal. Without loss of generality, assume that $r = s$. Then

$$
\begin{aligned}
&mt \sum_{r,t=1}^{3} a_r^2 a_t \Big((i_r \times i_r) \times i_t \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{r,(m)} \tilde{\mathbb{T}}_h^{t,(m)} \\
&\quad + (i_r \times i_t) \times i_r \tilde{\mathbb{T}}_g^{r,(m)} \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} + (i_t \times i_r) \times i_r \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{r,(m)} \Big) \\
&\overset{(1)}{=} m \sum_{r,t=1}^{3} a_r^2 a_t \Big((i_r \times i_t) \times i_r \tilde{\mathbb{T}}_g^{r,(m)} \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \\
&\quad + (i_t \times i_r) \times i_r \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{r,(m)} \Big) \\
&\overset{(2)}{=} mt \sum_{r,t=1}^{3} a_r^2 a_t \Big((i_r \times i_t) \times i_r \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_g^{r,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \\
&\quad - (i_r \times i_t) \times i_r \tilde{\mathbb{T}}_h^{t,(m)} \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{r,(m)} \Big) \\
&\overset{(3)}{=} mt \sum_{r,t=1}^{3} a_r^2 a_t i_t \tilde{\mathbb{T}}_h^{t,(m)} \left(\tilde{\mathbb{T}}_g^{r,(m)} \tilde{\mathbb{T}}_f^{r,(m)} - \tilde{\mathbb{T}}_f^{r,(m)} \tilde{\mathbb{T}}_g^{r,(m)} \right) \\
&= -mt \sum_{r,t=1}^{3} a_r^2 a_t i_t \tilde{\mathbb{T}}_h^{t,(m)} \left[\tilde{\mathbb{T}}_f^{r,(m)}, \tilde{\mathbb{T}}_g^{r,(m)} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\tilde{\mathbb{T}}_h^{(m)}\left(\sum_{r=1}^{3} m a_r^2 t\left[\tilde{\mathbb{T}}_f^{r,(m)}, \tilde{\mathbb{T}}_g^{r,(m)}\right]\right) \\
&\overset{(4)}{\sim} -\tilde{\mathbb{T}}_h^{(m)}\left(\sum_{r=1}^{3} a_r^2 \tilde{\mathbb{T}}_{\{f,g\}}^{r,(m)}\right) \\
&= \tilde{\mathbb{T}}_h^{(m)}\left(s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)}\right),
\end{aligned}
$$

where

(1) uses that $i_r \times i_r$ is zero.

(2) uses anticommutativity of cross product and that operators with different index commute.

(3) uses that for orthonormal i_1 and i_2,

$$
(i_1 \times i_2) \times i_1 = i_2.
$$

(4) uses Theorem 3.4.3.

Adding the terms for $r = t$ and $s = t$ gives the desired result.

□

The following theorem is a direct consequence of the three previous propositions:

Theorem 6.16.
Assume that i_n, $n = 1, 2, 3$ are orthonormal. The Hyperkähler Berezin-Toeplitz operators have the following properties:

1. *For every $f \in C^\infty(M)$*

$$
\|s \times \tilde{\mathbb{T}}_f^{(m)}\| \leq C \mid f \mid_\infty,
$$

where

$$
C = \left(\sum_{n=1}^{3} a_n\right)^2 - 1.
$$

2. *For every $f, g, h \in C^\infty(M)$,*

$$
\left\|\left(\tilde{\mathbb{T}}_f^{(m)} \times \tilde{\mathbb{T}}_g^{(m)}\right) \times \tilde{\mathbb{T}}_h^{(m)} + \left(\tilde{\mathbb{T}}_g^{(m)} \times \tilde{\mathbb{T}}_h^{(m)}\right) \times \tilde{\mathbb{T}}_f^{(m)} + \left(\tilde{\mathbb{T}}_h^{(m)} \times \tilde{\mathbb{T}}_f^{(m)}\right) \times \tilde{\mathbb{T}}_g^{(m)}\right\| = O\left(m^{-1}\right).
$$

