N-point Virasoro algebras are multi-point Krichever–Novikov type algebras

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Book: Krichever–Novikov type algebras. Theory and Applications. DeGruyther, 2014

Introduction

- the classical genus zero (two point) algebras (Witt algebra, Virasoro algebra, affine Kac-Moody algebras of untwisted type, ...) are well-established and of relevance e.g. in CFT
- but from the application there is a need for the multi-point algebras in every genus (of course including genus zero)
- higher genus and still two points this was done by Krichever and Novikov
- the multi-point theory was done by the speaker
- importance for KZ equations for genus zero in CFT is nowadays classical
- for higher genus KZ connections in the context of $M_{g,n}$ see joint work of the speaker with Oleg Sheinman
- recently revived interest in genus zero multi-point quantum field theory (N-point Virasoro algebra)

- Goal: show that the recently discussed N-point Virasoro algebras (Cox, Jurisisch, Martins, and others) are special examples of the multi-point KN type algebras
- Gain: gives useful structural insights and an easier approach to calculations
- removes some misconceptions about certain observed phenomena

What I will do here:

- recall the geometric setup for KN type algebras
- introduce the algebras
- almost-grading including triangular decomposition
- determine “all” central extensions

What will be the outcome for KN type, genus zero:

- all cohomology (cocycle) classes (2nd Lie algebra cohomology with values in the trival module) for vector field algebra and the differential operator algebras are geometric
- give the universal central extensions for them explicitly
- the same for the current algebra, yielding affine algebras
- Heisenberg algebra obtained by cocycles for the function algebra which are multiplicative
- give access to easy calculations of structure constants and cocycle values for these algebras
- As illustration: three point genus zero situation.

- purely algebraic terms the Virasoro algebra
  generators $\{e_n (n \in \mathbb{Z}), t\}$ and relations

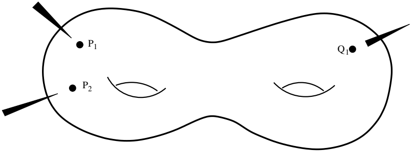
$$[e_n, e_m] = (m-n)e_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m} \cdot t.$$

- without central term: Witt algebra
- $\mathfrak{g}$ a finite-dimensional simple Lie algebra,
  β the Cartan–Killing form,

$$[\widehat{x \otimes z^n}, \widehat{y \otimes z^m}] := \widehat{[x,y] \otimes z^{n+m}} - \beta(x,y) \cdot n\,\delta_m^{-n} \cdot t.$$

$\widehat{\mathfrak{g}}$ is called affine Lie algebra.

Geometric Set-up (KN type algebras)



- Σ_g be a compact Riemann surface of genus $g = g(\Sigma_g)$.
- A be a finite subset of Σ_g, $A = I \cup O$, both non-empty, disjoint, $I = (P_1, \ldots, P_K)$ in-points and $O = (Q_1, \ldots, Q_M)$ out-points
- genus zero: $A = \{P_1, P_2, \ldots, P_N\}$, P_N can be brought to ∞ by fractional linear transformation
- $P_i = a_i, \quad a_i \in \mathbb{C}, \; i = 1, \ldots, N-1, \quad P_N = \infty$
- local coordinates $z - a_i, \; i = 1, \ldots, N-1, \qquad w = 1/z$
- classical situation: $\Sigma_0 = S^2, \quad I = \{0\}, \quad O = \{\infty\}$

GEOMETRIC REALIZATIONS OF THE KN TYPE ALGEBRAS

- $\mathcal{K}$ is the canonical bundle, i.e. local sections are the holomorphic differentials
- $\mathcal{K}^\lambda := \mathcal{K}^{\otimes\lambda}$ for $\lambda \in \mathbb{Z}$
- the sections are the forms of weight λ, e.g. $\lambda = -1$ are vector fields, $\lambda = 0$ are functions,
- for half-integer λ we need to fix a square root L of $\mathcal{K}$ (also called theta characteristics, or spin structure)
- for $g = 0$ only one square-root, the tautological bundle U
- we ignore in this presentation the half-forms (e.g. the supercase)

