Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices

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Submitted 6 May 2004; accepted 19 January 2005

Abstract

In the framework of cooperative game theory, the concept of interaction index, which can be regarded as an extension of that of value, has been recently proposed to measure the interaction phenomena among players. Axiomatizations of two classes of interaction indices, namely probabilistic interaction indices and cardinal-probabilistic interaction indices, generalizing probabilistic values and semivalues, respectively, are first proposed. The axioms we utilize are based on natural generalizations of axioms involved in the axiomatizations of values. In the second half of the paper, existing instances of cardinal-probabilistic interaction indices encountered thus far in the literature are also axiomatized.

Keywords: Cooperative games, interaction among players, values and interaction indices.
Journal of Economic Literature classification numbers: C71, C44.

1 Introduction

The study of the notion of interaction among players is relatively recent in the framework of cooperative game theory. The first attempt is probably due to Owen [27, §5] for superadditive games. More recent developments are due to Murofushi and Soneda [25], Roubens

[29], Grabisch [9], and Marichal and Roubens [23], and led successively to the concepts of *Shapley interaction index*, *Banzhaf interaction index*, and *chaining interaction index*. First axiomatic characterizations of the Shapley and Banzhaf interaction indices have been recently proposed by Grabisch and Roubens [13].

The concept of *interaction index*, which can be seen as an extension of the notion of *value* [2, 4, 16, 30, 31], is fundamental for it makes it possible to measure the interaction phenomena modelled by a game on a set of players. The expression "interaction phenomena" refers to either complementarity or redundancy effects among players of coalitions resulting from the non additivity of the underlying game.

Thus far, the notion of interaction index has been primarily applied to multicriteria decision making in the framework of aggregation by the Choquet integral. In this context, it is used to appraise the overall interaction among criteria [8, 9, 15, 20, 21, 22, 24], thereby giving more insight into the decision problem. Other natural applications concern statistics and data analysis [7, 10, 19].

In this paper we propose axiomatizations of two families of interaction indices introduced by Grabisch and Roubens [12, 14], namely the broad class of *probabilistic interaction indices* and the narrower subclass of *cardinal-probabilistic interaction indices* obtained by additionally imposing the symmetry axiom. Probabilistic interaction indices can be seen as extensions of *probabilistic values* studied by Weber [31]. Cardinal-probabilistic interaction indices are generalizations of *semivalues*, which were axiomatized by Dubey et al. [4]. We show that this latter subclass encompasses the Shapley, Banzhaf, and chaining interaction indices, but also the Möbius transform of a game, which is constructed from the Harsanyi *dividends* [16]. We also axiomatize these particular instances of cardinal-probabilistic interaction indices.

Besides classical axioms such as *linearity* and *additivity*, the axioms involved in the characterizations we present can be regarded as natural generalizations of those used in the axiomatizations of values. Two of the most important axioms in the proposed characterizations of probabilistic and cardinal-probabilistic interaction indices are the *k-monotonicity* axiom, which generalizes the *monotonicity* axiom [31, §4] (called *positivity* in Kalai and Samet [17, §4]), and the *dummy partnership* axiom, which extends the *dummy player* axiom through the concept of *partnership* (see e.g. [17]). The notion of partnership is also at the root of some of the axioms we additionally impose to characterize the Shapley, Banzhaf, and chaining interaction indices.

Note that the main feature of the new characterizations we offer for the Shapley and Banzhaf interaction indices is that we avoid the *recursive axiom*, which was a rather technical and "tailor-made" condition in Grabisch-Roubens characterizations [13]. Here, the new set of axioms we propose for those indices is recursive free and essentially based on a consistency property for "reduced" partnerships, which is natural and easily interpretable.

In addition to those characterization results we provide a representation theorem for cardinal-probabilistic interaction indices, generalizing that of semivalues by Dubey et al. [4]. We also provide a compact formula linking any cardinal-probabilistic interaction index with the well-known Owen *multilinear extension* of a game [27].

This paper is organized as follows. In Section 2 we recall some basic definitions and results we use in this paper. Section 3 is devoted to the concept of interaction index. An intuitive approach is adopted to present this notion and the axiomatizations by Grabisch and Roubens [13] are recalled. In the last section we present our characterization results. Probabilistic and cardinal-probabilistic interaction indices are first axiomatized. Then, the

Shapley, Banzhaf, and chaining interaction indices as well as the Möbius transform are characterized by imposing additional axioms. Previous to this, we also yield representation theorems for cardinal-probabilistic interaction indices.

In order to avoid a heavy notation, we adopt that used in [13]. Thus, we will often omit braces for singletons, e.g., by writing $v(i)$, $U \setminus i$ instead of $v(\{i\})$, $U \setminus \{i\}$. Similarly, for pairs, we will write ij instead of $\{i, j\}$. Furthermore, cardinalities of subsets $S, T, \ldots$, will often be denoted by the corresponding lower case letters $s, t, \ldots$, otherwise by the standard notation $|S|, |T|, \ldots$

2 Preliminary definitions

We consider an infinite set U, the universe of players. As usual, a *game* on U is a set function $v : 2^U \to \mathbb{R}$ such that $v(\varnothing) = 0$, which assigns to each *coalition* $S \subseteq U$ its *worth* $v(S)$.

In this section we recall some concepts and results we will use throughout.

2.1 Carriers

A set $N \subseteq U$ is said to be a *carrier* (or *support*) of a game v when $v(S) = v(N \cap S)$ for all $S \subseteq U$. Thus, a game v with carrier $N \subseteq U$ is completely defined by the knowledge of the coefficients $\{v(S)\}_{S \subseteq N}$ and the players outside N have no influence on the game since they do not contribute to any coalition.

In this paper, we restrict our attention to finite games, that is, games that possess finite carriers. We denote by $\mathcal{G}$ the set of finite games on U and by $\mathcal{G}^N$ the set of games with finite carrier $N \subseteq U$.

2.2 Möbius and co-Möbius transforms

Let us recall two equivalent representations of a game (see e.g. [11]). Any game $v \in \mathcal{G}^N$ can be uniquely expressed in terms of its *dividends* $\{m(v, S)\}_{S \subseteq N}$ (see e.g. [16]) by

$$v(T) = \sum_{S \subseteq T} m(v, S), \qquad \forall\, T \subseteq N.$$

In combinatorics, the set function $m(v, \cdot\,) : 2^U \to \mathbb{R}$ is called the *Möbius transform* [28] of v and is given by

$$m(v, S) := \sum_{T \subseteq S} (-1)^{s-t} v(T), \qquad \forall\, S \subseteq U.$$

Another equivalent representation is the *co-Möbius transform* [11] of v which can be defined by

$$m^*(v, S) := \sum_{T \subseteq S} (-1)^t v(N \setminus T), \qquad \forall\, S \subseteq U.$$

We have the following result.

Proposition 2.1. *For any $v \in \mathcal{G}^N$ and any $S \nsubseteq N$, we have $m(v, S) = 0$ and $m^*(v, S) = 0$.*

Proof. Let $S \not\subseteq N$. Then there exists $i \in S \setminus N$. Setting $K := S \setminus i$, we can write

$$\begin{aligned} m(v,S) &= \sum_{L \subseteq K \cup i} (-1)^{k+1-l} v(L) \\ &= \sum_{L \subseteq K} (-1)^{k+1-l} v(L) + \sum_{L \subseteq K} (-1)^{k-l} v(L \cup i) \\ &= \sum_{L \subseteq K} (-1)^{k+1-l} v(L) - \sum_{L \subseteq K} (-1)^{k+1-l} v(L) = 0, \end{aligned}$$

since, as $i \notin N$, $v(L \cup i) = v(L)$, for all $L \subseteq K$.

On the other hand, from [11, Eq. (13)], we have

$$m^*(v,S) = \sum_{T \supseteq S} m(v,T)$$

and hence $m^*(v,S) = 0$. □

2.3 Discrete derivatives

Given a game $v \in \mathcal{G}^N$ and finite coalitions $S, T \subseteq U$, we denote by $\Delta_S v(T)$ the S-*derivative* of v at T, which is recursively defined by

$$\Delta_i v(T) := v(T \cup i) - v(T \setminus i), \qquad \forall\, i \in U,$$

and

$$\Delta_S v(T) := \Delta_i [\Delta_{S \setminus i} v(T)], \qquad \forall\, i \in S,$$

with convention $\Delta_\varnothing v(T) := v(T)$; see [6, §1] and [11, §2].

We can easily prove by induction on s that

$$\Delta_S v(T) = \Delta_S v(T \setminus S) = \sum_{L \subseteq S} (-1)^l v[(T \cup S) \setminus L], \qquad \forall\, S, T \subseteq U,$$

and hence

$$\Delta_S v(T) = \sum_{L \subseteq S} (-1)^{s-l} v(T \cup L), \qquad \forall\, S \subseteq U,\ \forall\, T \subseteq U \setminus S.$$

It is also easy to show (see e.g. [11, §2]) that, in terms of dividends, we have

$$\Delta_S v(T) = \sum_{L \subseteq T} m(v, L \cup S), \qquad \forall\, S \subseteq U,\ \forall\, T \subseteq U \setminus S. \tag{1}$$

In particular,

$$m(v,S) = \Delta_S v(\varnothing) = \Delta_S v(S), \qquad \forall\, S \subseteq U.$$

Now, it follows from Eq. (1) and Proposition 2.1 that

$$\Delta_S v(T) = 0, \qquad \forall\, S \not\subseteq N,\ \forall\, T \subseteq U \setminus S. \tag{2}$$

2.4 k-monotonicity

Let $k \geq 2$ be an integer. A game $v \in \mathcal{G}^N$ is said to be *k-monotone* (see e.g. [3, §2]) if, for any k coalitions $A_1, A_2, \ldots, A_k \subseteq U$, we have

$$v\Big(\bigcup_{i=1}^{k} A_i\Big) \geq \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{j+1} v\Big(\bigcap_{i\in J} A_i\Big). \tag{3}$$

It is easy to verify [3, §2] that k-monotonicity, with any $k \geq 2$, implies l-monotonicity for all $l \in \{2, \ldots, k\}$. By extension, 1-monotonicity (which does not correspond to $k = 1$ in Eq. (3)) is defined as standard monotonicity, i.e.,

$$v(S) \leq v(T) \quad \text{whenever} \quad S \subseteq T \subseteq U.$$

Clearly, a game $v \in \mathcal{G}$ is 1-monotone if and only if $\Delta_i v(T) \geq 0$ for all $i \in U$ and all $T \subseteq U \setminus i$. For k-monotonicity ($k \geq 2$) we have the following result, which immediately follows from Eqs. (1)–(2) and [3, Proposition 4].