3. For every $f, g, h \in C^\infty(M)$,

$$\Bigg\| mt\left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\times, \tilde{\mathbb{T}}_h^{(m)}\right]^\times - 4\tilde{\mathbb{T}}_f^{(m)}\left(s \cdot \tilde{\mathbb{T}}_{\{g,h\}}^{(m)}\right)$$
$$+ mt\left[\left[\tilde{\mathbb{T}}_g^{(m)}, \tilde{\mathbb{T}}_h^{(m)}\right]^\times, \tilde{\mathbb{T}}_f^{(m)}\right]^\times - 4\tilde{\mathbb{T}}_g^{(m)}\left(s \cdot \tilde{\mathbb{T}}_{\{h,f\}}^{(m)}\right)$$
$$+ mt\left[\left[\tilde{\mathbb{T}}_h^{(m)}, \tilde{\mathbb{T}}_f^{(m)}\right]^\times, \tilde{\mathbb{T}}_g^{(m)}\right]^\times - 4\tilde{\mathbb{T}}_h^{(m)}\left(s \cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)}\right)\Bigg\| = O\left(m^{-1}\right).$$

Corollary 6.17. *Theorem 6.16 holds for a K3 surface and the three quantum line bundles L_I, L_J and L_K.*

Remark. Theorem 6.16 looks again similar to of Theorem 3.4. However, one should note that when working with only one Kähler form, i.e., when $a_1 = 1$, $a_2 = a_3 = 0$, all expressions of the previous theorem are identically zero. Hence, this theorem shows new properties that do not have an equivalent for the original Berezin-Toeplitz operators.

6.2.3 Quaternionic product

Before explaining the results for the quaternionic product, we are showing some extra properties which relate the behavior of the Hyperkähler Berezin-Toeplitz operators with respect to the scalar and cross product which will be useful to relate the properties of the operators with respect to the different products.

Lemma 6.18.

$$\left[\left[\tilde{\mathbb{T}}_f^{(k)}, \tilde{\mathbb{T}}_g^{(k)}\right]^\times, \tilde{\mathbb{T}}_h^{(k)}\right]^\cdot = 0.$$

Proof.

$$\begin{aligned}
\left[\left[\tilde{\mathbb{T}}_f^{(k)}, \tilde{\mathbb{T}}_g^{(k)}\right]^\times, \tilde{\mathbb{T}}_h^{(k)}\right]^\cdot &= 2\left[\sum_{l,m=1}^{3} a_l a_m i_l \times i_m \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)}, \tilde{\mathbb{T}}_h^{(k)}\right]^\cdot \\
&= 2\sum_{l,m,n=1}^{3} a_l a_m a_n (i_l \times i_m) \cdot i_n [\tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)}, \tilde{\mathbb{T}}_h^{n,(k)}]^\cdot \\
&\overset{(1)}{=} 2\sum_{\substack{n\neq m\\ m\neq l\\ l\neq n}} a_l a_m a_n (i_l \times i_m) \cdot i_n [\tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)}, \tilde{\mathbb{T}}_h^{n,(k)}]^\cdot \\
&\overset{(2)}{=} 0,
\end{aligned}$$

where we use the following facts:

(1) $(i_l \times i_m) \cdot i_n$ is the signed volume of the parallelepiped with edges $\{i_l, i_m, i_n\}$. Therefore it is zero if two are equal.

(2) Two Toeplitz operators acting on different spaces commute.

$\square$

Recall that, for purely imaginary quaternionic numbers i_1, i_2, i_3:

$$i_1 \times (i_2 \times i_3) = -i_2(i_1 \cdot i_3) + i_3(i_1 \cdot i_2). \tag{6.10}$$

Remark. The usual form of the last expression differs from the one here written by a sign. This is due to the fact that we have considered the scalar product to have signature $(1,3)$.

Similarly:

Lemma 6.19.

$$\|\tilde{\mathbb{T}}_f^{(k)} \times \left(\tilde{\mathbb{T}}_g^{(k)} \times \tilde{\mathbb{T}}_h^{(k)}\right) + \tilde{\mathbb{T}}_g^{(k)} \left(\tilde{\mathbb{T}}_f^{(k)} \cdot \tilde{\mathbb{T}}_h^{(k)}\right) - \tilde{\mathbb{T}}_h^{(k)} \left(\tilde{\mathbb{T}}_f^{(k)} \cdot \tilde{\mathbb{T}}_g^{(k)}\right) \| = O(m^{-1}).$$

