Higher algebra over the Leibniz operad

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Outline*

1. The supergeometry of Loday algebroids (J. Geo. Mech., 2013)
2. Free Courant and derived Leibniz algebroids (J. Geo. Mech., 2016)
3. Infinity category of homotopy Leibniz algebras (Theo. Appl. Cat., 2014)
4. A tale of three homotopies (Appl. Cat. Struct., 2016)

*Joint with V. Dotsenko, J. Grabowski, B. Jubin, D. Khudaverdian, J. Qiu, K. Uchino

The supergeometry of Loday algebroids (J. Geo. Mech., 2013)

Motivations

- Double of a Lie bialgebra $\mathfrak{g}$ is a Lie algebra: $\mathfrak{g} \oplus \mathfrak{g}^*$
- Double of a Lie bialgebroid is a Courant algebroid: $\mathrm{T}\,M \oplus \mathrm{T}^*\,M$, E
- Leibniz bracket – **derived brackets** $\rightsquigarrow$ Leibniz algebroids?
- $[X, fY] = f[X, Y] + \lambda(X)f\;Y$ $\rightsquigarrow$ classical Leibniz algebroids

Requirements

Find a concept of Leibniz/Loday algebroid that

- is close to the notion of Lie algebroid
- contains Courant algebroids as special case
- reduces to a Leibniz algebra over a point
- includes a locality condition on both arguments

Loday algebroids: first attempt

'Definition': A Loday algebroid is a Leibniz bracket $[-,-]$ on sections of a vb E together with a left and right anchor

- If $\mathrm{rk}(E) = 1$, $[-,-]$ is AS and 1st order
- If $\mathrm{rk}(E) > 1$, $[-,-]$ is 'locally' a LAD bracket

'No' new examples $\rightsquigarrow$ *modify 'definition'*

Loday algebroids: second attempt

$[X, fY] = f[X, Y] + \lambda(X)f\ Y$

$[X^i e_i, fY^j e_j] = X^i a_{ij}^k fY^j\ e_k + X^i \lambda_i^a \partial_a f\ Y^j e_j + X^i \lambda_i^a\ f\ \partial_a Y^j\ e_j - Y^i \lambda_i^a \partial_a X^j\ e_j$

$\lambda(X)(\mathrm{d}\, f \otimes Y) = X^i \lambda_{ij}^{ak} \partial_a f\ Y^j e_k$

Derivation in f, $\mathcal{C}^\infty(M)$-linear in X and Y, valued in sections

$\lambda : \Gamma(E) \stackrel{\mathcal{C}^\infty(M)-\text{lin}}{\longrightarrow} \Gamma(TM) \otimes_{\mathcal{C}^\infty(M)} \mathrm{End}_{\mathcal{C}^\infty(M)} \Gamma(E)$

$\lambda : E \to TM \otimes \mathrm{End}\, E \rightsquigarrow$ generalized anchor

Cohomology theory $\rightsquigarrow$ traditional left anchor λ

Definition [Grabowski, Khudaverdian, P, ’13]

Definition

A Loday algebroid (LodAD) is a Leibniz bracket on sections of a vb $E \to M$ together with two bundle maps $\lambda : E \to TM$ and $\rho : E \to TM \otimes \mathrm{End}\, E$ such that

$$[X, fY] = f[X, Y] + \lambda(X)f\, Y$$

and

$$[fX, Y] = f[X, Y] + \rho(Y)(\mathrm{d}\, f \otimes X).$$

Examples

- Leibniz algebra
- (twisted) Courant-Dorfman ($\mathrm{T}\, M \oplus \mathrm{T}^*\, M$)
- Grassmann-Dorfman ($\mathrm{T}\, M \oplus \wedge \mathrm{T}^*\, M$ or $E \oplus \wedge E^*$)
- classical Leibniz algebroid associated to a Nambu-Poisson structure
- Courant algebroid
- ...

Courant: $f \in \mathcal{C}^\infty(M)\,,\, X, Y \in \Gamma(E)$

$D \in \mathrm{Der}(\mathcal{C}^\infty(M), \Gamma(E))$: $(Df|Y) = \frac{1}{2}\lambda(Y)f$

$D(fX|Y) = [fX, Y] + [Y, fX] \rightsquigarrow \rho(Y)(\mathrm{d}\, f \otimes X) = D(f)(X|Y)$

Derivation in f, $\mathcal{C}^\infty(M)$-linear in X and Y, valued in sections

Supergeometric interpretation

$$(E, [-,-], \lambda) \rightleftharpoons (\Gamma(\wedge E^*), \mathrm{d}) \rightleftharpoons \boxed{\mathrm{d} \in \mathrm{Der}_1(\Gamma(\wedge E^*), \wedge), \mathrm{d}^2 = 0}$$

Lie algebroids $\rightleftharpoons$ homological vfs on supermfds

Loday algebroids $\rightleftharpoons$?