Proposition 2.2. *Let $k \geq 2$. A game $v \in \mathcal{G}$ is k-monotone if and only if, for all $S \subseteq U$ such that $2 \leq s \leq k$ and all $T \subseteq U \setminus S$, we have $\Delta_S v(T) \geq 0$.*

2.5 Unanimity games

The *unanimity game* for $T \subseteq N$, $T \neq \varnothing$, is defined as the game $u_T \in \mathcal{G}^N$ such that, for all $S \subseteq N$, $u_T(S) := 1$ if and only if $S \supseteq T$ and 0 otherwise. It is easy to check that T is a carrier of u_T and that its Möbius transform is given, for all $S \subseteq N$, by $m(u_T, S) = 1$ if and only if $S = T$ and 0 otherwise.

Following Dubey et al. [4, §1], for any $T \subseteq N$, we also consider the game $\hat{u}_T \in \mathcal{G}^N$, defined for all $S \subseteq N$, by $\hat{u}_T(S) := 1$ if and only if $S \supsetneq T$ and 0 otherwise. It can be easily proved that its Möbius transform is given by

$$m(\hat{u}_T, S) = \begin{cases} (-1)^{s-t+1}, & \text{if } S \supsetneq T, \\ 0, & \text{otherwise,} \end{cases} \qquad \forall\, S \subseteq N. \tag{4}$$

2.6 Permuted games

Following Shapley [30, §2], given a game $v \in \mathcal{G}^N$ and a permutation π on U (i.e., a one-to-one mapping from U onto itself), we denote by πv the game defined by

$$\pi v[\pi(S)] := v(S), \qquad \forall\, S \subseteq N,$$

where $\pi(S) := \{\pi(i) \mid i \in S\}$. Note that $\pi(N)$ is a carrier of πv.

2.7 Restricted and reduced games

Given a game $v \in \mathcal{G}^N$ and a coalition $A \subseteq N$, the *restriction* of v to A [13] is a game of $\mathcal{G}^A$ defined by

$$v^A(S) := v(S), \qquad \forall\, S \subseteq A.$$

This is equivalent to considering for v only coalitions containing players of A.

Given a coalition $B \subseteq N \setminus A$, the *restriction* of v to A *in the presence of* B [13] is a game of $\mathcal{G}^A$ defined by

$$v^A_{\cup B}(S) := v(S \cup B) - v(B), \qquad \forall\, S \subseteq A.$$

This is equivalent to considering for v only coalitions containing coalition B and some players of A. The subtraction of $v(B)$ is introduced only to satisfy the condition

$$v^A_{\cup B}(\varnothing) = 0.$$

Given a game $v \in \mathcal{G}^N$ and a coalition $T \subseteq N$, $T \neq \varnothing$, the *reduced game* with respect to T [13, 26], denoted $v_{[T]}$, is a game of $\mathcal{G}^{(N \setminus T) \cup [T]}$ where $[T]$ indicates a single hypothetical player, which is the representative (or macro player) of the players in T. It is defined by

$$\begin{aligned} &v_{[T]}(S) := v(S), \\ &v_{[T]}(S \cup [T]) := v(S \cup T), \end{aligned}$$

for all $S \subseteq N \setminus T$.

2.8 Dummy coalitions and partnerships

A coalition $S \subseteq U$ is said to be *dummy* in a game $v \in \mathcal{G}^N$ if $v(T \cup S) = v(T) + v(S)$ for all $T \subseteq U \setminus S$. In other words, the marginal contribution of a dummy coalition S to any coalition T not containing elements of S is simply its worth $v(S)$.

A coalition $S \subseteq U$ in a game $v \in \mathcal{G}^N$ is said to be *null* if it is a dummy coalition in v such that $v(S) = 0$.

A *dummy* (resp. *null*) player is a *dummy* (resp. *null*) one-membered coalition.

A coalition $P \subseteq U$, $P \neq \varnothing$, is said to be a *partnership* [17, §4] in a game $v \in \mathcal{G}^N$ if $v(S \cup T) = v(T)$ for all $S \subsetneq P$ and all $T \subseteq U \setminus P$. In other words, as long as all the members of a partnership P are not all in coalition, the presence of some of them only leaves unchanged the worth of any coalition not containing elements of P. In particular $v(S) = 0$ for all $S \subsetneq P$.

Notice that, thus defined, a partnership behaves like a single hypothetical player, that is, the game v and its reduced version $v_{[P]}$ can be considered as equivalent.

Now, a *dummy partnership* is simply a partnership $P \subseteq U$ that is dummy. Thus, a dummy partnership can be regarded as a single hypothetical dummy player. It is easy to verify that any coalition $P \subseteq U$ is a dummy partnership in the corresponding unanimity game u_P.

3 The concept of interaction index

3.1 Probabilistic and cardinal-probabilistic values

As mentioned in the introduction, interaction indices can be seen as extensions of values. In turn, a value can be seen as a function $\phi : \mathcal{G} \times U \to \mathbb{R}$ that assigns to every player $i \in U$ in a game $v \in \mathcal{G}$ his/her prospect $\phi(v, i)$ from playing the game. The exact form of a value depends on the axioms that are imposed on it. For instance, the well-known Shapley value can be defined as the sole value that satisfies the *linearity*, *dummy player*, *symmetry*, and *efficiency* axioms [31, Theorem 15].

Given a game $v \in \mathcal{G}^N$, the Shapley value of a player $i \in N$ is given by

$$\phi_{Sh}(v,i) := \sum_{T \subseteq N \setminus i} \frac{1}{n} \binom{n-1}{t}^{-1} \Delta_i v(T),$$

where $\Delta_i v(T) = v(T \cup i) - v(T)$ is the i-derivative of v at $T \not\ni i$. If $i \notin N$, we set $\phi_{Sh}(v,i) := 0$, which is in accordance with Eq. (2); see also [30, Lemma 1].

Another frequently encountered value is the Banzhaf value [2, 5]. The Banzhaf value of a player $i \in N$ in a game $v \in \mathcal{G}^N$ is defined by

$$\phi_B(v,i) := \sum_{T \subseteq N \setminus i} \frac{1}{2^{n-1}} \Delta_i v(T).$$

Here also, if $i \notin N$, we set $\phi_B(v,i) := 0$.

The Shapley and Banzhaf values are instances of *probabilistic values* [31] and, more precisely, of *cardinal-probabilistic values* also known as *semivalues* [4].

A *probabilistic value* ϕ_p of a player $i \in N$ in a game $v \in \mathcal{G}^N$ is a value of the form

$$\phi_p(v,i) := \sum_{T \subseteq N \setminus i} p_T^i(N)\, \Delta_i v(T),$$

where the family of coefficients $\{p_T^i(N)\}_{T \subseteq N \setminus i}$ forms a probability distribution on $2^{N \setminus i}$. Again, if $i \notin N$, we naturally set $\phi_p(v,i) := 0$.

Thus defined, $\phi_p(v,i)$ can be interpreted as the mathematical expectation on $2^{N \setminus i}$ of the marginal contribution $\Delta_i v(T) = v(T \cup i) - v(T)$ of player i to a coalition $T \subseteq N \setminus i$ with respect to the probability distribution $\{p_T^i(N)\}_{T \subseteq N \setminus i}$.

A *cardinal-probabilistic value* is a probabilistic value such that, additionally, for all $i \in N$, the coefficients $p_T^i(N)$ $(T \subseteq N \setminus i)$ depend only on the cardinalities of the coalitions i, T, and N, i.e., there exist n nonnegative numbers $\{p_t(n)\}_{t=0,\ldots,n-1}$ fulfilling

$$\sum_{t=0}^{n-1} \binom{n-1}{t} p_t(n) = 1$$

such that, for any $i \in N$ and any $T \subseteq N \setminus i$, we have $p_T^i(N) = p_t(n)$.

3.2 Intuitive presentation of interaction indices

As noticed by Grabisch and Roubens [13], the fact that in general, for a player $i \in N$ in a game $v \in \mathcal{G}^N$, $\phi(v,i)$ is not equal to the coefficient $v(i)$ shows that players in N have some interest in forming coalitions. For instance, consider another player $j \in N$ and assume that $v(i)$ and $v(j)$ are small whereas $v(ij)$ is large. Then, i and j have clearly a strong interest in joining together. Conversely, it may happen that $v(i)$ and $v(j)$ are large whereas $v(ij)$ is small, in which case i and j have no interest in joining together.

In order to intuitively approach the concept of interaction, consider two players i and j such that

$$v(ij) > v(i) + v(j).$$

Clearly, the above inequality seems to model a *positive* interaction or *complementary* effect between i and j. Similarly, the inequality

$$v(ij) < v(i) + v(j)$$

suggests considering that i and j interact in a *negative* or *redundant* way. Finally, if

$$v(ij) = v(i) + v(j),$$

it seems natural to consider that players i and j do not interact, i.e., that they have *independent* roles in the game.