Proof. First, let's compute each term:

- $$\tilde{\mathbb{T}}_f^{(k)} \times \left(\tilde{\mathbb{T}}_g^{(k)} \times \tilde{\mathbb{T}}_h^{(k)}\right) = \sum_{l,m,n=1}^{3} a_l a_m a_n i_l \times (i_m \times i_n)\, \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)}.$$

- $$\tilde{\mathbb{T}}_g^{(k)} \left(\tilde{\mathbb{T}}_f^{(k)} \cdot \tilde{\mathbb{T}}_h^{(k)}\right) = \sum_{l,m,n=1}^{3} a_l a_m a_n i_m \,(i_l \cdot i_n)\, \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_h^{n,(k)}.$$

- $$\tilde{\mathbb{T}}_h^{(k)} \left(\tilde{\mathbb{T}}_f^{(k)} \cdot \tilde{\mathbb{T}}_g^{(k)}\right) = \sum_{l,m,n=1}^{3} a_l a_m a_n i_n \,(i_l \cdot i_m)\, \tilde{\mathbb{T}}_h^{n,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)}.$$

Fix l, m, n and compare the terms without the a_i constants:

$$\begin{aligned}
-i_l \times (i_m \times i_n) & \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} - i_m \,(i_l \cdot i_n)\, \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_h^{n,(k)} \\
&+ i_n \,(i_l \cdot i_m)\, \tilde{\mathbb{T}}_h^{n,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \\
&\overset{(1)}{=} (i_m \,(i_l \cdot i_n) - i_n \,(i_l \cdot i_m)) \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} \\
&\quad - i_m \,(i_l \cdot i_n)\, \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_h^{n,(k)} + i_n \,(i_l \cdot i_m)\, \tilde{\mathbb{T}}_h^{n,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \\
&= i_m \,(i_l \cdot i_n) \left(\tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} - \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_h^{n,(k)}\right) \\
&\quad - i_n \,(i_l \cdot i_m) \left(\tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} - \tilde{\mathbb{T}}_h^{n,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)}\right) \\
&\sim 0,
\end{aligned}$$

where at (1) we use Relation 6.10 for quaternionic numbers. $\square$

Lemma 6.20.

$$\|\tilde{\mathbb{T}}_f^{(m)} \cdot \left(\tilde{\mathbb{T}}_g^{(m)} \times \tilde{\mathbb{T}}_h^{(m)}\right) - \tilde{\mathbb{T}}_g^{(m)} \cdot \left(\tilde{\mathbb{T}}_h^{(m)} \times \tilde{\mathbb{T}}_f^{(m)}\right)\| = O(m^{-1}).$$

Proof. Denote by $V_{l,m,n}$ the signed volume of the parallelepiped formed by $\{i_l, i_m, i_n\}$. Then

$$\begin{aligned}\tilde{\mathbb{T}}_f^{(k)} \cdot \left(\tilde{\mathbb{T}}_g^{(k)} \times \tilde{\mathbb{T}}_h^{(k)}\right) &= \sum_{l,m,n} a_l a_m a_n i_l \cdot (i_m \times i_n) \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} \\ &= \sum_{\substack{n \neq m \\ m \neq l \\ l \neq n}} a_l a_m a_n V_{l,m,n} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)}\end{aligned}$$

$$\begin{aligned}\tilde{\mathbb{T}}_g^{(k)} \cdot \left(\tilde{\mathbb{T}}_h^{(k)} \times \tilde{\mathbb{T}}_f^{(k)}\right) &= \sum_{l,m,n} a_l a_m a_n i_m \cdot (i_n \times i_l) \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} \tilde{\mathbb{T}}_f^{l,(k)} \\ &= \sum_{\substack{n \neq m \\ m \neq l \\ l \neq n}} a_l a_m a_n V_{l,m,n} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} \tilde{\mathbb{T}}_f^{l,(k)}.\end{aligned}$$

Fix l, m, n. Since operators acting on different spaces commute, one has

$$a_l a_m a_n V_{l,m,n} \tilde{\mathbb{T}}_f^{l,(k)} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} = a_l a_m a_n V_{l,m,n} \tilde{\mathbb{T}}_g^{m,(k)} \tilde{\mathbb{T}}_h^{n,(k)} \tilde{\mathbb{T}}_f^{l,(k)}.$$

□

To finish this section, we present the properties of the Hyperkähler Berezin-Toeplitz operators with respect to the quaternionic product.

