ADS MANIFOLDS WITH PARTICLES AND EARTHQUAKES ON SINGULAR SURFACES

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Abstract. We prove two related results. The first is an "Earthquake Theorem" for closed hyperbolic surfaces with cone singularities where the total angle is less than π: any two such metrics in are connected by a unique left earthquake. The second result is that the space of "globally hyperbolic" AdS manifolds with "particles" – cone singularities (of given angle) along time-like lines – is parametrized by the product of two copies of the Teichmüller space with some marked points (corresponding to the cone singularities). The two statements are proved together.

1. Introduction and results

The Earthquake Theorem. Let Σ be a closed surface with a hyperbolic metric h, and let λ be a measured lamination on Σ. Then λ can be uniquely realized as a measured geodesic lamination for h. Thurston [18, 19] defined the image of h by the *right earthquake* along λ, called $E^r_\lambda(h)$ here, in a way which can be described simply when the support of λ is a disjoint union of closed curves: it is obtained by cutting Σ along each geodesics in the support of λ, doing a fractional Dehn twist by the length corresponding to the weight assigned to the curve by λ, and gluing back. This defines a map:

$$E^r : \mathcal{ML}_\Sigma \times \mathcal{T}_\Sigma \to \mathcal{T}_\Sigma \ .$$

Thurston then proved that for any given h the map $E^r_.(h) : \mathcal{ML}_\Sigma \to \mathcal{T}_\Sigma$ is bijective. In other terms, given $h, h' \in \mathcal{T}_\Sigma$, there is a unique $\lambda \in \mathcal{ML}_\Sigma$ such that $E^r_\lambda(h) = h'$. A different proof was given by Kerckhoff [10].

The Mess proof of the Earthquake Theorem. Yet another proof of the Earthquake Theorem was later discovered by Mess [14] as a by-product of the geometric properties of globally hyperbolic maximal compact (GHMC) Anti-de Sitter (AdS) manifolds. Mess discovered that such manifolds share several remarkable properties with quasifuchsian hyperbolic manifolds. In particular those manifolds (containing a space-like surface diffeomorphic to Σ) are uniquely determined by two hyperbolic metrics on Σ, called their "left" and "right" representations, which are analogs in the AdS context of the conformal metrics at infinity for quasifuchsian manifolds. This result of Mess can be interpreted as an analog of the Bers double uniformization theorem for quasifuchsian 3-manifolds.

GHMC AdS manifolds also have a convex core, with a boundary which has a hyperbolic induced metric and which is "pleated" along a measured geodesic laminations. Earthquakes are natural analogs in AdS geometry of grafting in quasifuchsian geometry, and some geometric properties of GHMC AdS manifold then yield the Earthquake Theorem as a consequence of the Mess parametrization of the space of those manifolds by $\mathcal{T}_\Sigma \times \mathcal{T}_\Sigma$.

Surfaces with cone singularities. From this point on we consider a closed surface Σ with n distinct marked points $x_1, \cdots, x_n$. We are interested in the hyperbolic metrics on Σ with cone singularities at the x_i. Given such a metric, we call θ_i the angle at x_i, $1 \leq i \leq n$. It follows from a result of Troyanov [20] and McOwen [12] (see also [9, 13]) that, given the θ_i, those metrics are in one-to-one correspondence with the conformal structures on Σ, so that, considered up to the isotopies fixing the x_i, those metrics are parametrized by $\mathcal{T}_{\Sigma,n}$, the Teichmüller space of Σ with n marked points. Setting $\theta := (\theta_1, \cdots, \theta_n)$, we will often denote by $\mathcal{T}_{\Sigma,n,\theta}$ the space of hyperbolic metrics on Σ with cone singularities at the x_i of angles given by the θ_i (considered up to isotopies fixing the marked points).

It is interesting to note at this point that the theory of geodesic laminations works on hyperbolic surfaces with cone singularities quite like it does on closed hyperbolic surfaces, as long as the cone angles θ_i are less than π.

Lemma 1.1. *Suppose that $\theta_i < \pi, 1 \leq i \leq n$. Then each measured lamination in the complement of the x_i in Σ can be realized uniquely as a geodesic lamination.*

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This statement (as much of the material used here) was discovered by Thurston. The proof (which is elementary) can be found in section 3. In addition, still under the hypothesis that the angles are less than π, the geodesic laminations cannot come too close to the singular points, and it follows that earthquakes along measured geodesic laminations do not change the angles at the cone points.

We will call $\mathcal{ML}_{\Sigma,n}$ the space of measured laminations on the complement of the x_i in Σ. It is well known (see [8]) that $\mathcal{ML}_{\Sigma,n}$ is homeomorphic to a ball of dimension $6g-6+2n$, where g is the genus of Σ.

Earthquakes on singular surfaces. The first result of this paper is an extension of the Earthquake Theorem to hyperbolic surfaces with cone singularities.

Theorem 1.2. *For all $h, h' \in \mathcal{T}_{\Sigma,n,\theta}$, there is a unique $\lambda \in \mathcal{ML}_{\Sigma,n}$ such that $E^r_\lambda(h) = h'$.*

AdS manifolds with particles. The second theme considered here — which is strongly related to the first — concerns 3-dimensional AdS manifolds with cone singularities along time-like lines. Such cone singularities are called "particles" here, since they are sometimes used in the $2+1$-gravity community to model massive, spin-less point particles. A precise definition is given in section 3. We are in particular interested in "globally hyperbolic compact maximal" AdS manifolds with "particles", extending those considered by Mess [14]. An AdS manifold with "particles" (cone singularities along time-like lines) M is GHMC if:

- it contains a closed, oriented, locally convex space-like surface S which is "orthogonal to the singular line" (in a manner which is described in section 3),
- every inextensible time-like curve in M intersects S exactly once,
- if M' is another AdS manifold with particles satisfying the first two properties in which M can be isometrically embedded, then $M' = M$.

Definition 1.3. We call $\mathcal{GH}_{\Sigma,n}$ the space of GHMC AdS metrics on $\Sigma \times \mathbb{R}$, with cone singularities at the lines $\{x_i\} \times \mathbb{R}$ for $1 \leq i \leq n$, considered up to isotopies fixing the singular lines. Given $\theta := (\theta_1, \cdots, \theta_n) \in (0,\pi)^n$, we also call $\mathcal{GH}_{\Sigma,n,\theta}$ the subspace of those metrics for which the angle at the line $\{x_i\} \times \mathbb{R}$ is θ_i.

Note that we will sometimes abuse notation and write about a GHMC AdS metric or a GHMC AdS manifold indifferently.

The right and left metrics associated to a GHMC manifold. Let $\theta := (\theta_1, \cdots, \theta_n) \in (0,\pi)^n$, and let $M \in \mathcal{GH}_{\Sigma,n,\theta}$. By definition M contains an oriented, space-like, convex surface S which is "orthogonal" to the singular lines. Let I be the induced metric on S, let J be the complex structure associated to I, and let B be the shape operator of S. It is then possible to define two metrics μ_l and μ_r on S as:

$$\mu_l := I((E+JB)\cdot, (E+JB)\cdot)\ ,\ \mu_r := I((E-JB)\cdot, (E-JB)\cdot)\ . \tag{1}$$

This corresponds to the metrics $I^\#_\pm$ defined in [11], the notation presented here is better suited for our needs. It is proved in [11] that those two metrics are hyperbolic, with cone singularities at the intersections of S with the "particles", where their angle is equal to the angle of M at the corresponding "particle". Moreover it is also proved in [11] that μ_l and μ_r (considered up to isotopy) do not depend on the choice of S. So this construction defines two maps

$$\mu_l, \mu_r : \mathcal{GH}_{\Sigma,n,\theta} \to \mathcal{T}_{\Sigma,n,\theta}\ .$$

When no "particle" is present, μ_l and μ_r are the hyperbolic metrics corresponding to the right and left representations considered by Mess [14].

The second main result here is that, as for non-singular GHMC AdS manifolds, the maps μ_l and μ_r provide a parametrization of $\mathcal{GH}_{\Sigma,n,\theta}$.

Theorem 1.4. *The map $(\mu_l, \mu_r) : \mathcal{GH}_{\Sigma,n,\theta} \to \mathcal{T}_{\Sigma,n,\theta} \times \mathcal{T}_{\Sigma,n,\theta}$ is one-to-one.*

This statement can be construed as an extensions of Mess' AdS version of the Bers double uniformization theorem to AdS manifolds with "particles". Note that on the hyperbolic side an extension of the Bers double uniformization theorem to quasifuchsian manifolds with "particles" – cone singularities of angle less than π along lines going from one boundary at infinity to the other – might well hold but it has not been proved yet (a first step is made in [15]).

The structure of GHMC AdS manifolds with particles. The proof of Theorem 1.2 uses Theorem 1.4 along with some properties which were discovered by Mess for non-singular GHMC AdS manifolds, which extend directly to GHMC AdS manifolds with particles. In particular, those manifolds contain a smallest convex subset — where "smallest" is understood with respect to the inclusion — called its convex core. We call $CC(M)$ the convex core of a GHMC AdS manifold M.

Lemma 1.5. *The boundary of $CC(M)$ has two connected components, called $\partial_+ CC(M)$ and $\partial_- CC(M)$. Each is a space-like surface (outside its intersection with the singular set of M), which is "orthogonal" to M_s (as defined in section 3). Its induced metric is hyperbolic, and it is "pleated" along a measured geodesic lamination.*

The structure of M can be readily understood from its convex core, exactly as for non-singular GHMC AdS manifolds.

Lemma 1.6.
1. *Let $x \in \partial_- CC(M)$, and let H be a space-like plane containing x which is a support plane of $CC(M)$ at x. Let n be the future-oriented unit vector at x which is orthogonal to H. Then the geodesic maximal segment starting from x in the direction of n has length $\pi/2$.*
2. *M is the union of the future of $\partial_- CC(M)$ and of the past of $\partial_+ CC(M)$, and their intersection is equal to $CC(M)$.*

Moreover, the metric on the future of the past boundary of the convex core can expressed, in a fairly simple way, in terms of the induced metric and the measured bending lamination, on the past boundary of the convex core.

Outline of the proofs. The basic idea of the proof of Theorem 1.2 is to use a "deformation" argument, in which we fix $\theta \in (0, \pi)^n$. We fix a hyperbolic $h \in \mathcal{T}_{\Sigma,n,\theta}$ and consider the map

$$\begin{array}{cccc} E^r_\cdot(h): & \mathcal{ML}_{\Sigma,n} & \to & \mathcal{T}_{\Sigma,n,\theta} \\ & \lambda & \mapsto & E^r_\lambda(h)\ . \end{array}$$

Our goal is to show that this map is homeomorphism. This follows from some basic points:

1. $E^r_\cdot(h)$ is a local homeomorphism,
2. it is proper,
3. its target space $\mathcal{T}_{\Sigma,n,\theta}$ is simply connected, while $\mathcal{ML}_{\Sigma,n}$ is connected.

A key point of the proof, however, is to use the settings of both Theorem 1.2 and Theorem 1.4. The first point is to prove the equivalence between those two statements; the fact that $E^r_\cdot(h)$ is a local homeomorphism can be proved in the setting of Theorem 1.4, while the fact that $E^r_\cdot(h)$ is proper can be shown fairly easily on the side of Theorem 1.2, where it appears as the following compactness statement.

Lemma 1.7. *Let $\theta = (\theta_1, \cdots, \theta_n)$ be fixed. Let $\mu \in \mathcal{T}_{\Sigma,n,\theta}$. The map $E^r_\cdot(\mu) : \mathcal{ML}_{\Sigma,n} \to \mathcal{T}_{\Sigma,n,\theta}$ is proper.*

The equivalence between Theorem 1.2 and Theorem 1.4 uses crucially the existence of a "convex core" in the GHMC AdS manifolds with particles considered here, as proved in section 5. It is then possible to prove, in section 6, that the two main theorems are equivalent, using the relations between the representation of the fundamental group of a GHMC AdS manifold with particles, on one hand, and the induced metrics and measured lamination on the boundary of the convex core, on the other.

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2. Background material

This section contains a number of important facts which are presumably well-known (or close to facts which are well-known), mostly concerning hyperbolic surfaces or AdS manifolds.

Geodesic laminations on cone surfaces. It was mentioned in the introduction that the theory of geodesic laminations work well on hyperbolic surfaces with cone singularities, provided that the singular angles are less than π. A key reason for this is that embedded geodesics cannot come too close to the singularities.

Remark 2.1. There is a decreasing function $\rho : (0, \pi) \to \mathbb{R}_{>0}$ as follows. Let S be a surface with a complete hyperbolic metric with cone singularities, with positive singular curvature, and let x be one of the singular points, with cone angle $\theta \in (0, \pi)$. Then any complete embedded geodesic in S is at distance at least $\rho(\theta)$ from x.

The proof is left to the reader, the basic idea is that, in a neighborhood of x, the metric on S is isometric to the metric on a hyperbolic cone of angle θ near the vertex. However, in such cones, the geodesics which are too close to the vertex cannot be embedded.