A coefficient measuring the interaction between i and j should therefore depend on the difference

$$v(ij) - [v(i) + v(j)].$$

However, as discussed by Grabisch and Roubens [13], the intuitive concept of interaction requires a more elaborate definition. Clearly, one should not only compare $v(ij)$ and $v(i) + v(j)$ but also see what happens when i, j, and ij join coalitions. In other words, an index of interaction between i and j in the game $v \in \mathcal{G}^N$ should take into account all the coefficients of the form $v(T \cup i)$, $v(T \cup j)$, and $v(T \cup ij)$, with $T \subseteq N \setminus ij$.

Owen [27, §5] defined an interaction index between two players $ij \subseteq N$ in a game $v \in \mathcal{G}^N$ by

$$I(v, ij) := \sum_{T \subseteq N \setminus ij} \frac{1}{n-1} \binom{n-2}{t}^{-1} \Delta_{ij} v(T).$$

Notice that, for a coalition T not containing i and j, the expression

$$\Delta_{ij} v(T) = v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T)$$

can be regarded as the difference between the marginal contributions

$$\Delta_j v(T \cup i) = v(T \cup ij) - v(T \cup i) \quad \text{and} \quad \Delta_j v(T) = v(T \cup j) - v(T).$$

Following Grabisch et al. [11, §2], we shall call it the *marginal interaction between i and j in the presence of T*. Indeed, it seems natural to consider that if

$$\Delta_j v(T \cup i) > \Delta_j v(T) \qquad (\text{resp. } <)$$

then i and j interact positively (resp. negatively) in the presence of T since the presence of player i increases (resp. decreases) the marginal contribution of j to coalition T.

The interaction index proposed by Owen, which was actually rediscovered twenty years later by Murofushi and Soneda [25], can thus be regarded as a weighted average of the marginal interactions between i and j in the presence of T, all coalitions T not containing i and j being considered.

Grabisch [9] recently extended the above interaction index to coalitions containing more than two players. The *Shapley interaction index* [9] of a coalition $S \subseteq N$ in a game $v \in \mathcal{G}^N$ is defined by

$$I_{Sh}(v, S) := \sum_{T \subseteq N \setminus S} \frac{1}{n-s+1} \binom{n-s}{t}^{-1} \Delta_S v(T).$$

This index is an extension of the Shapley value in the sense that $I_{Sh}(v, i)$ and $\phi_{Sh}(v, i)$ coincide for all $i \in U$ and all $v \in \mathcal{G}$.

Two similar indices are due to Roubens [29] and Marichal and Roubens [23] and are known as the *Banzhaf interaction index* and the *chaining interaction index*, respectively. The former extends the Banzhaf value, while the latter extends the Shapley value.

The *Banzhaf interaction index* [29] and the *chaining interaction index* [23] of a coalition $S \subseteq N$ in a game $v \in \mathcal{G}^N$ are respectively defined by

$$\begin{aligned} I_B(v,S) &:= \sum_{T\subseteq N\setminus S} \frac{1}{2^{n-s}}\,\Delta_S v(T), \\ I_{ch}(v,S) &:= \sum_{T\subseteq N\setminus S} \frac{s}{s+t}\binom{n}{s+t}^{-1}\Delta_S v(T). \end{aligned}$$

If $S \nsubseteq N$, we naturally set $I_{Sh}(v,S) := 0$, $I_B(v,S) := 0$, and $I_{ch}(v,S) := 0$.

It was shown [11, 23] that, for any $v \in \mathcal{G}^N$, the restriction of $I_{Sh}(v,\cdot)$ (resp. $I_B(v,\cdot)$, $I_{ch}(v,\cdot)$) to 2^N (resp. 2^N, $2^N \setminus \{\varnothing\}$) is an equivalent representation of the restriction of v to 2^N (resp. 2^N, $2^N \setminus \{\varnothing\}$).

3.3 Probabilistic and cardinal-probabilistic interaction indices

By analogy with the works of Dubey et al. [4] and Weber [31] on values, Grabisch and Roubens [12, 14] defined the class of probabilistic interaction indices and the subclass of cardinal-probabilistic interaction indices.

A *probabilistic interaction index* of a coalition $S \subseteq N$ in a game $v \in \mathcal{G}^N$ is of the form

$$I_p(v,S) := \sum_{T\subseteq N\setminus S} p_T^S(N)\,\Delta_S v(T),$$

where, for any $S \subseteq N$, the family of coefficients $\{p_T^S(N)\}_{T\subseteq N\setminus S}$ forms a probability distribution on $2^{N\setminus S}$. Here again, if $S \nsubseteq N$, we naturally set $I_p(v,S) := 0$.

A *cardinal-probabilistic interaction index* is a probabilistic interaction index such that, additionally, for any $S \subseteq N$, the coefficients $p_T^S(N)$ $(T \subseteq N \setminus S)$ depend only on the cardinalities of the coalitions S, T, and N, i.e., for any $s \in \{0,\ldots,n\}$, there exists a family of nonnegative numbers $\{p_t^s(n)\}_{t=0,\ldots,n-s}$ fulfilling

$$\sum_{t=0}^{n-s}\binom{n-s}{t}p_t^s(n) = 1,$$

such that, for any $S \subseteq N$ and any $T \subseteq N \setminus S$, we have $p_T^S(N) = p_t^s(n)$.

The Shapley, Banzhaf, and chaining interaction indices defined above are clearly cardinal-probabilistic interaction indices. Furthermore, it is interesting to notice that the Möbius and co-Möbius transforms of a game $v \in \mathcal{G}^N$ are also cardinal-probabilistic interaction indices. The coefficients of $m(v,S)$ are defined by $p_t^s(n) = 1$ if $t = 0$ and 0 otherwise, while the coefficients of $m^*(v,S)$ are defined by $p_t^s(n) = 1$ if $t = n-s$ and 0 otherwise ; see [11, §2]. These indices will be called *internal* and *external* interaction indices, respectively. This terminology will be justified in Section 4.5.

3.4 Interpretation of probabilistic interaction indices

Similarly to probabilistic values, a probabilistic interaction index of a coalition $S \subseteq N$, $s \geq 1$, in a game $v \in \mathcal{G}^N$ can be regarded as the mathematical expectation on $2^{N\setminus S}$ of $\Delta_S v(T)$ with respect to the probability distribution $\{p_T^S(N)\}_{T\subseteq N\setminus S}$. The interpretation of $I_p(v,S)$ when $s = 1$ has already been discussed in Section 3.1. In this subsection we provide an interpretation of $I_p(v, S)$ when $s \geq 2$ based on an interpretation of $\Delta_S v(T)$ ($T \subseteq N \setminus S$) as the *marginal interaction among players in* S *in the presence of* T [11].

As discussed in [18], let us show that, for any coalition $S \subseteq N$, $s \geq 2$, the quantity $\Delta_S v(T)$ can be interpreted in the same way as $\Delta_{ij} v(T)$.

We have seen that, for two players i and j and a coalition $T \subseteq N \setminus ij$, $\Delta_{ij} v(T)$ corresponds to the difference between the marginal contributions $\Delta_i v(T \cup j)$ and $\Delta_i v(T)$ or equivalently, by symmetry, to the difference between $\Delta_j v(T \cup i)$ and $\Delta_j v(T)$, i.e.,

$$\begin{aligned} \Delta_{ij} v(T) &= \Delta_j v(T \cup i) - \Delta_j v(T) \\ &= \Delta_i v(T \cup j) - \Delta_i v(T). \end{aligned}$$

It seems then sensible to define the marginal non-interaction (in this case marginal independence) between i and j in the presence of T as the equality of the marginal contributions in the equations above. Similarly, marginal positive (resp. negative) interaction between i and j in the presence of T occurs when, for example, $\Delta_i v(T \cup j) > \Delta_i v(T)$ (resp. $<$).

For three players $ijk \subseteq N$ and a coalition $T \subseteq N \setminus ijk$, it is easy to verify that $\Delta_{ijk} v(T)$ can be rewritten as

$$\begin{aligned} \Delta_{ijk} v(T) &= \Delta_{ij} v(T \cup k) - \Delta_{ij} v(T), \\ &= \Delta_{ik} v(T \cup j) - \Delta_{ik} v(T), \\ &= \Delta_{jk} v(T \cup i) - \Delta_{jk} v(T). \end{aligned}$$

Now assume for instance that $\Delta_{jk} v(T \cup i) = \Delta_{jk} v(T)$, in others words, that the presence of player i does not affect the marginal interaction between players j and k. From the equations above, this is equivalent to

$$\begin{cases} \Delta_{ij} v(T \cup k) = \Delta_{ij} v(T), \\ \Delta_{ik} v(T \cup j) = \Delta_{ik} v(T), \\ \Delta_{ijk} v(T) = 0. \end{cases}$$

Thus, we see that if the presence of one of the three players does not affect the marginal interaction between the two others, then, by symmetry, the marginal interaction between any pair of players is not affected by the presence of the remaining player. In this case, it seems natural to consider that the three players *do not interact simultaneously* in the presence of T. Similarly, when, for example, $\Delta_{ij} v(T \cup k) > \Delta_{ij} v(T)$ (resp. $<$), we shall consider that ijk interact *positively* (resp. *negatively*) in the presence of T.

In the general case, it is easy to verify from the definition of the discrete derivative that, for any $S \subseteq N$, $s \geq 1$, any $i \in N \setminus S$, and any $T \subseteq N \setminus (S \cup i)$,

$$\Delta_{S \cup i} v(T) = \Delta_S v(T \cup i) - \Delta_S v(T).$$

Using the previous result and pursuing the same reasoning as above, for any $S \subseteq N$, $s \geq 2$, any $T \subseteq N \setminus S$, when $\Delta_S v(T) > 0$ (resp. < 0), it seems sensible to consider that there exists a positive (resp. negative) *simultaneous* interaction among the players in S in the presence of T. However, $\Delta_S v(T) = 0$ should obviously not be interpreted as an absence of interactions among players in S in the presence of T but as an absence of simultaneous interaction among the players in S in the presence of T.