Note that, by linearity

$$\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\star = \left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\times + \left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\cdot.$$

We will show that this *splitting* also works when composing two brackets (with some modification), but it does not work when using a higher number of brackets. Then we will use the splitting to deduce the properties of the Hyperkähler Berezin-Toeplitz operators with respect to the quaternionic product from Theorems 6.9 and 6.16.

Lemma 6.21.

$$\left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\times, \tilde{\mathbb{T}}_h^{(m)}\right]^\star = \left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\times, \tilde{\mathbb{T}}_h^{(m)}\right]^\times.$$

Proof. We use that, for purely imaginary numbers, $a \star b = a \cdot b + a \times b$ and Lemma 6.18. □

Lemma 6.22.

$$\left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\cdot},\tilde{\mathbb{T}}_h^{(m)}\right]^{\star} = -\sum_{l=1}^{3} i_l a_l^3 \left[\left[\tilde{\mathbb{T}}_f^{l,(k)},\tilde{\mathbb{T}}_g^{l,(k)}\right]\tilde{\mathbb{T}}_h^{l,(k)}\right].$$

Proof. Fix l, m, n. We will compute the term

$$\left[\left[i_l\tilde{\mathbb{T}}_f^{l,(k)}, i_m\tilde{\mathbb{T}}_g^{m,(k)}\right]^{\cdot}, i_n\tilde{\mathbb{T}}_h^{n,(k)}\right]^{\star}.$$

Remember that by Lemma 6.6

$$\left[i_l\tilde{\mathbb{T}}_f^{l,(k)}, i_m\tilde{\mathbb{T}}_g^{m,(k)}\right]^{\cdot} = 0$$

whenever $l \neq m$. For $l = m$

$$\begin{aligned}\left[\left[i_l\tilde{\mathbb{T}}_f^{l,(k)}, i_l\tilde{\mathbb{T}}_g^{l,(k)}\right]^{\cdot}, i_n\tilde{\mathbb{T}}_h^{n,(k)}\right]^{\star} &= \left[-\left[\tilde{\mathbb{T}}_f^{l,(k)},\tilde{\mathbb{T}}_g^{l,(k)}\right], i_n\tilde{\mathbb{T}}_h^{n,(k)}\right]^{\star}\\ &\overset{(1)}{=} -i_n\left[\left[\tilde{\mathbb{T}}_f^{l,(k)},\tilde{\mathbb{T}}_g^{l,(k)}\right],\tilde{\mathbb{T}}_h^{n,(k)}\right].\end{aligned}$$

where at (1) we use that $\star$ is commutative when one of the numbers is real. This expression is zero unless $l = m = n$ since operators acting on different spaces commute. □

Remark. Note that

$$\left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\cdot},\tilde{\mathbb{T}}_h^{(m)}\right]^{\star} = \left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\cdot},\tilde{\mathbb{T}}_h^{(m)}\right].$$

Since $\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\cdot}$ has real coefficients, this is just a matter of notation.

Proposition 6.23.

$$\left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\star},\tilde{\mathbb{T}}_h^{(m)}\right]^{\star} = \left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\cdot},\tilde{\mathbb{T}}_h^{(m)}\right] + \left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\times},\tilde{\mathbb{T}}_h^{(m)}\right]^{\times}.$$

