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Global and Invariant Aspects of Consensus on the n-Sphere*

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Abstract— **This paper concerns two aspects of the multi-agent consensus problem on the n-sphere. Firstly, it proves that a standard consensus protocol, in a certain sense, yields asymptotical stability on a global level for a nontrivial class of graph topologies. Secondly, it provides a novel consensus protocol that leaves the centroid of agent states in $\mathbb{R}^{n+1}$ projected back to the sphere invariant. It hence becomes possible to determine the consensus point as a function of the initial states. Much of the stability analysis has an intuitive geometric appeal since it is based on the symmetries of the n-sphere rather than generic Lyapunov theory.**

I. Introduction

Consider a system consisting of N agents, each of which is equipped with some limited communication and sensing capabilities. The goal is for all agents to converge to the same state, *i.e.*, to reach a consensus. This type of problem is widely studied in the literature, see *e.g.*, [14] and the references therein. As the field of networked and multi-agent system has matured, research focus shifts from linear dynamics to more realistic models such as switched and highly nonlinear systems including those featured in the attitude synchronization problem [16], [19], [20], [22], [25]. Research on attitude synchronization is motivated by applications such as satellite formation flying [7], [11], co-operative robotic manipulation [17], multi-camera networks [23], and distributed rotation averaging [2], [4].

The reduced attitude is a property of objects that for various reasons, such as task redundancy, cylindrical symmetry, or actuator failure, lack one degree of rotational freedom in three-dimensional space and whose orientation corresponds to a pointing direction with no regard for the rotation about the axis of pointing [9]. The reduced attitude synchronization problem is equivalent to the consensus problem on the sphere. The problem of cooperative control on the sphere has received some attention in the literature [3], [15], but comparatively less than the full attitude synchronization problem. Other applications of consensus on the sphere include planetary scale mobile sensing networks [5].

This paper concerns two aspects of the consensus problem for a multi-agent system on the n-sphere. One problem concerns the possibility of finding a control law that makes the consensus manifold almost globally asymptotically stable. A basic control law is shown to render all equilibrium manifolds except the consensus manifold unstable for a nontrivial class of graph topologies. The question of whether this result holds for general graph topologies is explored in simulation. Another problem concerns the determination of the final state, *i.e.*, the particular point on the consensus manifold to which the agents converge. Applications can be found in the field of distributed computation. A control law is provided that leave the centroid of agent states in $\mathbb{R}^{n+1}$ projected back to the n-sphere invariant.

Related research concerns the problem of almost global consensus on $\mathcal{S}^1$ [18], on SO(3) [24], and on $\mathcal{S}^n$ in the special case of a complete graph [3], [15]. The work [24] makes use of a particular reshaping function to establish almost global consensus for general graph topologies. The authors apply an optimization based method to characterize the stability of all equilibria. Their approach is roughly equivalent to the direct method of Lyapunov. By shifting consideration from SO(3) to the n-sphere, this paper uses an analysis based on the symmetries of the n-sphere to establish stability results like those in [24] without the use of any reshaping function.

The work [24] divides the literature on attitude consensus into two categories: extrinsic and intrinsic algorithms. An algorithm is said to be extrinsic if it makes use of a parametrization that embeds SO(3) in some Euclidean space. There are algorithms in this class that provide consensus on a global level. For the second category of algorithms that work with SO(3) directly, global level results had not been obtained prior to the publication of [24]. To the best of our knowledge, previous work concerning intrinsic cooperative control on the n-sphere only regards the case of a complete graph [3], [15].

The 2-sphere is akin to a subset of SO(3), and as such many results obtained for SO(3) also applies to $\mathcal{S}^2$. Special cases sometimes allow for stronger results. The findings of this paper indicates that the conditions for achieving almost global consensus are, in a certain sense, more favorable on the 2-sphere than the particularization of previously known results concerning SO(3) would imply. There are graph topologies for which standard consensus protocols render all equilibria except the consensus point unstable on $\mathcal{S}^2$ whereas simulation results indicate that some of the corresponding equilibria are asymptotically stable on SO(3) [24].

The main contributions of this paper are summarized as follows. Firstly, it provides a convergence results on a global level for topologies apart from trees and the complete graph, *i.e.*, for a larger class of topologies than what has previously been found on the n-sphere [3], [15],

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[25]. Secondly, it provides a novel consensus protocol that leaves the centroid of agent states invariant. There is, to the best of our knowledge, no such previous result, nor has the problem of determining the consensus point of a continuous time multi-agent system on the n-sphere been addressed previously.

II. Problem Description

The following notation is used throughout this paper. The inner product and outer product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\mathbf{x} \otimes \mathbf{y}$ respectively. The Euclidean norm is used for vectors and the Frobenius norm is used for matrices. The general linear group of invertible $n \times n$ matrices over the field $\mathbb{R}$ is written $\mathsf{GL}(n)$. The special orthogonal group is $\mathsf{SO}(n) = \{\mathbf{R} \in \mathsf{GL}(n) \,|\, \mathbf{R}^{-1} = \mathbf{R}^\top, \det \mathbf{R} = 1\}$. The Lie algebra of $\mathsf{SO}(n)$ is $\mathsf{so}(n) = \{\mathrm{S} \in \mathbb{R}^{n \times n} \,|\, \mathrm{S}^\top = -\mathrm{S}\}$. The n-sphere is denoted by $\mathcal{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \,|\, \|\mathbf{x}\| = 1\}$, where $n \geq 2$. A graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. For any graph $\mathcal{G}$, the notation $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ refers to its node and edge sets respectively. The union of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as the graph $(\mathcal{V}(\mathcal{G}_1) \cup \mathcal{V}(\mathcal{G}_2), \mathcal{E}(\mathcal{G}_1) \cup \mathcal{E}(\mathcal{G}_2))$. Some special graphs that will be used are the complete graphs $\mathcal{K}_0 = (\emptyset, \emptyset)$, $\mathcal{K}_1 = (\{1\}, \emptyset)$, $\mathcal{K}_2 = ([2], \{(1, 2)\})$, and $\mathcal{K}_n = \{[n], \{(i, j) \,|\, i, j \in [n], i \neq j\}\}$ where $n \in \mathbb{N}$ and $[\,] : n \mapsto \{k \in \mathbb{N} \,|\, k \leq n\}$. The symbol $\tilde{\in}$ denotes an isomorphism between a graph and an element of a set of graphs, *i.e.*, $\mathcal{G} \,\tilde{\in}\, \{\mathcal{H}_1, \ldots, \mathcal{H}_m\}$ if $\mathcal{G} \simeq \mathcal{H}_i$ for some $i \in [m]$. The symbol $\tilde{\subset}$ denotes an isomorphism relation between sub- and supersets, namely $\mathcal{S}_1 \,\tilde{\subset}\, \mathcal{S}_2$ implies that $\mathcal{S}_1 \subset \mathcal{S}_3$ and $\mathcal{S}_3 \simeq \mathcal{S}_2$. In particular, $\mathcal{S} \,\tilde{\subset}\, \mathcal{S}^1$ denotes that $\mathcal{S}$ belongs to a great circle.

An equilibrium manifold is, roughly speaking, a set of equilibria of a system that also constitute a manifold. The concepts of stability of an equilibrium can be extended to sets and hence to manifolds [13], [21]. This paper introduces the following terminology to describe stability properties of manifolds.

Definition 1. An equilibrium manifold is said to be maximal if it is connected and not a strict subset of any other connected equilibrium manifold.

Definition 2. A maximal equilibrium manifold is said to be uniquely stable if it is stable and no other maximal equilibrium manifold is stable, uniquely attractive if it is attractive and no other maximal equilibrium manifold is attractive, and uniquely asymptotically stable if it is both uniquely stable and uniquely attractive.