Coming back to the interaction indices, we thus have the following interpretation : for a coalition $S \subseteq N$, $s \geq 2$, the inequality $I_p(v, S) > 0$ (resp. < 0) can be interpreted as a positive (resp. negative) *simultaneous* interaction among all the players in S *on average*. However, $I_p(v,S)=0$ should not be interpreted as an *absence of interaction* among players in S *on average* but as an absence of *simultaneous* interaction among all the players in S *on average*.

Remark. From the above interpretation and Proposition 2.2, it is important to notice that for a k-monotone game v ($k \geq 2$), the marginal interaction among players of a coalition S, $2 \leq s \leq k$, is necessarily positive, and thus that $I_p(v,S) \geq 0$. This observation is at the root of the *k-monotonicity* axiom that we shall state in the next section.

3.5 Existing axiomatic characterizations

Setting $\mathcal{U} := 2^U \setminus \{\varnothing\}$, an interaction index can be regarded as a function $I : \mathcal{G} \times \mathcal{U} \to \mathbb{R}$ such that, for any $v \in \mathcal{G}$ and any $i \in U$, $I(v,i)$ is the value of player i in the game v, and for any $S \subseteq U$ such that $s \geq 2$, $I(v,S)$ is a measure of the (simultaneous) interaction among players in S in the game v.

Grabisch and Roubens recently proposed an axiomatic characterization of the Shapley and the Banzhaf interaction indices [13, §3]. We present their results hereafter, with the only difference that here we force the second argument of I to be nonempty.

The following axioms have been considered by Grabisch and Roubens :

- *Linearity axiom (L)* : I is a linear function with respect to its first argument.
- *Dummy player axiom (D)* : If $i \in U$ is a dummy player in a game $v \in \mathcal{G}$, then
  (i) $I(v,i) = v(i)$,
  (ii) for all $S \subseteq U \setminus i$, $S \neq \varnothing$, we have $I(v, S \cup i) = 0$.
- *Symmetry axiom (S)* : For any permutation π on U, and any $v \in \mathcal{G}$, we have $I(v,S) = I(\pi v, \pi(S))$ for all $S \subseteq U, S \neq \varnothing$.
- *Recursive axiom (R)* : For any finite $N \subseteq U$, $n \geq 2$, and any $v \in \mathcal{G}^N$, we have

$$I(v,S) = I(v^{N\setminus j}_{\cup j}, S \setminus j) - I(v^{N\setminus j}, S \setminus j), \qquad \forall\, S \subseteq N, s \geq 2, \forall\, j \in S.$$

- *Efficiency (E)* : For any finite $N \subseteq U$, $n \geq 1$, and any $v \in \mathcal{G}^N$, we have

$$\sum_{i \in N} I(v,i) = v(N).$$

- *2-efficiency (2-E)* : For any finite $N \subseteq U$, $n \geq 2$, and any $v \in \mathcal{G}^N$, we have

$$I(v,i) + I(v,j) = I(v_{[ij]}, [ij]), \qquad \forall\, ij \subseteq N.$$

Axiom (L) implies that values and interaction indices are linear combinations of the basic information related to the game : the worth of each coalition of players. Axiom (D) states that a dummy player has a value equal to its worth and that he does not interact with any outsider coalition (see [13]). Axiom (S) indicates that the names of the players play no role in determining the values and interaction indices. Axiom (R) postulates that interaction at level s is linked to the difference of interactions defined at level $(s-1)$

(more details can be found in [13, §3]). Axiom (E), initially considered by Shapley [30], is dedicated to values and ensures the players in a game $v \in \mathcal{G}^N$ share the total amount $v(N)$ among them in terms of their respective values. Axiom (2-E), initially considered by Nowak [26], expresses the fact that the sum of the values of two players should be equal to the value of these players considered as twins in the corresponding reduced game.

The following theorem was shown by Grabisch and Roubens [13, §3].

Theorem 3.1. *Let I be a function from $\mathcal{G} \times \mathcal{U}$ to $\mathbb{R}$.*

(i) I satisfies axiom (L) if and only if, for any finite $N \subseteq U$, $n \geq 1$, and any $S \subseteq U$, $s \geq 1$, there exists a family of real constants $\{\alpha_T^S(N)\}_{T \subseteq N}$ such that, for any $v \in \mathcal{G}^N$, we have

$$I(v, S) = \sum_{T \subseteq N} \alpha_T^S(N) v(T).$$

(ii) I satisfies axioms (L) and (D), if and only if, for any finite $N \subseteq U$, $n \geq 1$, and any $S \subseteq N$, $s \geq 1$, there exists a family of constants $\{p_T^S(N)\}_{T \subseteq N \setminus S}$ such that, for any $v \in \mathcal{G}^N$, we have

$$I(v, S) = \sum_{T \subseteq N \setminus S} p_T^S(N)\, \Delta_S v(T),$$

and for any $S \nsubseteq N$ and any $v \in \mathcal{G}^N$, we have $I(v, S) = 0$.

(iii) I satisfies axioms (L), (D), and (S), if and only if, for any finite $N \subseteq U$, $n \geq 1$, and any $S \subseteq N$, $s \geq 1$, there exists a family of constants $\{p_t^s(n)\}_{t=0,\ldots,n-s}$ such that, for any $v \in \mathcal{G}^N$, we have

$$I(v, S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \Delta_S v(T),$$

and for any $S \nsubseteq N$ and any $v \in \mathcal{G}^N$, we have $I(v, S) = 0$.

(iv) I satisfies axioms (L), (D), (S), (R), and (E) if and only if $I = I_{Sh}$.

(v) I satisfies axioms (L), (D), (S), (R), and (2-E) if and only if $I = I_B$.

Parts (iv) and (v) of Theorem 3.1 thus provide axiomatic characterizations of the Shapley and Banzhaf interaction indices respectively. It is noteworthy that, since axiom (R) determines uniquely $I(v, S)$, $s \geq 2$, from the values $I(v, i)$, $i \in N$, the axioms (L), (D), and (S) are somewhat redundant in parts (iv) and (v) and are needed only for values $I(v, i)$, $i \in N$.

4 New axiomatic characterizations

4.1 Characterizations of probabilistic and cardinal-probabilistic interaction indices

We shall now axiomatize the class of probabilistic interaction indices and that of cardinal-probabilistic interaction indices. The following axioms are first considered :

- *Additivity axiom (A)* : I is an additive function with respect to its first argument.

- *Monotonicity axiom (M)* : For any monotone game $v \in \mathcal{G}$, we have $I(v,i) \geq 0$ for all $i \in U$.
- *k-monotonicity axiom (M^k)* : For any $k \geq 2$ and any k-monotone game $v \in \mathcal{G}$, we have $I(v,S) \geq 0$ for all coalition $S \subseteq U$ such that $2 \leq s \leq k$.

Axiom (A) indicates that interaction indices should be decomposable additively whenever games are decomposable additively. Axiom (M), used by Weber [31, §4] to characterize probabilistic values, concerns only the value part of I and states that, since in a monotone game the marginal contributions of a player are necessarily positive, its value should be positive. Axiom (M^k) can be seen as a generalization of axiom (M) and concerns the interaction part of I. As discussed in Section 3.4, in a k-monotone game ($k \geq 2$), it seems sensible to consider that there are necessarily complementarity effects among players of coalitions containing between 2 and k players. Axiom (M^k) then simply states that these effects should be represented as positive interactions.

We now present the following important lemma.

Lemma 4.1. *If $I : \mathcal{G} \times \mathcal{U} \to \mathbb{R}$ satisfies axioms (A), (M), and (M^k), then it also satisfies axiom (L).*

Proof. We only need to show that, for any $v \in \mathcal{G}$, any $\lambda \in \mathbb{R}$, and any $S \subseteq U$, $S \neq \varnothing$, we have $I(\lambda v, S) = \lambda I(v,S)$. Since the family of unanimity games $\{u_T\}_{T \subseteq U, T \neq \varnothing}$ is a basis of $\mathcal{G}$, it suffices to prove the equality when v is an arbitrary unanimity game.

Let $S, T \subseteq U$, with $S, T \neq \varnothing$, and let $\lambda \in \mathbb{R}$. Consider sequences r_m and s_m of rational numbers converging to λ and such that $s_m \leq \lambda \leq r_m$ for all m. By (A), we have

$$I(r_m u_T, S) = r_m I(u_T, S) \quad \text{and} \quad I(s_m u_T, S) = s_m I(u_T, S).$$

Since u_T is k-monotone for every $k \geq 1$, by (M) and (M^k) it follows that the following real sequences

$$I((r_m - \lambda)u_T, S) \quad \text{and} \quad I((\lambda - s_m)u_T, S)$$

are nonnegative. On the other hand, by (A), the first one converges to

$$l := \lambda I(u_T, S) - I(\lambda u_T, S) \geq 0$$

and the second one converges to $-l \geq 0$. It follows that $l = 0$. □

We also consider the following fundamental axiom :

- *Dummy partnership axiom (DP)* : For any $v \in \mathcal{G}$, if $P \neq \varnothing$ is a dummy partnership in v, then
  (i) $I(v,P) = v(P)$,
  (ii) for all $S \subseteq U \setminus P$, $S \neq \varnothing$, we have $I(v, S \cup P) = 0$.

Axiom (DP) is a natural generalization of axiom (D). As discussed in Weber [31, §3], the first part of axiom (D) is based on the following intuition : since the marginal contribution of a dummy player to any coalition not containing it is simply its worth, its value should be its worth as well. Similarly, the first part of axiom (DP) states that the interaction index of a dummy partnership P in a game v should be its worth since the marginal interaction among the players in P in the presence of any coalition T not containing elements of P is its worth, that is, $\Delta_P v(T) = v(P)$.

The second part of axiom (DP) is a natural extension of the second part of axiom (D) and says that there should be no simultaneous interaction among players of coalitions containing dummy partnerships.

We now provide axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. The proofs are given in Appendix A.