Proof. Apply bilinearity and Lemma 6.21. □

Remark. This splitting does not work for higher number of brackets. In fact,

$$\begin{aligned}\left[\left[\left[\tilde{\mathbb{T}}_{f_1}^{(m)},\tilde{\mathbb{T}}_{f_2}^{(m)}\right]^{\star},\tilde{\mathbb{T}}_{f_3}^{(m)}\right]^{\star},\tilde{\mathbb{T}}_{f_4}^{(m)}\right]^{\star} &= \left[\left[\left[\tilde{\mathbb{T}}_{f_1}^{(m)},\tilde{\mathbb{T}}_{f_2}^{(m)}\right]^{\cdot},\tilde{\mathbb{T}}_{f_3}^{(m)}\right],\tilde{\mathbb{T}}_{f_4}^{(m)}\right]^{\cdot}\\ &+ \left[\left[\left[\tilde{\mathbb{T}}_{f_1}^{(m)},\tilde{\mathbb{T}}_{f_2}^{(m)}\right]^{\cdot},\tilde{\mathbb{T}}_{f_3}^{(m)}\right],\tilde{\mathbb{T}}_{f_4}^{(m)}\right]^{\times}\\ &+ \left[\left[\left[\tilde{\mathbb{T}}_{f_1}^{(m)},\tilde{\mathbb{T}}_{f_2}^{(m)}\right]^{\times},\tilde{\mathbb{T}}_{f_3}^{(m)}\right]^{\times},\tilde{\mathbb{T}}_{f_4}^{(m)}\right].\end{aligned}$$

Remark. Similarly,

$$\left(\tilde{\mathbb{T}}^{(m)}_{f_1} \star \tilde{\mathbb{T}}^{(m)}_{f_2}\right) \star \tilde{\mathbb{T}}^{(m)}_{f_3} = \left(\tilde{\mathbb{T}}^{(m)}_{f_1} \times \tilde{\mathbb{T}}^{(m)}_{f_2}\right) \times \tilde{\mathbb{T}}^{(m)}_{f_3} + \left(\tilde{\mathbb{T}}^{(m)}_{f_1} \cdot \tilde{\mathbb{T}}^{(m)}_{f_2}\right) \tilde{\mathbb{T}}^{(m)}_{f_3}.$$

Remark.

$$\left\| \left[\left[\tilde{\mathbb{T}}^{(m)}_{f}, \tilde{\mathbb{T}}^{(m)}_{g} \right]^{\cdot}, \tilde{\mathbb{T}}^{(m)}_{h} \right] \right\| = O\left(m^{-1}\right).$$

While the quaternionic product does not split into scalar and cross product in general, it splits when we multiply up to three purely imaginary quaternionic numbers. This corresponds the cases we have used on Theorems 6.9 and 6.16 and allows one to merge the results in a more general form:

Theorem 6.24.
Assume that i_1, i_2 and i_3 are orthogonal. Let $f, g, h \in C^\infty(M)$. The Hyperkähler Berezin-Toeplitz operators have the following properties:

1. *There exist a $C > 0$ such that:*

$$\left(\mid f \mid_\infty - \frac{C}{m}\right) \leq \| s \star \tilde{\mathbb{T}}^{(m)}_{f} \| \leq \left(\sum_{n=1}^{3} a_n\right)^2 \mid f \mid_\infty .$$

2.

$$\left\| mt \left[\tilde{\mathbb{T}}^{(m)}_{f}, \tilde{\mathbb{T}}^{(m)}_{g}\right]^{\star} - mt \left[\tilde{\mathbb{T}}^{(m)}_{f}, \tilde{\mathbb{T}}^{(m)}_{g}\right]^{\times} - s \cdot \tilde{\mathbb{T}}^{(m)}_{\{f,g\}} \right\| = O\left(m^{-1}\right).$$

3.

$$\left\| \tilde{\mathbb{T}}^{(m)}_{f} \star \tilde{\mathbb{T}}^{(m)}_{g} + \tilde{\mathbb{T}}^{(m)}_{g} \star \tilde{\mathbb{T}}^{(m)}_{f} - 2s \cdot \tilde{\mathbb{T}}^{(m)}_{fg} \right\| = O\left(m^{-1}\right).$$

4.

$$\begin{aligned} &\left\| \left(\tilde{\mathbb{T}}^{(m)}_{f} \star \tilde{\mathbb{T}}^{(m)}_{g}\right) \star \tilde{\mathbb{T}}^{(m)}_{h} - \left(s \cdot \tilde{\mathbb{T}}^{(m)}_{fg}\right) \tilde{\mathbb{T}}^{(m)}_{h} \right. \\ &+ \left(\tilde{\mathbb{T}}^{(m)}_{g} \star \tilde{\mathbb{T}}^{(m)}_{h}\right) \star \tilde{\mathbb{T}}^{(m)}_{f} - \left(s \cdot \tilde{\mathbb{T}}^{(m)}_{gh}\right) \tilde{\mathbb{T}}^{(m)}_{f} \\ &+ \left. \left(\tilde{\mathbb{T}}^{(m)}_{h} \star \tilde{\mathbb{T}}^{(m)}_{f}\right) \star \tilde{\mathbb{T}}^{(m)}_{g} - \left(s \cdot \tilde{\mathbb{T}}^{(m)}_{fh}\right) \tilde{\mathbb{T}}^{(m)}_{g} \right\| = O\left(m^{-1}\right). \end{aligned}$$