Remark 3. Note that for an equilibrium manifold to be uniquely asymptotically stable is stronger than for it to be the only asymptotically stable maximal equilibrium manifold since it also excludes any other maximal equilibrium manifold being either stable or attractive.

A. Distributed Control Design

Consider a multi-agent system where each agent is identified by an index $i \in [N]$. The agents are capable of limited pairwise communication and local sensing. The topology of the communication network is described by an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = [N] = \{1, 2, \ldots, N\}$, and the presence of an edge $(i, j) \in \mathcal{E}$ implies that agent i and j can communicate, or equivalently that agent i and j can sense the so-called local or relative information regarding the displacement of their states. Control is assumed to be based on a minimal amount of relative information and to be carried out on a kinematic level.

The information $\mathcal{I}_{ij}$ that agent i has access to regarding one of its neighboring agents j includes

$$\mathcal{I}_{ij} \supset \operatorname{span}\{(\mathbf{I} - \mathbf{X}_i)(\mathbf{x}_j - \mathbf{x}_i)\} = \operatorname{span}\{\mathbf{x}_j - \langle \mathbf{x}_j, \mathbf{x}_i \rangle \mathbf{x}_i\}, \quad (1)$$

where $\mathbf{X}_i = \mathbf{x}_i \otimes \mathbf{x}_i$ and $\mathbf{I} - \mathbf{X}_i : \mathbb{R}^{n+1} \to \mathsf{T}_{\mathbf{x}_i} \mathcal{S}^n$ is a projection. The space of relative information known to any agent $i \in \mathcal{V}$ is $\cup_{j \in \mathcal{N}_i} \mathcal{I}_{ij}$. The subset of $\mathcal{I}_{ij}$ given by (1) corresponds to the customary relative information in linear spaces $\mathcal{I}_{ij} \supset \operatorname{span}\{\mathbf{x}_j - \mathbf{x}_i\}$ projected on $\mathsf{T}_{\mathbf{x}_i} \mathcal{S}^n$. An agent can calculate this aspect of $\mathcal{I}_{ij}$ based on local sensing since all it needs to discern is the direction towards its neighbor along its tangent space.

System 4. The system is given by N agents, an undirected and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, agent states $\mathbf{x}_i \in \mathcal{S}^n$, where $n \geq 2$, and dynamics

$$\dot{\mathbf{x}}_i = \mathbf{u}_i - \langle \mathbf{u}_i, \mathbf{x}_i \rangle \mathbf{x}_i, \quad (2)$$

where $\mathbf{u}_i = \sum_{j \in \mathcal{N}_i} f_{ij} \mathbf{x}_j \in \mathbb{R}^{n+1}$ for some continuously differentiable $f_{ij} : \mathcal{S}_i \to (0, \infty)$, where $\mathcal{S}_i \subseteq \prod_{j \in \mathcal{N}_i} \mathcal{I}_{ij}$, such that $\dot{f}_{ij} \leq 0$ for all $j \in \mathcal{N}_i$, for all $i \in \mathcal{V}$.

Note that the dynamics (2) projects the input $\mathbf{u}_i$ on the space of relative information. While some agent $i \in \mathcal{V}$ may not be able to calculate $\mathbf{u}_i$ based on the information (1) obtained from all its neighbors, that agent can still calculate an input $\mathbf{v}_i$ whose projection on $\mathsf{T}_{\mathbf{x}_i} \mathcal{S}^n$ is identical to that of $\mathbf{u}_i$. This holds for any input that belongs to $\operatorname{span}\{\mathbf{x}_j \,|\, j \in \mathcal{N}_i\}$, and in particular for any element of the positive cone $\operatorname{pos}\{\mathbf{x}_j \,|\, j \in \mathcal{N}_i\}$. Intuitively speaking, it is reasonable to assume that agent i should be able to move towards any point in $\operatorname{pos}\{\mathbf{x}_j \,|\, j \in \mathcal{N}_i\}$, *i.e.*, that $\operatorname{pos}\{\mathbf{x}_j \,|\, j \in \mathcal{N}_i\} \subset \mathcal{I}_{ij}$.

All vectors in this section are defined in the world frame $\mathcal{W}$. To implement the control law in a distributed fashion, $\mathbf{u}_i$ must be transfered to the body frame $\mathcal{B}_i$ of agent i. Let $[\mathbf{x}]_\mathcal{F}$ denote that the coordinates of vector $\mathbf{x}$ are given with respect to the $\mathcal{F}$. Suppose $\mathcal{B}_i$ is related to $\mathcal{W}$ by means of a rotation $\mathbf{R}_i : [\mathbf{v}]_\mathcal{W} \mapsto [\mathbf{v}]_{\mathcal{B}_i}$. The control law in $\mathcal{W}$ is given by $[\mathbf{u}_i]_\mathcal{W} = \sum_{j \in \mathcal{N}_i} f_{ij} [\mathbf{x}_j]_\mathcal{W}$. Hence $[\mathbf{u}_i]_{\mathcal{B}_i} = \sum_{j \in \mathcal{N}_i} f_{ij} \mathbf{R}_i [\mathbf{x}_j]_\mathcal{W} = \sum_{j \in \mathcal{N}_i} f_{ij} [\mathbf{x}_j]_{\mathcal{B}_i}$. Moreover,

$$\begin{aligned} [\dot{\mathbf{x}}_i]_{\mathcal{B}_i} &= \mathbf{R}_i [\mathbf{u}_i]_\mathcal{W} + \langle [\mathbf{u}_i]_\mathcal{W}, \mathbf{R}_i^\top \mathbf{R}_i [\mathbf{x}_i]_\mathcal{W} \rangle \mathbf{R}_i [\mathbf{x}_i]_\mathcal{W} \\ &= [\mathbf{u}_i]_{\mathcal{B}_i} + \langle [\mathbf{u}_i]_{\mathcal{B}_i}, [\mathbf{x}_i]_{\mathcal{B}_i} \rangle [\mathbf{x}_i]_{\mathcal{B}_i}, \end{aligned} \quad (3)$$

due to inner products being invariant under orthonormal changes of coordinates. From the perspective of stability analysis, (3) is the same as equation (2).

The problem of multi-agent consensus on the n-sphere concerns the design of distributed control protocols $\{\mathbf{u}_i\}_{i=1}^N$ based on relative information, as discussed in the above paragraphs, that stabilize the consensus manifold

$$C = \{\{\mathbf{x}_i\}_{i=1}^N \in (\mathcal{S}^n)^N \mid \mathbf{x}_i = \mathbf{x}_j,\ \forall\, i, j \in \mathcal{V}\} \tag{4}$$

of System 4. If all agents converge to one point on the n-sphere, then they are said to reach consensus. For all connected graphs, it can easily be shown to suffice that the states of any pair of neighboring agents are equal.

B. Problem Statement

This paper concerns two aspects of consensus on the n-sphere.

Problem 5. Design a consensus protocol, *i.e.*, input signals $\mathbf{u}_i$ for all $i \in \mathcal{V}$, for System 4 such that the consensus manifold is uniquely asymptotically stable.

Problem 6. Design a consensus protocol, $\mathbf{u}_i$ for all $i \in \mathcal{V}$, for System 4 such that the system converges asymptotically to a point on the consensus manifold for which an explicit expression can be given in terms of the initial states of the agents.

