Exact Solutions to a Class of Feedback Systems on $\mathsf{SO}(n)$ ⋆

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Abstract

This paper provides a novel approach to the problem of attitude tracking for a class of almost globally asymptotically stable feedback laws on $\mathsf{SO}(n)$. The closed-loop systems are solved exactly for the rotation matrices as explicit functions of time, the initial conditions, and the gain parameters of the control laws. The exact solutions provide insight into the transient dynamics of the system and can be used to prove almost global attractiveness of the identity matrix. Applications of these results are found in model predictive control problems where detailed insight into the transient attitude dynamics is utilized to approximately complete a task of secondary importance. Knowledge of the future trajectory of the states can also be used as an alternative to the zero-order hold in systems where the attitude is sampled at discrete time instances.

Key words: Attitude control; global stability; Lie groups; nonlinear systems; predictive control; sampled-data systems.

1 Introduction

The nonlinear control problem of stabilizing the attitude dynamics of a rigid body has a long history of study and is important in a diverse range of engineering applications related to *e.g.* quadrotors (Lee et al., 2010), inverted 3-D pendulums (N.A. Chaturvedi et al., 2009), and robotic manipulators (Hu et al., 2009). It is interesting from a theoretical point of view due to the nonlinear state equations and the topology of the underlying state space $\mathsf{SO}(3)$. An often cited result states that global asymptotical stability on $\mathsf{SO}(3)$ cannot be achieved by means of a continuous, time-invariant feedback (S.P. Bhat and D.S. Bernstein, 2000). The literature does however provide results such as almost global asymptotical stability through continuous time-invariant feedback (N.A. Chaturvedi et al., 2011; Sanyal et al., 2009), almost semi-global stability (Lee, 2012), or global stability by means of a hybrid control approach (C.G. Mayhew et al., 2011b). The parameterizations used to represent $\mathsf{SO}(3)$ has important implications for the limits of control performance (N.A. Chaturvedi et al., 2011; S.P. Bhat and D.S. Bernstein, 2000; C.G. Mayhew et al., 2011a). In particular, the use of local representations yields local results. In most cases, it is preferable to either use global representations such as the unit quaternions or to work with the space of rotation matrices directly (N.A. Chaturvedi et al., 2011).

The exact solutions of a closed-loop system gives a detailed picture of both its transients and asymptotical behaviour and can hence be of use in control applications. The literature on solutions to attitude dynamics can be divided into two categories. Firstly, in a number of works the solutions are obtained during the control design process, *e.g.* using exact linearization (Dwyer III, 1984) or optimal control design techniques such as the Pontryagin maximum principle (Spindler, 1998). Secondly, there are works whose main focus is solving the equations defining rigid-body dynamics under a set of specific assumptions (Elipe and Lanchares, 2008; M.A. Ayoubi and J.M. Longuski, 2009; A.V. Doroshin, 2012). This paper falls into the second category.

There is a considerable literature on the kinematics and dynamics of n-dimensional rigid-bodies. This literature includes works on attitude stabilization (D.H.S. Maithripala et al., 2006), attitude synchronization (Lageman et al., 2009), distributed averaging (Matni and Horowitz, 2014), and generalized Newtonian equations of motion (J.E. Hurtado and A.J. Sinclair, 2004). It also includes the authors previous work (Markdahl et al., 2013; Markdahl and Hu, 2014), which we shall comment on shortly. A key difference between the study of $\mathsf{SO}(3)$ and $\mathsf{SO}(n)$ is that parameterizations such as the unit quaternions cannot be used. Another is the motivation: work on $\mathsf{SO}(3)$ is usually motivated by applications concerning

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the attitude of rigid bodies. Work on $\mathsf{SO}(n)$ is not only of theoretical concern however, it also finds applications in the visualization of high-dimensional data (Thakur, 2008).

The main contribution of this paper is to provide exact solutions to differential equations representing closed feedback loops on $\mathsf{SO}(n)$. Recent work on this problem include (Markdahl et al., 2012, 2013; Markdahl and Hu, 2014). Other works such as (Elipe and Lanchares, 2008; M.A. Ayoubi and J.M. Longuski, 2009; A.V. Doroshin, 2012) are related in spirit but address somewhat different problems. The work (Markdahl et al., 2012) considers the solutions to closed-loop kinematics on $\mathsf{SO}(3)$. An application towards model predictive control (MPC) is proposed but left unexplored. The more general problem of solving two differential equations on $\mathsf{SO}(n)$ is treated in (Markdahl et al., 2013). An application towards the problem of continuous actuation under discrete-time sensing is considered. The work (Markdahl and Hu, 2014) generalizes the results of (Markdahl et al., 2013) to a greater class of feedback laws. This paper in turn generalizes (Markdahl and Hu, 2014) and explores the applications proposed in (Markdahl et al., 2012) and (Markdahl et al., 2013). Note that many of the results of this paper easily extends to the case of $\mathsf{SE}(n)$ and may be combined with position control laws in an inner-outer loop feedback scheme to achieve pose stabilization on $\mathsf{SE}(n)$ (Roza and Maggiore, 2012).

This paper is structured as follows. Section 2 recalls the notation and some basic properties of matrix analysis, it can be skipped if the reader is familiar with that topic. Section 3 presents the attitude stabilization problem and introduces Problem 1, the problem of solving the closed-loop state equations. Section 4.1 generalizes a class of known control laws on $\mathsf{SO}(3)$ to the case of $\mathsf{SO}(n)$. It contains the main result of this paper, the solution to Problem 1. It also makes use of the exact solutions to prove that the proposed algorithms stabilize System 1 almost globally. Section 6 explores practical applications of the exact solutions to problems of model predictive control and continuous feedback in sampled systems. Section 8 provides some brief concluding remarks.

2 Preliminaries

Let $\mathbf{A},\mathbf{B}\in\mathbb{R}^{n\times n}$. The spectrum of $\mathbf{A}$ is written $\sigma(\mathbf{A})$. The transpose and conjugate transpose of $\mathbf{A}$ is written $\mathbf{A}^\top$ and $\mathbf{A}^*$ respectively. The commutator of $\mathbf{A}$ and $\mathbf{B}$ is defined by $[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA}$. Their inner product is defined by $\langle\mathbf{A},\mathbf{B}\rangle=\operatorname{tr}(\mathbf{A}^\top\mathbf{B})$ and the Frobenius norm by $\|\mathbf{A}\|_F=\langle\mathbf{A},\mathbf{A}\rangle^{\frac{1}{2}}$.

The set of nonsingular matrices over a field $\mathcal{F}$ is denoted by $\mathsf{GL}(n,\mathcal{F})$. The unitary group is denoted by $\mathsf{U}(n)=\{\mathbf{U}\in\mathsf{GL}(n,\mathbb{C})\,|\,\mathbf{U}^{-1}=\mathbf{U}^*\}$. The orthogonal group is $\mathsf{O}(n)=\{\mathbf{Q}\in\mathsf{GL}(n,\mathbb{R})\,|\,\mathbf{Q}^{-1}=\mathbf{Q}^\top\}$. The special orthogonal group is denoted by $\mathsf{SO}(n)=\{\mathbf{R}\in\mathsf{O}(n)\,|\,\det\mathbf{R}=1\}$. In this paper we define $\mathcal{R}(n)=\{\mathbf{R}\in\mathsf{SO}(n)\,|\,-1\in\sigma(\mathbf{R})\}$. It can be shown that $\{\mathbf{R}\in\mathsf{SO}(n)\,|\,\mathbf{R}^\top=\mathbf{R}\}/\{\mathbf{I}\}\subset\mathcal{R}(n)$. Equality holds in the cases of $n\in\{2,3\}$. The Lie algebra of $\mathsf{SO}(n)$ is denoted by $\mathsf{so}(n)=\{\mathbf{S}\in\mathbb{R}^{n\times n}\,|\,\mathbf{S}^\top=-\mathbf{S}\}$. In this paper, we use $\mathbf{S}$ to denote the matrix $\operatorname{Log}\mathbf{R}\in\mathsf{so}(n)$ for $\mathbf{R}\in\mathsf{SO}(n)\backslash\mathcal{R}(n)$.