5.

$$\begin{aligned} &\left\| mt \left[\left[\tilde{\mathbb{T}}^{(m)}_{f}, \tilde{\mathbb{T}}^{(m)}_{g}\right]^{\star}, \tilde{\mathbb{T}}^{(m)}_{h}\right]^{\star} - 4\tilde{\mathbb{T}}^{(m)}_{f} \left(s \cdot \tilde{\mathbb{T}}^{(m)}_{\{g,h\}}\right) \right. \\ &+ mt \left[\left[\tilde{\mathbb{T}}^{(m)}_{g}, \tilde{\mathbb{T}}^{(m)}_{h}\right]^{\star}, \tilde{\mathbb{T}}^{(m)}_{f}\right]^{\star} - 4\tilde{\mathbb{T}}^{(m)}_{g} \left(s \cdot \tilde{\mathbb{T}}^{(m)}_{\{h,f\}}\right) \\ &+ \left. mt \left[\left[\tilde{\mathbb{T}}^{(m)}_{h}, \tilde{\mathbb{T}}^{(m)}_{f}\right]^{\star}, \tilde{\mathbb{T}}^{(m)}_{g}\right]^{\star} - 4\tilde{\mathbb{T}}^{(m)}_{h} \left(s \cdot \tilde{\mathbb{T}}^{(m)}_{\{f,g\}}\right) \right\| = O\left(m^{-1}\right). \end{aligned}$$

Proof. 1. For the second inequality, add propositions 6.5 and 6.13. For the first one, repeat the same argument.

2. By linearity,

$$\begin{aligned} mt\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\star =& mt\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\times mt\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\cdot \\ \overset{(1)}{=}& mt\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\times + s\cdot \tilde{\mathbb{T}}_{\{f,g\}}^{(m)}, \end{aligned}$$

where at (1) we use Theorem 6.9.2.

3. By linearity,

$$\begin{aligned} \tilde{\mathbb{T}}_f^{(m)} \star \tilde{\mathbb{T}}_g^{(m)} + \tilde{\mathbb{T}}_g^{(m)} \star \tilde{\mathbb{T}}_f^{(m)} =& \tilde{\mathbb{T}}_f^{(m)} \times \tilde{\mathbb{T}}_g^{(m)} + \tilde{\mathbb{T}}_g^{(m)} \times \tilde{\mathbb{T}}_f^{(m)} \\ &+ \tilde{\mathbb{T}}_f^{(m)} \cdot \tilde{\mathbb{T}}_g^{(m)} + \tilde{\mathbb{T}}_g^{(m)} \cdot \tilde{\mathbb{T}}_f^{(m)} \\ \overset{(1)}{=}& \tilde{\mathbb{T}}_f^{(m)} \cdot \tilde{\mathbb{T}}_g^{(m)} + \tilde{\mathbb{T}}_g^{(m)} \cdot \tilde{\mathbb{T}}_f^{(m)} \\ \overset{(2)}{\sim}& 2s\cdot \tilde{\mathbb{T}}_{fg}^{(m)}, \end{aligned}$$

where we use

(1) Antisymmetry of cross product.

(2) Theorem 6.9.3 twice.