Theorem 4.1. *A function $I : \mathcal{G} \times \mathcal{U} \to \mathbb{R}$ satisfies axioms (A), (M), (M^k), and (DP) if and only if, for any finite $N \subseteq U$, $n \geq 1$, and any $S \subseteq N$, $s \geq 1$, there exists a family of nonnegative constants $\{p_T^S(N)\}_{T \subseteq N \setminus S}$ satisfying $\sum_{T \subseteq N \setminus S} p_T^S(N) = 1$ such that, for any $v \in \mathcal{G}^N$, we have*

$$I(v, S) = \sum_{T \subseteq N \setminus S} p_T^S(N)\, \Delta_S v(T),$$

and for any $S \nsubseteq N$ and any $v \in \mathcal{G}^N$, we have $I(v, S) = 0$.

Theorem 4.2. *A function $I : \mathcal{G} \times \mathcal{U} \to \mathbb{R}$ satisfies axioms (A), (M), (M^k), (DP), and (S) if and only if, for any finite $N \subseteq U$, $n \geq 1$, and any $S \subseteq N$, $s \geq 1$, there exists a family of nonnegative constants $\{p_t^s(n)\}_{t=0,\ldots,n-s}$ satisfying $\sum_{t=0}^{n-s} \binom{n-s}{t} p_t^s(n) = 1$, such that, for any $v \in \mathcal{G}^N$, we have*

$$I(v, S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \Delta_S v(T),$$

and for any $S \nsubseteq N$ and any $v \in \mathcal{G}^N$, we have $I(v, S) = 0$.

4.2 Representation theorems for cardinal-probabilistic interaction indices

We now present a generalization of the representation theorem given by Dubey et al. [4, Theorem 1(a)] for semivalues. The proof, given in Appendix B, is mainly based on the so-called power moment problem on $[0, 1]$, also known as the Hausdorff's moment problem (see e.g. [1, Chapter 2, §6.4]).

Theorem 4.3. *If I_p is a cardinal-probabilistic interaction index given in the form of Theorem 4.2, then, for any finite $N \subseteq U$, $n \geq 1$, and any $s \in \{1, \ldots, n\}$, there exists a uniquely determined cumulative density function (CDF) F_s on $[0, 1]$ such that*

$$p_t^s(n) = \int_0^1 x^t (1-x)^{n-s-t} dF_s(x), \qquad \forall t \in \{0, \ldots, n-s\},$$

where the integral is to be understood in the sense of Riemann-Stieltjes.

Thus, with each cardinal-probabilistic interaction index I_p is associated a unique denumerable family of CDFs $\mathcal{F} := \{F_s \mid s \geq 1\}$ and we will write $I_{\mathcal{F}} := I_p$.

Remark. It is easy to see that the CDFs corresponding to the Shapley, Banzhaf, and chaining interaction indices as well as for the Möbius and co-Möbius transforms are given in the following table, where, for any $E \subseteq [0, 1]$, $\mathbf{1}_E$ denotes the characteristic function of E.

	I_{Sh}	I_B	I_{ch}	m	m^*
$F_s(x) =$	x	$\mathbf{1}_{[1/2,1]}$	$x^s\,\mathbf{1}_{]0,1]}$	$\mathbf{1}_{]0,1]}$	$\mathbf{1}_{\{1\}}$

The following result will be useful as we go on. For the proof, see first remark following Proposition 2.1 in [11].

Proposition 4.1. *For any $N \subseteq U$ finite, $n \geq 1$, any cardinal-probabilistic interaction index can be rewritten in terms of the Möbius transform as*

$$I_p(v, S) = \sum_{T \supseteq S} q_t^s(n)\, m(v, T), \qquad \forall\, v \in \mathcal{G}^N, \forall\, S \subseteq U,\ s \geq 1, \tag{5}$$

where, for all $s \in \{1, \ldots, n\}$ and all $t \in \{s, \ldots, n\}$,

$$q_t^s(n) = \sum_{k=0}^{n-t} \binom{n-t}{k} p_{k+t-s}^s(n).$$

By combining Theorem 4.3 and Proposition 4.1, it is easy to see that

$$I_{\mathcal{F}}(v, S) = \sum_{T \supseteq S} \Big[\int_0^1 x^{t-s}\, dF_s(x)\Big]\, m(v, T), \qquad \forall\, v \in \mathcal{G}, \forall\, S \subseteq U,\ s \geq 1, \tag{6}$$

which shows that the coefficients $q_t^s(n)$ of Proposition 4.1 do not depend on n. More precisely, we have

$$q_t^s(n) = q_t^s = p_{t-s}^s(t) \qquad \forall\, t \geq s \geq 1. \tag{7}$$

The coefficients q_t^s for the particular interaction indices introduced thus far are given in the following table.

	I_{Sh}	I_B	I_{ch}	m	m^*
$q_t^s =$	$\dfrac{1}{t-s+1}$	$\dfrac{1}{2^{t-s}}$	$\dfrac{s}{t}$	$\begin{cases} 1, & \text{if } t = s \\ 0, & \text{else} \end{cases}$	1

It is noteworthy that, since $q_s^s = p_0^s(s) = 1$ for all $s \geq 1$, the coefficient matrix of the linear system (5) is invertible and hence, that the restriction of $I_p(v, \cdot\,)$ to $2^N \setminus \{\varnothing\}$ is an equivalent representation of the restriction of v to $2^N \setminus \{\varnothing\}$.

Now, let $N \subseteq U$ finite, $n \geq 1$, and $v \in \mathcal{G}^N$. Consider the so-called *Owen multilinear extension* $g : [0,1]^n \to \mathbb{R}$ of v [27], namely

$$g(x) := \sum_{T \subseteq N} v(T) \prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i), \qquad \forall\, x \in [0,1]^n$$

and, for any $S \subseteq N$, let $\Delta_S g$ denote its classical S-derivative, that is,

$$\Delta_S g(x) := \frac{\partial^s g(x)}{\partial x_{i_1} \cdots \partial x_{i_s}}\,, \quad \text{with } S = \{i_1, \ldots, i_s\}.$$

It has been proved [11, Eq. (26)] (see Owen [27] for the case $s = 1$) that

$$I_{Sh}(v, S) = \int_0^1 (\Delta_S g)(x, \ldots, x)\, dx \qquad \forall\, S \subseteq N,$$

which means that the Shapley interaction index related to S can be obtained by integrating the S-derivative of g along the main diagonal of the unit hypercube.

By combining Eq. (6) and [11, Eq. (22)], we can immediately generalize this latter formula to any cardinal-probabilistic interaction index, thus also generalizing [11, Eqs. (23)–(26)]. The result can be stated as follows.

Theorem 4.4. *Let $N \subseteq U$ finite, $n \geq 1$, and $v \in \mathcal{G}^N$. For any cardinal-probabilistic interaction index $I_{\mathcal{F}}$, associated with the family of CDFs $\mathcal{F} := \{F_s \mid s \geq 1\}$, we have*

$$I_{\mathcal{F}}(v,S) = \int_0^1 (\Delta_S g)(x,\ldots,x)\, dF_s(x) \qquad \forall\, S \subseteq N,$$

where $g : [0,1]^n \to \mathbb{R}$ is the multilinear extension of v.

We can observe that Theorem 4.4 leads to generalizations of formulas presented in [11] and [23]. Indeed, in addition to Eq. (6), for any $v \in \mathcal{G}$ and any $S \subseteq U$, $s \geq 1$, we have the following formulas (cf. [11, Eqs. (32) and (40)])

$$\begin{aligned}
I_{\mathcal{F}}(v,S) &= \sum_{T \supseteq S} \Big[\int_0^1 (x-1)^{t-s}\, dF_s(x)\Big]\, m^*(v,T) \\
&= \sum_{T \supseteq S} \Big[\int_0^1 (x-1/2)^{t-s}\, dF_s(x)\Big]\, I_B(v,T) \\
&= \sum_{T \supseteq S} \Big[\int_0^1 B_{t-s}(x)\, dF_s(x)\Big]\, I_{Sh}(v,T) \\
&= \sum_{T \supseteq S} \Big[\int_0^1 (x-1)^{t-s-1}(x-s/t)\, dF_s(x)\Big]\, I_{ch}(v,T),
\end{aligned}$$

where, for any integer $k \geq 0$, $B_k(x)$ represents the kth Bernoulli polynomial.

We shall now proceed with the characterizations of the Shapley, Banzhaf, and chaining interaction indices as well as the Möbius and co-Möbius transforms, which are instances of cardinal-probabilistic interaction indices. As mentioned in the introduction, these characterizations have the advantage of being free of the recursive axiom.

4.3 Characterizations of the Shapley and Banzhaf interaction indices by means of the reduced-partnership-consistency axiom

The following axiom is first additionally considered :

- *Reduced-partnership-consistency axiom (RPC)* : If P is a partnership in a game $v \in \mathcal{G}$ then
$$I(v,P) = I(v_{[P]},[P]).$$

Recall that a partnership can be considered as behaving as a single hypothetical player. Furthermore, it is easy to verify that the marginal interaction among the players of a partnership P in a game $v \in \mathcal{G}^N$ in the presence of a coalition $T \subseteq N \setminus P$ is equal to the marginal contribution of P to coalition T, i.e.,

$$\Delta_P v(T) = v(T \cup P) - v(T).$$

In other words, when we measure the interaction among the players of a partnership, it is as if we were measuring the value of a hypothetical player. Axiom (RPC) then simply states that the interaction among players of a partnership P in a game v should be regarded as the value of the reduced partnership $[P]$ in the corresponding reduced game $v_{[P]}$.

We then have the following interesting result.