The principal matrix logarithm Log is defined on the set $\{\mathbf{A}\in\mathsf{GL}(n,\mathbb{R})\,|\,\sigma(\mathbf{A})\cap(-\infty,0]=\varnothing\}$ (Culver, 1966). It satisfies $\operatorname{Im}\sigma(\operatorname{Log}\mathbf{A})\subset\{z\in i\mathbb{R}\,|\,|z|<\pi\}$ (Higham, 2008). Since any rotation matrix $\mathbf{R}$ is normal, it follows that $\mathbf{R}=\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*$ and the logarithm of $\mathbf{R}$ may be calculated as $\operatorname{Log}\mathbf{R}=\mathbf{U}\operatorname{Log}(\boldsymbol{\Lambda})\mathbf{U}^*$. Moreover, $\boldsymbol{\Lambda}=\exp(i\boldsymbol{\Theta})$ for a diagonal matrix $\boldsymbol{\Theta}$ which satisfies $\boldsymbol{\Theta}_{ii}\in(-\pi,\pi)$ for all $\mathbf{R}\in\mathsf{SO}(n)\backslash\mathcal{R}(n)$. Hence $\operatorname{Log}(\boldsymbol{\Lambda})=i\boldsymbol{\Theta}$ and $\operatorname{Log}\mathbf{R}=i\mathbf{U}\boldsymbol{\Theta}\mathbf{U}^*$. The matrix logarithm allows us to calculate the geodesic distance between $\mathbf{R}_1,\mathbf{R}_2\in\mathsf{SO}(n)$ using the Riemannian metric $d_R(\mathbf{R}_1,\mathbf{R}_2)=\frac{1}{\sqrt{2}}\|\operatorname{Log}(\mathbf{R}_1^\top\mathbf{R}_2)\|_F$. The set of symmetric matrices is $\mathcal{S}(n)=\{\mathbf{P}\in\mathbb{R}^{n\times n}\,|\,\mathbf{P}^\top=\mathbf{P}\}$. The set of positive-semidefinite matrices is denoted by $\mathcal{P}(n)=\{\mathbf{P}\in\mathcal{S}(n)\,|\,\sigma(\mathbf{P})\subset[0,\infty)\}$. The set of positive-definite matrices is $\mathcal{P}(n)\cap\mathsf{GL}(n,\mathbb{R})$.

By the kth root of a normal matrix $\mathbf{A}=\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*$ we refer to its principal root, the normal matrix $\mathbf{A}^{\frac{1}{k}}=\mathbf{U}\boldsymbol{\Lambda}^{\frac{1}{k}}\mathbf{U}^*$. The principal root satisfies $\mathbf{R}^{\frac{1}{k}}=\exp(\frac{1}{k}\mathbf{S})\in\mathsf{SO}(n)$. Moreover, $\mathbf{R}^{\frac{1}{k}}\notin\mathsf{SO}(n)\backslash\mathcal{R}(n)$ if $\mathbf{R}\notin\mathsf{SO}(n)\backslash\mathcal{R}(n)$.

The solution to a differential equation $\dot{\mathbf{X}}=\mathbf{F}(t,\mathbf{X})$ is denoted $\mathbf{X}(t;t_0,\mathbf{X}_0)$ where t is the time, t_0 is the initial time, and $\mathbf{X}_0$ is the initial condition. If the system is time independent we set $t_0=0$ and omit this dependence. Furthermore, let $\boldsymbol{\Phi}(\mathbf{X}_0,t)$ denote the flow of $\mathbf{F}(t,\mathbf{X})$, *i.e.* $\boldsymbol{\Phi}(\mathbf{X}_0,t)=\mathbf{X}(t;t_0,\mathbf{X}_0)$.

3 Problem Statement

From a mathematical perspective it is appealing to strive for generalization. Consider the evolution of a positively oriented n-dimensional orthogonal frame represented by $\mathbf{R}\in\mathsf{SO}(n)$. The dynamics on $\mathsf{SO}(n)$ are given by $\dot{\mathbf{R}}=\boldsymbol{\Omega}\mathbf{R}$. This paper concerns the following system.

System 1 *Consider the system* $\dot{\mathbf{R}}=\boldsymbol{\Omega}(\mathbf{R})\mathbf{R}$ *where* $\mathbf{R}\in\mathsf{SO}(n)$ *and* $\boldsymbol{\Omega}:\mathsf{SO}(n)\to\mathsf{so}(n)$. *The input is given by* $\boldsymbol{\Omega}$, i.e. *the system is fully actuated on a kinematic level.*

The kinematic level stabilization problem on $\mathsf{SO}(n)$ concerns the design of an $\boldsymbol{\Omega}$ that stabilizes the identity matrix. The matrix $\mathbf{R}$ can be actuated along any direction of $\mathsf{so}(n)$, its tangent space at the identity. Note that $\mathsf{SO}(n)$ is invariant under the kinematics $\dot{\mathbf{R}}=\boldsymbol{\Omega}(\mathbf{R})\mathbf{R}$, *i.e.* any solution $\mathbf{R}(t;\mathbf{R}_0)$ remains in $\mathsf{SO}(n)$ for all $t\in$

$[0, \infty)$ if $\mathbf{R}_0 \in \mathsf{SO}(n)$. This paper concerns a class of almost globally stabilizing feedback laws $\mathbf{\Omega}$ that allow the closed-loop equations to be solved for $\mathbf{R}$ as a function of time, any design parameters, and the initial conditions. It analyzes the stability of said class of control laws and discusses possible applications of these results.

An equilibrium of is said to be almost globally asymptotically stable if it is asymptotically stable and the region of attraction is all of $\mathsf{SO}(n)$ except for a set of measure zero. A set $\mathcal{N} \subset \mathsf{SO}(n)$ has measure zero if for every chart $\phi : \mathcal{S} \to \mathbb{R}^{\frac{1}{2}n(n-1)}$ in some atlas of $\mathsf{SO}(n)$, it holds that $\phi(\mathcal{S} \cap \mathcal{N})$ has Lebesgue measure zero.

Problem 1 *For a given almost globally stabilizing feedback law* $\mathbf{\Omega} : \mathsf{SO}(n) \to \mathsf{so}(n)$*, solve System 1 for* $\mathbf{R}(t; \mathbf{R}_0)$*,* i.e. *for* $\mathbf{R}$ *as function of the time* $t \in [0, \infty)$ *and all initial conditions* $\mathbf{R}_0 \in \mathsf{SO}(n)$ *belonging to the region of attraction of the identity matrix.*

Previous work on global level attitude stabilization apply the stable-unstable manifold theorem (N.A. Chaturvedi et al., 2011; Sanyal et al., 2009; Lee, 2012) or use Lyapunov function arguments (C.G. Mayhew et al., 2011b) to establish the region of attraction of the identity matrix. The stable-unstable manifold theorem (S.S. Sastry, 1999) is however ineffective to prove almost global asymptotical stability for systems that are actuated on a kinematic level when the unstable equilibrium manifold corresponds to the uncountable set $\{\mathbf{R} \in \mathsf{SO}(n) \,|\, \mathbf{R}^\top = \mathbf{R}\} \backslash \{\mathbf{I}\} \subset \mathcal{R}(n)$.

This paper presents a novel approach to establishing almost global asymptotical stability by means of exact solutions to the closed-loop system kinematics. It is possible to establish global existence and uniqueness of the solutions, see Lemma 1 in Appendix A. Statements regarding control performance can hence be based on the properties of the exact solutions. This paper uses the solutions to show that the region of attraction of the identity matrix for the closed-loop systems generated by Algorithm 1–2 below is $\mathsf{SO}(n) \backslash \mathcal{R}(n)$. The desired result follows since $\mathcal{R}(n)$ is a set of measure zero in $\mathsf{SO}(n)$.

Remark 1 *The attitude dynamics of a rigid body is often described by a second order system consisting of a kinematic equation coupled with Euler's equation of motion. In that case, the input signal is a torque vector. Kinematic level control design may however be preferable under certain circumstances, for example when an application programming interface restricts actuation to velocity level control commands or as a prerequisite in applying the backstepping control design technique (Krstic et al., 1995). Models with kinematic level actuation are also common in certain fields such as visual servo control (Chaumette and Hutchinson, 2006, 2007). What is more, there is no compelling reason to impose Newtonian mechanics in the general* $\mathsf{SO}(n)$ *case.*

4 Main Results

This section contains the main results of the paper, the exact solutions to the closed-loop systems resulting from feedback by Algorithm 1 and 2 defined below.

4.1 Closed-Loop System 1

The following algorithm is well-known in the literature.

Algorithm 1 (Positive-semidefinite gain matrix) *The input signal* $\mathbf{\Omega} : \mathsf{SO}(n) \times \mathcal{P}(n) \to \mathsf{so}(n)$ *is given by* $\mathbf{\Omega} = \mathbf{P}\mathbf{R}^\top - \mathbf{R}\mathbf{P}$*, where* $\mathbf{P} \in \mathcal{P}(n)$ *is either a rank* $n-1$ *or a rank* n *matrix.*

The closed-loop system resulting from Algorithm 1 is $\dot{\mathbf{R}} = \mathbf{P} - \mathbf{R}\mathbf{P}\mathbf{R}$.

Theorem 1 *The trajectory of the closed-loop system generated by Algorithm 1 is given by* $\mathbf{R}(t; \mathbf{R}_0) = (\sinh(\mathbf{P}t) + \cosh(\mathbf{P}t)\mathbf{R}_0)(\cosh(\mathbf{P}t) + \sinh(\mathbf{P}t)\,\mathbf{R}_0)^{-1}$.