Lemma 2.2. *Let S be a closed hyperbolic surface with cone singularities, with angles in $(0, \pi)$ at the singular points. Let S_0 be the complement of the singular points. Then any lamination in S_0 can be realized as a geodesic lamination for the hyperbolic metric on S_0.*

Proof. Let $x_1, \cdots, x_n$ be the singular points on S, and let g be the hyperbolic metric on the complement of the x_i. It follows for instance from [20, 12] that there exists a one-parameter family of hyperbolic metrics $(g_t)_{t\in[0,1]}$ with cone singularities (or cusps) at the x_i such that $g_0 = g$ and that g_1 is a complete hyperbolic metric with cusps at the x_i. Moreover it is possible to demand that the angle at the cone singularities remain in $(0, \pi - \epsilon)$ for all $t \in (0, 1)$, where ϵ is some strictly positive constant.

Let λ be a lamination on S_0. It is well-known (see [8]) that it can be realized uniquely as a lamination which is geodesic for g_1. Let E be the set of all $t \in [0, 1]$ such that, for all $s \in [t, 1]$, λ can be realized uniquely as a geodesic lamination λ_s in g_s. Then E is open (because geodesic laminations can be deformed to "follow" a deformation of the underlying hyperbolic metric) and closed by the previous remark. So $0 \in E$, which proves the statement. ☐

Particles in AdS manifolds. The Anti-de Sitter space H^3_1 — called the AdS space here — is a Lorentz space of constant curvature -1. We only consider it here in dimension 3. It is not simply connected. It can be obtained as the quadric:

$$H^3_1 := \{x \in \mathbb{R}^4_2 \mid \langle x, x\rangle = -1\}$$

in the 4-dimensional flat space of index 2, $\mathbb{R}^4_2$, with the induced metric.

There is a useful projective model of the AdS space, obtained by considering the interior of a quadric of signature $(1, 1)$ in the sphere S^3, with its "Hilbert metric". We do not elaborate much here on the geometry of the AdS space and refer the reader to e.g. [16, 14].

Remark 2.3. Let us just remark that there is a natural identification of the isometry group of H^3_1 with $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. Moreover there is an isometric embedding of the hyperbolic plane $\mathbb{H}^2$ in H^3_1 such that

1) the image is a spacelike totally geodesic plane.

2) the embedding is $PSL(2,\mathbb{R})$ equivariant, where $PSL(2,\mathbb{R})$ acts on H^3_1 by the diagonal action.

We call such embedding the *standard* embedding.

An AdS manifold is a manifold locally modeled on the AdS space. We are interested here in AdS cone-manifolds of a special kind, which have "conical" singularities along time-like lines; such singularities will be called "particles". They have a simple local description.

Recall that a geodesic c in H^3_1 is called *time-like* if the restriction to c of the ambient metric is negative definite. Time-like geodesics in H^3_1 are closed curves of length 2π, see e.g. [16]. A totally geodesic plane P in H^3_1 is called time-like when the restriction to P of the ambient metric is Lorentzian. Totally geodesic time-like planes in H^3_1 are topologically cylinders, isometric to the 2-dimensional AdS space H^2_1. If c is a time-like geodesic contained in P, it bounds to half-planes. Any two such half-planes H and H', bounded by time-like geodesics, are isometric. More precisely, given an isometric ϕ from the boundary of H to the boundary of H', it has a unique extension Φ which is an isometry from H to H'.

Consider a time-like geodesic c in the AdS space, and let D be a domain in H^3_1 bounded by two totally geodesic time-like half-planes H and H', both bounded by c, with an angle θ between them (the angle is well-defined since c is supposed to be time-like). Because of the property recalled above concerning the extensions to half-planes of isometries on their boundaries, there is a unique way to glue isometrically H to H' so that the identification is the identity on c, and we call $H^3_{1,\theta}$ the resulting space. The complement of c in $H^3_{1,\theta}$ is locally modeled on the AdS space — there is no singularity at the gluing — while c corresponds to a cone singularity with angle θ, which is the local model we use for "particles".

We then define an *AdS manifold with particles* to be a manifold such that the complement of a disjoint union of open curves is endowed with a Lorentz metric, and such that each point has a neighborhood isometric either to an open subset of the AdS space or to a neighborhood of a point of c in $H^3_{1,\theta}$, for some value of θ in $(0, \pi)$.

Note that by construction the angle θ is constant along a "particle".

GHMC AdS manifolds with particles. In the local model described above for the neighborhood of the "particles" there is a natural notion of "horizontal" totally geodesic planes: those are the images, under the gluing construction, of the restriction to D of the totally geodesic planes orthogonal to c in H^3_1. Note that, under the gluing construction, thoses planes are sent to totally geodesic surfaces, i.e., the intersection of those planes with the two half-planes bounding D are glued together and no singularity occurs (except the cone point). By construction there is a unique horizontal plane containing each point of c.

We define a *space-like surface* in $H^3_{1,\theta}$ to be a surface which:

- intersects the singular set c at exactly one point x,
- is space-like outside x and locally convex,

- is such that the tangent plane at a sequence of points converging to x converges to the tangent plane to the (unique) horizontal plane at x.

In other terms, what we call "space-like surfaces" are really surfaces which, in addition to being "space-like" outside the singular set, are "orthogonal" to the singular set in a natural way.

Returning to an AdS manifold with particles M, recall that any point in M has a neighborhood which is isometric to an open set in some $H^3_{1,\theta}$, for some $\theta \in (0, \pi)$. It is therefore quite natural to define a *space-like* surface in M as a subset which corresponds, in each of the neighborhoods, to a space-like surface in $H^3_{1,\theta}$.

There is also a natural notion of *time-like curves* in AdS manifolds with particles; they are curves which are time-like (in the usual sense) outside the singular set, and which might follow segments of the singular set, but in such a way that any time function is monotonous along them.

Globally hyperbolic AdS manifolds. We are now almost ready to define a GHMC AdS manifold with particles. Given an AdS manifold with particles M, a *Cauchy surface* in M is a closed, space-like surface such that any inextensible time-like curve in M intersects S exactly once.

Proposition 2.4. *If Σ is a Cauchy surface of a AdS manifold with particles M, then topologically $M = \Sigma \times \mathbb{R}$.*

This proposition can be proved by adapting the general result for globally hyperbolic spacetimes without singularities to the case with particles: one constructs a timelike vector-field on M that is tangent to the singular locus. The flow of such a field restricted to S realizes a diffeomorphism between a regular neighbourhood of $S \times \{0\}$ in $S \times \mathbb{R}$ and M.

Definition 2.5. Let M be an AdS manifold with particles. It is **convex globally hyperbolic** (called GH here) if it contains a Cauchy surface which is locally convex. It is **convex globally hyperbolic maximal** (or GHM) if moreover any GH AdS manifold with particles M' containing a subset isometric to M is itself isometric to M.

Proposition 2.6. *Let M be an AdS GH spacetime with particles. There exists a GHM AdS spacetime with particles M', called the maximal extension of M , in which M isometrically embeds. Moreover, two maximal extensions are isometric.*

Sketch of the proof. The proof of this proposition follows the same steps of the analogous result for GH spacetimes without singularities [5]. Actually the way to adapt the original proof of [5] to the case without singularity was suggested to the authors by Thierry Barbot.

The existence of the maximal extension is an application of Zorn's Lemma. We order the set of extensions of M by isometric inclusions. Such an order turns out to be inductive so a maximal element exists.

The uniqueness of the maximal extension is more delicate. One proves that given two extensions M_1, M_2, there is another extension containing both of them. The idea is to consider the pairs (N_1, N_2) such that

(1) N_i is a GH spacetime contained in M_i and containing M.
(2) the isometry of $M \to M_2$ extends to an isometry $N_1 \to M_2$ sending N_1 to N_2.

Clearly there is a natural order on such pairs. Again an application of Zorn's Lemma ensures that there is a maximal element among those pairs, say (N_1, N_2). The idea is to consider the space $\hat{M}$ obtained by gluing M_1 and M_2 identifying N_1 to N_2. If we prove that $\hat{M}$ is topologically a manifold, then it is clear that the Lorentzian structures of M_1 and M_2 induce a Lorentzian structure on $\hat{M}$ in such a way that the projection $M_1 \cup M_2 \to \hat{M}$ is an isometric embedding on each component. Moreover simple arguments show that a Cauchy surface of M is a Cauchy surface for $\hat{M}$, thus $\hat{M}$ is GH extension of M containing both M_1 and M_2.

Let us prove that $\hat{M}$ is a manifold. Since the projection π is open, every point has a neighbourhood homeomorphic to $\mathbb{R}^3$. The only point to check is that $\hat{M}$ is a Hausdorff space. By contradiction, assume there exist points $x, y \in \hat{M}$ whose neighbourhoods cannot be taken disjoint. The only possibility is that x lies on the frontier of N_1 in M_1 and y lies on the frontier of N_2. Since N_i is GH it is a general fact that its boundary in M_i is achronal (see [17]).

Thus, up to reversing time-orientation we may suppose that small timelike curves starting from x must be contained in N_1. Denote by $I^+_\epsilon(x) = \{\exp_x tv | t \in (0, \epsilon), v \text{ future-pointing unit timelike vector at } x\}$ the set of geodesics of length less than ϵ starting from x. For ϵ small, $I^+_\epsilon(x) \subset N_1$, thus $I^+_\epsilon(x)$ isometrically embeds in M_2.

Since every neighbourood of x meets every neighbourhood of y in $\hat{M}$, and since $\lim_{t \to 0} \exp_x tv = x$, we have that the image of $\exp_x tv$ in M_2 goes to y as $t \to 0$. Thus the identification between N_1 with N_2 sends $I^+_\epsilon(x)$ to $I^+_\epsilon(y)$. In particular such identification continuously extends at x, sending x to y. Now, since M_1 and M_2 are

AdS, we can choose small neighbourhoods U, V of x and y respectively, such that the isometry between $I^+_\epsilon(x)$ and $I^+_\epsilon(y)$ extends to an isometry between $I^+_\epsilon(x) \cup U$ and $I^+_\epsilon(y) \cup V$.

This induces an isometry between $N_1 \cup U$ and $N_2 \cup V$. Up to taking smaller U and V, the spaces $N_1 \cup U$ and $N_2 \cup V$ are GH, but this contradicts the maximality of (N_1, N_2). $\square$

In this paper we will deal with convex globally hyperbolic AdS structures with particles on a fixed topological support $\Sigma \times \mathbb{R}$ and fixed singular set equal to $\{p_1, \ldots, p_n\} \times \mathbb{R}$ (where p_i's are fixed points on Σ) with cone angles $\theta_1, \ldots, \theta_n$ respectively. Thanks to Proposition 2.6 we can restrict to *maximal* convex GH AdS spacetimes. The corresponding "Teichmüller space" — that is the space of GHM AdS structures on $\Sigma \times \mathbb{R}$ up to diffeomorphisms isotopic to the identity — will be $\mathcal{GH}_{\Sigma,n,\theta}$.

Remark 2.7. The condition that the Cauchy surface is locally convex is perhaps not necessary; it is conceivable that all maximal globally hyperbolic AdS manifolds — containing a Cauchy surface which is perhaps not locally convex — actually contain another Cauchy surface which is locallly convex. We do not elaborate on this point here and simply assume that this condition is satisfied, since it is necessary in the sequel.

The left and right representations of a GHMC AdS manifold. We have already mentioned in the introduction that $SO_0(2,2)$ is the product of two copies of $PSL(2,\mathbb{R})$. It follows that the representation $\phi_M : \Gamma \to SO_0(2,2)$ can be identified with a pair of representations (ϕ_l, ϕ_r) of Γ in $PSL(2,\mathbb{R})$.

Lemma 2.8. *ϕ_l and ϕ_r are the holonomy representations of the hyeprbolic metrics defined in (1) μ_l and μ_r.*

The proof can be found in [11].

3. Earthquakes on surfaces with particles

Let S denote a hyperbolic structure on Σ with cone singularities at $x_1, \ldots, x_n$, with cone angles $\theta = (\theta_1, \ldots, \theta_n) \in (0, \pi)$.

If λ is a weighted multicurve on S then *the right earthquake* along λ is the fractional negative Dehn twist along each curve with shear factor equal to the corresponding weight. The corresponding point in $\mathcal{T}_{\Sigma,n,\theta}$ will be denoted by $E^r_\lambda(S)$.

The following proposition ensures that the definition of $E^r_\lambda(S)$ can be extended by continuity to every measured geodesic lamination.

Proposition 3.1. *Let (λ_k) be a sequence of weighted multicurves converging to a measured geodesic lamination λ. Then the sequence $E^r_{\lambda_k}(S)$ of hyperbolic surfaces with cone singularities is convergent in $\mathcal{T}_{\Sigma,n,\theta}$.*

Given a measured geodesic lamination λ, the surface $E^r_\lambda(S)$ is the limit of $E^r_{\lambda_k}(S)$, where λ_k is any sequence of weighted multicurve converging to λ.

Corollary 3.2. *The map*

$$\begin{array}{rccc} E^r_\cdot(S): & \mathcal{ML}_{\Sigma,n} & \to & \mathcal{T}_{\Sigma,n,\theta} \\ & \lambda & \mapsto & E^r_\lambda(S) \end{array}$$

is continuous.