Proposition 4.2. *A function $I : \mathcal{G} \times \mathcal{U} \to \mathbb{R}$ that satisfies axioms (L), (D) and (RPC) also satisfies axiom (DP).*

Proof. Let $v \in \mathcal{G}$ and let $P \neq \varnothing$ be a dummy partnership in v. By (RPC) and (D), we immediately have

$$I(v, P) = I(v_{[P]}, [P]) = v_{[P]}([P]) = v(P).$$

Let us now show that the second part of axiom (DP) is also true. Let $S \subseteq U \setminus P$, $S \neq \varnothing$, and let N be a carrier of v. By Theorem 3.1(ii), we have $I(v, S \cup P) = 0$ if $S \cup P \nsubseteq N$ and

$$I(v, S \cup P) = \sum_{L \subseteq N \setminus (S \cup P)} p_T^{S \cup P}(N)\, \Delta_{S \cup P} v(T)$$

otherwise. However, since P is a dummy partnership, we have, for any $T \subseteq N \setminus (S \cup P)$,

$$\begin{aligned}
\Delta_{S \cup P} v(T) &= \sum_{L \subseteq S} \sum_{K \subseteq P} (-1)^{s+p-l-k} v(T \cup L \cup K) \\
&= \sum_{L \subseteq S} \Big[\sum_{K \subseteq P} (-1)^{s+p-l-k} v(T \cup L) + (-1)^{s-l} v(P) \Big] \\
&= 0,
\end{aligned}$$

and hence $I(v, S \cup P) = 0$. □

We now state axiomatic characterizations of the Shapley and Banzhaf interaction indices. The proofs of the theorems are given in Appendix C.

Theorem 4.5. *The Shapley interaction index is the only cardinal-probabilistic interaction index additionally satisfying axioms (E) and (RPC). As a consequence, the Shapley interaction index is the only interaction index satisfying axioms (A), (M), (M^k), (D or DP), (S), (E), and (RPC).*

Theorem 4.6. *The Banzhaf interaction index is the only cardinal-probabilistic interaction index additionally satisfying axioms (2-E) and (RPC). As a consequence, the Banzhaf interaction index is the only interaction index satisfying axioms (A), (M), (M^k), (D or DP), (S), (2-E), and (RPC).*

The following interesting result can be used to obtain additional characterizations of the two interaction indices under consideration. It is a direct consequence of Lemma C.3 (see Appendix C) and [13, Proposition 8].

Proposition 4.3. *Under axioms (L), (DP), and (S), axioms (R) and (RPC) are equivalent.*

4.4 Characterizations of the Banzhaf and chaining interaction indices by means of the partnership-allocation axiom

We consider the following additional axiom :

- *Partnership-allocation axiom (PA)* : For any partnership P in $v \in \mathcal{G}$ there exists $\alpha_{|P|} \in \mathbb{R}$ such that

$$I(v, P) = \alpha_{|P|}\, I(v, i), \qquad \forall\, i \in P. \tag{8}$$

Let I_p be a cardinal-probabilistic interaction index, P be a partnership in a game $v \in \mathcal{G}^N$, and i be a member of P. Axiom (PA) is based on the following intuitive reasoning.

1. It is easy to verify that $I_p(v, P)$ is a weighted arithmetic mean of the marginal contributions $v(T \cup P) - v(T)$ ($T \subseteq N \setminus P$) and that $I_p(v, i)$ is a weighted sum of these same marginal contributions. In other words, both $I_p(v, P)$ and $I_p(v, i)$ can be considered as measuring the value in the game v of the hypothetical macro player corresponding to P.

2. Let α be a real number such that $I_p(v, P) = \alpha\, I_p(v, i)$. Notice this equality still holds if i is replaced with any other player $j \in P$, since all players in a partnership play symmetric roles. Hence, the coefficient α depends only on v and P and can then be seen as determining the way $I_p(v, P)$ is calculated from the value of any of the players of the partnership, quantity that contains all the "relevant information" as discussed in Point 1.

3. It could then be required that the way the value of P is determined from the value of any player of the partnership does not depend on the underlying game. Therefore, it depends only on $|P|$, which justifies axiom (PA).

For cardinal-probabilistic interaction indices I_p, the expression of $\alpha_{|P|}$ in Eq. (8) can be easily obtained. We simply have

$$1 = \alpha_{|P|} I_p(u_P, i), \qquad \forall\, i \in P,$$

since coalition P is always a dummy partnership in the unanimity game u_P.

We now state another characterization of the Banzhaf interaction index and a characterization of the chaining interaction index. The proofs of the theorems are given in Appendix D.

Theorem 4.7. *The Banzhaf interaction index is the only cardinal-probabilistic interaction index additionally satisfying axioms (2-E) and (PA). As a consequence, the Banzhaf interaction index is the only interaction index satisfying axioms (A), (M), (M^k), (DP), (S), (2-E), and (PA).*

Theorem 4.8. *The chaining interaction index is the only cardinal-probabilistic interaction index additionally satisfying axioms (E) and (PA). As a consequence, the chaining interaction index is the only interaction index satisfying axioms (A), (M), (M^k), (DP), (S), (E), and (PA).*

4.5 Characterizations of the internal and external interaction indices

Finally, we consider two last axioms in order to characterize the internal and external interaction indices (i.e., the Möbius and the co-Möbius transforms) :

- *Internal interaction axiom (II)* : For any $S \subseteq U$, $s \geq 1$, and any $v \in \mathcal{G}$, we have $I(v, S) = I(v^S, S)$.

- *External interaction axiom (EI)* : For any $N \subseteq U$ finite, $n \geq 1$, any $S \subseteq N$, $s \geq 1$, and any $v \in \mathcal{G}^N$, we have $I(v, S) = I(v^S_{\cup N \setminus S}, S)$.

These two axioms are based on the idea that, for any $S \subseteq N$, $s \geq 1$, the quantity $m(v, S) = \Delta_S v(\varnothing)$ (resp. $m^*(v, S) = \Delta_S v(N \setminus S)$) is exactly the marginal interaction among players in S in the presence of no (resp. all) outside players. Thus, axiom (II) simply states that the interaction index among players of a nonempty coalition S in a game v should be independent of players that are outside S. At the opposite, axiom (EI) states the interaction index among players of a nonempty coalition S in a game v should be measured in the presence of all outside players. Notice that, by replacing $N \setminus S$ with any $T \subseteq N \setminus S$ in axiom (EI), one could characterize $\Delta_S v(T)$ as a particular interaction index.

We can then state the following two characterizations. The proofs are given in Appendix E.

Theorem 4.9. *The internal interaction index is the only cardinal-probabilistic interaction index additionally satisfying axiom (II). As a consequence, the Möbius transform is the only interaction index satisfying axioms (A), (M), (M^k), (DP), (S), and (II).*

Theorem 4.10. *The external interaction index is the only cardinal-probabilistic interaction index additionally satisfying axiom (EI). As a consequence, the co-Möbius transform is the only interaction index satisfying axioms (A), (M), (M^k), (DP), (S), and (EI).*

Remark. Eq. (5) shows that any cardinal-probabilistic interaction index I_p is a weighted sum over $T \supseteq S$ of internal interaction indices $m(v, T)$. In this sum, each coefficient q_t^s represents the extent to which $m(v, T)$ contributes in the computation of the index I_p. For example, for the chaining and Banzhaf interaction indices, these coefficients are respectively given by

$$\frac{s}{t} = \frac{|S|}{|T|} \quad \text{and} \quad \frac{1}{2^{t-s}} = \frac{|2^S|}{|2^T|},$$

which shows that $m(v, T)$ is weighted by the contribution of S as a subset of T, where we reason on the elements in the first case and on the subsets in the second case. Similarly, for the Shapley and Banzhaf interaction indices, the coefficients are respectively given by

$$\frac{1}{t-s+1} = \frac{|[S]|}{|[S] \cup (T \setminus S)|} \quad \text{and} \quad \frac{1}{2^{t-s}} = \frac{|2^{[S]}|}{|2^{[S]\cup(T\setminus S)}|},$$

again showing that $m(v, T)$ is weighted by the contribution of S in T except that, this time, S is regarded as a single representative $[S]$.

5 Conclusion

Axiomatic characterizations of the broad class of probabilistic interaction indices and of the narrower subclass of cardinal-probabilistic interaction indices have been proposed. The presented characterizations are based on natural generalizations of the monotonicity and dummy player axioms, namely, the k-monotonicity and the dummy partnership axioms. Then, by further imposing classical axioms such as efficiency, 2-efficiency, and additional axioms based on the concept of partnership, we have characterized the Shapley, Banzhaf, and chaining interaction indices, which are the three best-known instances of cardinal-probabilistic interaction indices.

A Proofs of Theorems 4.1 and 4.2

Let us first prove Theorem 4.1.

Proof. (Necessity) From Lemma 4.1, I satisfies axiom (L). From Theorem 3.1(ii), we know that for any finite $N \subseteq U$, $n \geq 1$, and any $S \subseteq N$, $s \geq 1$, there exists a family of real constants $\{p_T^S(N)\}_{T \subseteq N \setminus S}$ such that, for any $v \in \mathcal{G}^N$, we have

$$I(v,S) = \sum_{T \subseteq N \setminus S} p_T^S(N)\, \Delta_S v(T),$$

and, for any $S \nsubseteq N$, $s \geq 1$, and any $v \in \mathcal{G}^N$, we have $I(v,S) = 0$.

The case $S \nsubseteq N$ being immediate, consider $S \subseteq N$, $s \geq 1$, and let us first show that

$$\sum_{T \subseteq N \setminus S} p_T^S(N) = 1. \tag{9}$$

For $v = u_S \in \mathcal{G}^N$, we have

$$I(u_S, S) = \sum_{T \subseteq N \setminus S} p_T^S(N)\, \Delta_S u_S(T) = \sum_{T \subseteq N \setminus S} p_T^S(N)$$

since $\Delta_S u_S(T) = \sum_{L \subseteq T} m(u_S, L \cup S) = 1$ (cf. Eq. (1) and Section 2.5). On the other hand, from axiom (DP), we have that $I(u_S, S) = u_S(S) = 1$ since S is a dummy partnership in the game u_S. Hence Eq. (9) is proved.