PROOF. Equation (4.1) is a matrix valued differential Ricatti equation that can be solved using the adjoint equations technique. Introduce two matrices $\mathbf{X}, \mathbf{Y} \in \mathsf{GL}(n, \mathbb{R})$ that satisfy $\dot{\mathbf{X}} = \mathbf{P}\mathbf{Y}, \quad \dot{\mathbf{Y}} = \mathbf{P}\mathbf{X}$ with initial conditions $\mathbf{X}(0; \mathbf{R}_0) = \mathbf{R}_0$, $\mathbf{Y}(0; \mathbf{R}_0) = \mathbf{I}$. Note that $\mathbf{R} = \mathbf{X}\mathbf{Y}^{-1}$ since $\mathbf{R}(0; \mathbf{R}_0) = \mathbf{X}(0; \mathbf{R}_0)\mathbf{Y}^{-1}(0;) = \mathbf{R}_0$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{X}\mathbf{Y}^{-1}) = \dot{\mathbf{X}}\mathbf{Y}^{-1} - \mathbf{X}\mathbf{Y}^{-1}\dot{\mathbf{Y}}\mathbf{Y}^{-1} = \mathbf{P} - \mathbf{R}\mathbf{P}\mathbf{R} = \dot{\mathbf{R}}$. The state equation of $\mathbf{X}$ and $\mathbf{Y}$ is linear and has the transition matrix

$$\exp\left(\begin{bmatrix} \mathbf{0} & \mathbf{P} \\ \mathbf{P} & \mathbf{0} \end{bmatrix} t\right) = \begin{bmatrix} \cosh(\mathbf{P}t) & \sinh(\mathbf{P}t) \\ \sinh(\mathbf{P}t) & \cosh(\mathbf{P}t) \end{bmatrix}.$$

By reversing the change of variables we find $\mathbf{R}(t; \mathbf{R}_0)$.■

Proposition 1 *The identity matrix is an almost globally asymptotically stable equilibrium of System 1 under Algorithm 1. The rate of convergence is locally exponential and the region of attraction is* $\mathsf{SO}(n) \backslash \mathcal{R}(n)$.

PROOF. The proof for the cases of $\operatorname{rank} \mathbf{P} = n$ and $\operatorname{rank} \mathbf{P} = n - 1$ are carried out separately. The proofs rely on the uniqueness property established in Lemma 1 which allows us to draw conclusions regarding control performance based on the exact solutions.

Consider the positive-definite case. The Frobenius norm is submultiplicative whereby $\|\mathbf{X}\mathbf{Y}^{-1} - \mathbf{I}\|_F = \|(\mathbf{X} - \mathbf{Y})\mathbf{Y}^{-1}\|_F \leq \|\mathbf{X} - \mathbf{Y}\|_F \cdot \|\mathbf{Y}^{-1}\|_F$. That $\lim_{t\to\infty} \|\mathbf{X}\mathbf{Y}^{-1} - \mathbf{I}\|_F = 0$ hence follows from $\lim_{t\to\infty} \mathbf{X} - \mathbf{Y} = \lim_{t\to\infty} \exp(-\mathbf{P}t)(\mathbf{R}_0 - \mathbf{I}) = \mathbf{0}$ and $\lim_{t\to\infty} \mathbf{Y}^{-1} = \lim_{t\to\infty}(\mathbf{I} + \tanh(\mathbf{P}t)\mathbf{R}_0)^{-1}\cosh^{-1}(\mathbf{P}t) = \mathbf{0}$. The last limit is given by Lemma 2 in Appendix A. It requires the assumption of $\mathbf{R}_0 \notin \mathcal{R}(n)$, *i.e.* $-1 \notin \sigma(\mathbf{R})$. Hence we have shown that $\mathbf{I}$ attracts all system trajectories such that $\mathbf{R}_0 \in \mathsf{SO}(n)\backslash\mathcal{R}(n)$. That $\mathcal{R}(n)$ does not belong to the region of attraction of $\mathbf{I}$ follows from Lemma 3 in Appendix A.

Use the first method of Lyapunov to show that $\mathbf{I}$ is a locally exponentially stable equilibrium of $\mathbf{R}$. Take $\mathbf{Z}$ to be the matrix corresponding to the linearization of $\mathbf{R} - \mathbf{I}$ around $\mathbf{0}$. Then $\dot{\mathbf{Z}} = -\mathbf{P}\mathbf{Z} - \mathbf{Z}\mathbf{P}$, with $\mathbf{Z}(0) = \mathbf{Z}_0 = \mathbf{R}_0 - \mathbf{I}$. Hence $\mathbf{Z}(t; \mathbf{Z}_0) = \exp(-\mathbf{P}t)\mathbf{Z}_0 \exp(-\mathbf{P}t)$, *i.e.* the linearized system is exponentially stable.

Consider the positive-semidefinite case. The eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbf{P} \in \mathcal{P}(n)$ form an orthogonal basis of $\mathbb{R}^n$ by virtue of the spectral theorem. Let 0 be the eigenvalue corresponding to $\mathbf{v}_n$. Let $\mathbf{V} = [\mathbf{n}_1 \cdots \mathbf{v}_n$ and denote $\mathbf{P}$ expressed in the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ by

$$\mathbf{Q} = \mathbf{V}^\top \mathbf{P} \mathbf{V} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}.$$

Denote $\mathbf{R}$ expressed in the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ by $\mathbf{X}$, and write

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{x}_{12} \\ \mathbf{x}_{21} & x_{22} \end{bmatrix}.$$

It can be shown that $\dot{\mathbf{X}}_{11} = \mathbf{Q}_{11} - \mathbf{X}_{11}\mathbf{Q}_{11}\mathbf{X}_{11}$, whereby Theorem 1 gives $\mathbf{X}_{11} = (\sinh(\mathbf{Q}_{11}t) + \cosh(\mathbf{Q}_{11}t))(\cosh(\mathbf{Q}_{11}t) + \sinh(\mathbf{Q}_{11}t)\mathbf{X}_{11,0})^{-1}$.

Since $\operatorname{rank}\mathbf{Q} = \operatorname{rank}\mathbf{P}$, we find that $\mathbf{Q}_{11} \in \mathcal{P}(n-1) \cap \mathsf{GL}(n-1, \mathbb{R})$. What is more, $-1 \notin \sigma(\mathbf{X}_{11,0})$ follows from $\mathbf{R} \notin \mathcal{R}(n)$ by Lemma 6. The almost global attractiveness and stability of $\mathbf{I}$ as an equilibrium of $\mathbf{X}_{11}$ follows by reasoning analogously as done in the case of a positive definite $\mathbf{P}$. The corresponding properties of $\mathbf{x}_{12}, \mathbf{x}_{21}$, and x_{22} follow from the constraints on $\mathbf{X} \in \mathsf{SO}(n)$. This carries over to $\mathbf{R}$. ■

Remark 2 *A key step in the above proof makes use of the constraints on $\mathbf{X} \in \mathsf{SO}(n)$ to conclude the attractiveness and stability properties of $\mathbf{x}_{12}, \mathbf{x}_{21}$, and x_{22} based on those of $\mathbf{X}_{11}$. This would not be possible if* $\operatorname{rank}\mathbf{P} \leq n-2$.

4.2 Closed-Loop System 2

The following closed-loop systems are generated by a class of control laws which all share the property that the state and the input signals commute. This class is of interest since it reduces Problem 1 to Problem 2 (see below). Instead of solving a system with n^2 variables and $\frac{1}{2}n(n-1)$ degrees of freedom on the Lie group $\mathsf{SO}(n)$, a system that evolves on the Lie algebra $\mathsf{so}(n)$ is solved. The Lie algebra is a linear space where the number of variables equals the number of degrees of freedom.

Problem 2 *For a given almost globally stabilizing feedback law $\mathbf{\Omega} : \mathsf{SO}(n) \to \mathsf{so}(n)$, solve the autonomous system $\dot{\mathbf{S}} = \mathbf{\Omega}(\exp(\mathbf{S}))$ for $\mathbf{S}(t; \mathbf{S}_0)$,* i.e. *for $\mathbf{S}$ as function of the time $t \in [0, \infty)$, and any initial condition $\mathbf{S}_0 \in \mathsf{so}(n)$.*

Algorithm 2 (Input and state commutes) *Let $\mathbf{F} : \mathsf{so}(n) \to \mathsf{so}(n)$ be a mapping that satisfies $[\mathbf{F}(\mathbf{S}), \mathbf{S}] = \mathbf{0}$. Moreover, suppose that the zero matrix is a globally asymptotically stable equilibrium of $\dot{\mathbf{S}} = \mathbf{F}(\mathbf{S})$, which is required to have a known, unique, and continuously differentiable solution $\mathbf{S}(t; \mathbf{S}_0)$. The input matrix $\mathbf{\Omega} : \mathsf{SO}(n)\backslash\mathcal{R}(n) \times \mathcal{P}(n) \to \mathsf{so}(n)$ is given by $\mathbf{\Omega}(\mathbf{R}) = \mathbf{F}(\operatorname{Log}\mathbf{R})$, where $\operatorname{Log} : \mathsf{SO}(n)\backslash\mathcal{R}(n) \to \mathsf{so}(n)$ denotes the principal matrix logarithm.*

The resulting closed-loop system is $\dot{\mathbf{R}} = \mathbf{F}(\operatorname{Log}\mathbf{R})\mathbf{R}$.