Problem 5 concerns the global behavior of the system. Under certain assumptions regarding the connectivity of $\mathcal{G}$, local consensus on SO(3) can be established with the region of attraction being the largest geodesically convex set on $\mathcal{S}^n$, *i.e.*, open hemispheres. See for example [25] in the case of an undirected graph and [22] in the case of a directed and time-varying graph. A global stability result for discrete-time consensus on SO(3) is provided in [24]. The algorithm of [24] does however require the use of a reshaping function which depends on an unknown parameter. Almost global asymptotical stability of the consensus manifold on the n-sphere is known to hold when the graph is a tree [25] or is complete in the case of first- and second-order models [3], [15]. The author of [15] conjectures that global stability also holds for a larger class of topologies.

Problem 6 concerns the invariant state information that can be extracted by means of local communication, and is of importance in the field of distributed computing. This problem is well-known in the linear case [14] but it does not appear to have been given much attention in the n-sphere case; see [2], [4], [25] for treatments of related problems in the SO(n) case. The problem of calculating averages on sphere has been studied in the context of computer graphics [12]. Splines are used to port these results to systems that evolve in continuous time. The authors of [3] remark that initial conditions at rest for a second order system on a hemisphere results in consensus on said hemisphere. Note that although it may be argued that establishing a consensus is trivial in the case of a complete graph, it may still be nontrivial to calculate the final state. Moreover, previous works on cooperative control on the sphere [3], [15] focus on the complete graph case.

The next section presents solutions to Problem 5 and 6. The following result, Theorem 7, may be considered as one of the many known facts concerning consensus on convex subsets of manifolds [25]. To solve Problem 5 this paper provides a weaker companion to Theorem 7 that holds for a larger class of graph topologies.

Theorem 7. *Consider System 4. The consensus manifold* (4) *is stable. Suppose that the positions of all agents belong to an open hemisphere, then they reach consensus asymptotically.*

Remark 8. A proof of this result—or generalizations thereof such as consensus over directed graph topologies—can be obtained by following lines of reasoning that is found in many works within the consensus literature, and is therefore omitted here. Theorem 7 is similar to a result which establishes asymptotical convergence for a class of discrete-time consensus protocols on convex subsets of Riemannian manifolds [25]. The findings of [25] can also be used to show almost global asymptotical consensus in case the graph $\mathcal{G}$ is a tree. Other results of note include proofs that the consensus manifold is almost globally asymptotically stable when $\mathcal{G}$ is complete [3], [15].

III. Main Results

A. Instability of Undesired Equilibria

This section concerns System 4 governed by Algorithm 9 which provides a basic protocol for consensus on the n-sphere.

Algorithm 9. The feedback is given by $\mathbf{u}_i = \sum_{j \in \mathcal{N}_i} w_{ij}\mathbf{x}_j$, where $w_{ij} \in (0, \infty)$ and $w_{ij} = w_{ji}$ for all $(i, j) \in \mathcal{E}$.

Algorithm 9 can be derived by taking the gradient of the potential function

$$V(\mathbf{x}_i, \mathbf{x}_j) = \tfrac{1}{2} \sum_{(i,j) \in \mathcal{E}} f_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|^2, \tag{5}$$

in the case that $f_{ij} = w_{ij}$ for all $(i, j) \in \mathcal{E}$, see the proof of Proposition 10. It is possible to work with more general functions f_{ij} than those of Algorithm 9, but we prefer constant weights for ease of notation. Proposition 10 implies that there are no limits cycles in System 4 under Algorithm 9. If the graph in System 4 under Algorithm 9 were directed, then there would exist examples of topologies that result in limit cycles.

Proposition 10. *System 4 converges to an equilibrium.*

Proof. Consider the potential function (5). It holds that

$$\begin{aligned}
\dot{V}(\mathbf{x}_i, \mathbf{x}_j) &= \sum_{(i,j) \in \mathcal{E}} f_{ij} \langle \mathbf{x}_i - \mathbf{x}_j, \dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j \rangle + \tfrac{1}{2} \dot{f}_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|^2 \\
&= \sum_{(i,j) \in \mathcal{E}} f_{ij} \langle \mathbf{x}_i - \mathbf{x}_j, \mathbf{u}_i - \mathbf{u}_j - \langle \mathbf{u}_i, \mathbf{x}_i \rangle \mathbf{x}_i \\
&\qquad + \langle \mathbf{u}_j, \mathbf{x}_j \rangle \mathbf{x}_j \rangle + \tfrac{1}{2} \dot{f}_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|^2 \\
&= \sum_{(i,j) \in \mathcal{E}} f_{ij} (\langle \mathbf{u}_j, \mathbf{x}_j \rangle \langle \mathbf{x}_i, \mathbf{x}_j \rangle + \langle \mathbf{u}_i, \mathbf{x}_i \rangle \langle \mathbf{x}_i, \mathbf{x}_j \rangle \\
&\qquad - \langle \mathbf{u}_j, \mathbf{x}_i \rangle - \langle \mathbf{u}_i, \mathbf{x}_j \rangle) + \tfrac{1}{2} \dot{f}_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|^2 \\
&= \sum_{(i,j) \in \mathcal{E}} f_{ij} [\langle \mathbf{u}_i, (\mathbf{x}_i \otimes \mathbf{x}_i - \mathbf{I}) \mathbf{x}_j \rangle
\end{aligned}$$

$$
\begin{aligned}
&+\langle \mathbf{u}_j, (\mathbf{x}_j \otimes \mathbf{x}_j - \mathbf{I})\mathbf{x}_i\rangle] + \tfrac{1}{2}\dot{f}_{ij}\|\mathbf{x}_i - \mathbf{x}_j\|^2 \\
&= \sum_{i\in V}\sum_{j\in\mathcal{N}_i} f_{ij}\langle \mathbf{u}_i, (\mathbf{x}_i \otimes \mathbf{x}_i - \mathbf{I})\mathbf{x}_j\rangle + \tfrac{1}{2}\dot{f}_{ij}\|\mathbf{x}_i - \mathbf{x}_j\|^2 \\
&= -\sum_{i\in V}\langle \mathbf{u}_i, (\mathbf{I} - \mathbf{x}_i \otimes \mathbf{x}_i)\mathbf{u}_i\rangle + \tfrac{1}{2}\dot{f}_{ij}\|\mathbf{x}_i - \mathbf{x}_j\|^2 \\
&\le -\sum_{i\in V}\langle \mathbf{u}_i, (\mathbf{I} - \mathbf{x}_i \otimes \mathbf{x}_i)\mathbf{u}_i\rangle,
\end{aligned}
$$

since $\dot{f}_{ij} \le 0$ by definition of System 4. System 4 converges to the set $\{\{\mathbf{x}_i\}_{i=1}^N \,|\, \|\mathbf{u}_i \parallel \mathbf{x}_i\}$ by LaSalle's theorem, *i.e.*, the input and state of each agent align up to sign. This implies $\dot{\mathbf{x}}_i = \mathbf{0}$ for all $i \in \mathcal{V}$ by inspection of (2).