Let us now show that $p_T^S(N) \geq 0$ for all $T \subseteq N \setminus S$. To do so, we also fix T in $N \setminus S$ and we consider the game v_T^S of $\mathcal{G}^N$ defined through its dividends by

$$m(v_T^S, L) = \begin{cases} 0, & \text{if } L = \varnothing, \\ 0, & \text{if } L \supseteq S \text{ and } L \nsupseteq S \cup T \ (T \neq \varnothing), \\ (-1)^{l-s-t}, & \text{if } L \supseteq S \cup T, \\ 1, & \text{otherwise.} \end{cases}$$

Then, for any $C \subseteq N$ such that $1 \leq c \leq s$ and for any $K \subseteq N \setminus C$, we have

$$\Delta_C v_T^S(K) \geq 0.$$

Indeed, we have

$$\Delta_C v_T^S(K) = \sum_{L \subseteq K} m(v_T^S, L \cup C) = \sum_{\substack{L \subseteq K \\ L \cup C \supseteq S \cup T}} (-1)^{l+c-s-t} + \sum_{\substack{L \subseteq K \\ L \cup C \nsupseteq S}} 1,$$

where

$$\sum_{\substack{L \subseteq K \\ L \cup C \supseteq S \cup T}} (-1)^{l+c-s-t} = \begin{cases} (-1)^{c+k-s-t}, & \text{if } K = (S \cup T) \setminus C, \\ 0, & \text{else,} \end{cases}$$

and, if $K = (S \cup T) \setminus C$, we have

$$\sum_{\substack{L \subseteq K \\ L \cup C \nsupseteq S}} 1 = 0$$

only when $C \supseteq S$, that is, $C = S$ (since $1 \leq c \leq s$), in which case $K = T$ and $(-1)^{c+k-s-t} = 1$.

It is now clear that v_T^S is s-monotone (cf. Proposition 2.2). Finally, by axioms (M) and (M^k), we have

$$0 \leq I(v_T^S, S) = \sum_{K \subseteq N \setminus S} p_K^S(N)\, \Delta_S v_T^S(K) = p_T^S(N).$$

(Sufficiency) Straightforward. □

The proof of Theorem 4.2 follows from Theorem 4.1 and Theorem 3.1(iii).

B Proof of Theorem 4.3

We proceed nearly as in [4, Theorem 1(a)].

Proof. Let $N \subseteq U$ finite, with $n \geq 1$, let $T \subseteq N$, and consider the simple game $\hat{u}_T \in \mathcal{G}^N$. Then, from Eqs. (1) and (4), it follows that, for any $S \subseteq N \setminus T$, with $s \geq 1$,

$$\Delta_S\, \hat{u}_T(K) = \begin{cases} (-1)^{s+1}, & \text{if } K = T, \\ 0, & \text{otherwise,} \end{cases} \qquad \forall\, K \subseteq N \setminus S. \tag{10}$$

We know from Theorem 4.2 that, for each $K \subseteq N$, $k \geq 1$, there exists a family of nonnegative numbers $\{p_l^k(n)\}_{l=0,\ldots,n-k}$ such that, for any $v \in \mathcal{G}^N$,

$$I_p(v, K) = \sum_{L \subseteq N \setminus K} p_l^k(n)\, \Delta_K v(L).$$

Let $S \subseteq N \setminus T$, $s \geq 1$. For the game $\hat{u}_T$, it is easy to verify from Eq. (10) that

$$I_p(\hat{u}_T, S) = (-1)^{s+1} p_t^s(n). \tag{11}$$

Let $i \in U \setminus N$. N being a carrier of $\hat{u}_T$, $N \cup i$ is also a carrier of $\hat{u}_T$ in which i is a null player. From Eq. (4), we have

$$m(\hat{u}_T, K) = \begin{cases} (-1)^{k-t+1}, & \text{if } K \supsetneq T, \\ 0, & \text{otherwise,} \end{cases} \qquad \forall\, K \subseteq N \setminus S,$$

and, from Proposition 2.1,

$$m(\hat{u}_T, K \cup i) = 0, \qquad \forall\, K \subseteq N \setminus S.$$

Hence, from Eq. (1), we have that

$$\Delta_S \hat{u}_T(K) = \Delta_S \hat{u}_T(K \cup i) = \begin{cases} (-1)^{s+1}, & \text{if } K = T, \\ 0, & \text{otherwise.} \end{cases}$$

We know that, for each $K \subseteq N \cup i$, $k \geq 1$, there exists a family of nonnegative numbers $\{p_l^k(n+1)\}_{l=0,\ldots,n+1-k}$ such that, for any $v \in \mathcal{G}^{N \cup i}$,

$$I_p(v, K) = \sum_{L \subseteq (N \cup i) \setminus K} p_l^k(n+1)\, \Delta_K v(L).$$

For the game $\hat{u}_T$ seen as an element of $\mathcal{G}^{N\cup i}$ and the coalition S, it is easy to verify that

$$I_p(\hat{u}_T, S) = (-1)^{s+1}p_t^s(n+1) + (-1)^{s+1}p_{t+1}^s(n+1).$$

From Eq. (11), it follows that the coefficients $p_t^s(n)$ obey the recurrence relation

$$p_t^s(n) = p_t^s(n+1) + p_{t+1}^s(n+1).$$

Setting $\alpha_l^s := p_l^s(s+l)$ for all $l \in \mathbb{N}$, we can prove by induction that

$$\begin{aligned} p_t^s(n) &= (-1)^{n-s-t}\sum_{i=0}^{n-s-t}(-1)^i\binom{n-s-t}{i}\alpha_{n-s-i}^s \\ &= (-1)^{n-s-t}\nabla^{n-s-t}\alpha_{n-s}^s, \end{aligned} \tag{12}$$

for all $n \geq 1$, all $s \in \{1,\ldots,n\}$, and all $t \in \{0,\ldots,n-s\}$, where ∇^k denotes the kth iterate of the standard backward difference operator ∇, which is defined as

$$\nabla\alpha_m^s = \alpha_m^s - \alpha_{m-1}^s.$$

Clearly, the sequence $(\alpha_m^s)_{m\geq 0}$ is nonnegative and $\alpha_0^s = p_0^s(s) = 1$. Moreover, from Eq. (12), we have $(-1)^k\nabla^k\alpha_m^s \geq 0$ for all $k \leq m$. Then, according to Hausdorff's moment problem (see e.g. [1, Theorem 2.6.4]), we know that $\alpha_0^s, \alpha_1^s, \ldots$ are the moments of a uniquely determined CDF F_s on $[0,1]$, that is, we have

$$\alpha_m^s = \int_0^1 x^m dF_s(x), \qquad \forall\, m \geq 0.$$

Therefore, for each $t \in \{0,\ldots,n-s\}$, we have, by Eq. (12),

$$\begin{aligned} p_t^s(n) &= \int_0^1 x^t \sum_{i=0}^{n-s-t}\binom{n-s-t}{i}(-x)^{n-s-t-i}dF_s(x) \\ &= \int_0^1 x^t(1-x)^{n-s-t}dF_s(x). \end{aligned}$$

□

C Proofs of Theorems 4.5 and 4.6

The following three lemmas are used to prove Theorems 4.5 and 4.6.

Lemma C.1. *If I_p is a cardinal-probabilistic interaction index additionally satisfying axiom (E), then, for any $v \in \mathcal{G}$ and any $i \in U$, $I_p(v,i)$ is the Shapley value of i in the game v.*

Proof. The result follows from Lemma 4.1 and [31, Theorem 15]. □

Lemma C.2. *If I_p is a cardinal-probabilistic interaction index additionally satisfying axiom (2-E), then, for any $v \in \mathcal{G}$ and any $i \in U$, $I_p(v,i)$ is the Banzhaf value of i in the game v.*

Proof. The result follows from Lemma 4.1 and [13, Theorem 2]. □

Lemma C.3. *A cardinal-probabilistic interaction index I_p given in the form of Theorem 4.2 additionally satisfies axiom (RPC) if and only if, for any $n \geq 1$, any $s \in \{1, \ldots, n\}$, and any $t \in \{0, \ldots, n-s\}$, we have*

$$p_t^s(n) = p_t^1(n-s+1).$$

Proof. (Necessity) Let $N \subseteq U$ finite, $n \geq 1$, let $v \in \mathcal{G}^N$, and let $S \subseteq N$, $s \geq 1$, be a partnership in v. Then, it is easy to verify that

$$\Delta_S v(T) = \Delta_{[S]} v_{[S]}(T), \qquad \forall\, T \subseteq N \setminus S. \tag{13}$$

The function I_p being a cardinal-probabilistic interaction index, we can write

$$I_p(v, S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \Delta_S v(T). \tag{14}$$

Furthermore, we have

$$I_p(v_{[S]}, [S]) = \sum_{T \subseteq ((N \setminus S) \cup [S]) \setminus [S]} p_t^1(n-s+1) \Delta_{[S]} v_{[S]}(T).$$

Since I_p also satisfies axiom (RPC), we obtain

$$\sum_{T \subseteq N \setminus S} [p_t^s(n) - p_t^1(n-s+1)][v(T \cup S) - v(T)] = 0. \tag{15}$$

Now, it is easy to verify that, for each $T \subseteq N \setminus S$, S is a partnership in the unanimity game $u_{T \cup S}$ and that $u_{T\cup S}(K \cup S) - u_{T \cup S}(K) = 0$ for $K \subsetneq T$ and 1 for $K \subseteq N \setminus S$, $K \supseteq T$. Thus, using the family of games $\{u_{T \cup S}\}_{T \subseteq N \setminus S}$ (starting with u_N) in conjunction with Eq. (15), we obtain that

$$p_t^s(n) = p_t^1(n-s+1), \qquad \forall\, t \in \{0, \ldots, n-s\}.$$

(Sufficiency) Starting from Eq. (14) and using Eq. (13) and the fact that $p_t^s(n) = p_t^1(n-s+1)$ for all $t \in \{0, \ldots, n-s\}$, we immediately obtain that I_p satisfies (RPC). □

We can now prove Theorems 4.5 and 4.6.