Theorem 2 *The trajectories generated by Algorithm 2 are given by $\mathbf{R}(t; \mathbf{R}_0) = \exp(\mathbf{S}(t; \operatorname{Log}(\mathbf{R}_0)))$.*

PROOF. Note that $\mathbf{R}(0; \mathbf{R}_0) = \mathbf{R}_0$. Since $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$, it follows that $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$, see Lemma 4 in Appendix A. Hence $\mathbf{\Omega}\mathbf{R} = \dot{\mathbf{R}} = \frac{\mathrm{d}}{\mathrm{d}t}\exp(\mathbf{S}) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\frac{1}{i!}\mathbf{S}^i = \dot{\mathbf{S}}\mathbf{R}$. By multiplying the above identity by $\mathbf{R}^{-1}$ from the right, we are left with $\dot{\mathbf{S}} = \mathbf{\Omega} = \mathbf{F}$. Also note that $\mathbf{R}(0; \mathbf{R}_0) = \mathbf{R}_0$. The expression for $\mathbf{R}(t; \mathbf{R}_0)$ is obtained from the exponential mapping. ■

Proposition 2 *Algorithm 2 stabilizes System 1 almost globally. The region of attraction is $\mathsf{SO}(n)\backslash\mathcal{R}(n)$.*

PROOF. The exact solution is unique by Lemma 1. Since the zero matrix is a globally asymptotically stable equilibrium of the system on $\mathsf{so}(n)$, we find that $\lim_{t\to\infty}\mathbf{S}(t; \operatorname{Log}(\mathbf{R}_0)) = \mathbf{0}$, $\lim_{t\to\infty}\mathbf{R}(t; \mathbf{R}_0) = \mathbf{I}$,*i.e.* the identity matrix is almost globally attractive. The region of attraction is $\mathsf{SO}(n)\backslash\mathcal{R}(n)$ since Algorithm 2 is restricted to this domain.

The identity matrix being a stable equilibrium follows from the stability of the system on $\mathsf{so}(n)$ and the continuity of the exponential mapping. More precisely, we require a pair (δ, ε) such that $d_R(\mathbf{I}, \mathbf{R}(t; \mathbf{R}_0)) \leq \varepsilon$ for all $t \in [0, \infty)$ when $d_R(\mathbf{I}, \mathbf{R}_0) \leq \delta$. Note that $d_R(\mathbf{I}, \mathbf{R}(t; \mathbf{R}_0)) = \frac{1}{\sqrt{2}}\|\log\mathbf{R}(t; \mathbf{R}_0)\|_F = \frac{1}{\sqrt{2}}\|\mathbf{S}(t; \mathbf{S}_0) - \mathbf{0}\|_F$. The stability of $\mathbf{0}$ as an equilibrium of $\mathbf{S}$ implies the existence of a pair (δ', ε') such that $\|\mathbf{S}(t; \mathbf{S}_0) - \mathbf{0}\|_F \leq \varepsilon'$ for all $t \in [0, \infty)$ when $\|\mathbf{S}_0 - \mathbf{0}\|_F \leq \delta'$. Hence we may take $(\delta, \varepsilon) = 1/\sqrt{2}(\delta', \varepsilon')$. ■

5 Examples

Algorithm 2 cannot be implemented without choosing a specific function $\mathbf{F}$ which satisfies the stated requirements. The class of feedback laws satisfying the commutativity requirement includes any $\mathbf{F} : \mathsf{so}(n) \to \mathsf{so}(n)$ that extends an analytic function $f : \mathbb{R} \to \mathbb{R}$ so that $\mathbf{F}(\mathbf{S})$ can be defined in terms of the Taylor expansion of f (Higham, 2008). Since it can be a nontrivial task to find such $\mathbf{F}$ that also stabilize the zero matrix, we provide the following three control laws, Algorithm 2.A–2.C, which are special cases of Algorithm 2.

5.1 Geodesic Feedback

An important special case of Algorithm 2 is the geodesic feedback based on the matrix logarithm (Bullo and Murray, 1995).

Algorithm 2.A *The geodesic feedback* $\mathbf{\Omega} = -\operatorname{Log}\mathbf{R}$ *yields* $\mathbf{F}(\mathbf{S}) = -\mathbf{S}$ *on* $\mathsf{so}(n)$.

The closed-loop system resulting from use of the feedback in Algorithm 2.A is $\dot{\mathbf{R}} = -\operatorname{Log}(\mathbf{R})\mathbf{R}$.

Theorem 2.A *The trajectories generated by Algorithm 2.A are given by* $\mathbf{R}(t;\mathbf{R}_0) = \exp(\mathrm{e}^{-t}\operatorname{Log}(\mathbf{R}_0))$.

PROOF. Note that $[\log\mathbf{R},\mathbf{R}] = \mathbf{0}$ (Higham, 2008). Moreover, the solution to $\dot{\mathbf{S}} = -\mathbf{S}$ is given by $\mathbf{S}(t;\mathbf{S}_0) = \mathrm{e}^{-t}\,\mathbf{S}_0$. Algorithm 2.A is hence a special case of Algorithm 2 wherefore the desired result follows by Theorem 2. ∎

5.2 Matrix Root

Algorithm 1 with $\mathbf{P} = \mathbf{I}$ satisfies $[\mathbf{\Omega},\log\mathbf{R}] = \mathbf{0}$. This also holds when $\mathbf{R}$ is replaced by its kth root $\mathbf{R}^{\frac{1}{k}}$ for $k \in \mathbb{N}$.

Algorithm 2.B (Matrix root) *The input matrix for this control law is given by* $\mathbf{\Omega} = k(\mathbf{R}^{-\frac{1}{k}} - \mathbf{R}^{\frac{1}{k}})$, *where the proportional gain factor* k *is used to scale the time dependence of* $\mathbf{R}$, i.e. $\mathbf{F}(\mathbf{S}) = -2k\sinh(\frac{1}{k}\mathbf{S})$.

The closed-loop system generated by Algorithm 2.B is $\dot{\mathbf{R}} = k(\mathbf{R}^{1-\frac{1}{k}} - \mathbf{R}^{1+\frac{1}{k}})$. The scalar gain $k \in \mathbb{N}$ is introduced so that the limit $\lim_{k\to\infty}\mathbf{\Omega} = -2\operatorname{Log}\mathbf{R}$; without it the limit would be zero.

Theorem 2.B *The trajectories generated by Algorithm 2.B are given by* $\mathbf{R}(t;\mathbf{R}_0) = (\tanh(t)\,\mathbf{I} + \mathbf{R}_0^{\frac{1}{k}})^k(\mathbf{I} + \tanh(t)\,\mathbf{R}_0^{\frac{1}{k}})^{-k}$.

PROOF. Introduce the variable $\mathbf{X} = \mathbf{R}^{\frac{1}{k}} \in \mathsf{SO}(n)$. Then $\dot{\mathbf{X}} = \frac{1}{k}\dot{\mathbf{R}}\mathbf{R}^{\frac{1}{k}-1} = \frac{1}{k}k(\mathbf{R}^{1-\frac{1}{k}} - \mathbf{R}^{1+\frac{1}{k}})\mathbf{R}^{\frac{1}{k}-1} = \mathbf{I} - \mathbf{R}^{\frac{2}{k}} = \mathbf{I} - \mathbf{X}^2$, which also results from setting $\mathbf{P} = \mathbf{I}$ in Algorithm 1. Reversing the change of variables in the solution for $\mathbf{X}$ given by Theorem 1 yields the desired expression. ∎

5.3 Cayley Transform

Another special case of Algorithm 2 is the Cayley transform and the higher order Cayley transforms.

Algorithm 2.C (Cayley transform) *The input matrix is given by* $\mathbf{\Omega} = k(\mathbf{I} - \mathbf{R}^{\frac{1}{k}})(\mathbf{I} + \mathbf{R}^{\frac{1}{k}})^{-1}$, i.e. *the kth order Cayley transform up to a scalar gain factor* $k \in \mathbb{N}$. *It yields* $\mathbf{F}(\mathbf{S}) = -k\tanh(\frac{1}{2k}\mathbf{S})$.

The closed-loop system resulting from use of the feedback in Algorithm 2.C is $\dot{\mathbf{R}} = k(\mathbf{I} - \mathbf{R}^{\frac{1}{k}})(\mathbf{I} + \mathbf{R}^{\frac{1}{k}})^{-1}\mathbf{R}$. The scalar gain $k \in \mathbb{N}$ is introduced so that the limit $\lim_{k\to\infty}\mathbf{\Omega} = -\frac{1}{2}\operatorname{Log}\mathbf{R}$; without it the limit would be the zero matrix.

Theorem 3 *The trajectories generated by Algorithm 2.C are given by* $\mathbf{R}(t;\mathbf{R}_0) = \exp(2k\operatorname{Atanh}\mathbf{Y}(t,\mathbf{X}_0))$, *where* $\mathbf{Y}(t;\mathbf{X}_0) = \sinh(\mathbf{X}_0)\left(\sinh^2(\mathbf{X}_0) + \mathrm{e}^t\,\mathbf{I}\right)^{-\frac{1}{2}}$, *and* $\mathbf{X}_0 = \frac{1}{2k}\operatorname{Log}\mathbf{R}_0$.