Consider the final state of System 4 under Algorithm 9 given in the proof of Proposition 10. The following equilibrium configurations may be attractive:

$$
(\mathbf{x}_i, \mathbf{u}_i) \in \left\{ \left(-\frac{\mathbf{s}_i}{\|\mathbf{s}_i\|}, \mathbf{s}_i\right), \left(\frac{\mathbf{s}_i}{\|\mathbf{s}_i\|}, \mathbf{s}_i\right), (\mathbf{x}_i, \mathbf{0}) \right\},
$$

where $\mathbf{s}_i = \sum_{j\in\mathcal{N}_i} w_{ij}\mathbf{x}_j$. The global behavior of the system is hence determined by the stability of these equilibria. It can be showed that any equilibrium where $\mathbf{x}_i = {-\mathbf{s}_i}/{\|\mathbf{s}_i\|}$ or $\mathbf{s}_i = \mathbf{0}$ for some i is unstable. An example of $\mathbf{x}_i = {-\mathbf{s}_i}/{\|\mathbf{s}_i\|}$ is given by a tetrahedron formation with a tetrahedral graph, *i.e.*, $\mathcal{G} = \mathcal{K}_4$. An example of $\mathbf{x}_i = \mathbf{0}$ for all $i \in \mathcal{V}$ in the case of all weights being equal is provided by the octahedral graph, see Figure 1. The remaining case of $\mathbf{x}_i = {\mathbf{s}_i}/{\|\mathbf{s}_i\|}$ for all $i \in \mathcal{V}$ poses a more difficult challenge.

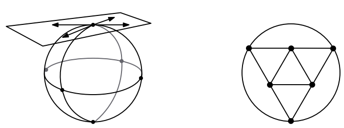

Fig. 1. An unstable equilibrium of a system on $\mathcal{S}^2$ (left) with an octahedral graph (right).

Proposition 11. *Any equilibrium $\{\mathbf{x}_i\}_{i=1}^N$ of System 4 under Algorithm 9 where $\mathbf{x}_i = {-\mathbf{s}_i}/{\|\mathbf{s}_i\|}$ or $\mathbf{s}_i = \mathbf{0}$ for some $i \in \mathcal{V}$ is unstable.*

Proof. The proof makes use of the linearization provided by Lemma 27 in Appendix A. Further details are omitted.

Proposition 12. *Take any equilibrium of System 4 which is not part of the consensus manifold. If all agents belong to a hyperplane in $\mathbb{R}^{n+1}$, then the equilibrium is unstable.*

Proof. Let $\{\mathbf{x}_i\}_{i=1}^N$ denote the equilibrium. The tangent space $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \,|\, \langle \mathbf{x}_i, \mathbf{x}\rangle = 0\}$ contains any normal $\mathbf{n}$ of the hyperplane. Consider a sequence of initial conditions $\{\mathbf{x}_i(\varepsilon, \mathbf{n})\}_{i\in\mathcal{V}}$ depending on a parameter $\varepsilon \in [0, \infty)$. For any equilibrium $\{\mathbf{x}_i\}_{i\in\mathcal{V}}$ on the hyperplane it holds that

$$
\mathbf{x}_i(\varepsilon, \mathbf{n}) = \frac{\mathbf{x}_i + \varepsilon\mathbf{n}}{\|\mathbf{x}_i + \varepsilon\mathbf{n}\|} \in \mathcal{S}^n
$$

is a continuous function of ε that satisfies $\mathbf{x}_i(0, \mathbf{n}) = \mathbf{x}_i$. Consider $\varepsilon \in (0, \infty]$. Note that $\langle \mathbf{x}_i(\varepsilon, \mathbf{n}), \mathbf{n}\rangle > 0$, *i.e.*, the initial states $\mathbf{x}_i(\varepsilon)$ belong to an open hemisphere whereby the agents reach consensus by Theorem 7.

Any tree that branches out from a root node on the graph turns out to have little influence on the possible equilibrium configurations of the system. This notion is made precise by the next proposition.

Proposition 13. *Suppose $\mathcal{G} = \mathcal{H} \cup \mathcal{T}$ where $\mathcal{T}$ is a tree and $\mathcal{T} \cap \mathcal{H} \simeq \mathcal{K}_1$. At any equilibrium of System 4 under Algorithm 9, the states of all agents in $\mathcal{T}$ are equal up to sign.*

Proof. The proof is by induction. Take any leaf node l and let p denote its parent node. Consider the state $\mathbf{x}_l$ at an equilibrium,

$$
\dot{\mathbf{x}}_l = w_{lp}(\mathbf{x}_p - \langle \mathbf{x}_p, \mathbf{x}_l\rangle\mathbf{x}_l) = \mathbf{0},
$$

i.e., $\mathbf{x}_l \in \{-\mathbf{x}_p, \mathbf{x}_p\}$. Suppose this holds for all nodes from the leaves down to depth d in the tree. For any agent i at depth $d-1$ in the tree, let C denote its set of children and p its parent. It follows that

$$
\begin{aligned}
\dot{\mathbf{x}}_i &= \sum_{j\in C} w_{ij}(\mathbf{x}_j - \langle \mathbf{x}_j, \mathbf{x}_i\rangle\mathbf{x}_i) + w_{ip}(\mathbf{x}_p - \langle \mathbf{x}_p, \mathbf{x}_i\rangle\mathbf{x}_i) \\
&= w_{ip}(\mathbf{x}_p - \langle \mathbf{x}_p, \mathbf{x}_i\rangle\mathbf{x}_i) = \mathbf{0}.
\end{aligned}
$$

Hence $\mathbf{x}_i \in \{-\mathbf{x}_p, \mathbf{x}_p\}$.

Remark 14. By Proposition 13, any analysis of the equilibria of System 4 under Algorithm 9 can essentially be reduced to considerations of graph topologies where the leaf nodes of $\mathcal{G}$ have been removed in a recursive fashion to yield a graph where every node belongs to the node set of at least one cycle.

Consider the case when the agents' states are confined to a great circle. The great circle is invariant. Proposition 16 states that if a graph consists of a cycle then, at any equilibrium, all agents belong to a great circle. Moreover, if all agents belong to a great circle, and all agents but one are at rest, then that last agent is also at rest. This result is the first of two concerning equilibria that belong to great circles.

Definition 15. The term $w_{ij}(\mathbf{x}_j - \langle \mathbf{x}_j, \mathbf{x}_i\rangle\mathbf{x}_i)$ is referred to as the contribution to $\dot{\mathbf{x}}_i$ from agent j. A set $\mathcal{S} \subset \mathcal{V}$ is said to contribute zero to the velocity of agent i if

$$
\sum_{j\in\mathcal{S}\cap\mathcal{N}_i} w_{ij}(\mathbf{x}_j - \langle \mathbf{x}_j, \mathbf{x}_i\rangle\mathbf{x}_i) = \mathbf{0}.
$$

Proposition 16. *Suppose the graph can be decomposed as $\mathcal{G} = \mathcal{H} \cup C \cup \mathcal{F}$ where C is a cycle graph on k nodes. Assume that $\mathcal{H} \cap C \simeq \mathcal{K}_1$ and that the agents in $\mathcal{V}(\mathcal{F}\backslash C)$ contribute zero to the derivatives $\dot{\mathbf{x}}_i$ for all $i \in C$ at any equilibrium. Then, at any equilibrium of System 4 under Algorithm 9, $\{\mathbf{x}_i \,|\, i \in C\} \,\tilde{\subset}\, \mathcal{S}^1$. Suppose $\{\mathbf{x}_i \,|\, i \in \mathcal{V}(C)\} \,\tilde{\subset}\, \mathcal{S}^1$ and the agents in $\mathcal{V}(C)\backslash\{k\}$ are at rest. Suppose that the agents in $\mathcal{V}(\mathcal{F}\backslash C)$ contribute zero to $\dot{\mathbf{x}}_i$ for all $i \in C\backslash\{k\}$. At that moment the agents in $\mathcal{V}(C)$ contribute zero to the velocity of agent k.*

Proof. Denote the agents on the cycle by $1, \ldots, k$. Let both 0 and k denote the same agent. Consider any agent $i \in \mathcal{V}(C)$.