The remaining part of this Section will be devoted to proving Proposition 3.1.

Denote by $\tilde\Sigma$ the universal covering of $\Sigma \setminus \{x_1, \ldots, x_n\}$ and by $dev : \tilde\Sigma \to \mathbb{H}^2$ a developing map of S. Given a weighted multicurve λ, let $\tilde\lambda$ be its lifting on $\tilde\Sigma$. Given an oriented arc c in $\tilde\Sigma$ transverse to $\tilde\lambda$, consider the leaves of $\tilde\lambda$, say $l_1, \ldots, l_k$, cutting c. An orientation is induced on each l_i, by requiring that, at the intersection point with c, a positive tangent vector of c and a positive tangent vector of l_i form a positive basis.

For each i, denote by $u(l_i, a_i)$ the element of $PSL(2,\mathbb{R})$, representing a negative translation along $dev(l_i)$ with translation length equal to the weight a_i of l_i. A simple argument shows that the element

$$u(l_1, a_1) \circ \cdots \circ u(l_k, a_k)$$

actually depends only on the endpoints x, y of c. This element will be denoted by $\beta_\lambda(x, y)$.

In such a way a map

$$\beta_\lambda : \tilde\Sigma \times \tilde\Sigma \to PSL(2,\mathbb{R})$$

is defined. Such a map is a $\pi_1(\Sigma \setminus \{x_1, \ldots, x_n\})$-invariant $PSL(2,\mathbb{R})$-valued cocycle, that is

$$\begin{array}{ll} 1. & \beta_\lambda(x,y) \circ \beta_\lambda(y,z) = \beta_\lambda(x,z) \\ 2. & \beta_\lambda(\gamma x, \gamma y) = h(\gamma)\beta_\lambda(x,y)h(\gamma)^{-1} \end{array}$$

where $h : \pi_1(\Sigma \setminus \{x_1, \ldots, x_n\}) \to PSL(2,\mathbb{R})$ is the holonomy representation of S.

Moreover, given a point $x_0 \in \tilde{\Sigma} \setminus \tilde{\lambda}$ the map

$$\begin{array}{rccc} dev_\lambda : & \tilde{\Sigma} & \to \mathbb{H}^2 & \\ & x & \mapsto & \beta_\lambda(x, x_0) dev(x) \end{array}$$

is a kind of developing map for $E^r_\lambda(S)$: although it is not in general continuous, the pull-back by this map of the hyperbolic metric (defined in the complement of the support of lift of λ) extends to a hyperbolic metric for which $\tilde{S}$ is the metric covering of $E^r_\lambda(S)$. Furthermore

$$\begin{array}{rccc} h_\lambda : & \pi_1(\Sigma \setminus \{x_1, \ldots, x_n\}) & \to & PSL(2, \mathbb{R}) \\ & \gamma & \mapsto & \beta_\lambda(\gamma x_0, x_0) \circ h(\gamma) \end{array}$$

is the corresponding holonomy representation.

To prove Proposition 3.1 it is sufficient to show that for a sequence λ_n of weighted multicurves converging to a measured geodesic lamination λ, the sequence dev_{λ_n} converges to a developing map. This fact is an easy consequence of the following lemma.

Lemma 3.3. *If λ_k is a sequence of weighted multicurves converging to a measured geodesic lamination λ then β_{λ_k} converges to a $\pi_1(\Sigma \setminus \{x_1, \ldots, x_n\})$-invariant $PSL(2, \mathbb{R})$-valued cocycle.*

Proof. Since β_{λ_k} are cocycles it is sufficient to prove that $\beta_{\lambda_k}(x, y)$ converges if a geodesic segment c joins x to y.

The proof is based on the following lemma.

Lemma 3.4. *Let λ be a measured geodesic lamination. Let c be a geodesic arc in S joining two leaves l_1, l_2 of λ. Then either there exists an isometric immersion of a hyperbolic triangle with an ideal vertex in S, sending the compact edge on c and the ideal edges on l_1 and l_2 respectively or there exists a geodesic arc c' that satisfies the following properties:*

1. *It is homotopic to c through a family of arcs joining l_1 to l_2 in $S \setminus \{x_1, \ldots, x_n\}$.*
2. *It is orthogonal to both l_1 and l_2.*

The proof of this lemma will be postponed until the end of this proof. An easy consequence of Lemma 3.4 is that two leaves l and l' of λ cutting c are sent by dev to disjoint geodesics.

Take a partition of c in segments $c_1, \ldots, c_N$ with endpoints x_i, x_{i+1} such that the length of each c_i is less than ε. For every $k > 0$ and $i = 1, \ldots, N$ denote by $u_{k,i} = \beta_{\lambda_k}(x_i, x_{i+1})$, $m_{k,i}$ the mass of c_i with respect to λ_k and $l_{k,i}$ a leaf of λ_k meeting c_i. If ε is sufficiently small there exists a constant C such that

$$||u_{k,i} - u(l_{k,i}, m_{k,i})|| \leq C\varepsilon m_{k,i}$$

(the norm we consider is the operator norm of $PSL(2, \mathbb{R})$ and the inequality is a consequence of the fact that dev sends leaves of $\tilde{\lambda}$ cutting c to disjoint geodesics, see Chapter 3 of [7]).

If c_i intersects $\tilde{\lambda}$ then choose a leaf l_i of $\tilde{\lambda}$ and $l_{k,i}$ can be chosen converging to l_i as $k \to +\infty$. It follows that up to changing C

$$||u_{k,i} - u(l_i, m_{k,i})|| \leq C\varepsilon m_{k,i}\,.$$

Since $\beta_{\lambda_k}(x, y) = u_{k,1} \circ \cdots \circ u_{k,N}$ and the $u_{k,i}$ run in a compact set of $PSL(2, \mathbb{R})$, there exists a constant C' such that

$$||\beta_{\lambda_k}(x, y) - \beta_{\lambda_h}(x, y)|| \leq C' \sum_{i=1}^{N} ||u_{k,i} - u_{h,i}||\,.$$

Now if c_i intersects $\tilde{\lambda}$ then

$$||u_{k,i} - u_{h,i}|| \leq C\varepsilon(m_{k,i} + m_{h,i}) + |u(l_i, m_{k,i}) - u(l_i, m_{h,i})| \leq C\varepsilon(m_{k_i} + m_{h,i}) + C''|m_{k,i} - m_{h,i}|$$

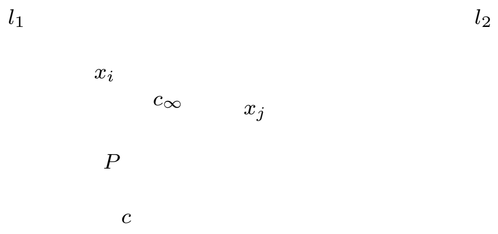
for some constant C''. Otherwise

$$||u_{k,i} - u_{h,i}|| \leq C'''(m_{k,i} + m_{h,i})$$

for some $C''' > 0$. If A denotes the union of the c_i that do not meet $\tilde{\lambda}$ then

$$||\beta_{\lambda_k}(x, y) - \beta_{\lambda_h}(x, y)|| \leq K(\varepsilon(\tilde{\lambda}_k(c) + \tilde{\lambda}_h(c)) + \sum_{i=1}^{N} |m_{k,i} - m_{h,i}| + \tilde{\lambda}_k(A) + \tilde{\lambda}_h(A))$$

for some $K > 0$. Since $\tilde{\lambda}_k \to \tilde{\lambda}$ as $k \to +\infty$ and since the mass of A with respect to $\tilde{\lambda}$ is 0 by definition, it follows that $\beta_{\lambda_k}(x, y)$ is a Cauchy sequence. □



Proof of Lemma 3.4. Consider the family $\mathcal{S}$ of arcs in $S \setminus \{x_1, \dots, x_n\}$ homotopic to c through arcs joining l_1 to l_2 and avoiding the singularities.

Consider a sequence $c_k \in \mathcal{S}$ minimizing the length. Up to passing to a subsequence, c_k converges either to a point $\hat{p}$ or to an arc c_∞ in S.

First consider the case that c_k converges to $\hat{p}$. Notice that either $c \cup c_k \cup l_1 \cup l_2$ bounds a hyperbolic quadrilateral, or they bound two hyperbolic triangles. In the latter situation the length of c_k would be bounded by the length of c multiplied by some constant depending only by the angles that c forms with l_1 and l_2. Thus we can suppose that for all k there exists an isometric immersion of a hyperbolic quadrilateral Q_k in S such that two opposite edges are sent respectively to c and c_k and the other two edges, say u_k, v_k, are sent to l_1 and l_2. Let a_k and b_k be the lengths of u_k and v_k. Denote by p and q the endpoints of c and by v, w the vectors tangent to l_1 and l_2 respectively at p and q. We have that $\exp_p a_k v$ and $\exp_p b_k w$ are connected by geodesics shorter and shorter. So, if both a_k and b_k remained finite, we whould find an intersection point between l_1 and l_2.

In the Poincaré model we can choose Q_k in such a way that the vertex sent to p is 0 and the edges sent to c are contained in a fixed geodesic, $\hat{c}$. Notice that there are two geodesics, say $\hat{l}_1, \hat{l}_2$ of $\mathbb{H}^2$ containing all u_k and v_k respectively. Thus they cannot intersect (otherwise a_k and b_k should be bounded) and cannot be ultraparallel (otherwise the length of c_k cannot go to 0). Thus they are parallel, and there is a triangle T with an ideal vertex bounded by $\hat{l}_1, \hat{l}_2$ and $\hat{c}$. Moreover, T is the union of Q_k. Since the immersions of Q_k's into F coincide on their intersections, it is possible to define an isometric immersion of T in S as stated in the Lemma.

Consider now the case c_k goes to an arc c_∞. If c_∞ avoids the singularities then it is clear that it belongs to $\mathcal{S}$. Moreover, by standard variational arguments, it is geodesic and orthogonal to both l_1 and l_2.

Suppose by contradiction that c_∞ intersects some singularity. Then it is not difficult to see that it is piecewise geodesic with vertices at singular points. Since it is homotopic to c there exists an embedded hyperbolic polygon P whose boundary is contained in $c \cup c' \cup l_1 \cup l_2$. Moreover, at least one vertex of this polygon can be supposed to lie on a singular point. Since the cone angles are less than π, P is convex. It follows that the length of c' can be shortened. □

4. Convex subsets in AdS manifolds with particles

Globally hyperbolic convex subsets. Let M be an AdS manifold with particles. We are interested in convex subsets in M, and in particular in the "convex core" which will be defined below. This convex core always contains a closed space-like surface, and this is also the case of many (perhaps all) convex subsets. For technical reasons, we are lead to include this property in the definition of convex subsets, so that we consider here "globally hyperbolic" convex subsets.

Definition 4.1. Let Ω be a non-empty, connected subset of M. It is *GH convex* if:

- Ω contains a closed, space-like surface S,
- Ω has a space-like, locally convex boundary.

It follows from the considerations made below that this definition is equivalent to a "global" definition: any geodesic segment in M, with endpoints in Ω, is actually contained in Ω. However the definition given here is more convenient here. We will often write "convex" instead of "GH convex", hoping that it does not confuse the reader.

Convex GHM AdS manifolds contain a compact GH convex subset. The reason why we consider only *convex* GHM manifolds is that they always contain a compact, GH convex subset. This will be used below to show that such a convex GHM manifold actually contains a smallest such convex subset, its "convex core".

Lemma 4.2. *Suppose that M contains a closed, space-like, locally convex surface S_0. Then M contains another such surface, arbitrarily close to S_0, which in addition is smooth.*

For the proof of this lemma, we will use the notion of the distance from S_0. It is defined at p as the *maximum* of the Lorentzian length of causal segments connecting p to S_0.

Lemma 4.3. *Let S be a convex spacelike surface in M, and let d be the distance from S. Then*

1. *d is continuous;*
2. *for every point p there is a geodesic segment joining p to S that avoids the singularities and realizes the distance;*
3. *if $d(p) < \pi/2$ and p is in the convex region bounded by S, then this segment is unique;*
4. *the function d is $C^{1,1}$ on the set of points in the convex region bounded by S satisfying $d(p) < \pi/2$;*
5. *if S is in the class C^k, then $C^{1,1}$ can be replaced by C^k in the previous point.*

Proof. The continuity of d is an easy consequence of the compactness of S.

For point (2), it is well known that the space of causal curves joining p to S is compact and the length function is upper-semicontinuous (for a proof see Sections 6 and 7 of [17]). Thus a causal path realizing the distance exists. A priori this path is piecewise geodesic with vertices on the singular locus. Suppose that a vertex, say r, on a cone singularity of angle θ_i, occurs. Then take points on the curve, r_-, r_+ respectively in the past and the future of r. If r_-, r_+ are chosen in some neighbourhood of r isometric to a convex neighbourhood of H^3_{1,θ_i}, then the segment joining r_- and r_+ has length bigger than the sum of the lengths joining r_- to r and r to r_+. Thus the curve can be lengthened.