PROOF. That $\mathbf{Y}(t;\mathbf{X}_0)$ and $\operatorname{Atanh}\mathbf{Y}(t;\mathbf{X}_0)$ are well-defined follows from Lemma 5 in Appendix A. Change variables from $\mathbf{R}$ to $\mathbf{X} = \frac{1}{2k}\operatorname{Log}\mathbf{R}$ where the scaling is just a matter of notational convenience. Note that $\mathbf{\Omega} = -k\tanh\mathbf{X}$, whereby $[\mathbf{X},\mathbf{\Omega}] = \mathbf{0}$ and $\dot{\mathbf{X}} = \frac{1}{2k}\mathbf{\Omega}$ by Lemma 4.

Note that $\dot{\mathbf{Y}}(t;\mathbf{X}_0) = -\frac{1}{2}\sinh(\mathbf{X}_0)(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I})^{-\frac{3}{2}}$ $\mathrm{e}^t = -\frac{1}{2}\mathbf{Y}(t;\mathbf{X}_0)(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I})^{-1}\,\mathrm{e}^t = -\frac{1}{2}\mathbf{Y}(t;\mathbf{X}_0)$ $(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I})^{-1}(\sinh^2\mathbf{X}_0 + \mathrm{e}^t\,\mathbf{I} - \sinh^2\mathbf{X}_0) = -\frac{1}{2}\mathbf{Y}(t;\mathbf{X}_0)(\mathbf{I} - \mathbf{Y}(t;\mathbf{X}_0)^2)$ as required.

It remains to verify that $\mathbf{X}(t;\mathbf{X}_0) = \operatorname{Atanh}\mathbf{Y}(t;\mathbf{X}_0)$ solves $\dot{\mathbf{X}} = -\frac{1}{2}\tanh\mathbf{X}$. Note that $\mathbf{X}(0;\mathbf{X}_0) = \operatorname{Atanh}(\tanh(\mathbf{X}_0)) = \mathbf{X}_0$. Moreover, $\dot{\mathbf{X}}(t;\mathbf{X}_0) = (\mathbf{I} - \mathbf{Y}^2(t;\mathbf{X}_0))^{-1}\dot{\mathbf{Y}}(t;\mathbf{X}_0) = -\frac{1}{2}\mathbf{Y}(t;\mathbf{X}_0) = -\frac{1}{2}\tanh\mathbf{X}(t;\mathbf{X}_0)$, where the dynamics of $\mathbf{Y}(t;\mathbf{X}_0)$ are used. ∎

Proposition 2 reduces the stability analysis for Algorithm 2.A–2.C to proving the global asymptotical stability of the zero matrix on $\mathsf{so}(n)$. Further details are omitted for the sake of brevity.

5.4 Discussion

Algorithm 1 and 2 differ in several respects. Algorithm 1 have a constant positive-definite gain matrix that can be tuned for desired performance. It provides a continuous feedback but has a low input norm for rotations that are far from the identity, the disadvantage of which is slow convergence in the case of large errors (Lee, 2012). Algorithm 2.A provides a geodesic control law. The feedback laws of Algorithm 2.A and 2.B have input norms that are increasing functions of $\|\mathbf{S}\|_F$. This property can for example be useful in attitude control of satellites that are required to make large angle manoeuvres (Lee, 2012). The input norm of Algorithm 2.C diverges as the error is maximized. Althought such behaviour is normally undesired in practice, it does exemplify the width of the class of algorithms contained in Algorithm 2. The disadvantage of Algorithm 2.A–2.B as compared to Algorithm 1 is the discontinuity when $\mathbf{R} \in \mathcal{R}(n)$. Fig 1 illustrates some of these considerations for $\mathbf{R} \in \mathsf{SO}(3)$.

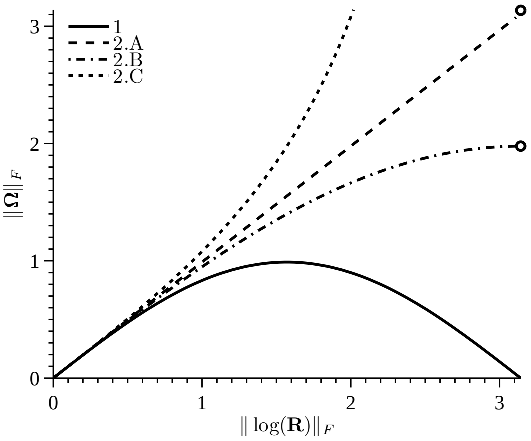

Fig. 1. Input norms for Algorithm 1, 2.A–2.C. The parameter $\mathbf{P} = \mathbf{I}$ in Algorithm 1, $k = 2$ in Algorithm 2.B, and $k = 1$ in Algorithm 2.C. The gains have been scaled to have equal slope at the origin.

6 Applications

The results of this paper have applications in the field of visual servo control with regards to model predictive control problems and sampled systems.

6.1 Model Predictive Control

The exact solutions can be used to pose a model predictive control (MPC) problem in terms of the feedback gain parameters of the control law. Algorithm 1 provide a gain matrix, $\mathbf{P} \in \mathcal{P}(n) \cap \mathsf{GL}(n, \mathbb{R})$. The potential benefit using optimization techniques in lieu with the solutions provided in this paper is hence greater than in (Markdahl et al., 2012) where only two parameters are available for tuning. This problem is of interest in visual servo control for the case of $n = 3$ and in applications that require the visualization of high dimensional data for the case of general n (Thakur, 2008).

Before turning to the MPC problem, consider a switched feedback control based on the extension of Algorithm 1 to the case of $\mathsf{SO}(n)$, where a time-dependence is introduced by replacing the gain matrix $\mathbf{P}$ by a piece-wise constant matrix valued function of time.

Algorithm 3 *Consider a feedback* $\mathbf{\Omega} = \mathbf{\Sigma}\mathbf{R}^\top - \mathbf{R}\mathbf{\Sigma}$, *where* $\mathbf{\Sigma} : [0, \infty) \to \mathcal{P}(n)$ *is a matrix valued switching signal. The matrix* $\mathbf{\Sigma} \in \mathcal{P}(n)$ *is piece-wise constant, right-continuous, has a strictly positive dwell time* Δt, *and satisfies* $\mathbf{\Sigma} \geq \varepsilon\mathbf{I}$ *for some constant* $\varepsilon \in (0, \infty)$. *The closed-loop system is* $\dot{\mathbf{R}} = \mathbf{\Sigma} - \mathbf{R}\mathbf{\Sigma}\mathbf{R}$.

Proposition 3 (Stability under switches) *Suppose System 1 with* $\mathbf{R}_0 \in \mathsf{SO}(n)/\mathcal{R}(n)$ *is governed by Algorithm 3. The identity matrix is a uniformly asymptotically stable equilibrium of* $\mathbf{R}$. *Its region of attraction is* $\mathsf{SO}(n)/\mathcal{R}(n)$.

PROOF. Consider the Lyapunov function $V = \operatorname{tr}(\mathbf{I} - \mathbf{R}) = n - \operatorname{tr}\mathbf{R}$. It satisfies $\dot{V} = -\operatorname{tr}\dot{\mathbf{R}} = -\operatorname{tr}(\mathbf{\Sigma} - \mathbf{R}\mathbf{\Sigma}\mathbf{R}) = -\operatorname{tr}(\mathbf{\Sigma}(\mathbf{I} - \mathbf{R}^2)) = -\langle\mathbf{\Sigma}, \mathbf{I} - \mathbf{R}^2\rangle = -\left\langle\mathbf{\Sigma}, \mathbf{I} - \frac{1}{2}\left(\mathbf{R}^2 + \mathbf{R}^{-2}\right) - \frac{1}{2}\left(\mathbf{R}^2 - \mathbf{R}^{-2}\right)\right\rangle = -\langle\mathbf{\Sigma}, \mathbf{I} - \frac{1}{2}(\mathbf{R}^2 + \mathbf{R}^{-2})\rangle = -\langle\mathbf{\Sigma}, -\frac{1}{2}(\mathbf{R} - \mathbf{R}^\top)^2\rangle \leq -\varepsilon\langle\mathbf{I}, -\frac{1}{2}(\mathbf{R} - \mathbf{R}^\top)^2\rangle = \frac{\varepsilon}{2}\langle -(\mathbf{R} - \mathbf{R}^\top), \mathbf{R} - \mathbf{R}^\top\rangle = -\frac{\varepsilon}{2}\|\mathbf{R} - \mathbf{R}^\top\|_F^2$, where the inequality follows from utilizing that $\mathbf{\Sigma} = \mathbf{X} + \varepsilon\mathbf{I} \geq$ for some $\mathbf{X} \geq \mathbf{0}$.