Then

$$\begin{aligned}\dot{\mathbf{x}}_i &= \sum_{j\in\mathcal{N}_i} w_{ij}\mathbf{x}_j - w_{ij}\langle \mathbf{x}_j, \mathbf{x}_i\rangle \mathbf{x}_i \\ &= w_{ii-1}\mathbf{x}_{i-1} + w_{ii+1}\mathbf{x}_{i+1} - w_{ii-1}\langle \mathbf{x}_{i-1}, \mathbf{x}_i\rangle \mathbf{x}_i - \\ &\quad w_{ii+1}\langle \mathbf{x}_{i+1}, \mathbf{x}_i\rangle \mathbf{x}_i,\end{aligned}$$

which can only sum to zero if $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, and $\mathbf{x}_{i+1}$ are coplanar, *i.e.*, if they lie on a great circle. This holds for any three consecutive numbers in $[k]$ whereby $\{\mathbf{x}_i \,|\, i \in C\} \,\tilde{\subset}\, \mathcal{S}^1$.

Assume $\{\mathbf{x}_i \,|\, i \in C\} \,\tilde{\subset}\, \mathcal{S}^1$. Let $\vartheta_1, \ldots, \vartheta_k$ be angle coordinates on the great circle. As shown in [15], it holds that $\dot{\vartheta}_i = \sum_{j\in\mathcal{N}_i} w_{ij}\sin(\vartheta_j - \vartheta_i)$. Add the derivatives of the states on the cycle,

$$\begin{aligned}\sum_{i=1}^{k-1}\dot{\vartheta}_i &= \sum_{i=1}^{k-1}\sum_{j\in\mathcal{N}_i} w_{ij}\sin(\vartheta_j - \vartheta_i) = \sum_{i=1}^{k-1} w_{ii-1}\sin(\vartheta_{i-1} - \vartheta_i) \\ &\quad + w_{ii+1}\sin(\vartheta_{i+1} - \vartheta_i) \\ &= w_{10}\sin(\vartheta_0 - \vartheta_1) + w_{k-1k}\sin(\vartheta_k - \vartheta_{k-1}) + \\ &\quad \sum_{i=2}^{k-2} w_{i-1i}\sin(\vartheta_i - \vartheta_{i-1}) + w_{ii-1}\sin(\vartheta_{i-1} - \vartheta_i) \\ &= w_{10}\sin(\vartheta_0 - \vartheta_1) + w_{k-1k}\sin(\vartheta_k - \vartheta_{k-1}) \\ &= \sum_{j\in\mathcal{N}_k\cap\mathcal{V}(C)} w_{kj}\sin(\vartheta_j - \vartheta_k).\end{aligned}$$

The sum of derivatives is zero, and so is the contribution to the velocity of agent k.

Proposition 17. *Suppose the graph can be decomposed as $\mathcal{G} = \mathcal{H}\cup C_1\cup C_2\cup\mathcal{F}$ where C_1, C_2 are cycles, $C_1\cap C_2 \simeq \mathcal{K}_2$, $\mathcal{H}\cap C_1 \simeq \mathcal{K}_0$, and $\mathcal{H}\cap C_2 \,\tilde{\in}\, \{\mathcal{K}_1, \mathcal{K}_2\}$. Furthermore assume that the agents in $\mathcal{V}[\mathcal{F}\backslash(C_1\cup C_2)]$ contribute zero to $\dot{\mathbf{x}}_i$ for all $i\in\mathcal{V}(C)$ at any equilibrium of System 4 under Algorithm 9. That equilibrium then satisfies $\{\mathbf{x}_i \,|\, i\in\mathcal{V}(C_1\cup C_2)\} \,\tilde{\subset}\, \mathcal{S}^1$.*

Proof. By reasoning as in the proof of Proposition 16, it is first shown that $\{\mathbf{x}_i \,|\, i\in\mathcal{V}(C_1)\} \,\tilde{\subset}\, \mathcal{S}^1$. It can then be shown that $\{\mathbf{x}_i \,|\, i\in\mathcal{V}(C_1\cup C_2)\} \,\tilde{\subset}\, \mathcal{S}^1$.

Remark 18. The case of $C_1\cap C_2$ being a path graph with $k \geq 3$ vertices is qualitatively different since it admits equilibria where the states on $\mathcal{V}(C_1\backslash C_2)$, $\mathcal{V}(C_2\backslash C_1)$, and $\mathcal{V}(C_1\cap C_2)$ are spread out over three great circles. The approach of this paper can sometimes be used in that case, but it does not apply to all such graphs.

The next results states that certain cycles in a graph have little influence, relative to the rest of the graph, on the stability of any equilibrium manifolds. It amounts to the main technique of this paper for showing the instability of equilibrium manifolds. The intuitive idea is to show that the states of the agents are distributed over a number of great circles and then to rotate the circles one by one into the others, thereby obtaining an equilibrium where all agent states belong to the same great circle. The instability of such an equilibrium manifold follows from Proposition 12.

Proposition 19. *Suppose $\mathcal{G} = \mathcal{H}\cup\mathcal{F}$ where $\mathcal{H}\cap\mathcal{F} = (\{k\},\emptyset)$. Consider an equilibrium $\{\mathbf{x}_i\}_{i=1}^N$ of System 4 under Algorithm 9 such that $\{\mathbf{x}_i \,|\, i\in\mathcal{F}\} \,\tilde{\subset}\, \mathcal{S}^1$. Let $\mathcal{M}$ denote the maximal equilibrium manifold that contains $\{\mathbf{x}_i\}_{i=1}^N$. Suppose the contribution to $\dot{\mathbf{x}}_k$ from agents in $\mathcal{V}(\mathcal{F})$ is zero. Then $\{\{\mathbf{R}_i(\vartheta)\mathbf{x}_i\}_{i=1}^N \,|\, \vartheta\in(-\pi,\pi]\}\subset\mathcal{M}$ where $\mathbf{R}_i(\vartheta) = \mathbf{I}$ if $i\notin\mathcal{F}$, $\mathbf{R}_i(\vartheta) = \mathbf{R}(\vartheta)$ if $i\in\mathcal{F}$, and $\mathbf{R}(\vartheta)\in\mathrm{SO}(n)$ is any rotation of ϑ radians that leaves the state of agent k invariant.*

Proof. Since $\mathbf{R}(\vartheta)$ acts on the agents in $\mathcal{V}(\mathcal{F})$ as a change of coordinates, they remain at an equilibrium. By assumption, the contribution to $\dot{\mathbf{x}}_k$ from its neighbors belonging to $\mathcal{V}(\mathcal{F})$ is zero. The other neighbors of k are unaffected by $\mathbf{R}(\vartheta)$ wherefore $\dot{\mathbf{x}}_k = \mathbf{0}$ after the rotation. The agents in $\mathcal{V}(\mathcal{H})$ remain at an equilibrium for all $\vartheta\in(-\pi,\pi]$.

Theorem 20. *Suppose $\mathcal{G} = \cup_{i=1}^m C_i\cup\mathcal{F}$ for some choice of cycle graphs C_i such that $C_i\cap C_j \,\tilde{\in}\, \{\mathcal{K}_0,\mathcal{K}_1,\mathcal{K}_2\}$ for all $i,j\in[m]$ such that $i\neq j$. Furthermore assume that the agents in $\mathcal{V}(\mathcal{F})\backslash\cup_{i=1}^m\mathcal{V}(C_i)$ contribute zero to the derivatives of the states of agents in $\cup_{i=1}^m\mathcal{V}(C_i)$ at any equilibrium. Introduce the graph*

$$\mathcal{T} = ([m], \{(u,v)\in\mathcal{E}(\mathcal{G}) \,|\, u\in\mathcal{V}(C_i), v\notin\mathcal{V}(C_i)\}).$$

The consensus manifold of System 4 under Algorithm 9 is uniquely asymptotically stable if $\mathcal{T}$ is a tree.