4. By Remark 6.2.3 and 6.16.2,

$$\begin{aligned} &\left(\tilde{\mathbb{T}}_f^{(m)} \star \tilde{\mathbb{T}}_g^{(m)}\right) \star \tilde{\mathbb{T}}_h^{(m)} + \left(\tilde{\mathbb{T}}_g^{(m)} \star \tilde{\mathbb{T}}_h^{(m)}\right) \star \tilde{\mathbb{T}}_f^{(m)} + \left(\tilde{\mathbb{T}}_h^{(m)} \star \tilde{\mathbb{T}}_f^{(m)}\right) \star \tilde{\mathbb{T}}_g^{(m)} \\ &\sim \left(\tilde{\mathbb{T}}_f^{(m)} \cdot \tilde{\mathbb{T}}_g^{(m)}\right) \tilde{\mathbb{T}}_h^{(m)} + \left(\tilde{\mathbb{T}}_g^{(m)} \cdot \tilde{\mathbb{T}}_h^{(m)}\right) \tilde{\mathbb{T}}_f^{(m)} \\ &+ \left(\tilde{\mathbb{T}}_h^{(m)} \cdot \tilde{\mathbb{T}}_f^{(m)}\right) \tilde{\mathbb{T}}_g^{(m)}. \end{aligned}$$

Then use Theorem 6.9.3.

5. Apply splitting from Proposition 6.23. By Theorem 6.16.3, one only needs to proof that

$$\begin{aligned} &\Big\| mt\left[\left[\tilde{\mathbb{T}}_f^{(m)}, \tilde{\mathbb{T}}_g^{(m)}\right]^\cdot, \tilde{\mathbb{T}}_h^{(m)}\right]^\star \\ &+ mt\left[\left[\tilde{\mathbb{T}}_g^{(m)}, \tilde{\mathbb{T}}_h^{(m)}\right]^\cdot, \tilde{\mathbb{T}}_f^{(m)}\right]^\star \\ &+ mt\left[\left[\tilde{\mathbb{T}}_h^{(m)}, \tilde{\mathbb{T}}_f^{(m)}\right]^\cdot, \tilde{\mathbb{T}}_g^{(m)}\right]^\star \Big\| = O\left(m^{-1}\right). \end{aligned}$$

Then

$$mt\left[\left[\tilde{\mathbb{T}}_f^{(m)},\tilde{\mathbb{T}}_g^{(m)}\right]^{\cdot},\tilde{\mathbb{T}}_h^{(m)}\right]^{\star}\overset{(1)}{=}-mt\sum_{l=1}^{3}i_l a_l^3\left[\left[\tilde{\mathbb{T}}_f^{l,(k)},\tilde{\mathbb{T}}_g^{l,(k)}\right]\tilde{\mathbb{T}}_h^{l,(k)}\right]$$
$$\overset{(2)}{\sim}-\sum_{l=1}^{3}i_l a_l^3\left[\tilde{\mathbb{T}}_{\{f,g\}}^{l,(k)},\tilde{\mathbb{T}}_h^{l,(k)}\right]$$
$$\overset{(3)}{\sim}0,$$

where we use

(1) Lemma 6.22.

(2) Theorem 3.4.2.

(3) Corollary 3.6.

□

Corollary 6.25. *Theorem 6.24 holds for a K3 surface and the three quantum line bundles L_I, L_J and L_K.*

Example 6.7. (Berezin-Toeplitz operators) Consider the case where $a_1 = 1$, $a_2 = a_3 = 0$. The first three properties of Theorem 6.24 read as follows:

1. For every $f \in C^\infty(M)$, there exists a $C > 0$ such that

$$(\mid f\mid_\infty - \frac{C}{m}) \leq \|T_f^{(m)}\| \leq \mid f\mid_\infty .$$

2. For every $f, g \in C^\infty(M)$,

$$\left\|mt\left[T_f^{(m)},T_g^{(m)}\right]-T_{\{f,g\}}^{(m)}\right\| = O\left(m^{-1}\right).$$

3. For every $f, g \in C^\infty(M)$,

$$\left\|T_f^{(m)}T_g^{(m)}+T_g^{(m)}T_f^{(m)}-2T_{fg}^{(m)}\right\| = O\left(m^{-1}\right).$$

In particular (1) implies that

$$\lim_{m\to\infty}\|T_f^{(m)}\| = \mid f\mid_\infty .$$

Remark. Changing Theorem 6.24.3 to

$$\left\|\tilde{\mathbb{T}}_f^{(m)}\star\tilde{\mathbb{T}}_g^{(m)}-\tilde{\mathbb{T}}_g^{(m)}\times\tilde{\mathbb{T}}_f^{(m)}-s\cdot\tilde{\mathbb{T}}_{fg}^{(m)}\right\| = O\left(m^{-1}\right)$$

allows one to recover the exact formulation of Theorem 3.4.3.

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