For point (3) we consider the unit normal bundle of the surface S, i.e. the set of unit vectors $n \in T_xS$, for $x \in S$, such that the plane orthogonal to n at x is a support plane of S and n is towards the convex side of S. This unit normal bundle, N^1S, is a submanifold of the tangent bundle of M. The normal exponential map of S is the map $\exp : N^1S \times \mathbb{R} \to M$ sending a unit vector $n \in N^1S$ and a $t \in \mathbb{R}_{>0}$ to the endpoint of the time-like geodesic segment of length t starting from the basepoint of n with initial velocity equal to n. We denote this endpoint by $\exp_t(n)$.

Let $t \in (0, \pi/2)$, and let $S_t := \exp_t(N^1S)$. Clearly any point on the convex side of S which can be connected to S by a maximizing (time-like) geodesic segment g of length t has to be in S_t, since the plane orthogonal to g at its intersection with S is a support plane of S (my maximality of g).

We claim that $\exp_t$ is a homeomorphism from N^1S to S_t. The fact that $\exp_t$ is a local homeomorphism follows from the properties of Jacobi fields along time-like geodesics in AdS^3. It is not difficult to check that such Jacobi fields are of the form

$$J(s) = \cos(s)v_0 + \sin(s)v_1 \ ,$$

where v_0 and v_1 are parallel vector fields along the time-like geodesic. Given $n \in N^1S$, a first-order displacement of n on N^1S induces a Jacobi field $J(s)$ along the geodesic segment with initial velocity equal to n which is of the form above, with $\langle v_0, v_1\rangle \geq 0$ – because S is convex – but either v_0 or v_1 non-zero. If $v_0 \neq 0$, then

$$\langle J(s), v_0\rangle = \cos(s)\|v_0\|^2 + \sin(s)\langle v_0, v_1\rangle > 0$$

for all $s \in (0, \pi/2)$, while if $v_1 \neq 0$ then $\langle J(s), v_1\rangle > 0$ for all in $s \in (0, \pi/2)$. Therefore $J(s)$ does not vanish for $s \in (0, \pi/2)$, which means that $\exp_t$ is a local homeomorphism from N^1S to S_t for $t \in (0, \pi/2)$.

Suppose that $\exp_t$ is not a global homeomorphism for some $t \in (0, \pi/2)$. Let t_0 be the infimum of all $t \in (0, \pi/2)$ for which $\exp_t$ is not a global homeomorphism, then there are points $n_1, n_2 \in N^1S$ such that $\exp_{t_0}(n_1) = \exp_{t_0}(n_2)$. It follows from the definition of t_0 that the geodesic segments $\exp_{[0,t_0]}(n_1)$ and $\exp_{[0,t_0]}(n_2)$ are parallel at their common endpoint. This is clearly impossible unless $n_1 = n_2$. So $\exp_t$ is a global homeomorphism from N^1S to S_t for all $t \in (0, \pi/2)$, and this proves point (3).

For point (4), because of point (3) one can adapt to the AdS context a nice argument developed for the corresponding hyperbolic situation by Bowditch (see [7] and [14, 3] for the Lorentzian case). We include an outline here for completeness. Given a point p in the convex region bounded by S, let c be the timelike segment that realizes the distance from p to S (with endpoints p and $r \in S$). Let U be a convex neighbourhood of c that isometrically embeds in the Anti de Sitter space. With a little abuse we will confuse U with its image in H^3_1. Moreover, up to changing the time-orientation, we may suppose p to be the future endpoint of c.

In H^3_1 let us consider the function $d_1 : I^+(r) \to \mathbb{R}$ defined at x as the Lorentzian length of the timelike geodesic joining r to x. Clearly $d_1(x) \leq d(x)$ for any $x \in U$. Let P be the plane passing through x and orthogonal to c, and let d_2 be the distance from P. By the definition of d it is easy to see that P is a support plane for $S \cap U$ at r. Thus one sees that $d(x) \leq d_2(x)$ in a neighborhood of p (every timelike curve from x to S is extendible to a timelike arc joining x to P).

Moreover, an explicit computation shows that d_1 and d_2 are smooth and that

$$\begin{aligned} &d_1(p) = d(p) = d_2(p) \\ &\nabla d_1(p) = \nabla d_2(p) = v \end{aligned}$$

where v is the unit timelike vector tangent to c at p. This shows that d is differentiable at p and that $\nabla d(p) = v$. ☐

Proof of Lemma 4.2. For each $\pi/2 > \epsilon > 0$, let S_ϵ be the set of points at distance ϵ from S_0 on its concave side.

Then S_ϵ is not only convex, but it is uniformly convex: given a point $p \in S_\epsilon$ let q be a point on S such that the distance between p, q is ϵ (in general such a point is not unique). The germ of plane P through q and orthogonal to the geodesic segment between p and q, turns out to be a support plane for S. Moreover the set P_ϵ of points at distance ϵ from P is tangent to S_ϵ at p and in fact S_ϵ is contained in the convex region bounded by P_ϵ in a neighbourhood of p. The uniform convexity of S_ϵ follows then since the set of points at distance ϵ from a geodesic plane in H^3_1 is uniformly strictly convex (actually even umbilic). However S_ϵ is not smooth.

Now choose $\epsilon' \in (0, \epsilon)$ small enough, and let $S'_{\epsilon,\epsilon'}$ be the surface at constant distance ϵ' from S_ϵ on the *convex* side. Note that $S'_{\epsilon,\epsilon'}$ is *not* the surface at constant distance $\epsilon - \epsilon'$ from S.

Notice that not only $S'_{\epsilon,\epsilon'}$ converges to S_ϵ as $\epsilon' \to 0$, but also the unit normal bundle of $S'_{\epsilon,\epsilon'}$ converges to the unit normal bundle of S_ϵ in TM. Thus the unifom convexity of S_ϵ easily implies that $S'_{\epsilon,\epsilon'}$ is convex (and also uniformly convex) for ϵ' close to 0.

The last step is to smoothe $S'_{\epsilon,\epsilon'}$ to obtain a surface S as needed. Standard arguments (based on convolution and partitions of unity) can be used here to obtain a smooth embedding which is C^1-close to $S'_{\epsilon,\epsilon'}$; since the question is of a local nature, the corresponding (classical) results in the Euclidean 3-space can actually be used here through the projective model of AdS. ☐

Lemma 4.4. *Let M be a convex GHM AdS manifold with particles. Then M contains a compact, GH convex subset.*

Proof. By definition of a convex GHM manifold, M contains a closed, space-like, locally convex surface S_0 which is orthogonal to the particles. By the previous remark M also contains a closed, space-like, locally convex surface S, which in addition is smooth.

Let I and B be the induced metric and shape operator of S, respectively. Consider the manifold $S \times [0, \pi/2)$ with the metric:

$$g_S := -dt^2 + I((\cos(t)E + \sin(t)B)\cdot, (\cos(t)E + \sin(t)B)\cdot) ~,$$

where t is the coordinate in $[0, \pi/2)$ and E is the identity morphism on TS. A simple computation shows that the metric g_S is locally modeled on H^3_1 — except at the points which project to singular points of S, where there are of course cone singularities. Also by construction (and because M is maximal), $(S \times [0, \pi/2), g_S)$ embeds isometrically into M, with the surface $S \times \{t\}$ sent to the set of points at distance t from S on the convex side. Finally note that $(S \times [0, \pi/2), g_S)$ has locally convex boundary, and that its shape operator at $t = \pi/2$ is simply B^{-1} (unless B is positive semi-definite but not positive definite, in which case the boundary has "pleating lines"). So the closure of the image of $S \times [0, \pi/2)$ in M is a GH convex subset which is compact, as required. ☐

The distance to a convex subset is bounded. Another key property of GHM AdS manifolds is that the distance from a GH convex subset to the boundary is always less than $\pi/2$.

Lemma 4.5. *Let M be a convex GHM AdS manifold, and let K be a GH convex, compact subset of M. For each $x \in M \setminus K$, the maximal time-like geodesic segment(s) joining x to K has (have) length less than $\pi/2$.*

The proof is based on a simple proposition concerning the closest point projection on a convex subset in an AdS manifold. We call $\exp_K$ the normal exponential map, which is defined on the unit normal bundle of ∂K, and $\exp^r_K$ the restriction to the set of vectors of norm equal to r.

Proposition 4.6. *Let M be a GHM AdS manifold, and let K be a GH convex subset of M. Let $x \in M \setminus K$ be a regular point of M at distance less than $\pi/2$ from K.*

1. *There exists a (time-like) geodesic segment γ which has maximal length going from x to ∂K, which connects x to a point $y \in \partial K$. Let r be its length.*

(2) *∂K is $C^{1,1}$ smooth at y, with principal curvatures bounded from below, and its shape operator B is such that $\cos(r)E - \sin(r)B$ is non-negative.*
(3) *Let P be the support plane of K at y, let $\Pi' : T_yM \to T_xM$ be the parallel transport along γ, and let Π be its restriction from P to its image. Then the map $\Pi^{-1} \circ d\exp_K^r : P \to P$ is equal to:*

$$\Pi^{-1} \circ d\exp_K^r = \cos(r)E - \sin(r)B \ .$$

Proof. We suppose without loss of generality that x is in the future of K. Since M is GH and K contains a Cauchy surface, any past-oriented time-like curve starting from x intersects K. In particular this holds for all time-like geodesics, so a simple compactness argument based on the light cone of x shows that there exists a time-like geodesic segment of maximal length connecting x to K. This segment is not necessarily unique.

For point (2) note first that, since γ has maximal length, the plane P orthogonal to γ at y is a support plane of K. Fix a small neighborhood U of y, and let H be the set of points in U which can be joined to x by a time-like geodesic segment of length r. Since $r \in (0, \pi/2)$, H is a space-like umbilic surface of principal curvatures equal to $\text{cotan}(r)$. The definition of y shows that H is tangent to P at y. But y maximizes the distance from x to ∂K, it follows that, in U, ∂K is in the future of H (because any geodesic segment going from x to H has to intersect ∂K before H). So ∂K is "pinched" at y between P and H, which implies that it is $C^{1,1}$ with principal curvatures bounded between 0 — the principal curvatures of P — and $\text{cotan}(r)$ — the principal curvatures of H. This proves point (2).

For point (3) consider a geodesic segment $\gamma : [0, r] \to H^3_1$, and let u be a unit vector field along $\gamma([0, r])$ which is parallel and orthogonal to $\gamma([0, r])$. For all $v, w \in \mathbb{R}$ there is a unique Jacobi field Y along $\gamma([0, r])$ with $Y(0) = vu$ and $Y'(0) = wu$. It is equal to $Y(s) = (v\cos(s) + w\sin(s))u$ at $\gamma(r), 0 \leq s \leq r$. We can apply this computation with u equal to a principal vector at x, $v = 1$, and w equal to the principal curvature corresponding to x. This yields point (3). □

Proof of Lemma 4.5. We suppose again, still without loss of generality, that x is in the future of K.

The first point is that there exists a space-like curve $c : [0, L] \to M$ which begins on ∂K and ends at x. This follows again from the global hyperbolicity of M; since every past-directed time-like and light-like curve starting from x intersects K, it is also true for some space-like curves starting from x.

Consider the distance to K, it is a function, which we still call r, defined in $M \setminus K$. We suppose (by contradiction) that $x = c(L)$ is at distance at least $\pi/2$ from K, and let t_0 be the minimum of all $t \in [0, L]$ for which $c(t)$ is at distance at least $\pi/2$ from K.

The function r is continuous but not C^1, since there are points which are joined to K by two maximizing (time-like) geodesic segments. Notice that the function r could be regarded as the cosmological time of the spacetime $I^+(K)$ – that is $r(x)$ is the sup of the Lorentzian lengths of the timelike curves contained in $I^+(K)$ with future endpoint at x. Thus, from the general result in [2], r is a semi-convex function, i.e. in local charts it is the sum of a convex function and a smooth function. In particular r is twice differentiable almost everywhere. It follows that it is possible to choose a generic space-like curve c, which intersects the points where r is not differentiable on a set E of measure 0, so that r is continuous and, for $s \notin E$, r is differentiable at s and there is a unique maximizing segment between $c(s)$ and K. At such points, we call $\pi(s)$ the endpoint on ∂K of the maximizing curve between x and K. Let us stress that, since r is semi-convex, then $(r \circ c)' \in L^\infty([0, L])$, that implies that $r \circ c$ is an absolutely continuous function (it coincides with the integral of its derivative).

So $\pi : [0, t_0] \to \partial K$ is well-defined and continuous except on E, and it is Lipschitz on the complement of E; its derivative $\pi'(s)$ is defined almost everywhere. We now suppose that c is parametrized in such a way that π is parametrized at speed 1 on the complement of E.