C-E operator restricted to $\wedge_{\mathcal{C}^\infty(M)}(\Gamma(E), \mathcal{C}^\infty(M)) = \Gamma(\wedge E^*)$

Leibniz operator restricted to

$$\mathrm{Lin}_{\mathcal{C}^\infty(M)}\, D(\Gamma(E), \mathcal{C}^\infty(M)) = \Gamma(\otimes E^*) =: D(E)$$

$$(D \pitchfork \Delta)(X_1, \ldots, X_{p+q})$$

$$= \sum_{\sigma \in \mathrm{sh}(p,q)} \mathrm{sign}(\sigma) D(X_{\sigma_1}, \ldots, X_{\sigma_p})\, \Delta(X_{\sigma_{p+1}}, \ldots, X_{\sigma_{p+q}})$$

Supergeometric interpretation

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LodADs as hom vfs [Grabowski, Khudaverdian, P, ’13]

Theorem

There is a 1-to-1 correspondence between LodAD structures $(E, [-,-], \lambda, \rho)$ *and equivalence classes of homological vfs*

$$\boxed{\mathrm{d} \in \mathcal{D}er_1(\mathcal{D}(E), \pitchfork), \mathrm{d}^2 = 0}$$

of the supercommutative space $(\mathcal{D}(E), \pitchfork)$*.*

Cartan calculus

Free Courant and derived Leibniz algebroids (J. Geo. Mech., ’16)

Koszul duality for operads

Ginzburg-Kapranov, '94:

P_∞-algebra on $V \rightleftharpoons \mathrm{d} \in \mathrm{Der}_1\left(\mathbf{F}_{P^!}(sV^*)\right), \mathrm{d}^2 = 0$

Example:

L_∞-algebra on $V \rightleftharpoons$ homological vf on the formal smfd V

Geometric extensions:

L_∞-algebroid $\rightleftharpoons$ homological vf on a $\mathbb{N}$-smfd (Bonavolontà, P, '12)

LAD $\rightleftharpoons$ homological vf on a smfd

LodAD $\rightleftharpoons$ homological vf on a supercommutative space

Derived brackets induced by the homological vf

Courant algebroid

Classical LeiAD $(E, [-,-], \lambda)$ with a scalar product $(-|-)$

Invariance relations:

$$\lambda(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z])$$

$$\lambda(X)(Y|Z) = (X|[Y,Z] + [Z,Y])$$

Compatibility condition:

$$([X,Y]|Z) + (Y|[X,Z]) = (X|[Y,Z] + [Z,Y])$$

$\Gamma(E)$: $\mathcal{C}^\infty(M)$-module, $\mathcal{C}^\infty(M)$: commutative $\mathbb{R}$-algebra, $\mathbb{R}$: field $\rightsquigarrow$

$\mathcal{E}$: $\mathcal{A}$-module, $\mathcal{A}$: commutative R-algebra; R: commutative ring

Free Courant algebroid

Free Courant AD over an anchored $\mathcal{A}$-module $(\mathcal{E}, \lambda)$?

Free Leibniz algebra over the R-module $\mathcal{E}$: $(\underline{\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}})$

Free LeiAD over $(\mathcal{E}, \lambda)$: $(\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}, \underline{\mathcal{F}(\lambda)}) \rightsquigarrow (-|-)_{\text{USP}}$?

$(\mathcal{E}_0, [-,-]_0, \lambda_0, (-|-)_0)$, $f : \mathcal{E} \to \mathcal{E}_0$, $f_1 : \mathcal{F}(\mathcal{E}) \to \mathcal{E}_0$, $X, Y \in \mathcal{F}(\mathcal{E})$

'$(X|Y)_{\text{USP}} =$' $(f_1(X)|f_1(Y))_0 = f_2(X \odot Y)$

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Invariance $\rightsquigarrow \mathcal{Q}(\mathcal{F}(\mathcal{E}))$ must have $\mathcal{F}(\mathcal{E})$-actions $\underline{\mu^\ell}$ and $\underline{\mu^r}$

$$\boxed{(\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{F}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{\text{USP}})}$$