Proof. Take some $r\in\mathcal{V}(\mathcal{T})$ to be the root of the tree. For any $i\in\mathcal{V}(\mathcal{T})$ let $\mathcal{T}_i\subset\mathcal{T}$ denote the maximal subtree branching out of node i away from the the root, *i.e.*, $\mathcal{V}(\mathcal{T}_i) = \{j\in\mathcal{T} \,|\, d(j,r)\}$, where $d:\mathcal{V}\times\mathcal{V}\rightarrow\mathbb{N}\cup\{0\}$ denotes the graph distance. Suppose that the equilibrium manifold which contains $\{\mathbf{x}_j\}_{j=1}^N$ also contains an equilibrium $\{\mathbf{y}_j\}_{j=1}^N$ such that $\mathcal{S}_i = \{\mathbf{y}_j \,|\, j\in\mathcal{V}(C_k),\, k\in\mathcal{V}(\mathcal{T}_i)\}$ belongs to a great circle for some $i\in\mathcal{T}$. Moreover, suppose the agents in $\mathcal{V}(\mathcal{T}_i)$ contribute zero to $\{\dot{\mathbf{y}}_j \,|\, j\in\mathcal{V}(G)\backslash\cup_{k\in\mathcal{T}_i}\mathcal{V}(C_k)\}$. The proof is by induction over decreasing values of the graph distance $d(r,i)$. An induction basis is provided by the set $\mathcal{V}(C_l)$ where l is a leaf of $\mathcal{T}$ that satisfies $l = \mathrm{argmax}_{j\in\mathcal{T}}\, d(r,j)$. The truthfulness of this follows by Proposition 16 or 17 since C_l is a cycle that only intersects one other cycle of the graph, C_j, and $C_l\cap C_j \,\tilde{\in}\, \{\mathcal{K}_1,\mathcal{K}_2\}$.

Suppose the induction hypothesis holds for some value k of the graph distance and let $c\in\mathcal{V}(\mathcal{T})$ be such that $d(r,c) = k$. If there is no such c aside from r, then the induction hypothesis holds for all $i\in\mathcal{V}(\mathcal{T})$ and we are done. The agents in $\{\mathcal{V}(C_i) \,|\, i\in\mathcal{V}(\mathcal{T}_c)\}$ belong to a great circle and contribute zero to the derivatives of all other agents. It follows by Proposition 16 and 17 that the agents in $\mathcal{V}(C_p)$ belong to a great circle. Rotate the equilibrium continuously as described in Proposition 19 so that $\{\mathbf{x}_i \,|\, i\in\mathcal{V}(C_j),\, j\in\mathcal{V}(\mathcal{T}_c)\}$ belongs to the same great circle as $\{\mathbf{x}_i \,|\, i\in\mathcal{V}(C_p)\}$. This shows that p, where $d(r,p) = k-1$, satisfies the induction hypothesis. Proceed in the same manner with all subtrees $\mathcal{T}_i$ such that $d(r,i) = k$ to complete the induction step.

There is hence a path in the equilibrium manifold which leads to a configuration where all agents belong to a great circle. But any such configuration is unstable by Proposition 12. If all agents are perturbed from the great circle into a hypersphere, then they reach consensus by Theorem 7. It follows that the equilibrium manifold which the equilibrium under consideration belongs to is unstable. Together with Theorem 7, this implies that the consensus manifold is uniquely asymptotically stable.

Theorem 20 implies that the consensus manifold is uniquely asymptotically stable for the class of graphs containing exactly one and exactly two cycles as stated in the following corollary. These two cases can be considered the next level of generality as compared to trees and the complete graphs, for which almost global asymptotical stability has either been shown or is implied by results in the literature [3], [15], [25]. Theorem 20 also applies to certain graph topologies containing an arbitrary number of cycles, such as the one in Figure 2. Note that the decomposition $\mathcal{G} = \cup_{i=1}^m \mathcal{C}_i \cup \mathcal{F}$ in Theorem 20 may not be unique. It is only required that one decomposition exists which results in $\mathcal{T}$ being a tree.

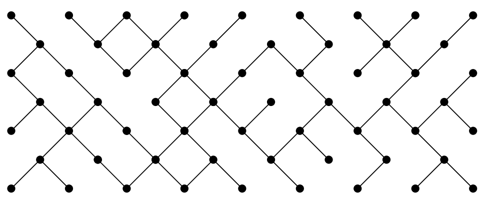

Fig. 2. A graph for which the consensus manifold is uniquely asymptotically stable.

Corollary 21. *Consider the case of $n \geq 2$. Suppose $\mathcal{G}$ contains exactly one or exactly two cycles. The consensus manifold is then uniquely asymptotically stable.*

Proof. If $\mathcal{C}_1 \cap \mathcal{C}_2 \notin \{\mathcal{K}_0, \mathcal{K}_1\}$, then there would be more than two cycles in $\mathcal{G}$.

Remark 22. The results of Theorem 20 and its corollary do not hold when $n = 1$. It is the fact that a great circle is the boundary of an asymptotically stable region, *i.e.*, a hemisphere, that gives rise to the instability of equilibrium formations that belong to great circles. On the 1-sphere, or circle, there is no degrees of freedom such as the ones used in the proof of Theorem 20. The circle case is a subject for separate study. It is sometimes referred to as a Kuramoto model in the literature, see *e.g.*, [1], [15], [18].

B. Calculation of the Final State

This section presents a solution to Problem 6 in the case of a complete graph.

Algorithm 23. The feedback is given by $\mathbf{u}_i = \frac{1}{\langle \mathbf{s}, \mathbf{x}_i \rangle}\mathbf{s}$, $\mathbf{s} = \sum_{j\in\mathcal{V}} w_j \mathbf{x}_j$, where $w_i \in (0,\infty)$ for all $i \in \mathcal{V}$.

The closed loop system is $\dot{\mathbf{x}}_i = \frac{1}{\langle \mathbf{s}, \mathbf{x}_i \rangle}\mathbf{s} - \mathbf{x}_i$.

Remark 24. The results of this section still hold if $\mathbf{s}$ is replaced by $\mathbf{s}_i = \sum_{j\in\mathcal{V}\backslash\{i\}} w_j \mathbf{x}_j$. The difference lies in whether the relative information is defined as the salient cone spanned by the states of agents in $\mathcal{V}$ or $\mathcal{V}/\{i\}$ respectively. It is preferred to use $\mathbf{s}$ over $\mathbf{s}_i$ for ease of notation.