- $\mathcal{F}^\lambda := \mathcal{F}^\lambda(A) := \{f$ is a global meromorphic section of $\mathcal{K}^\lambda$ such that f is holomorphic over $\Sigma \setminus A\}$.
- infinite dimensional vector spaces
- meromorphic forms of weight λ
- $$\mathcal{F} := \bigoplus_{\lambda \in \frac{1}{2}\mathbb{Z}} \mathcal{F}^\lambda.$$

- We define an associative structure

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}.$$

- in local representing meromorphic functions

$$(s\, dz^\lambda, t\, dz^\nu) \mapsto s\, dz^\lambda \cdot t\, dz^\nu = s \cdot t\; dz^{\lambda+\nu}.$$

- $\mathcal{F}$ is an associative and commutative graded algebra.
- $\mathcal{F}^0 =: \mathcal{A}$ is a subalgebra and $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.

- Lie algebra structure:

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (s,t) \mapsto [s,t],$$

- in local representatives of the sections

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto [s\,dz^\lambda, t\,dz^\nu] := \left((-\lambda)s\frac{dt}{dz} + \nu\,t\frac{ds}{dz}\right) dz^{\lambda+\nu+1},$$

- $\mathcal{F}$ with $[.,.]$ is a Lie algebra
- $\mathcal{F}$ with respect to $\cdot$ and $[.,.]$ is a Poisson algebra
- $\mathcal{L} := \mathcal{F}^{-1}$ is a Lie subalgebra (the algebra of vector fields), and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.
- $\mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L} =: \mathcal{D}^1$ is also a Lie subalgebra of $\mathcal{F}$, it is the Lie algebra of differential operators of degree ≤ 1

Almost-graded structure

- $\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$ is a vector space direct sum, then $\mathcal{L}$ is called an almost-graded (Lie-) algebra if
  - (I) $\dim \mathcal{L}_n < \infty$,
  - (II) There exist constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

- introduce an almost-grading for $\mathcal{F}^\lambda$ by exhibiting certain elements $f^\lambda_{n,p} \in \mathcal{F}^\lambda$, $p = 1, \ldots, K$ which constitute a basis of the subspace $\mathcal{F}^\lambda_n$ of homogeneous elements of degree n.
- the basis element $f^\lambda_{n,p}$ of degree n is of order

$$\mathrm{ord}_{P_i}(f^\lambda_{n,p}) = (n + 1 - \lambda) - \delta^p_i$$

at the point $P_i \in I$, $i = 1, \ldots, K$.

- prescription at the points in O is made in such a way that the element $f^\lambda_{n,p}$ is essentially unique
- Warning: The decomposition depends on the splitting of A into $I \cup O$.

Genus zero – standard splitting

- standard splitting: $I = \{P_1, P_2, \dots, P_{N-1}\}$ and $O = \{\infty\}$, we have $K = N - 1$
- it is enough to construct a basis $\{A_{n,p}\}$ of $\mathcal{A}$
- then $\mathcal{F}_n^\lambda = \mathcal{A}_{n-\lambda} dz^\lambda, \quad f_{n,p}^\lambda = A_{n-\lambda,p} dz^\lambda$
- $A_{n,p}(z) := (z - a_p)^n \cdot \prod_{\substack{i=1 \\ i \neq p}}^K (z - a_i)^{n+1} \cdot \alpha(p)^{n+1}$, $p = 1, \dots, K$
- $\alpha(p)$ normalization factor such that $A_{n,p}(z) = (z - a_p)^n (1 + O(z - a_p))$
- the order at ∞ is fixed as $-(Kn + K - 1)$
- $e_{n,p} = f_{n,p}^{-1} = A_{n+1,p} \frac{d}{dz}, \quad p = 1, \dots, K$

- The above algebras are almost-graded algebras.
- the almost-grading depends on the splitting of the set A into I and O.
- $\mathcal{F}^\lambda = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}^\lambda_m, \qquad \text{with} \quad \dim \mathcal{F}^\lambda_m = K.$
- there exist R_1, R_2 (independent of n and m) such that

$$\mathcal{A}_n \cdot A_m \subseteq \bigoplus_{h=n+m}^{n+m+R_1} \mathcal{A}_h\ , \qquad [\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{L}_h\ ,$$

- for genus zero and standard splitting

$$R_1 = \begin{cases} 0, & N = 2, \\ 1, & N > 2, \end{cases} \qquad R_2 = \begin{cases} 0, & N = 2, \\ 1, & N = 3, \\ 2, & N > 3\,. \end{cases}$$

- triangular decomposition $\mathcal{U} = \mathcal{U}_{[-]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[+]}$ with

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m.$$

Here $\mathcal{U}$ is any of the above algebras $\mathcal{A}$, $\mathcal{L}$, ...