Proof. (Necessity) The result follows immediately from Lemmas C.1 and C.3 for the Shapley interaction index and from Lemmas C.2 and C.3 for the Banzhaf interaction index.

(Sufficiency) The Shapley and Banzhaf interaction indices clearly satisfies axiom (RPC). The satisfaction of axioms (E) and (2-E) follows for example from parts (iv) and (v), respectively, of Theorem 3.1. □

D Proofs of Theorems 4.7 and 4.8

A lemma is used to prove Theorems 4.7 and 4.8.

Lemma D.1. *Let I_p be a cardinal-probabilistic interaction index given in the form of Proposition 4.1. Then, I_p additionally satisfies axiom (PA) if and only if,*

$$q_t^s q_s^1 = q_t^1, \qquad \forall\, t \geq s \geq 1.$$

Proof. (Necessity) Let $S \subseteq U$, $s \geq 1$, and let $T \supseteq S$. Clearly, S is a partnership in the game u_T. Then, since I_p satisfies axiom (PA), we have

$$I_p(u_T, S)I_p(u_S, i) = I_p(u_T, i), \qquad \forall\, i \in S.$$

Let $i \in S$. On the one hand, we have

$$I_p(u_T, S)I_p(u_S, i) = q_t^s q_s^1,$$

from the results given in Section 2.5 on the Möbius transform of a unanimity game. On the other hand,

$$I_p(u_T, i) = q_t^1,$$

which implies the necessity.

(Sufficiency) Let $S \subseteq U$, $s \geq 1$, and let $v \in \mathcal{G}$ be a game in which S is a partnership, let $i \in S$, $T \ni i$, and $K = T \setminus S$. Then,

$$m(v, T) = \sum_{R \subseteq K} \sum_{L \subseteq T \cap S} (-1)^{t-l-r} v(R \cup L).$$

If $T \not\supseteq S$, then, for all $L \subseteq T \cap S$, $v(R \cup L) = v(R)$, since S is a partnership in v. It follows that

$$m(v, T) = \sum_{R \subseteq K} (-1)^{t-r} v(R) \sum_{L \subseteq T \cap S} (-1)^l = 0.$$

Hence, $m(v, L) = 0$ for all $L \ni i$ such that $L \not\supseteq S$. Then, using Proposition 4.1 and the results given in Section 2.5, we have

$$\begin{aligned} I_p(v, S)I_p(u_S, i) &= \sum_{T \supseteq S} q_t^s m(v, T) \sum_{L \supseteq S} q_l^1 m(u_S, L) \\ &= \sum_{T \supseteq S} q_t^s q_s^1 m(v, T) \\ &= \sum_{T \supseteq S} q_t^1 m(v, T) \\ &= I_p(v, i), \end{aligned}$$

and thus the sufficiency. □

We now prove Theorem 4.8. The proof of Theorem 4.7 is analogue.

Proof. (Necessity) Let I_p be a cardinal-probabilistic interaction index given in the form adopted in Proposition 4.1 and let $v \in \mathcal{G}^N$. According to Eq. (7), it suffices to prove that

$$q_t^s = \frac{s}{t} \qquad \forall\, t \geq s \geq 1.$$

Since I_p satisfies axiom (E), from Lemma C.1 and Eq. (7), we have

$$q_t^1 = \frac{1}{t} \qquad \forall\, t \geq 1.$$

We then conclude by Lemma D.1.

(Sufficiency) The chaining interaction index clearly satisfies axiom (E) since it coincides with I_{Sh} on singletons. From Lemma D.1 and Eq. (7), we immediately have that I_p satisfies (PA). □

E Proofs of Theorems 4.9 and 4.10

Let us first prove Theorem 4.9.

Proof. Let $N \subseteq U$ finite, $n \geq 1$. It is easy to verify that, for any $v \in \mathcal{G}^N$ and any $S \subseteq U$, we have $\Delta_S v(\varnothing) = \Delta_S v^S(\varnothing)$. Let I_p be a cardinal-probabilistic interaction index given in the form of Theorem 4.2 additionally satisfying axiom (II) and consider $S \subseteq N$, $s \geq 1$, and $v \in \mathcal{G}^N$. Then, from axiom (II), we immediately have

$$I(v, S) - I(v^S, S) = (p_0^s(n) - 1)\Delta_S v(\varnothing) + \sum_{\substack{T \subseteq N \setminus S \\ T \neq \varnothing}} p_t^s(n) \Delta_S v(T) = 0.$$

The result is then immediately obtained by using the family of games $\{u_{T \cup S}\}_{T \subseteq N \setminus S}$ (starting with u_N) in the equation above. □

Let us now prove Theorem 4.10.

Proof. Let $N \subseteq U$ finite, $n \geq 1$. It is easy to verify that, for any $v \in \mathcal{G}^N$, any $S \subseteq U$, $\Delta_S v(N \setminus S) = \Delta_S v^S_{\cup N \setminus S}(\varnothing)$. Let I_p be a cardinal-probabilistic interaction index given in the form of Theorem 4.2 additionally satisfying axiom (EI). Then, we can write

$$\sum_{T \subsetneq N \setminus S} p_t^s(n) \Delta_S v(T) + (p_{n-s}^s(n) - 1)\Delta_S v(N \setminus S) = 0.$$

The result is then immediately obtained by using the game u_N in the equation above. □

References

[1] N. Akhiezer. *The classical moment problem and some related questions in analysis.* Edinburgh-London: Oliver & Boyd, 1965.

[2] J. F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.

[3] A. Chateauneuf and J.-Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Math. Soc. Sci.*, 17(3):263–283, 1989.

[4] P. Dubey, A. Neyman, and R. J. Weber. Value theory without efficiency. *Math. Oper. Res.*, 6:122–128, 1981.

[5] P. Dubey and L. S. Shapley. Mathematical properties of the Banzhaf power index. *Math. Oper. Res.*, 4:99–131, 1979.

[6] S. Foldes and P. L. Hammer. Submodularity, supermodularity, higher order monotonicities. *Math. Oper. Res.*, To appear.

[7] M. Grabisch. The representation of importance and interaction of features by fuzzy measures. *Pattern Recognition Letters*, 17(6):567–575, 1996.

[8] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 5:587–607, 1997.

[9] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92(2):167–189, 1997.

[10] M. Grabisch. Modelling data by the Choquet integral. In *Information Fusion in Data Mining*, volume 123 of *Stud. Fuzziness Soft Comput.*, pages 135–148. Physica, 2003.

[11] M. Grabisch, J.-L. Marichal, and M. Roubens. Equivalent representations of set functions. *Math. Oper. Res.*, 25(2):157–178, 2000.

[12] M. Grabisch and M. Roubens. An axiomatic approach of interaction in multicriteria decision making. In *Proc. 5th European Congr. on Intelligent Techniques and Soft Computing (EUFIT'97)*, pages 81–85, Aachen, Germany, September 1997.

[13] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. J. Game Theory*, 28(4):547–565, 1999.

[14] M. Grabisch and M. Roubens. Probabilistic interactions among players of a cooperative game. In *Beliefs, interactions and preferences in decision making (Mons, Belgium, 1997)*, pages 205–216. Kluwer Acad. Publ., Dordrecht, 1999. Vol. 40 of Theory Decis. Libr. Ser. B.

[15] M. Grabisch and M. Roubens. Application of the Choquet integral in multicriteria decision making. In *Fuzzy measures and integrals*, volume 40 of *Stud. Fuzziness Soft Comput.*, pages 348–374. Physica, Heidelberg, 2000.

[16] J. Harsanyi. A simplified bargaining model for the n-person cooperative game. *Int. Econ. Rev.*, 4:194–220, 1963.

[17] E. Kalai and D. Samet. Weighted Shapley values. In *The Shapley value. Essays in honor of Lloyd S. Shapley*, pages 83–99. Cambridge University Press, 1988.

[18] I. Kojadinovic. Modeling interaction phenomena using fuzzy measures: on the notions of interaction and independence. *Fuzzy Sets and Systems*, 135(3):317–340, 2002.

[19] I. Kojadinovic. *Modeling interaction phenomena using non additive measures : applications in data analysis.* PhD thesis, Université de La Réunion, France, 2002.

[20] I. Kojadinovic. Estimation of the weights of interacting criteria from the set of profiles by means of information-theoretic functionals. *European Journal of Operational Research*, 155:741–751, 2004.

[21] J.-L. Marichal. An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria. *IEEE Trans. Fuzzy Syst.*, 8(6):800–807, 2000.

[22] J.-L. Marichal. Behavioral analysis of aggregation in multicriteria decision aid. In *Preferences and decisions under incomplete knowledge*, pages 153–178. Physica, Heidelberg, 2000.

[23] J.-L. Marichal and M. Roubens. The chaining interaction index among players in cooperative games. In *Advances in decision analysis (Mons, 1997)*, pages 69–85. Kluwer Acad. Publ., Dordrecht, 1999.

[24] J.-L. Marichal and M. Roubens. Determination of weights of interacting criteria from a reference set. *European J. Oper. Res.*, 124(3):641–650, 2000.

[25] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (iii): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[26] A. S. Nowak. On an axiomatization of the Banzhaf value without the additivity axiom. *Int. J. Game Theory*, 26(1):137–141, 1997.

[27] G. Owen. Multilinear extensions of games. *Management Sci.*, 18:P64–P79, 1972.

[28] G.-C. Rota. On the foundations of combinatorial theory. I: Theory of Möbius functions. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 2:340–368, 1964.

[29] M. Roubens. Interaction between criteria and definition of weights in MCDA problems. In *Proc. 44th Meeting of the European Working Group "Multiple Criteria Decision Aiding"*, October 1996.

[30] L. Shapley. A value for n-person games. In *Contrib. Theory of Games, II, Ann. Math. Stud. No. 28*, pages 307–317. 1953.

[31] R. J. Weber. Probabilistic values for games. In *The Shapley value. Essays in honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.