Theorem 25. *Suppose that $\mathcal{G}$ is a complete graph and that the initial conditions satisfy $\langle \mathbf{s}, \mathbf{x}_i \rangle > 0$ for all $i \in \mathcal{V}$. Apply Algorithm 23 to System 4. The projection of $\mathbf{s} = \sum_{j\in\mathcal{V}} w_j \mathbf{x}_j$ on the n-sphere is invariant. Moreover, the agents reach consensus in this point.*

Proof. First calculate

$$\dot{\mathbf{s}} = \sum_{j\in\mathcal{V}} w_j \dot{\mathbf{x}}_j = \sum_{j\in\mathcal{V}} \frac{w_j}{\langle \mathbf{s}, \mathbf{x}_j \rangle}\mathbf{s} - w_j \mathbf{x}_j = \left(\sum_{j\in\mathcal{V}} \frac{w_j}{\langle \mathbf{s}, \mathbf{x}_j \rangle} - 1\right)\mathbf{s}. \tag{6}$$

Assume $\langle \mathbf{s}, \mathbf{x}_i \rangle$ is positive for all $i \in \mathcal{V}$. It follows that the coefficient of $\mathbf{s}$ in the right-hand side of (6) is positive since

$$\sum_{j\in\mathcal{V}} \frac{w_j}{\langle \mathbf{s}, \mathbf{x}_j \rangle} \geq \sum_{j\in\mathcal{V}} \frac{w_j}{\|\mathbf{s}\|} \geq \sum_{j\in\mathcal{V}} \frac{w_j}{\sum_{j\in\mathcal{V}} w_j} = 1. \tag{7}$$

The set $\mathcal{S} = \{\{\mathbf{x}_i\}_{i=1}^N \,|\, \langle \mathbf{s}, \mathbf{x}_i \rangle > 0 \,\forall\, i \in \mathcal{V}\}$ is invariant since

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{s}, \mathbf{x}_i \rangle &= \left(\sum_{j\in\mathcal{V}} \frac{w_j}{\langle \mathbf{s}, \mathbf{x}_j \rangle} - 1\right)\langle \mathbf{s}, \mathbf{x}_i \rangle + \tfrac{1}{\langle \mathbf{s}, \mathbf{x}_i \rangle}\|\mathbf{s}\|^2 - \langle \mathbf{s}, \mathbf{x}_i \rangle \\ &\geq \left(\sum_{j\in\mathcal{V}} \frac{w_j}{\langle \mathbf{s}, \mathbf{x}_j \rangle} - 1\right)\langle \mathbf{s}, \mathbf{x}_i \rangle \end{aligned}$$

is positive. This establishes the desired invariance; since $\mathbf{s}$ is constant in direction, its projection on the n-sphere is constant.

Algorithm 23 satisfies

$$\dot{f}_{ij} = \tfrac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{s}, \mathbf{x}_i \rangle^{-1} = -\langle \mathbf{s}, \mathbf{x}_i \rangle^{-2}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{s}, \mathbf{x}_i \rangle,$$

which is negative over the set $\mathcal{S}$ which also belongs to an open hemisphere. The agents reach consensus asymptotically by Theorem 7.

Remark 26. A sufficient requirement for $\langle \mathbf{x}_i, \mathbf{s} \rangle > 0$ to hold is that all agents belong to an open ball with radius $\pi/4$ in terms of the geodesic distance on the n-sphere, *i.e.*,

$$\max_{(i,j)\in\mathcal{E}} \arccos\langle \mathbf{x}_i, \mathbf{x}_j \rangle < \pi/2.$$

Note that the exact region of attraction,

$$\mathcal{R} = \{(\mathbf{x}_1, \ldots, \mathbf{x}_m) \in (\mathcal{S}^n)^N \,|\, \langle \mathbf{s}, \mathbf{x}_i \rangle > 0,\, \forall\, i \in \mathcal{V}\},$$

allows for initial conditions where the agents are spread out over an open hemisphere, extending beyond an open ball with radius $\pi/4$.

IV. Discussion

This paper investigates two problems of multi-agent consensus on the sphere. The problems are prompted by the two questions: (*i*) is it possible to determine the consensus point and (*ii*) is it possible to establish consensus from almost all initial conditions using a basic consensus protocol. The answer to (*i*) is yes. As for (*ii*), the answer is known to be yes for graphs that are either trees or complete. This paper extends those results to a larger class of connected graphs including graphs with exactly one or two cycles and all graphs with up to four vertices. The result obtained is not almost global asymptotical stability but unique asymptotical stability of the consensus manifold. This result does not exclude almost global asymptotical stability, which appears to hold in simulation. The result concerning the exactly one cycle case on the manifold $\mathcal{S}^2$ is interesting since it differs from the Lie groups $\mathcal{S}^1$ and SO(3) where unique asymptotical stability is not generated by the corresponding basic consensus protocols [18], [24].

References

[1] Eduardo Canale and Pablo Monzón. Gluing Kuramoto coupled oscillators networks. In *Proceedings of 46th IEEE Conf. Decision and Control*, pages 4596–4601, 2007.

[2] Richard Hartley, Jochen Trumpf, Yuchao Dai, and Hongdong Li. Rotation averaging. *International journal of computer vision*, 103(3):267–305, 2013.

[3] Wei Li and M.W. Spong. Unified cooperative control of multiple agents on a sphere for different spherical patterns. *IEEE Transactions on Automatic Control*, 59(5):1283–1289, 2014.

[4] N. Matni and M.B. Horowitz. A convex approach to consensus on SO(n). In *Proceedings of the 52nd Annual Allerton Conference on Communication, Control, and Computing*, pages 959–966, 2014.

[5] D.A. Paley. Stabilization of collective motion on a sphere. *Automatica*, 45(1):212–216, 2009.

[6] H.K. Khalil. *Nonlinear systems.* Prentice Hall, 2002.

[7] J.R. Lawton and R.W. Beard. Synchronized multiple spacecraft rotations. *Automatica*, 38(8):1359–1364, 2002.

[8] M.E. Muller. A note on a method for generating points uniformly on n-dimensional spheres. *Communications of the ACM*, 2(4):19–20, 1959.

[9] N.A. Chaturvedi, A.K. Sanyal, and N.H. McClamroch. Rigid-body attitude control: Using rotation matrices for continuous singularity-free control laws. *IEEE Control Systems Magazine*, 31(3):30–51, 2011.

[10] R.A. Horn and C.R. Johnson. *Matrix analysis.* Cambridge University Press, 2012.

[11] R.W. Beard, J.R. Lawton, and F.Y. Hadaegh. A coordination architecture for spacecraft formation control. *IEEE Transactions on control systems technology*, 9(6):777–790, 2001.

[12] S.R. Buss and J.P. Fillmore. Spherical averages and applications to spherical splines and interpolation. *ACM Transactions on Graphics*, 20(2):95–126, 2001.

[13] Y.A. Kuznetsov. *Elements of applied bifurcation theory.* Springer, 2013.

[14] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multi-Agent Networks.* Princeton University Press, 2010.

[15] Reza Olfati-Saber. Swarms on sphere: A programmable swarm with synchronous behaviors like oscillator networks. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 5060–5066. IEEE, 2006.

[16] Wei Ren. Distributed cooperative attitude synchronization and tracking for multiple rigid bodies. *IEEE Transactions on Control Systems Technology*, 18(2):383–392, 2010.

[17] A Rodriguez-Angeles and Henk Nijmeijer. Mutual synchronization of robots via estimated state feedback: a cooperative approach. *IEEE Transactions on Control Systems Technology*, 12(4):542–554, 2004.

[18] Alain Sarlette. *Geometry and symmetries in coordination control.* PhD thesis, Liège University, 2009.

[19] Alain Sarlette, Silvere Bonnabel, and Rodolphe Sepulchre. Coordinated motion design on Lie groups. *IEEE Transactions on Automatic Control*, 55(5):1047–1058, 2010.

[20] Alain Sarlette, Rodolphe Sepulchre, and N.E. Leonard. Autonomous rigid body attitude synchronization. *Automatica*, 45(2):572–577, 2009.

[21] Michael Shub. *Global stability of dynamical systems.* Springer, 2013.

[22] Johan Thunberg, Wenjun Song, Eduardo Montijano, Yiguang Hong, and Xiaoming Hu. Distributed attitude synchronization control of multi-agent systems with switching topologies. *Automatica*, 50(3):832–840, 2014.

[23] R. Tron and R. Vidal. Distributed 3-D localization of camera sensor networks from 2-D image measurements. *IEEE Transactions on Automatic Control*, 59(12):3325–3340, 2014.