Well-DefNess of μ^ℓ and μ^r on $\mathcal{Q}(\mathcal{F}(\mathcal{E})) \rightsquigarrow$ $\boxed{\text{2 SymConds on } [-,-]_{\text{ULB}}}$

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Symmetric Leibniz algebroid [Jubin, P, Uchino, '16]

Definition

A symmetric LeiAD is a classical LeiAD $(\mathcal{E}, [-,-], \lambda)$ *s.th.*

$$X \circ fY - (fX) \circ Y = 0\ ,$$

$$([fX, Y] - f[X, Y]) \circ Z + Y \circ ([fX, Z] - f[X, Z]) = 0\ ,$$

where $X \circ Y := [X, Y] + [Y, X]\,.$

Examples

- Leibniz algebra
- (twisted) Courant-Dorfman
- Grassmann-Dorfman
- Courant algebroid

A LeiAD associated to Nambu-Poisson structure is NOT a symmetric LeiAD !

The free symmetric LeiAD over an anchored module is NOT Loday !

Generalized Courant AD [Jubin, P, Uchino, '16]

$$(\mathcal{SF}(\mathcal{E}), [-,-]_{\text{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{SF}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{\text{USP}})$$

Definition

Generalized Courant AD: $(\mathcal{E}_1, [-,-], \lambda, \mathcal{E}_2, \mu^\ell, \mu^r, (-|-))$

Invariance relations:

$$\mu^\ell(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z])$$

$$-\mu^r(X)(Y|Z) = ([Y,Z] + [Z,Y]|X)$$

Compatibility condition:

$$([X,Y]|Z) + (Y|[X,Z]) = ([Y,Z] + [Z,Y]|X)$$

Non-degeneracy $\Rightarrow$ symmetry AND $(\mathcal{C}^\infty(M), \lambda, -\lambda) \Rightarrow (\mathcal{E}_2, \mu^\ell, \mu^r)$

Generalized Courant AD [Jubin, P, Uchino, '16]

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Non-degeneracy $\Rightarrow$ symmetry AND $(\mathcal{C}^\infty(M), \lambda, -\lambda) \Rightarrow (\mathcal{E}_2, \mu^\ell, \mu^r)$

Application (I)

Derived brackets:

$(K,\{-,-\},\Delta)$: DGLA $\rightsquigarrow$ $\{k,\ell\}_\Delta = (-1)^{|k|+1}\{\Delta k,\ell\}$, $(K,\{-,-\}_\Delta)$: GLeiA

Question:

Which LeiAD brackets can be represented by a derived bracket ?

Application (II)

$(\mathcal{E}, \lambda)$: anchored module $\rightsquigarrow$

Free GCourant AD: $(\mathcal{SF}(\mathcal{E}), [-,-]_{\mathsf{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{SF}(\mathcal{E})), \mu^\ell, \mu^r, (-\|-)_{\mathsf{USP}})$

$(\mathcal{E}, [-,-], \lambda)$: symmetric LeiAD $\rightsquigarrow$

Associated GCourant AD: $(\mathcal{E}, [-,-], \lambda, \mathcal{Q}(\mathcal{E}), \mu^\ell, \mu^r, (-\|-))$

$\rightsquigarrow$

$(K, \{-,-\}_\Delta)$: universal derived bracket representation of $(\mathcal{E}, [-,-])$

Summary

Classical Leibniz AD

Loday AD

Symmetric Leibniz AD

Generalized Courant AD

Free generalized Courant AD

Associated generalized Courant AD

Universal derived bracket representation

Infinity category of homotopy Leibniz algebras (Theo. Appl. Cat., ’14)

∞-Homotopies

$$\boxed{P_\infty\text{-algebra on } V \rightleftharpoons \mathrm{d} \in \mathrm{Der}_1(\mathcal{F}_{P^!}(sV^*)),\ \mathrm{d}^2 = 0}$$

$$P_\infty\text{- algebra on } V \rightleftharpoons \mathrm{d} \in \mathrm{CoDer}_1(\mathcal{F}_{P^{¡}}(s^{-1}V)),\ \mathrm{d}^2 = 0$$

$$P_\infty\text{-}\mathtt{Alg} \simeq \mathtt{qfDGP^!A} \simeq \mathtt{qfDGP^{¡}C}$$

1. $\mathtt{qfDGP^!A}$-Ho

Concordances (Schlessinger, Stasheff): ‘($\mathtt{Man}$-Ho)*’

Problem for vertical $\circ$: Higher cat ?