It follows from point (3) of the previous proposition that for $s \notin E$ the norm of the image of $\pi'(s)$ by the map $\exp_K^{r(s)}$ is equal to $\|(\cos(r)E - \sin(r)B)\pi'(s)\|$, so that (using point (2) of the proposition) it is bounded by $\cos(r)$. But this vector is the "horizontal" component of $c'(s)$ (its projection on the kernel of dr in $T_{c(s)}M$). Since c is space-like, it follows that $|(r \circ c)'(s)| \leq \cos(r)$. So the function $r \circ c$ is well-defined on $[0, t_0]$, absolutely continuous, and it is a solution outside E of the differential inequality: $y'(s) \leq \cos(y(s))$, with $y(0) = 0$ by definition of c. It follows quite directly that $y(t_0) < \pi/2$, which contradicts the definition of t_0. This shows that x is at distance less than $\pi/2$ from K, as announced. □

The existence of the convex core. It is now possible, using in particular Lemma 4.5, to show that the intersection between two GH convex subsets of M is itself non-empty and GH convex. This is a key point in proving the existence of a "convex core" in M.

We will use the following simple remark.

Lemma 4.7. *Let Ω be a GH convex subset of an AdS spacetime M. Each boundary component of Ω is either convex in the past or convex in the future. (A locally convex spacelike surface S is said to be convex in the future (resp. past) if timelike unit vector tangent to S points towards the convex (resp. concave) side bounded by S.)*

Suppose Ω to be compact. Then there are 2 boundary components: the future boundary*, $\partial_+\Omega$ that is convex in the past, and the* past boundary*, $\partial_-\Omega$ that is convex in the future.*

Both $\partial_-\Omega$ and $\partial_+\Omega$ are Cauchy surfaces and

$$\Omega = I^+(\partial_-(\Omega)) \cap I^-(\partial_+(\Omega)) \tag{2}$$

Proof. For the first part, it is sufficient to notice that if the timelike vectors tangent to $x \in S$ points towards the convex side bounded by S, then the same holds for points y in a neighbourhood of x in S.

For the second part, the claim is that any inextensible causal curve intersect Ω in a compact interval whose future endpoint lies on $\partial_+\Omega$ and the past endpoint lies on $\partial_-\Omega$. Since Ω is compact in M then there is a Cauchy surface, say S_+, in the future of Ω and a Cauchy surface, say S_-, in the past of Ω. The curve c intersects S_- in a point x_- and S_+ in a point x_+. It is clear that $c \cap \Omega$ is contained in the segment bounded by x_- and x_+ on c. Now suppose I to be a connected component of $c \cap \Omega$. By definition we should have that the past endpoint of I lies on $\partial_-\Omega$ whereas the future endpoint lies on $\partial_+\Omega$. If $c \cap \Omega$ contained two connected components, then one could construct a causal curve in M with past endpoint, say x_P, on $\partial_+\Omega$ and future endpoint on $\partial_-\Omega$. Gluing c with a timelike geodesic with future endpoint at x_P and a timelike geodesic with past endpoint at x_F, produces a causal curve meeting S twice.

Consider now the flow ϕ of some future-oriented unit timelike vector-field X on M. It follows from the claim that the set, $\mathcal{S}$, of points $(x,t) \in S \times \mathbb{R}$ such that $\varphi_t(x) \in \Omega$ is a compact regular neighbourhood of S, that is $\mathcal{S} \cong S \times [-1,1]$ in such a way the image through ϕ of $S \times \{1\}$ is $\partial_+\Omega$ and the image of $S \times \{-1\}$ is $\partial_-\Omega$.

The identity (2) is a simple consequence of the claim. □

Proposition 4.8. *Let Ω, Ω' be two GH convex subsets of M. Then $\Omega \cap \Omega'$ is non-empty and GH convex.*

Proof. We claim that $\partial_+\Omega$ is contained in the future of $\partial_-\Omega'$ Suppose a point $x \in \partial_+\Omega$ is contained in the past of $\partial_-\Omega'$. Let M' be the AdS structure on $\partial_+\Omega \times [0, \pi/2)$ with metric given by

$$-dt^2 + I((\cos(t)E + \sin(t)B)\cdot, (\cos(t)E + \sin(t)B)\cdot)$$

where I is the first fundamental form on $\partial_+\Omega$, B is the shape operator, and $t \in [0, \pi/2)$. The past of $\partial_+\Omega$ is isometric to a regular neighbourhood of $S \times \{0\}$ in M'. If we glue the future of $\partial_+\Omega$ to M', we get a smooth spacetime M'' that contains M. The surface $\partial_+\Omega$ turns out to be a convex Cauchy surface so $\partial_+\Omega$ is convex GH.

Now in M'' there is a timelike geodesic with length equal to $\pi/2$ and future endpoint at x. Since x is in the past of $\partial_-\Omega'$, there is a timelike curve in the complement of Ω' with length bigger than $\pi/2$. This contradicts Lemma 4.5.

The fact that $\Omega \cap \Omega'$ is a non-empty convex subset follows directly from the claim and from (2). Moreover, the claim implies that every inextensible causal curve meets $\Omega \cap \Omega'$ in a non-empty interval. So the same arguments as in Lemma 4.7 show that $\Omega \cap \Omega' \cong S \times [0,1]$ and the boundary components of $\Omega \cap \Omega'$ are Cauchy surfaces. □

It follows from the previous lemma and from Lemma 4.4 that M contains a minimal GH convex subset.

Lemma 4.9. *Let M be a GHMC AdS manifold. Then M contains a non-empty GH convex subset $C(M)$ which is minimal: any non-empty GH convex subset Ω in M contains $C(M)$.*

Proof. We already know that M contains a GH convex subset (Lemma 4.4) and that the intersection of two GH convex subsets is GH convex. We can therefore consider the intersection K of all GH convex subsets in M, it is clear that it intersects all time-like curves in M and that it has locally convex boundary. The only point that remains to prove is that it has space-like boundary, since a limit of space-like surfaces could a priori be light-like. We do the proof here for the future boundary of K, denoted by ∂_+K, the same argument applies with obvious changes to the past boundary ∂_-K.

Let $S_\pm$ be respectivey the future and the past boundary of the GH convex compact subset appearing in the proof of Lemma 4.4. By construction, the distance between S_+ and S_- is $\pi/2$. Since K is contained in any GH convex sets, all points of ∂_+K are at distance less than $\pi/2$ from S_-.

Let Ω be the set of points at distance less than $\pi/2$ in the future of S_-. We consider the function u defined, on Ω, as the sine of the distance to S_-. By Lemma 4.3 u is a smooth function on Ω. Moreover it satisfies the equation:

$$\mathrm{Hess}(u) \leq ug ,$$

where g is the AdS metric on M. To check this equation note that it is satisfied (and is actually an equality) if S_- is totally geodesic. If $x \in \Omega$ and if $v \in T_x\Omega$ is the direction of the maximal geodesic segment from x to S_-, the Hessian of u behaves on $\mathbb{R}v$ as the Hessian of the distance to a geodesic plane, and on the plane orthogonal to v it is smaller since S_- is convex.

Now suppose that there exists a point $x \in \partial_+ K$ where $\partial_+ K$ is light-like. Then, since $\partial_+ K$ is convex, there exists a light-like past-oriented geodesic ray γ contained in a support plane of $\partial_+ K$ at x. Let (γ_k) be a sequence of space-like geodesic rays converging to γ, parametrized at unit speed. Then

$$\lim_{k\to\infty} (u \circ \gamma_k)'(0) = -\infty \ ,$$

and $(u\circ\gamma_k)'' \leq u \circ \gamma_k$ by the estimate on the Hessian of u. Therefore, for k large enough, γ_k intersects S_- at time t_k, with $\lim_{k\to\infty} t_k = 0$. It follows that γ intersects S_-, this contradicts the convexity of $\partial_+ K$. $\square$

As for non-singular AdS manifolds, we call $C(M)$ the *convex core* of M.

5. Pleated surfaces in AdS manifolds with particles

The geometry of the convex core. The boundary of the convex core of a GHM AdS manifold with particles shares all the important properties of the boundary of a non-singular GHM AdS manifold (as studied in [14], see also [1]), which are also the same as for quasifuchsian hyperbolic manifolds (including those with particles as in [11, 15]).

The first property is that boundary components of the convex core are pleated surfaces according to the following definition.

Definition 5.1. A convex **pleated surface** in M is a closed, convex, space-like surface, S orthogonal to the singular locus of M, which is developable: for any point $x \in S$, other than in the singular set of M, x is contained in the interior either of a geodesic segment of M contained in S, or of a geodesic disk contained in S.

Lemma 5.2. *Let M be a convex GHM AdS manifold with particles. Each boundary component of its convex core $C(M)$ is a pleated surface. If $p \in \partial C(M)$ lies on a singular line, then $C(M)$ has a unique support plane at p, say H, and $H \cap \partial C(M)$ contains a neighbourhood of p in H.*

Proof. Suppose by contradiction that some vertex occurs, that is, there exists a support plane P intersecting $\partial C(M)$ at exactly one point p. Without loss of generality we may suppose $p \in \partial_+ C(M)$.

Let v be the unit vector orthogonal to P at p and pointing in its past and consider the plane Q_ϵ orthogonal to the geodesic $\gamma(t) = \exp_p tv$ at $\gamma(\epsilon)$. For ϵ sufficiently small, then $Q_\epsilon \cap \partial_+ C(M)$ is topologically a circle and $Q \cap \partial_- C(M) = \varnothing$. Let Δ' the surface obtained by replacing in $\partial_+ C(M)$ the set $I^+(Q_\epsilon) \cap \partial_+ C(M)$ with $Q_\epsilon \cap C(M)$. Then Δ is a locally convex in the past surface. The convex domain $\Omega = I^-(\Delta) \cap I^+(\partial_- C(M))$ is then smaller than $C(M)$.

To show that $\partial C(M)$ is orthogonal to the singular locus of M, let $x \in \partial C(M)$ be a point contained in a singular curve of M. We consider the link $L_x(M)$ of M at x (the set of geodesic rays in M starting from x, with the natural angular metric). It is a (real) projective surface with two singular points. It is also endowed naturally with a "distance", coming from the angles between the geodesic rays starting from x, which is locally modeled on the de Sitter plane for space-like rays, and on the hyperbolic plane for the time-like rays (except the two rays which follow the singular line of M containing x). $L_x(M)$ also contains a closed curve γ_0, which is the union of rays orthogonal to the singular line containing x. It is geodesic for the metric just described (and a line for the real projective structure on $L_x(M)$). We call h the oriented distance to γ_0 (i.e., the length of the maximal geodesic connecting a point to γ_0), with a plus sign if this segment is past-oriented, and a minus sign if it is future-oriented). It is well-defined on the "de Sitter" part of $L_x(M)$ (corresponding to the space-like rays).

In $L_x(M)$ we consider the link $L_x(C(M))$ of $C(M)$ at x, namely, the set of geodesic rays starting from x for which a neighborhood of x is contained in $C(M)$. Since $\partial C(M)$ is space-like, $\partial L_x(C(M))$ is a space-like curve contained in the "de Sitter" part of $L_x(M)$. It follows from the convexity of $C(M)$ that $\partial L_x(C(M))$ is a locally convex curve.

Let y be point of $\partial L_x(C(M))$ where h attains its minimum. Then there is a geodesic segment γ_y in $L_x(M)$ containing y, which is a support line of $\partial L_x(C(M))$ at y, and such that the restriction of h to y is extremal at y. Considering the geometry of $L_x(M)$ shows that there is a maximal extension of γ_y as an embedded curve in $L_x(M)$ which is symmetric with respect to y (it has the same length on both sides of y).

The fact that the cone angle of M at the line containing x is less than π then shows that the restriction of h to γ_y has constant sign (positive, zero or negative), and that it has either a maximum (if its value is positive) or a minimum (if its value is negative) at y. However the convexity of $\partial L_x(C(M))$ shows that it is "below" γ_y:

FIGURE 1. The link of $C(M)$ at a singular point.

any geodesic orthogonal to γ_0 intersects γ_y and $\partial L_x(C(M))$ once, and the value of h at the intersection with γ_y is bigger than the value of h at the intersections with $\partial L_x(C(M))$.

This already shows that $h(y)$ cannot be positive: otherwise the restriction of h to $\partial L_x(C(M))$ would have a strict maximum at y, contradicting the definition of y. The same argument shows that if $h(y) = 0$ then $h = 0$ everywhere on $\partial L_x(C(M))$, i.e., $\partial L_x(C(M)) = \gamma_0$.

But $h(y)$ cannot be negative, otherwise h would be negative everywhere on γ_y, and therefore on $\partial L_x(C(M))$. This would mean that any plane intersecting the singular line containing x orthogonally a little "below" x would cut a small cone off $C(M)$, leaving a piece of $C(M)$ which would remain GH convex, and would thus contradict the definition of $C(M)$ as a minimal GH convex set.

So the only possibility is that $\partial L_x(C(M)) = \gamma_0$, so that $\partial C(M)$ is orthogonal at x to the singular locus of M. ☐

Let the bending locus of $\partial C(M)$, say L, be the complement of points that admit some support plane P such that $P \cap \partial C(M)$ is a neighbourhood of p in $\partial C(M)$. $L_\pm$ denote the intersection of L with $\partial_\pm C(M)$.