[24] Roberto Tron, Bijan Afsari, and René Vidal. Intrinsic consensus on SO(3) with almost-global convergence. In *Proceedings of the 51th IEEE Conference on Decision and Control*, pages 2052–2058, 2012.

[25] Roberto Tron, Bijan Afsari, and René Vidal. Riemannian consensus for manifolds with bounded curvature. *IEEE Transactions on Automatic Control*, 58(4):921–934, 2013.

A. Lemmas

Lemma 27. *The $(n+1)\times(n+1)$ blocks of the $N(n+1)\times N(n+1)$ matrix $\mathbf{A}$ that describes the linearized dynamics of System 4 under Algorithm 9 are given by*

$$\mathrm{A}_{ij} = \begin{cases} -(\mathbf{x}_i \otimes \mathbf{s}_i + \langle \mathbf{s}_i, \mathbf{x}_i \rangle \mathbf{I})(\mathbf{I} - \mathbf{X}_i) & \text{if } j = i, \\ w_{ij}(\mathbf{I} - \mathbf{X}_i)(\mathbf{I} - \mathbf{X}_j) & \text{if } j \neq i, \end{cases}$$

for $(i,j) \in \mathcal{E}$ and $\mathrm{A}_{ij} = \mathbf{0}$ otherwise. The matrix $\mathbf{A}$ is symmetric.

Proof. For systems evolving on manifolds, a perturbation technique is used to obtain the linearized dynamics. Let $\mathbf{x}_i$ for all $i \in \mathcal{V}$ be a solution to (2). Consider a perturbed solution $\mathbf{x}_i(\varepsilon, \mathbf{v}_i)$ given by

$$\mathbf{x}_i(\varepsilon, \mathbf{v}_i) = \frac{\mathbf{x}_i + \varepsilon \mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon \mathbf{v}_i\|},$$

where $\mathbf{v}_i$ is a smooth function. The perturbed solution is required to satisfy the differential equation

$$\dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i) = \mathbf{u}_i(\varepsilon, \mathbf{v}_i) - \langle \mathbf{u}_i(\varepsilon, \mathbf{v}_i), \mathbf{x}_i(\varepsilon, \mathbf{v}_i) \rangle \mathbf{x}_i(\varepsilon, \mathbf{v}_i).$$

The linearized dynamics on $\mathcal{S}^n$ can be derived by studying the linear effect of $\mathbf{v}_i$ on $\dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i)$. Define

$$\begin{aligned} \mathbf{w}_i &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbf{x}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \left.\frac{\mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon \mathbf{v}_i\|}\right|_{\varepsilon=0} - \\ &\qquad \left.\frac{\mathbf{x}_i + \varepsilon \mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon \mathbf{v}_i\|^3} \langle \mathbf{x}_i, \mathbf{v}_i \rangle \right|_{\varepsilon=0} \\ &= \mathbf{v}_i - \mathbf{x}_i \otimes \mathbf{x}_i \mathbf{v}_i = (\mathbf{I} - \mathbf{X}_i)\mathbf{v}_i, \end{aligned} \tag{8}$$

where $\mathbf{X}_i = \mathbf{x}_i \otimes \mathbf{x}_i$. The matrix $\mathbf{I} - \mathbf{X}_i$ projects onto the tangent space $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$ where $\mathbf{w}_i$ lives. Note that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbf{X}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i$$

by the product rule. Then

$$\begin{aligned} \dot{\mathbf{w}}_i &= \tfrac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}\varepsilon} \mathbf{x}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} \dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} (\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i)) \sum_{j \in \mathcal{N}_i} w_{ij} \mathbf{x}_j(\varepsilon, \mathbf{v}_j) \\ &= -\left[\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbf{X}_i(\varepsilon, \mathbf{v}_i)\right] \sum_{j \in \mathcal{N}_i} w_{ij} \mathbf{x}_j(\varepsilon, \mathbf{v}_j) \Big|_{\varepsilon=0} + \\ &\qquad (\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i)) \sum_{j \in \mathcal{N}_i} w_{ij} \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbf{x}_j(\varepsilon, \mathbf{v}_j) \Big|_{\varepsilon=0} \\ &= -(\mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i) \sum_{j \in \mathcal{N}_i} \mathbf{x}_j + \\ &\qquad (\mathbf{I} - \mathbf{X}_i) \sum_{j \in \mathcal{N}_i} w_{ij} \mathbf{w}_j \end{aligned}$$

$$
\begin{aligned}
&= -(\mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i)\mathbf{u}_i + (\mathbf{I} - \mathbf{X}_i) \sum_{j \in \mathcal{N}_i} w_{ij} \mathbf{w}_j \\
&= -(\langle \mathbf{u}_i, \mathbf{x}_i \rangle \mathbf{I} + \mathbf{x}_i \otimes \mathbf{u}_i)\mathbf{w}_i + \\
&\quad (\mathbf{I} - \mathbf{X}_i) \sum_{j \in \mathcal{N}_i} w_{ij} \mathbf{w}_j \\
&= -(\langle \mathbf{u}_i, \mathbf{x}_i \rangle \mathbf{I} + \mathbf{x}_i \otimes \mathbf{u}_i)(\mathbf{I} - \mathbf{X}_i)\mathbf{v}_i + \\
&\quad (\mathbf{I} - \mathbf{X}_i) \sum_{j \in \mathcal{N}_i} w_{ij} (\mathbf{I} - \mathbf{X}_j)\mathbf{v}_j, \qquad (9)
\end{aligned}
$$

where the relation $\langle \mathbf{x}, \mathbf{y} \rangle \mathbf{z} = (\mathbf{z} \otimes \mathbf{x})\, \mathbf{y} = (\mathbf{z} \otimes \mathbf{y})\mathbf{x}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n+1}$ is used. The vector $\mathbf{w} = [\mathbf{w}_1^\top \ldots \mathbf{w}_N^\top]^\top$ has $N(n+1)$ components whereas the linearized system actually evolves on an Nn-dimensional space that lies embedded in $\mathbb{R}^{N(n+1)}$. The dimension reduction is given implicitly by the definition of $\mathbf{w}_i$ which requires $\mathbf{w}_i \in \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$. This constraint is removed by using variables that are premultiplied by the projection matrices $\mathbf{I} - \mathbf{X}_i : \mathbb{R}^{n+1} \to \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$, *i.e.*, the variables $\mathbf{v}_i$ in (8). The matrix $\mathbf{A}$ is obtained by inspection of (9).

It remains to show that $\mathbf{A}$ is symmetric. Write

$$
\begin{aligned}
\mathrm{A}_{ii} &= -(\mathbf{x}_i \otimes \mathbf{u}_i + \langle \mathbf{u}_i, \mathbf{x}_i \rangle \, \mathbf{I})(\mathbf{I} - \mathbf{X}_i), \\
&= -(\mathbf{x}_i \otimes \mathbf{u}_i + \mathbf{u}_i \otimes \mathbf{x}_i + \langle \mathbf{u}_i, \mathbf{x}_i \rangle \, \mathbf{I})(\mathbf{I} - \mathbf{X}_i),
\end{aligned}
$$

which is clearly symmetric. Moreover,

$$
\begin{aligned}
\mathrm{A}_{ji}^\top - \mathbf{A}_{ij} = w_{ji}&[(\mathbf{I} - \mathbf{X}_j)(\mathbf{I} - \mathbf{X}_i)]^\top - \\
&w_{ij}(\mathbf{I} - \mathbf{X}_i)(\mathbf{I} - \mathbf{X}_j) = \mathbf{0},
\end{aligned}
$$

since $w_{ij} = w_{ji}$ for all $(i, j) \in \mathcal{E}$.