Concordances - all gory details

- $p, q : V \xrightarrow{\mathcal{C}^\infty} W$

  $p^*, q^* : \Omega(W) \xrightarrow{\text{Ch}} \Omega(V)$

  $\eta^* : \Omega(W) \xrightarrow{\text{Ch}} \Omega_1 \otimes \Omega(V)$

- $p, q : V \xrightarrow{P_\infty} W$

  $p^*, q^* : \mathcal{F}_{P^!}(W) \xrightarrow{\text{DGA}} \mathcal{F}_{P^!}(V)$

  $\eta^* : \mathcal{F}_{P^!}(W) \xrightarrow{\text{DGA}} \Omega_1 \otimes \mathcal{F}_{P^!}(V)$

- $\eta^*(w) = \varphi_w(t) + \mathrm{d}\, t\, \rho_w(t)$

  $\mathrm{d}_t\, \varphi = \mathrm{d}_V\ \rho(t) + \rho(t)\ \mathrm{d}_W$

- $\mathcal{C}^\infty$-case: $h := \int_0^1 \mathrm{d}\, t\, \rho(t)$: Ch-homotopy

  P_∞-case: $\varphi(t) : \mathcal{F}_{P^!}(W) \xrightarrow{\text{DGA}} \mathcal{F}_{P^!}(V)$, $\rho(t) : \mathcal{F}_{P^!}(W) \xrightarrow{\varphi-\text{Der}} \mathcal{F}_{P^!}(V)$

∞-Homotopies

$$P_\infty\text{-}\mathtt{Alg} \simeq \mathtt{qfDGP^!A} \simeq \mathtt{qfDGP^{¡}C}$$

2. $\mathtt{qfDGP^{¡}C}$-Ho

$$\mathrm{Hom}_{P_\infty}(V, W) \simeq \mathrm{Hom}_{\mathtt{DGP^{¡}C}}(\mathcal{F}_{P^{¡}}(V), \mathcal{F}_{P^{¡}}(W))$$

$$\simeq \mathrm{MC}(\mathrm{Hom}_{\mathbb{R}}(\mathcal{F}_{P^{¡}}(V), W)) =: \mathrm{MC}(\mathcal{C}) \quad (s^{-1} \text{ omitted})$$

Gauge Ho of MC$(\mathcal{C})$: IC of specific VF

Quillen Ho of MC$(\mathcal{C})$: MC$(\mathcal{C} \otimes \Omega_1)$

Gauge and Quillen homotopies - all gory details

$(\mathcal{C}, \ell_i)$, $r \in \mathcal{C}_0$

$V_r : \mathcal{C}_{-1} \ni \alpha \mapsto -\sum_i \frac{1}{i!}\ell_{i+1}(\alpha^{\otimes i}, r) = -\ell_1^{\alpha}(r) \in \mathcal{C}_{-1}$

$V_r|_{\mathrm{MC}(\mathcal{C})}$: vector field of $\mathrm{MC}(\mathcal{C})$

$\mathrm{MC}(\mathcal{C})$-elements gauge homotopic if related by integral curve of $V_r|_{\mathrm{MC}(\mathcal{C})}$

$\gamma \in \mathrm{MC}(\mathcal{C} \otimes \Omega_1)$

$\gamma = \gamma_1(t) + \mathrm{d}\, t\, \gamma_2(t), \quad \gamma_1(t) \in \mathcal{C}_{-1}$

$\mathrm{MC}(\mathcal{C})$-elements α, β Quillen homotopic if there is γ s.th. $\gamma_1(0) = \alpha, \gamma_1(1) = \beta$

Theorem

Concordances, Gauge and Quillen homotopies are equivalent concepts

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∞-Cat of P_∞-Alg [Khudaverdian, P, Qiu, '14]

P_∞ - Ho = P_∞ - 2 - Mor : MC$(\mathcal{C} \otimes \Omega_1)$

P_∞ - Mor = P_∞ - 1 - Mor : MC$(\mathcal{C} \otimes \Omega_0)$

Definition

$$P_\infty\text{-}(n+1)\text{-Mor} : \mathrm{MC}(\mathcal{C} \otimes \Omega_n), \; n \geq 0$$

Getzler, '09:

$$\int \mathcal{C} \xrightarrow{\sim} \mathrm{MC}(\mathcal{C} \otimes \Omega_\bullet) : \text{Kan complex}$$