Note that $\dot{V} \leq 0$, and $\dot{V} = 0$ if and only if $\mathbf{R} = \mathbf{R}^\top$, *i.e.* only if $\mathbf{R} \in \{\mathbf{I}\} \cup \mathcal{R}(n)$. The set $\mathsf{SO}(n)\backslash\mathcal{R}(n)$ is invariant under Algorithm 3 by Lemma 7. The function $\dot{V}$ is therefore negative-definite independently of $\mathbf{\Sigma}$ over $\mathsf{SO}(n)\backslash\mathcal{R}(n)$, making V a common Lyapunov function for all switching modes. It follows that the identity is uniformly asymptotically stable (Liberzon, 2003). The invariance implies that all trajectories starting in $\mathsf{SO}(n)\backslash\mathcal{R}(n)$ must converge to $\mathbf{I}$. ■

Let the switching times be given by $\{t_i\}_{i=0}^{\infty}$. Since $\mathbf{\Sigma}$ is constant on each interval $I_i = [t_i, t_{i+1})$, the switched system has a solution on I_i given by Theorem 1. Set $\mathbf{R}(t_i; \mathbf{R}_0) = \lim_{t \uparrow t_i} \mathbf{R}(t; \mathbf{R}_0)$ at isolated switching times. This yields left continuity. Piece together such solutions for$[0, \infty) = \{t_i\}_{i=0}^{\infty} \cup \bigcup_{i=0}^{\infty} I_i$ to find a solution to the switched system. The function thus obtained is not continuously differentiable at the switching times but it is a solution in the sense of Carathéodory (Filippov, 1988).

Problem 3 (MPC) *Let a set of time instances* $\{t_i\}_{i=0}^{m} \subset [0, \infty)$, $m \in \mathbb{N} \cup \{\infty\}$, $t_m = \infty$ *and an initial condition* $\mathbf{R}_0 \in \mathsf{SO}(n)$ *be given. Suppose System 1 is* *governed by Algorithm 3. Denote* $\tau = t - t_0$*. Consider the problem of minimizing a continuous function* f *with respect to the input* $\mathbf{\Sigma} = \mathbf{\Sigma}_i \in \mathcal{P}(n) \cap \mathsf{GL}(n, \mathbb{R})$, $t \in [t_i, t_{i+1})$, i.e. *to minimize* $f(\mathbf{R}, \mathbf{\Sigma})$ *with respect to* $\mathbf{\Sigma}$ *subject to the constraints* $\mathbf{R}(t; t_0, \mathbf{R}_0) = \mathbf{\Phi}(\mathbf{R}(t_i; t_0, \mathbf{R}_0), t)$, $\mathbf{\Sigma} = \mathbf{\Sigma}_i$, $\varepsilon \mathbf{I} \le \mathbf{\Sigma}$, *and* $\mathbf{\Sigma} \le \rho \mathbf{I}$ *for all* $t \in [t_i, t_{i+1})$ *and all* $i = 0, \ldots, m$.

The first constraint is obtained from solving the closed-loop system. The lower bound is imposed in Algorithm 3 to ensure convergence under arbitrary switching. It then follows that $\lim_{t \to \infty} \mathbf{R}(t, t_0; \mathbf{\Sigma}, \mathbf{R}_0) = \mathbf{I}$ for any feasible solution $\{\mathbf{\Sigma}_i\}_{i=0}^m$ to the MPC problem. This frees the specification of f from any concerns regarding the asymptotical stability of the system. The upper bound confine $\mathbf{\Sigma}$ to a compact set when $m \in \mathbb{N}$, thereby guaranteeing the existence of a solution to the MPC problem by virtue of Weierstrass' theorem. Note that the assumption of $m \in \mathbb{N}$ pose no restriction in practice (it does make the attractiveness property of Proposition 3 trivial).

The MPC problem utilizes the transient phase of the system's evolution to carry out a task of secondary importance. The MPC problem could also be posed with the first constraint replaced by the state-equations of the switched system. The benefit gained by using the solution obtained from Theorem 1 as compared to not having access to them is to eliminate the computational cost of solving the switched system numerically.

Example 1 *Consider the problem of stabilizing the orientation of a camera while at some points in time wishing to see a desired view corresponding to the camera orientation* $\mathbf{R}_d \in \mathsf{SO}(3)$*. A possible choice of* f *is* $f(\mathbf{R}, \mathbf{\Sigma}) = \min_{\mathbf{\Sigma}, t} d_R(\mathbf{R}_d, \mathbf{R})$ *for a constant switching matrix* $\mathbf{\Sigma} \in \mathcal{P}(3) \cap \mathsf{GL}(3, \mathbb{R})$, i.e. *the choice of* $\mathbf{\Sigma}$ *is made at time zero.*

Note that the problem addressed in Example 1 is not solved by tracking a curve in $\mathsf{SO}(3)$ that interpolates the points $\mathbf{R}_0, \mathbf{R}_d$, and $\mathbf{I}$. The key idea is to utilize the transient phase of the system for additional benefit. This can also be done for the case of trajectory tracking.

6.2 Sampled Systems

Consider the problem of continuous time actuation subject to sensing that is either piece-wise unavailable in time or discrete time. The relevance of this problem in the context of attitude stabilization may *e.g.* be motived by cases where the attitude is calculated from images obtained by a camera for which (i) the reference used to obtain the attitude from the image is temporarily obscured or outside the image, or (ii) images are shot at a slow frame rate. Problem (i) arise in the field of visual servo control. The approach of this paper is well-suited for applications in visual servo control since it adopts the same kinematic system model (Chaumette and Hutchinson, 2006, 2007). Problems of type (ii) are commonly addressed using piece-wise constant input signals (K.J. Åström and Wittenmark, 1997), *i.e.* by applying a zero-order hold (ZOH). This section discuss the ZOH approach and an approach based on the flow of $\mathbf{\Omega R}$, *i.e.* the exact solutions to the closed-loop kinematics.

Assume that an output $\mathbf{Y} = \mathbf{R} \in \mathsf{SO}(n)$ is available for use in feedback control at times t such that $t \in I$ and that it is unavailable when $t \notin I$, where $I \subset [0, \infty)$ is closed and contains 0 (*i.e.* a sample $\mathbf{Y}_0 = \mathbf{R}_0$ is taken at time $t_0 = 0$). In the case of continuous sensing, suppose that I is such that the corresponding switching sequence has a dwell-time. In the case of discrete time sensing the states are sampled at each time instance of a sequence $\{t_i\}_{i=0}^m$, $m \in \mathbb{N} \cup \{\infty\}$.

System 2 *Consider the system* $\dot{\mathbf{R}} = \mathbf{\Omega R}$ *where* $\mathbf{R}_0 \in \mathsf{SO}(n)$ *and* $\mathbf{\Omega} : \mathsf{SO}(n) \to \mathsf{so}(n)$ *is the input signal. An output given by* $\mathbf{Y} = \mathbf{R}$ *is available for use in any feedback loop when* $t \in I$.

Problem 4 *Design a feedback algorithm for System 2 that stabilizes the identity matrix.*

Algorithm 4 (Zero-order hold) *The zero-order hold control is a time-varying feedback law given by* $\mathbf{U}(\mathbf{Y}, t) = \mathbf{\Omega}(\mathbf{Y}(t))$ *for* $t \in I$ *and* $\mathbf{U}(\mathbf{Y}, t) = \mathbf{\Omega}(\mathbf{Y}(s))$ *otherwise, where* $\mathbf{\Omega}$ *is any control law that stabilizes System 1 and* $s = \max I \cap [0, t]$.

Proposition 4 *Consider System 2 under Algorithm 4 with sample times* $\{t_i\}_{i=0}^\infty$ *and* $\mathbf{U} = -k_i \operatorname{Log} \mathbf{R}$ *for* $t \in [t_i, t_{i+1})$*. The identity is attractive if and only if* $\prod_{i=0}^\infty 1 - k_i \Delta t_i = 0$*, where* $\Delta t_i = t_{i+1} - t_i$.

PROOF. The resulting closed-loop system is a switched linear system which can be integrated to yield $\mathbf{R}(t_{j+1}) = \exp(-k_j \Delta t_j \operatorname{Log}(\mathbf{R}(t_j)))\mathbf{R}(t_j) = \mathbf{R}^{1 - k_j \Delta t_j}(t_j) = \mathbf{R}_0^{\prod_{i=0}^j 1 - k_i \Delta t_i}$. ∎

Proposition 4 places requirements on $\{k_i\}_{i=0}^\infty$ and $\{\Delta t_i\}_{i=0}^\infty$. A deadbeat control, *i.e.* finite time convergence, is obtained if $k_i \Delta t_i = 1$ for at least one i. In practice however, $\{\Delta t_i\}_{i=0}^\infty$ is typically not a design parameter. Moreover, there are upper and lower bounds on $\{k_i\}_{i=0}^\infty$ due to requirements on the minimum and maximum angular speed that arise from time constraints and saturation effects. For large sample times, there may not be any choice of $\{k_i\}_{i=0}^\infty$ that both satisfies the requirements of Proposition 4 and accommodates the additional constraints.

Algorithm 5 (Flow) *The flow algorithm is a time-varying feedback law given by* $\mathbf{U}(\mathbf{Y}, t) = \mathbf{\Omega}(\mathbf{Y}(t))$ *for* $t \in I$ *and* $\mathbf{U}(\mathbf{Y}, t) = \mathbf{\Omega}(\mathbf{\Phi}(\mathbf{Y}(s), t))$ *otherwise, where* $\mathbf{\Omega}$

is a stabilizing feedback law under which the closed-loop system has a known solution and $s = \max I \cap [0, t]$.