Lemma 5.3. *If $L_+ = \varnothing$ (resp. L_-) then $C(M) = \partial C_-(M) = \partial C_+(M)$ is a totally geodesic surface orthogonal to the singular locus.*

If $L \neq \varnothing$ then L is foliated by complete spacelike geodesics of M.

Proof. If L_+ is empty then $\partial_+ C(M)$ is totally geodesic. In particular it is a convex subset, so it is contained in the convex core.

The second part is more delicate. Suppose $p \in L$. There exists a unique $v \in T_pM$ such that the segment γ defined by $\gamma(t) = \exp_p tv$ is contained in L for $t \in (-\epsilon, \epsilon)$. Now consider the set

$$A = \{t \in \mathbb{R} | \exp_p tv \in L\}\,.$$

Since L is closed in $\partial C(M)$ the set A is closed. We will show that is open too. In fact it is sufficient to show that $\exp_p tv \in L$ for t small, because the fact that A contains a neighborhood of any of its points, say t_0, then follows by substracting t_0 to t for t close to t_0.

Suppose by contradiction that there is support plane Q that intersects $\partial_+ C(M)$ in a neighbourhood of $\exp_p tv$. Since $\gamma(s)$ is contained in $\partial_+ C(M)$ for $s \in (0, t)$ we have that $\gamma'(t)$ is contained in Q. It follows that γ is contained in Q. Thus $Q \cap \partial_+ C(M)$ contains the convex hull of a ball of center $\gamma(t)$ in Q and $\gamma(-t)$, that is, a neighbourhood of p. This contradicts the assumption that $p \in L$. ☐

Notice that M contains a geodesic Cauchy surface iff $L = \varnothing$ and this is the only case where the convex core has empty interior. The leaves of the foliation of L pointed out in Lemma 5.3 will be called the bending lines of the convex core.

Proposition 5.4. *$\partial C(M)$ carries an intrinsic $\mathrm{C}^{0,1}$-hyperbolic structure. L is the support of a measured geodesic lamination λ on $\partial C(M)$, called the bending lamination. The hyperbolic structure on $\partial_+ C(M)$ and the measured lamination $\lambda_+ = \lambda|_{\partial_+ C(M)}$ determine M.*

This statement is the analog with cone singularities of a well-known fact for non-singular hyperbolic metrics. We include a proof for the reader's convenience because the proof in the non-singular case relies heavily on the use of the developing map and therefore does not extend to hyperbolic surfaces with cone singularities.

Proof. First notice that we can choose coordinates around a point of $p \in \partial C(M)$, such that $\partial C(M)$ looks like the boundary of a convex set of $\mathbb{R}^3$. This show that points of $\partial C(M)$ are locally connected by Lipschitz paths. Thus each component of $\partial C(M)$ is connected by Lipschitz paths. So we can consider the path distance d on $\partial C(M)$. Notice that a priori it is a pseudo-distance.

Given $p \in \partial C(M)$ we construct a map $\iota : U \to \partial C(M)$, where U is some open set of $\mathbb{H}^2$, ι is bi-Lipschitz and preserves the length of curves, the image of ι is a neighbourhood of p.

We take a small neighbourhood W of p in M and fix once and for all an isometric identification of W with some convex subset of H^3_1. Suppose without loss of generality that $p \in \partial_+ C(M)$. $\partial_+ C(M) \cap W$ can be regarded as the germ of a pleated surface of H^3_1, more precisely we claim that there exists a complete convex in the past pleated surface in $\mathbb{H}^3_1$, say Δ, such that $\Delta \cap W' = \partial_+ C(M) \cap W'$ where W' is a compact neighbourhood of p.

The existence of the map ι follows from the claim, thanks to the general results about pleated surfaces in H^3_1 proved in [3]. Let us prove the claim.

If $p \notin L$ then we can take W' such that $W' \cap \partial_+ C(M)$ is totally geodesic so the claim follows.

Suppose $p \in L$. We can choose $W_1 \subset W$ that is pre-compact and such that the leaves of L meeting W_1 are exactly the leaves intersecting a small path transverse to the leaf through p.

Consider the family $\mathcal{F}$ of spacelike planes of $\mathbb{H}^3_1$ that are support planes of $\partial_+ C(M) \cap W$ at some point $p \in W_1$. The family $\mathcal{F}$ is pre-compact in $\mathbb{H}^3_1$ and there is a plane P_0 that does not intersect any element of $\mathcal{F}$: indeed for a fixed $p_0 \in \partial_+ C(M)$, points of $P \in \mathcal{F}$ are connected to p_0 along spacelike geodesics so it is sufficient to set P_0 to be the set of points at distance $\pi/2$ from p_0.

Now for $P \in \mathcal{F}$ denote by $C(P)$ the convex set of $\mathbb{H}^3_1$ bounded by P and P_0 and containing $\partial_+ C(M) \cap W$. Let $\Omega = \cap_{P \in \mathcal{F}} C(P)$. Let us enumerate some easy properties of Ω.

1. Ω is a convex set of $\mathbb{H}^3_1$
2. Ω has two boundary components. One of them is P_0. Let us set Δ to be the other component.
3. $\Delta \cap W_1 = \partial_+ C(M) \cap W_1$.

The last property is a consequence of the compactness of $\mathcal{F}$.

The last point to check is that we can choose W_1 so that Δ is pleated.

Suppose that for some W_1 some vertex occurs. By property (4) it is not difficult to see that there are two bending lines in $\partial_+ C(M)$, say l_1, l_2 such that $l_i \cap W \neq \varnothing$ and the geodesics of $\mathbb{H}^3_1$ extending $l_i \cap W$ meet each other at a point q. Consider points $p_i \in W \cap\ l_i$ and let T be the geodesic triangle of $\mathbb{H}^3_1$ with vertices at p_1, p_2 and q. Denote by $\hat{l}_i$ the segments joining p_i to q.

Clearly $T \cap W_1$ embeds in M. Moreover the embedding $\sigma : T \cap W \to M$ extends on $(T \cap W) \cup \hat{l}_1$ sending $\hat{l}_1$ on l_1 (and also to $(T \cap W) \cup \hat{l}_2$, but a priori not on $(T \cap W) \cup \hat{l}_1 \cup \hat{l}_2$). If this isometric embedding extends to an embedding on the whole T we find a contradiction: the image of $\hat{l}_i$ would be contained in l_i and so the image of q would be contained in $l_1 \cap l_2$.

Let v_0 the tangent vector at p_1 to the segment $p_1 p_2$ and let v_t be the parallel transport of v_0 along $\hat{l}_1$. Consider the foliation of T by geodesics arcs starting from $\hat{l}_1$ with direction v_t (this is a foliation since T is hyperbolic). Denote by a_t the length of the segment c_t.

Let v^*_t the parallel transport of v_0 along l_1 in M. Since the triangle T does not embed in M there exists t_0 such that if p_{t_0} is the corresponding point in l_1 then $\exp_{p_{t_0}} s v^*_{t_0}$ is defined for $s < b < a_{t_0}$, that means that $\exp_{p_{t_0}} b v^*_{t_0}$ is a singular point.

Notice that for a suitable choice of W_1 the factor a_t can be close to 0. On the other hand the vector v_0 runs in a compact set of TM (independent of W_1), and so does the family $\{v^*_t | t > 0\}$. Thus for any choice of W_1 the pleated surface Δ contains some vertices. Furthermore some sequence in L converges to some singular point. This implies that the closure of L contains points on the singular locus and this contradicts Lemma 5.2.

Eventually we can choose W' such that Δ is a complete pleated surface in $\mathbb{H}^3_1$. Thus there is an isometry (that is a bijective map preserving the distance) $B : \mathbb{H}^2 \to \Delta$. Then $B : B^{-1}(\partial_+ C(M) \cap W) \to \partial_+ C(M) \cap W$ is the desired isometry.

In fact in [3] the map B is described in some more explicit way. It is shown that there is a measured geodesic lamination λ_Δ on $\mathbb{H}^2$ such that

(1) the bending locus of Δ is the image of the support of λ_Δ;
(2) Up to post-composing with an isometry of $\mathbb{H}^3_1$ we have
$$B(x) = (\beta^R(x_0, x), \beta^L(x_0, x))I(x)$$
where x_0 is a point, β^R and β^L are the right and left cocycles associated to λ_Δ as in Section 3 and I is the standard embedding of $\mathbb{H}^2$ in H^3_1 defined in Section 2.
(3) The lamination λ_Δ is determined by the bending: the bending of $\mathbb{H}^2$ along λ and λ' coincide on a neighbourhood U iff $\lambda|_U = \lambda'|_U$.

This shows that it is possible to equip L on $W \cap \partial_+ C(M)$ with a transverse measure that is the image of the transverse measure on the corresponding neighbourhood of $\mathbb{H}^2$. Notice that by property (3) the transverse measures defined on different neighbourhoods match on the intersecton, giving rise to a transverse measure on L on the whole of $\partial_+ C(M)$. Let λ_+ be the corresponding lamination. Let F_+ be the hyperbolic structure on $\partial_+ C(M)$ and λ_+ be the bending lamination. By point (3) these data determines the developing map of $\partial_+ C(M)$ and thus the germ of the structure around $\partial_+ C(M)$. From the uniqueness of the maximal extension, they determines the whole of M. ☐

Remark 5.5. Note that the arguments given here are almost the same as for the corresponding hyperbolic setting, as in the appendix of [15]. Note also that the condition that the cone angles at the singularities are less than π seems to be really necessary to ensure that the boundary of the convex core is orthogonal to the singular lines. An interesting example can be found in [4]; it has “particles” with cone angles equal to π and it seems that the boundary of the convex core is not “orthogonal” to those particles, and that it is bent along a geodesic segment joining its intersections with the two singular lines.

Reconstruction from the boundary of the convex core. Thanks to Proposition 5.4 there is a well-defined injective map
$$\mathcal{GH}_{\Sigma,r,\theta} \to \mathcal{T}_{\Sigma,r,\theta} \times \mathcal{ML}_{\Sigma,r}$$
associating to M the hyperbolic metric on the future boundary of the convex core, say h_+, and the bending lamination, say λ_+. The aim of this subsection is to show that this map is bijective, giving a parametrization of $\mathcal{GH}_{\Sigma,r,\theta}$ in terms of the embeding data of the future boundary of the convex core (an analogous parametrization is possible in terms of the embedding data of the past boundary of the convex core).

Lemma 5.6. *Let S be a pleated surface in M convex in the past. Then there is no point in the past of S at distance $\pi/2$ from S.*

Proof. Suppose by contradiction that there exists a point p at distance $\pi/2$. Let $r(p) \in S$ the point realizing the distance and let P be the set of points in M that can be joined to p by a timelike segment of length $\pi/2$. P is an immersed geodesic plane and clearly it is a support plane for S. Moreover $S \cap P$ is convex in P and without vertices. Thus the interior of $P \cap S$ is contained in $S \setminus L$ whereas the boundary of $P \cap S$ is contained in L. In particular $S \cap P$ contains a leaf, say l, of L.

If l were closed, l would be homotopic to the constant loop p in M. Since l is not trivial in S, it is not trivial in M and this gives a contradiction.

Suppose now that l is open. We know there is a another leaf, l', in the closure of l (the proof of this point can be done as for non-singular hyperbolic surfaces, see e.g. [8]). Moreover we can choose l' that is not a boundary leaf — that means that support planes for points in l' intersects S just in a geodesic segment.

Take a sequence $q_n \in l$ converging to $q_\infty \in l'$. Timelike geodesics connecting p to q_n go to a geodesic c connecting p to q_∞ that is not spacelike. If this geodesic segment is timelike then q_∞ lies on P, that is the plane orthogonal to the segment joining p to q_∞ is a support plane for q_∞ that contains many q_n. This contradicts the choice of l'.

Thus c is lightlike. But this holds for every point of l'. On the other hand it is not difficult to prove that on a spacelike geodesic there are only a finite number of points connected to p along a lightlike geodesic. This leads to a contradiction. ☐

Proposition 5.7. *Let M be a GHM AdS manifold with particles. The only convex pleated surfaces in M are the future and past boundary components of the convex core.*

Proof. If S is a pleated surface convex in the past, it is contained in the future of the convex core $C(M)$. Take $p \in S$ that is not in $C(M)$ and take a point $q \in \partial_+ C(M)$ such that $p \in I^+(q)$. Take a smooth convex surface S' in a neighbourhood of $\partial_+ C(M)$ and $q' \in S'$ such that $p \in I^+(q')$. By Lemma 4.4, M contains a timelike geodesic of length equal to $\pi/2$ arriving at q. So there is a timelike path of length bigger than $\pi/2$ arriving at p. This contradicts Lemma 5.6 □

Proposition 5.8. *Let h be a hyperbolic metric with cone singularities (of angles $\theta_1, \cdots, \theta_n \in (0, \pi)$) on S, and let λ be a measured bending lamination in the complement of the cone points. There is a unique GHM AdS metric with particles on $S \times (0,1)$ such that h and λ are the induced metric and measured bending lamination on the future boundary of the convex core.*

Proof. The hyperbolic metric h and the measured lamination λ determine an isometric embedding of the universal cover of the complement of the cone points in S into H^3_1 which is equivariant under an action of the fundamental group Γ of the complement of the cone points in S. More precisely, the developing map can be explicitly written in terms of the left and right cocycles β^l and β^r associated to λ. In fact

$$dev(x) = (\beta^r(x_0, x), \beta^l(x_0, x)) I(dev_0(x)) \tag{3}$$

where dev_0 is the developing map of h and I is the standard embedding $\mathbb{H}^2$ into H^3_1.