Theorem

$$P_\infty\text{-}\mathtt{Alg} : \infty\text{-Cat}$$

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Theorem

P_∞-`Alg`: *∞-Cat*

∞-Cat ET AL

$S \in$ `SSet` fibrant object, if

$S \to *$ has RLP with respect to $\Lambda^i[n] \hookrightarrow \Delta[n]$, i.e.,

$\Lambda^i[n] \to S$ extends to $\Delta[n] \to S$, i.e.,

Any horn in $S = (S_0, S_1, \ldots)$ has a filler: S is Kan complex

Any inner horn has a unique filler, any horn has a unique filler

Any inner horn in $S = (S_0, S_1, \ldots)$ has a filler: S is ∞-Cat

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Application [Khudaverdian, P, Qiu, '14]

P_∞-$\mathtt{Alg}$: ∞-Cat

$\mathtt{Lei}_\infty$-$\mathtt{Alg}$: ∞-Cat

$2\mathtt{Lei}_\infty$-$\mathtt{Alg}$: 2-Cat (badly understood)

Theorem

∞-*Cat in* $\mathtt{Lei}_\infty$-$\mathtt{Alg}$ $\twoheadrightarrow$ *2-Cat in* $2\mathtt{Lei}_\infty$-$\mathtt{Alg}$

Application of Getzler's integration technique

Baez-Crans and Schreiber-Stasheff

Projection – some details (I)

$$\mathrm{MC}(\mathcal{C}\otimes\Omega_n) \xrightarrow{B_n^i} \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}^i(\mathcal{C}\otimes\Omega_n)$$

$$\subset \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}(\mathcal{C}\otimes\Omega_n)$$

$$\mathrm{mc}^i(\mathcal{C}\otimes\Omega_n) = \{(\delta\otimes\mathrm{id} + \mathrm{id}\otimes d)\beta, \beta \in (\mathcal{C}\otimes\Omega_n)^0, \beta(\mathsf{V}_i) = 0\}$$

$\mathcal{B}_n^i$: inverse of B_n^i

$$\mathtt{SSet}(\Lambda^i[n], \mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)) \to \mathtt{SSet}(\Lambda^i[n], \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}(\mathcal{C}\otimes\Omega_\bullet)) \to$$

$$\mathrm{MC}(\mathcal{C}) \times \mathrm{mc}^i(\mathcal{C}\otimes\Omega_n) \to \mathrm{MC}(\mathcal{C}\otimes\Omega_n) = \mathtt{SSet}(\Delta[n], \mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet))$$

$\mathrm{MC}(\mathcal{C}\otimes\Omega_\bullet)$: Kan complex

Projection – some details (II)

∞-Cat in $\texttt{Lei}_\infty\texttt{-Alg}$ $\twoheadrightarrow$ strict 2-Cat in $\texttt{2Lei}_\infty\texttt{-Alg}$

α, β: 2-Cat-Maps $\rightsquigarrow$ α, β: ∞-Cat-Maps $\rightsquigarrow$ $\alpha, \beta \in \mathrm{MC}(\mathcal{C})$

$\gamma \in \mathrm{MC}(\mathcal{C} \otimes \Omega_1)$: ∞-Cat-Ho for α, β

$$B_1^0 : \mathrm{MC}(\mathcal{C} \otimes \Omega_1) \rightleftarrows \mathrm{MC}(\mathcal{C}) \times \mathrm{mc}^0(\mathcal{C} \otimes \Omega_1) : \mathcal{B}_1^0$$

$$\gamma = \mathcal{B}_1^0 B_1^0 \gamma = \alpha + \mathcal{E}(\alpha, \varepsilon)$$

$$\beta = \gamma(1) = \alpha + \mathcal{E}(\alpha, \varepsilon(1))$$

$\varepsilon(1)$: 2-Cat-Ho for α, β

$\gamma \mapsto \varepsilon(1)$: surjective

$$\varepsilon(1), \varepsilon'(1) \rightsquigarrow \gamma, \gamma',\ \gamma'' = \gamma' \circ \gamma \rightsquigarrow \varepsilon''(1) := \varepsilon'(1) \circ \varepsilon(1)$$

A tale of three homotopies
(Appl. Cat. Struct., ’16)

Equivalence of all ∞-homotopies [Dotsenko, P, '16]

Homotopy transfer theorem for homotopy cooperads

Explicit recipe to write a definition of operadic homotopy (nested trees in homotopy transfer formulas)

Theorem

Concordances, Quillen, gauge, cylinder, and operadic homotopies are $\simeq$

THANK YOU FOR YOUR INTEREST