Algorithm 2 generates the same system trajectory as the stabilizing feedback would subject to continuous time sensing for all $t \in [0, \infty)$. It is clear that the flow approach have advantages over the ZOH. Algorithm 2 may *e.g.* be applied as an open loop control based on a single measurement in which case the ZOH approach would fail. It is also clear that Algorithm 4 have problems with large hold times which are tolerable for Algorithm 2 (neither algorithm guarantees robustness under such circumstances but that is a different matter).

7 Numerical Example

Numerical quadrature transfers System 1 to a discrete-time system that generates a sequence $\{\mathbf{R}_i\}_{i=0}^m$. The use of Lie group variational integrators ensures that $\mathbf{R}_i \in \mathsf{SO}(n)$ at all discrete time instances $\{t_i\}_{i=0}^m$ of the simulation (Lee et al., 2009). This is accomplished by setting $\mathbf{R}_{i+1} = \exp(\mathbf{\Omega}_i(t_{i+1} - t_i))\mathbf{R}_i$, where $\mathbf{\Omega}_i = \mathbf{\Omega}(\mathbf{R}_i)$.

System 2 under Algorithm 4 and 5 with the negative matrix logarithm, *i.e.* Algorithm 2.A, as the underlying attitude control law is simulated on $\mathsf{SO}(3)$. The sample time is constant and the gains are set to one, *i.e.* $k_i = 1, \forall\, i \in \mathbb{N}$, in Proposition 4. The initial condition is

$$\mathbf{R}_0 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}.$$

The results are displayed in Fig. 2. Note that Algorithm 4 behaves as predicted by Proposition 4 with a deadbeat control for $\Delta t = 1$, asymptotical stability for $\Delta t = 1.5$, and critical stability for $\Delta t = 2$. The trajectory the system generated by Algorithm 5 is invariant of the sample time. Although the deadbeat control yields faster convergence than Algorithm 5 it is not robust to changes in the sample time. Moreover, Algorithm 5 yields a feedback that is continuous in time whereas Algorithm 4 is discontinuous in time and gives rise to chattering behaviour.

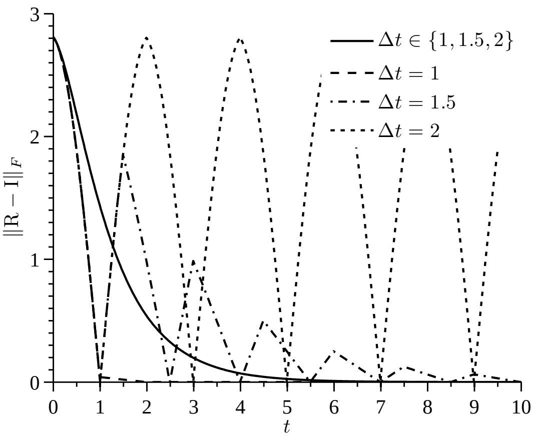

Fig. 2. The error $\|\mathbf{R} - \mathbf{I}\|_F$ for System 2 under Algorithm 4 (dashed lines) and Algorithm 5 (solid line).

8 Conclusions

This paper explores the question of whether it is possible to formulate a closed-loop system on $\mathsf{SO}(n)$ that is almost globally asymptotically stable and admits the exact solutions to be determined explicitly. The answer is yes, and it turns out to be possible for a large class of feedback laws. Moreover, the exact solutions can be expressed rather elegantly in terms of the matrix exponential. These expression provide complete knowledge of the transient and asymptotical behaviour of the system. Applications are found within the field of visual servo control in problems such as model predictive control and control of sampled systems.

References

F. Bullo and R. Murray. Proportional derivative control on the Euclidean group. In *Proceedings of the 3rd European Control Conference*, pages 1091–1097, 1995.

F. Chaumette and S. Hutchinson. Visual servo control. Part I: basic approaches. *IEEE Robotics & Automation Magazine*, 13(4):82–90, 2006.

F. Chaumette and S. Hutchinson. Visual servo control. Part II: advanced approaches. *IEEE Robotics & Automation Magazine*, 14(1):109–118, 2007.

W. Culver. On the existence and uniqueness of the real logarithm of a matrix. *Proceedings of the American Mathematical Society*, pages 1146–1151, 1966.

T. Dwyer III. Exact nonlinear control of large angle rotational maneuvers. *IEEE Transactions on Automatic Control*, 29(9):769–774, 1984.

A. Elipe and V. Lanchares. Exact solution of a triaxial gyrostat with one rotor. *Celestial Mechanics and Dynamical Astronomy*, 101:49–68, 2008.

A. Filippov. *Differential equations with discontinuous right-hand sides: control systems*. Kluwer, 1988.

N. Higham. *Functions of matrices: theory and computation*. Siam, 2008.

D. Hinrichsen and A. Pritchard. *Mathematical systems theory I: modelling, state space analysis, stability and robustness*. Springer, 2005.

G. Hu, W. MacKunis, N. Gans, W. Dixon, J. Chen, A. Behal, and D. Dawson. Homography-based visual servo control with imperfect camera calibration. *IEEE Transactions on Automatic Control*, 54(6):1318–1324, 2009.

M. Krstic, P.V. Kokotovic, and I. Kanellakopoulos. *Nonlinear and Adaptive Control Design*. Wiley, 1995.

C. Lageman, A. Sarlette, and R. Sepulchre. Synchronization with partial state feedback on SO(n). In *Proceedings of the 48th IEEE Conference on Decision and Control*, pages 1696–1701, 2009.

T. Lee. Exponential stability of an attitude tracking control system on SO(3) for large-angle rotational maneuvers. *Systems & Control Letters*, 61(1):231–237, 2012.

T. Lee, M. Leok, and N. H. McClamroch. Lagrangian mechanics and variational integrators on two-spheres. *International Journal for Numerical Methods in Engineering*, 79(9):1147–1174, 2009.

T. Lee, M. Leoky, and N.H. McClamroch. Geometric tracking control of a quadrotor UAV on SE(3). In *Proceedings of the 49th IEEE Conference on Decision and Control*, pages 5420–5425, 2010.

D. Liberzon. *Switching in systems and control.* Springer, 2003.

J. Markdahl and X. Hu. Analytical solutions to a class of feedback systems on SO(n). In *Proceedings of the 19th IFAC World Congress*, pages 445–450, 2014.

J. Markdahl, J. Hoppe, L. Wang, and X. Hu. Exact solutions to the closed loop kinematics of an almost globally stabilizing feedback law on SO(3). In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 2274–2279, 2012.

J. Markdahl, J. Thunberg, J. Hoppe, and X. Hu. Analytical solutions to feedback systems on the special orthogonal group SO(n). In *Proceedings of the 52th IEEE Conference on Decision and Control*, pages 5246–5251, 2013.

N. Matni and M. Horowitz. A convex approach to consensus on so(n). In *Proceedings of the 52nd Annual Allerton Conference on Communication, Control, and Computing*, pages 959–966, 2014.

A.V. Doroshin. Exact solutions for angular motion of coaxial bodies and attitude dynamics of gyrostat-satellites. *International Journal of Non-Linear Mechanics*, 50:68–74, 2012.

C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. On quaternion-based attitude control and the unwinding phenomenon. In *Proceedings of the 29th IEEE American Control Conference*, pages 299–304, 2011a.

C.G. Mayhew, R.G. Sanfelice, and A.R. Teel. Quaternion-based hybrid control for robust global attitude tracking. *IEEE Transactions on Automatic Control*, 56(11):2555–2566, 2011b.

D.H.S. Maithripala, J.M. Berg, and W.P. Dayawansa. Almost-global tracking of simple mechanical systems on a general class of lie groups. *IEEE Transactions on Automatic Control*, 51(2):216–225, 2006.

J.E. Hurtado and A.J. Sinclair. Hamel coefficients for the rotational motion of an n-dimensional rigid body. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 460(2052): 3613–3630, 2004.

K.J. Åström and B. Wittenmark. *Computer Controlled Systems: Theory and Design.* Prentice Hall, 1997.

M.A. Ayoubi and J.M. Longuski. Asymptotic theory for thrusting, spinning-up spacecraft maneuvers. *Acta Astronautica*, 64(7):810–831, 2009.

N.A. Chaturvedi, N.H. McClamroch, and D.S. Bernstein. Asymptotic smooth stabilization of the inverted 3-D pendulum. *IEEE Transactions on Automatic Control*, 54(6): 1204–1215, 2009.

N.A. Chaturvedi, A.K. Sanyal, and N.H. McClamroch. Rigid-body attitude control: Using rotation matrices for continuous singularity-free control laws. *IEEE Control Systems Magazine*, 31(3):30–51, 2011.

S.P. Bhat and D.S. Bernstein. A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon. *Systems & Control Letters*, 39(1):63–70, 2000.

S.S. Sastry. *Nonlinear systems: analysis, stability, and control.* Springer Verlag, 1999.

A. Roza and M. Maggiore. Position control for a class of vehicles in SE(3). In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 5631–5636. IEEE, 2012.