The fact that dev is locally injective and locally convex can be proved as in [3] in the non-singular case. The only point to check is that it induces a hyperbolic structure on the surface S with cone singularities. On the other hand, since the singular locus is far from the lamination, the cocycles β^r and β^l are trivial on $\pi^{-1}(U)$ for some neighbourhood U of the puncture. Thus on $\pi^{-1}(U)$ the map dev is conjugated with dev_0, so the induced metric on S looks like h in a neighbourhood of a cone point.

Consider the normal exponential map of S towards the convex side of S. It is the map:

$$G : N^1 S \times (0, \pi/2) \to M \ .$$

Here $N^1 S$ is the unit normal bundle of S, i.e., the set of unit vectors at points of S for which the oriented orthogonal plane is a support plane of S. The map is defined by sending (n, t), where n is a unit vector at $x \in S$, to $\exp_x(tn)$, where $\exp_x$ is the exponential map at x.

The convexity of S then shows that this map is locally injective on $S \times (0, \pi/2)$ (as seen in the proof of Lemma 4.3). So this map can be used to pull back the AdS metric to a locally AdS metric on $S \times (0, \pi/2)$, with cone singularities at the lines $x \times (0, \pi/2)$, where x is a cone point of S. This shows that S has an embedding into an AdS manifold N with image a convex pleated surface with induced metric h and measured pleating lamination λ. By definition N is contained in a GHMC AdS manifold M, also containing a pleated surface with induced metric h and measured pleating lamination λ.

In addition, Lemma 5.7 shows that S can only be a connected component of the convex core of M. Since a GHMC AdS manifold is obviously determined by the future boundary of its convex core, the lemma follows. □

From the convex core to earthquakes. There is an important relation between convex pleated surfaces in GHMC AdS manifolds and earthquakes on hyperbolic surfaces, which was discovered by Mess [14] (see also the annotations in [1]). It can be stated for convex cores of GHMC AdS manifolds with particles as follows.

Lemma 5.9. *Let M be a GHM AdS manifold with particles. Let h_+, h_- be the induced metrics on the upper and lower boundaries of the convex core, let λ_+, λ_- be the measured bending lamination of those upper and lower boundary components, and let μ_l, μ_r be the right and left hyperbolic metrics. Then*

$$\mu_l = E_l(\lambda_+)(h_+) = E_r(\lambda_-)(h_-) \ , \ \mu_r = E_l(\lambda_-)(h_-) = E_r(\lambda_+)(h_+) \ .$$

It follows that $\mu_l = E_l(2\lambda_+)(\mu_r) = E_r(2\lambda_-)(\mu_r)$.

$$\begin{tikzcd}
 & h_+ \arrow[dl, "E_r(\lambda_+)"'] \arrow[dr, "E_l(\lambda_+)"] & \\
\mu_l & & \mu_r \\
 & h_- \arrow[ul, "E_l(\lambda_-)"] \arrow[ur, "E_r(\lambda_-)"'] &
\end{tikzcd}$$

Proof. As stated in [11], the holonomy of μ_l is the projection of the holonomy of M on the first factor of $Isom(\mathbb{H}^3_1) = PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. On the other hand, by formula (3), such a representation is simply

$$\rho_l(\gamma) = \beta_r(x_0, \gamma x_0)\rho_+(\gamma)$$

where ρ_+ is the holonomy for h_+. Thus by formula (3), μ_+ and $E_r(h_+)$ share the same holonomy.

Now for hyperbolic surfaces with cone singularities of angle less than π, the holonomy determines the hyperbolic structure. This can be proved by the same argument, based on pants decompositions, as for non-singular hyperbolic surfaces. It is proved in [6] that any maximal set of pairwise non-homotopic simple closed curves in the complement of the singular points can be realized (uniquely) as a geodesic multicurves, and that the lengths of the curves of this "pants decomposition" uniquely determines the metric. Since the lengths of the geodesic curves are determined by the holonomy, it follows that the metric is determined by the holonomy.

Since we are assuming cone angles less than π, the holonomy determines the structure, and we conclude that μ_+ is equal to $E_r(h_+)$. □

We can conclude from Lemma 5.9 that Theorem 1.4 and Theorem 1.2 are equivalent. In fact, the composition

$$\mathcal{T}_{\Sigma,n,\theta} \times \mathcal{ML}_{\Sigma,n} \xrightarrow{I_+} \mathcal{GH}_{\Sigma,n,\theta} \xrightarrow{(\mu_l,\mu_r)} \mathcal{T}_{\Sigma,n,\theta} \times \mathcal{T}_{\Sigma,n,\theta}$$

is the map

$$(h,\lambda) \mapsto (E^r_\lambda(F), E^l_\lambda(F))$$

Since $E^l_\lambda = (E^r_\lambda)^{-1}$, it is easy to see that the (μ_l,μ_r) is bijective if and only if so is the map

$$(h,\lambda) \mapsto (h, E^r_\lambda(h)),$$

and that in turn is equivalent to requiring that for all h the map $E^r_\cdot(h) : \mathcal{ML}_{\Sigma,n} \to \mathcal{T}_{\Sigma,n,\theta}$ is bijective.

6. Local deformations

This section is devoted to local (or infinitesimal) deformations of GHM AdS manifolds with particles.

Convex space-like surfaces in GHMC manifolds. In this section we consider a closed, convex space-like surface $S \subset M$ which is orthogonal to the singularities. We call S_r the regular set of S — the complement in S of the set of singular points — and $\Gamma := \pi_1(S_r)$. There is a natural morphism

$$\phi_M : \Gamma \to SO_0(2,2)$$

obtained from the holonomy representation of M, because $\pi_1(S) = \pi_1(M_r)$.

Given S it is also possible to define two hyperbolic metrics on it, called μ_l and μ_r in the introduction. Note that this depends on the fact that S is convex (it is actually sufficient to suppose that the curvature of the induced metric on S does not vanish, but this happens to be true for all convex surfaces).

Those two metrics have cone singularities at the intersections of S with the singular lines of M, see [11]. Note that this point depends on the fact that S is orthogonal to the singular lines of M.

The left and right representations. We introduce a simple notation. For each cone point x_i of S, $1 \leq i \leq n$, we call γ_i the element of Γ corresponding to the simple closed curve going once around x_i.

A direct consequence of the Lemma 2.8 and of the following remark is that the images by ϕ_l and by ϕ_r of γ_i are hyperbolic rotations of angle θ_i, where θ_i is the angle of the singular curve of M which intersects S at x_i.

Remark 6.1. Let $\rho \in SO_0(2,2)$ be an AdS isometry, and let ρ_l, ρ_r be its left and right components. Let $\alpha \in (0, 2\pi)$. ρ is a (pure) rotation of angle α around a time-like geodesic in AdS if and only if ρ_l and ρ_r are both hyperbolic rotations of angle α.

Deformations of the holonomy representation of M.

Lemma 6.2. *The first-order deformations of M, among GHM AdS manifolds with particles with the same cone angles, are parametrized by the first-order deformations of the homomorphism ϕ_M from Γ to $SO_0(2,2)$, such that, for each $i \in \{1, \cdots, n\}$, the image of γ_i remains a pure rotation of angle θ_i.*

Proof. Consider a first-order deformation of the AdS metric on M, among GHMC AdS metrics with the same cone angles. The corresponding first-order variation of the holonomy representation of M is then a first-order deformation of ϕ_M, and the images of the γ_i remain a pure rotation of angle θ.

Consider now a one-parameter deformation ϕ_t of ϕ_M, $t \in [0, \epsilon]$, such that the images of each γ_i remains a pure rotations of angle θ_i. Let $\tilde{M}$ be the universal cover of the complement of the singular lines in M. There is natural local isometry from $\tilde{M}$ to H^3_1, the developing map of M. Let $\tilde{S}$ be the universal cover of the regular part of S. Then the developing map of M restricts to an immersion:

$$\psi_M : \tilde{S} \to H^3_1 ~,$$

which is equivariant under the action ϕ_M. Its image is a locally convex surface, which is ramified at the images of the cone points.

Given the one-parameter deformation ϕ_t, it is possible to construct a one-parameter deformation ψ_t of ψ_M, among embeddings of $\tilde{S}$ into H^3_1, such that ψ_t is equivariant under the action of ϕ_t. This can be achieved for instance by choosing a fundamental domain D in $\tilde{S}$ and constructing a deformation of ψ_M on D in such a way that D can be "glued" to its images under the action of a set of genetators of Γ. Moreover it is possible to choose this deformation so that, for t small enough, the image of $\tilde{S}$ remains locally convex.

Then for $\alpha > 0$ small enough and for t small enough, we can consider the normal exponential map:

$$\exp_n : \tilde{S} \times (-\alpha, \alpha) \to H^3_1 ~,$$

sending (x, t) to the image of tn_x by the exponential map at $\psi_t(x)$, where n_x is the future-oriented unit normal orthogonal to $\psi_t(\tilde{S})$ at $\psi_t(x)$. This map $\exp_n$ is a local homeomorphism, so that it can be used to pull back the AdS metric of the target space to an AdS metric on $\tilde{S} \times (-\alpha, \alpha)$, which has a natural isometric action of Γ through ϕ_t.

The quotient $\tilde{S} \times (-\alpha, \alpha)/\phi_t(\Gamma)$ is an AdS manifold with particles, which contains a closed, locally convex, space-like surface (the quotient of $\psi_t(\tilde{S})$. So its maximal extension is a GHMC AdS manifold with particles, with holonomy representation equal to ϕ_t, as needed.

Note that any one-parameter deformation of M (still under the same angle conditions) can be constructed in this manner, and that the resulting manifold depends only on the variation of the holonomy representation. This completes the proof of the lemma. □

A key infinitesimal rigidity lemma. We now have the tools necessary to prove the main lemma of this section: the first-order deformations of M are parametrized by the first-order deformations of its left and right hyperbolic metrics.

Lemma 6.3. *The map* $(\mu_l, \mu_r) : \mathcal{GH}_{\Sigma,n,\theta} \to \mathcal{T}_{\Sigma,n,\theta} \times \mathcal{T}_{\Sigma,n,\theta}$ *is a local homeomorphism.*

Proof. According to Lemma 6.2, the first-order deformations of M are parametrized by the first-order deformations of its holonomy representation, among morphisms of Γ in $SO_0(2,2)$ sending each γ_i to a pure rotation of angle θ_i. But Remark 6.1 shows that those deformations are characterized by the deformations of the left and right hyperbolic metrics, which can be any deformations of μ_l and μ_r among hyperbolic metrics with the same angle at the cone singularities of S. □

Consequences for earthquakes. The previous lemma has a direct application for earthquakes on hyperbolic surfaces with cone singularities.

Lemma 6.4. *Let* $h_0 \in \mathcal{T}_{\Sigma,n,\theta}$. *The map* $E^r_\cdot(h_0) : \lambda \mapsto E^r_\lambda(h)$ *is a local homeomorphism.*

Proof. Fix λ_0. Let $I : \mathcal{ML}_{\Sigma,n} \times \mathcal{T}_{\Sigma,n,\theta} \to \mathcal{GH}_{\Sigma,n,\theta}$ be the parameterization given by Lemma 5.9. Since I is continuous, thanks to Lemmas 6.3 and 5.9, we can take a neighbourhood U of (h_0, λ_0) such that the map

$$(h, \lambda) \mapsto (E^r_{\lambda/2}(h), E^l_{\lambda/2}(h))$$

is injective on U. The set V of laminations λ such that $(h_0, \lambda) \in U$ is a neighbourhood of (h_0, λ_0) and both $E^r_\cdot(h_0)$ and $E^l_\cdot(h_0)$ are injective on V. □

7. Compactness

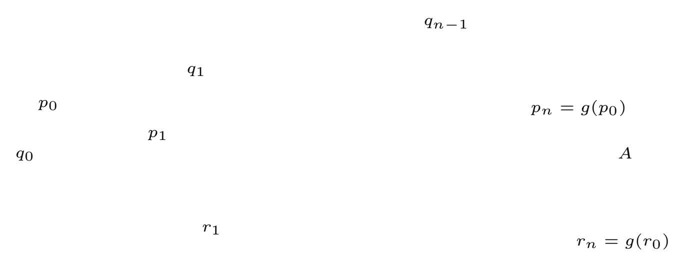
This section is devoted to the proof of Lemma 1.7, which states that, for a fixed element $\mu \in \mathcal{T}_{\Sigma,n,\theta}$, the map $E^r_\cdot(\mu) : \mathcal{ML}_{\Sigma,n} \to \mathcal{T}_{\Sigma,n,\theta}$ is proper. In the whole section we fix $\theta = (\theta_1, \cdots, \theta_n) \in [0, \pi)^n$.