A. Sanyal, A. Fosbury, N.A. Chaturvedi, and D.S. Bernstein. Inertia-free spacecraft attitude trajectory tracking with internal-model-based disturbance rejection and almost global stabilization. In *Proceedings of the 27th IEEE American Control Conference*, pages 4830–4835, 2009.

K. Spindler. Optimal control on lie groups with applications to attitude control. *Mathematics of Control, Signals, and Systems*, 11(3):197–219, 1998.

S. Thakur. *Framework for visualizing and exploring high-dimensional geometry.* PhD thesis, Indiana University, 2008.

A Lemmas

Lemma 1 *The closed-loop systems generated by Algorithm 1 and 2 have unique solutions that belong to* $\mathsf{SO}(n)$ *for all* $t \in \mathbb{R}^+$.

PROOF. The proof in the case of (4.1) is similar to that in Markdahl et al. (2013). The assumptions made in Algorithm 2 ensures uniqueness of the solution to the system on $\mathsf{so}(n)$ and hence also to that on $\mathsf{SO}(n)$.

Lemma 2 *The matrices* $\cosh^{-1}(\mathbf{P})$ *and* $\tanh(\mathbf{P})$ *are well defined for* $\mathbf{P} \in \mathcal{P}(n) \cap \mathsf{GL}(n, \mathbb{R})$ *and satisfies* $\lim_{t\to\infty} \cosh^{-1}(\mathbf{P}t) = \mathbf{0}$, $\lim_{t\to\infty} \tanh(\mathbf{P}t) = \mathbf{I}$.

PROOF. This follows from the fact that $\mathbf{P}$ is normal (*i.e.* unitarily diagonalizable) and calculating the corresponding scalar limits (Higham, 2008). ■

Lemma 3 *The set* $\mathcal{R}(n)$ *is invariant under the dynamics* (4.1).

PROOF. Consider the time-evolution of $\sigma(\mathbf{R})$. Take any eigenpair $(\lambda, \mathbf{v})$ of $\mathbf{R}$ and impose the constraint $\|\mathbf{v}\|_2 = 1$. Recall the following relations $\mathbf{R}\mathbf{v} = \lambda\mathbf{v}, \mathbf{R}^\top\mathbf{v} = \lambda^*\mathbf{v}, \mathbf{v}^*\mathbf{R} = \lambda\mathbf{v}^*, \mathbf{v}^*\mathbf{R}^\top = \lambda^*\mathbf{v}^*$, which hold due to $\lambda^{-1} = \lambda^*$ for any complex λ of unit length. The matrix $\mathbf{R}$ being normal and analytic implies, as a consequence of Rellich's Theorem, that its eigenpairs are locally analytic functions of the time (Hinrichsen and Pritchard, 2005). Note that $\dot{\lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}^*\mathbf{R}\mathbf{v} = \dot{\mathbf{v}}^*\mathbf{R}\mathbf{v} + \mathbf{v}^*\dot{\mathbf{R}}\mathbf{v} + \mathbf{v}^*\mathbf{R}\dot{\mathbf{v}} = \mathbf{v}^*\dot{\mathbf{R}}\mathbf{v} + \lambda(\mathbf{v}^*\dot{\mathbf{v}} + \dot{\mathbf{v}}^*\mathbf{v}) = \mathbf{v}^*\dot{\mathbf{R}}\mathbf{v} + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_2^2 = \mathbf{v}^*\dot{\mathbf{R}}\mathbf{v} = \mathbf{v}^*\mathbf{\Sigma}\mathbf{v} - \mathbf{v}^*\mathbf{R}\mathbf{P}\mathbf{R}\mathbf{v} = (1-\lambda^2)\|\mathbf{P}^{\frac{1}{2}}\mathbf{v}\|_2^2$. The eigenpair $(-1, \mathbf{v})$ hence constitute an equilibrium. ■

Lemma 4 *The statements* $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$ *and* $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$ *are equivalent. Moreover, they imply that* $\dot{\mathbf{S}} = \mathbf{\Omega}$.

PROOF. See Markdahl and Hu (2014). ∎

Remark 3 *Lemma 4 is important because it allows us to replace the assumption of* $[\dot{\mathbf{S}}, \mathbf{S}] = \mathbf{0}$ *with* $[\mathbf{\Omega}, \mathbf{S}] = \mathbf{0}$. *The latter assumption is preferable since we assume* $\mathbf{\Omega}$ *to be the control input,* i.e. *we can design* $\mathbf{\Omega}$. *It is not, however, possible to chose* $\dot{\mathbf{S}}$ *in general.*

Lemma 5 *The expression for* $\mathbf{Y}(t)$ *and* $\operatorname{Atanh}\mathbf{Y}(t)$ *given in Theorem 3 are well-defined for all* $t \in [0, \infty)$.

PROOF. Since $\sigma(\mathbf{R}) \subset \{z \in \mathbb{C} \,|\, |z| = 1\}$, we may obtain $\mathbf{S} = \operatorname{Log}\mathbf{R}$ for $\mathbf{R} \notin \mathcal{R}(n)$ using the principal logarithm. Then $\sigma(\mathbf{S}) = \{i\lambda \in i\mathbb{R} \,|\, |\lambda| < \pi, \mathrm{e}^{i\lambda} \in \sigma(\mathbf{R})\}$.4 Since $\mathbf{X} = \frac{1}{2k}\mathbf{S}$ we find that all $\lambda \in \sigma(\mathbf{X})$ satisfy $|\lambda| < \frac{1}{2}\pi$. It follows that $\sigma(\sinh^2 \mathbf{X} + \mathrm{e}^t \mathbf{I}) = \{-\sin^2\lambda + \mathrm{e}^t \,|\, i\lambda \in \sigma(\mathbf{X})\}$. These eigenvalues are strictly positive, *i.e.* $\sinh^2 \mathbf{X} + \mathrm{e}^t \mathbf{I}$ is nonsingular. It is also normal, whereby its principal square root can be calculated as detailed in Section 2. This shows $\mathbf{Y}(t)$ to be well-defined.

Recall the definition of Atanh given in Appendix A. Note that $\mathbf{Y}(t)$ is skew-symmetric, implying that $\sigma(\mathbf{Y}(t)) \subset i\mathbb{R}$. Hence $\operatorname{Atanh}(\mathbf{Y}(t))$ is well-defined. ∎

Lemma 6 *Let* $\mathbf{R} \in \mathsf{SO}(n)$ *be partitioned as*

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{r}_{12} \\ \mathbf{r}_{21} & r_{22} \end{bmatrix},$$

then the spectrum of $\mathbf{R}$ *belongs to the unit disc in* $\mathbb{C}$, *and in particular it holds that* $-1 \notin \sigma(\mathbf{R})$ *implies* $-1 \notin \sigma\mathbf{R}_{11}$.

PROOF. We prove that $-1 \in \sigma(\mathbf{R}_{11})$ implies $-1 \in \sigma(\mathbf{R})$. Take any $\mathbf{v} \in \mathbb{R}^{n-1}$. The matrix $\mathbf{R}$ being orthogonal gives

$$\left\| \begin{bmatrix} \mathbf{R}_{11} & \mathbf{r}_{12} \\ \mathbf{r}_{21} & r_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} \right\|^2 = \|\mathbf{R}_{11}\mathbf{v}\|^2 + \|\mathbf{r}_{21}\mathbf{v}\|^2 = \|\mathbf{v}\|^2,$$

Hence $\|\mathbf{R}_{11}\mathbf{v}\| \leq \|\mathbf{v}\|$ for all $\mathbf{v}$, *i.e.* the spectrum of $\mathbf{R}_{11}$ is a subset of the unit disc in $\mathbb{C}$. By supposing that $\mathbf{R}_{11}\mathbf{v} = -\mathbf{v}$ we obtain $\|\mathbf{r}_{21}\mathbf{v}\| = 0$ whereby

$$\begin{bmatrix} \mathbf{R}_{11} & \mathbf{r}_{12} \\ \mathbf{r}_{21} & r_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{r}_{12} \\ \mathbf{r}_{21} & r_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} = -\begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix},$$

i.e. $(-1, [\,\mathbf{v}^\top\, 0\,]^\top)$ is an eigenpair of $\mathbf{R}$. ∎

Lemma 7 *The set* $\mathsf{SO}(n)\backslash\mathcal{R}(n)$ *is invariant under the dynamics generated by Algorithm 1.*

PROOF. By reasoning as done in the proof of Lemma 3, it can be shown that $\operatorname{Re}\dot{\lambda} = \|\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{v}\|_2^2(1 - (\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2)$, $\operatorname{Im}\dot{\lambda} = -2\|\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{v}\|_2^2 \operatorname{Re}\lambda \operatorname{Im}\lambda$. Since $-2\|\mathbf{\Sigma}^{\frac{1}{2}}\mathbf{v}\|_2^2 \operatorname{Re}\lambda > \varepsilon > 0$ for $\operatorname{Re}\lambda < -\frac{1}{2}$ (due to $\mathbf{\Sigma} \geq \varepsilon\mathbf{I}$), it follows that λ cannot converge to -1. ∎