A compactness lemma for earthquakes.

Lemma 7.1. *Given* $\lambda \in \mathcal{ML}_{\Sigma,n}$ *and* $\mu \in \mathcal{T}_{\Sigma,n,\theta}$ *let* $\mu' = E^r_\lambda(\mu)$. *Then, for every closed geodesic* γ *of* Σ,

$$\ell_\mu(\gamma) + \ell_{\mu'}(\gamma) \geq \lambda(\gamma) ~,$$

where $\ell_\mu(\gamma)$ *denotes the length of* γ *with respect to* μ.



Proof. By a standard approximation argument, it is sufficient to prove the statement under the hypothesis that λ is a weighted multicurve. Moreover we can assume $\lambda(\gamma) > 0$.

Let $\tilde{\Sigma}_\mu$ and $\tilde{\Sigma}_{\mu'}$ the metric universal covering of the regular part of Σ with respect μ and μ' respectively (here Σ is regarded as a punctured surface). The lamination λ lifts to a lamination $\tilde{\lambda}$ of $\tilde{\Sigma}_\mu$ and the right earthquake along $\tilde{\lambda}$, say $\tilde{E}^r$, is the lifting of E^r. Thus for every covering transformation g of $\tilde{\Sigma}_\mu$ there exists a unique covering transformation, say $H(g)$, such that

$$\tilde{E}^r \circ g = H(g) \circ \tilde{E}^r \, .$$

Let g be a covering transformation of $\tilde{\Sigma}_\mu$ representing a loop of Σ freely homotopic to γ. There exists a g-invariant complete geodesic $A = A(g)$ in $\tilde{\Sigma}_\mu$ such that the projection of $A(g)$ on Σ is the geodesic representative of γ with respect to μ.

Analogously the projection of $A(H(g)) \subset \tilde{\Sigma}_{\mu'}$, is the geodesic representative of γ with respect to μ'. The inverse image A' through $\tilde{E}^r$ of $A(H(g))$ is a g-invariant union of disjoint geodesic segments whose endpoints lie on some leaves of $\tilde{\lambda}$. More precisely if $\{l_i\}_{i\in\mathbb{Z}}$ is the set of geodesics cutting A enumerated so that l_i and l_{i+1} intersect A in consecutive points p_i and p_{i+1} of $A \cap \tilde{\lambda}$, then A' is the union of geodesic segments joining a point $q_i \in l_i$ to a point $r_{i+1} \in l_{i+1}$.

Let A be oriented in such a way that g is a positive translation. The sequence p_i can be supposed to be increasing. Moreover each l_i can be oriented in such a way that the intersection of A with it is positive. Let x_i (resp. y_i) denote the signed distance of q_i (resp. r_i) from p_i on l_i. Since after the right earthquake A' becomes a continuous line, $x_i - y_i$ is equal to the weight of the leaf l_i.

If $g(p_0) = p_n$ for some $n > 0$ clearly we have $x_{i+n} = x_i$ and $y_{i+n} = y_i$. Moreover $\ell_\mu(\gamma)$ is equal to the sum of the lengths of the geodesic segments $[p_0, p_1], \ldots, [p_{n-1}, p_n]$, whereas $\ell_{\mu'}(\gamma)$ is equal to the sum of the geodesic segments $[q_0, r_0], \ldots, [q_{n-1}, r_n]$. From the triangle inequality,

$$x_i \leq y_{i+1} + \ell([p_i, p_{i+1}]) + \ell([q_i, r_{i+1}]) \ ,$$

that is,

$$x_i - y_{i+1} \leq \ell([p_i, p_{i+1}]) + \ell([q_i, r_{i+1}]) \, .$$

Summing the last inequality for $i = 0, \ldots, n-1$,

$$\sum_{i=0}^{n-1} x_i - y_i \leq \ell_\mu(\gamma) + \ell_{\mu'}(\gamma) \ .$$

Since the left hand of this inequality is the mass of γ with respect to λ, the proof is complete. $\square$

Proof of Lemma 1.7. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a divergent sequence in $\mathcal{ML}_{\Sigma,n}$ and $\mu_k = E^r_{\lambda_k}(\mu)$ for some fixed $\mu \in \mathcal{T}_{\Sigma,n,\theta}$. We have to prove that $(\mu_k)_{k\in\mathbb{N}}$ is a divergent sequence in $\mathcal{T}_{\Sigma,n,\theta}$.

Since $(\lambda_k)_{k\in\mathbb{N}}$ is divergent, there exists a closed geodesic γ such that $\lambda_n(\gamma) \to +\infty$. Then by Lemma 7.1, $\ell_{\mu_k}(\gamma) \to +\infty$, so $(\mu_k)_{k\in\mathbb{N}}$ does not admit a convergent subsequence.

8. Proof of the main results

Proof of Theorem 1.2. As mentioned in section 2 we fix $\theta = (\theta_1, \cdots, \theta_n) \in (0,\pi)^n$ and $h \in \mathcal{T}_{\Sigma,n,\theta}$. We then consider the map $E^r_{\cdot}(h) : \mathcal{ML}_{\Sigma,n} \to \mathcal{T}_{\Sigma,n,\theta}$. It is a local homeomorphism by Lemma 6.4, and is proper by Lemma 1.7. Therefore it is a covering. However $\mathcal{T}_{\Sigma,n,\theta}$ is simply connected and $\mathcal{ML}_{\Sigma,n}$ is connected. Therefore this map is a homeomorphism.

Proof of Theorem 1.4. Again we consider a fixed choice of $\theta = (\theta_1, \cdots, \theta_n) \in (0,\pi)^n$. Let $\mu_l, \mu_r \in \mathcal{T}_{\Sigma,n,\theta}$. By Theorem 1.2 there exists a unique $\lambda \in \mathcal{ML}_{\Sigma,n}$ such that $\mu_r = E^r_\lambda(\mu_l)$.

Let $\lambda_+ := \lambda/2$, and let $h_+ := E^r_{\lambda_+}(\mu_l)$. By Lemma 5.8 there exists a (unique) GHMC AdS metric g on $\Sigma \times (0,1)$ for which the induced metric and the measured bending lamination on the upper boundary of the convex core are h_+ and λ_+, respectively. It then follows from Lemma 5.9 that the left and right hyperbolic metrics of g are μ_l and μ_r, respectively.

Conversely, let g' be a GHMC AdS metric on $\Sigma \times (0,1)$ for which the left and right hyperbolic metrics are μ_l and μ_r, respectively. Let h'_+ and λ'_+ be the induced metric and measured bending lamination on the upper component of the boundary of the convex core of $(\Sigma \times (0,1), g')$. Then Lemma 5.9 shows that $\mu_r = E^r_{\lambda_+}(h'_+)$, while $\mu_l = E^l_{\lambda_+}(h'_+)$, so that $\mu_r = E^r_{2\lambda_+}(\mu_l)$. It follows that g' is the metric g constructed above, and this finishes the proof of Theorem 1.4.

9. Some concluding remarks

Reconstructing a GHMC AdS manifold from its convex core. The arguments developed above show that, given the convex core of a GHMC AdS manifold M, it is possible to understand the global geometry of M is a simple way. This is an immediate extension of statements already well-known in the non-singular case, see [3].

Lemma 9.1. *Let M be a GHM AdS manifold with particles, which is topologically $\Sigma \times (0,1)$, with cone angles $\theta_1, \cdots, \theta_n \in (0,\pi)$. Let Ω_+ be the set of points at distance at most $\pi/2$ in the past of the future boundary of the convex core, and let Ω_- be the set of points at distance at most $\pi/2$ in the future of the past boundary of the convex core. Then*

$$M = \Omega_+ \cup \Omega_- \ , \ \Omega_+ \cap \Omega_- = C(M) \ .$$

Moreover,

$$Vol(M) + Vol(C(M)) = \frac{\pi}{2}\left(2\pi\chi(\Sigma) + \sum_{i=1}^{n}(2\pi - \theta_i)\right) + \frac{L(\lambda)}{2} \ ,$$

where λ is the measured bending lamination of the boundary of the convex core and $L(\lambda)$ is its length.

Note that the quantity $2\pi\chi(\Sigma) + \sum_{i=1}^{n}(2\pi - \theta_i)$ is 2π times a natural "Euler characteristic" of a closed surface with cone singularities. It is equal for instance to the area of any hyperbolic metric with prescribed singular angles on such a surface.

Proof. Let $\partial_- C(M)$ and $\partial_+ C(M)$ be the past and future boundary components of $C(M)$, respectively. Since $\partial_- C(M)$ is a locally convex surface (with the convex part of its complement in the future direction) we can consider the normal exponential map of $\partial_- C(M)$, as in the proof of the previous lemma. Again it is locally injective on time $t \in (0, \pi/2)$, and can be used to pull back the AdS metric to a locally convex AdS metric (with particles) on a "slice" of width $\pi/2$ in the future of $\partial_- C(M)$.

This construction, and the definition of M as a *maximal* globally hyperbolic space, shows that M contains the image of this map, which corresponds to the space Ω_+ appearing in the lemma.

On the other hand, there is no point in M which is at distance larger than $\pi/2$ in the future of $\partial_- C(M)$. Indeed, suppose that some point $x \in M$ is at distance $\pi/2$ in the future of $\partial_- C(M)$. Let γ_0 be a maximizing geodesic in M from x to a point $y \in \partial_- C(M)$. y is contained in a maximal totally geodesic stratum σ of $\partial_- C(M)$, and, for all points $y' \in \sigma$, the geodesic orthogonal to σ at y' arrives at x after time exactly $\pi/2$.

But the universal cover $\tilde\sigma$ of σ is non-compact, and we can consider a sequence (y_n) of points in it, going to infinity. Let γ_n be the projection to M of the geodesic segment or length $\pi/2$ orthogonal to $\tilde\sigma$ at y_n, so that the other endpoint of γ_n is x. Then the sequence (γ_n) converges, in all compact sets containing x, to a light-like geodesic which does not intersect the universal cover of $\partial_- C(M)$, contradicting the global hyperbolicity of M.

This proves that the future of $\partial_- C(M)$ is equal to the image by G of $\partial_- C(M) \times (0, \pi/2)$, or in other terms Ω_+. The same argument applies for the past of $\partial_+ C(M)$, and this proves the first part of the lemma.

For the second point, let λ_+ and λ_- be the measured bending lamination of the future and past boundary components of $C(M)$. The statement will clearly follow if we prove that

$$\mathrm{Vol}(\Omega_+) = \frac{\pi}{4} A(\partial_- C(M)) + \frac{L(\lambda_-)}{2} ,$$

where $A(\partial_- C(M))$ is the area of the induced metric on $\partial_- C(M)$ and $L(\lambda_-)$ is the length of the measured lamination λ_-. We will prove that this relation holds when the support of λ_- is a disjoint union of closed curves; the general result for Ω_+ then follows by approximating $\partial_- \Omega_+$ by a sequence of pleated surfaces with such a measured bending lamination.

So suppose that the support of λ_- is the union of closed curves $\gamma_1, \cdots, \gamma_p$, each with a weight $\lambda_i, 1 \leq i \leq p$. Then Ω_+ can be decomposed as the union of two regions:

- the set Ω_0 of points $x \in \Omega_+$ which "project" to a point of $\partial_- C(M)$ which is not in the support of λ_-,
- the sets Ω_i of points $x \in \Omega_+$ which project to γ_i.

The volume of the first domain can be computed by integrating over the distance to $\partial_- C(M)$. It is equal to:

$$\mathrm{Vol}(\Omega_0) = \int_{\partial_- C(M) \setminus \cup_i \gamma_i} \int_{r=0}^{\pi/2} \cos(r)^2 dr da = \frac{\pi}{4} \int_{\partial_- C(M) \setminus \cup_i \gamma_i} da = \frac{\pi}{4} A(\partial_- C(M)) .$$

The same kind of computation shows that:

$$\mathrm{Vol}(\Omega_i) = L(\gamma_i)\lambda_i \int_0^{\pi/2} \cos(r)\sin(r) dr = \frac{L(\gamma_i)\lambda_i}{2} ,$$

and it follows that

$$\mathrm{Vol}(\Omega_+) = \frac{\pi}{4} A(\partial_- C(M)) + \frac{L(\lambda_-)}{2} ,$$

as needed. ☐

A note on the definition of GHMC manifolds used here. The reader might wonder why we consider here *convex* globally hyperbolic manifolds, i.e., AdS manifolds which contain a space-like surface which is convex (see Definition 2.5). It is quite possible that a weaker condition – the existence of a compact Cauchy surface, which is not necessarily convex – is sufficient, and that any AdS manifold containing such a Cauchy surface contains one which is convex. We do not further consider this question here since it is quite distinct from our main centers of interest.

Other possible proofs. It appears quite possible that arguments close to those used by Kerckhoff [10] can be applied to the setting of hyperbolic surfaces with cone singularities, however the extension is not completely clear. We suppose that the condition that the cone angles are less than π should appear also in such arguments.

Other arguments used on non-singular surfaces, however, make a stronger use of the geometry of the universal cover of the surface. Those are presumably less well adapted to the singular surfaces considered here.

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