Before central extensions

- C_i be positively oriented (deformed) circles around the points P_i in I, $i = 1, \ldots, K$
- C_j^* positively oriented circles around the points Q_j in O, $j = 1, \ldots, M$.
- A cycle C_S is called a separating cycle if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points.
- we will integrate meromorphic differentials on Σ_g without poles in $\Sigma_g \setminus A$ over closed curves C.
- hence, C and C' are equivalent if $[C] = [C']$ in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{Z})$.

- $[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]$
- given such a separating cycle C_S (respectively cycle class) we define $\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega$
- This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega \;=\; \sum_{i=1}^{K} \mathrm{res}_{P_i}(\omega) \;=\; -\sum_{l=1}^{M} \mathrm{res}_{Q_l}(\omega).$$

Central extensions

- A central extension of a Lie algebra $\mathcal{U}$ is defined on the vector space direct sum $\widehat{\mathcal{U}} = \mathbb{C} \oplus U$.
  $\hat{x} := (0, x)$, $t := (1, 0)$

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \Phi(x, y) \cdot t, \quad [t, \widehat{U}] = 0, \quad x, y \in U.$$

- $\widehat{U}$ will be a Lie algebra, if and only if Φ is antisymmetric and fulfills the Lie algebra 2-cocycle condition

$$0 = d_2\Phi(x, y, z) := \Phi([x, y], z) + \Phi([y, z], x) + \Phi([z, x], y).$$

- A 2-cocycles Φ is a coboundary if there exists a $\phi : \mathcal{U} \to \mathbb{C}$ such that

$$\Phi(x, y) = d_1\phi(x, y) = \phi([x, y]).$$

- the second Lie algebra cohomology $\mathrm{H}^2(\mathcal{U}, \mathbb{C})$ of $\mathcal{U}$ with values in the trivial module $\mathbb{C}$ classifies equivalence classes of central extensions.
- A Lie algebra $\mathcal{U}$ is called perfect if $[\mathcal{U}, \mathcal{U}] = \mathcal{U}$.
- perfect Lie algebras admit universal central extensions

LOCAL AND BOUNDED COCYCLES

- γ a cocycle for the almost-graded Lie algebra $\mathcal{U}$ is called a local cocycle if $\exists T_1, T_2$ such that $\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies T_2 \leq n + m \leq T_1$
- γ is called bounded (from above) if $\exists T_1$ such that $\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies n + m \leq T_1$
- for the classes it means that it contains one representing cocycle of this type.
- Importance: Local cocycles allow to extend the almost-grading to the central extension.
- The speaker classified for the above algebras local and bounded cocycle classes. They are given by geometric cocycles of certain type (see below).

GEOMETRIC COCYCLES

- A cocycle $\gamma : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$ is called a geometric cocycle if there is a bilinear map $\widehat{\gamma} : \mathcal{U} \times \mathcal{U} \to \mathcal{F}^1$, such that γ is the composition of $\widehat{\gamma}$ with an integration, i.e. $\gamma = \gamma_C := \frac{1}{2\pi \mathrm{i}} \int_C \widehat{\gamma}$ with C a curve on Σ_g.
- Given $\widehat{\gamma}$ only the class of C in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$ plays a role,
- $$\dim \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C}) = \begin{cases} 2g, & \#A = 0, 1, \\ 2g + (N-1), & \#A = N \geq 2\,. \end{cases}$$

- genus zero and $N \geq 1$: $\dim \mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C}) = (N-1)$
- basis e.g. given by circles C_i around the points P_i, where we leave out one of them.
  For example $[C_i]$, $i = 1, \ldots, N-1$.
- better choice: e.g. for the standard splitting take $[C_S] = -[C_\infty]$ and $[C_i]$, $i = 1, \ldots, N-2$

MAIN RESULT – PHILOSOPHY - (GENUS ZERO !!)

- we show that in genus zero our cocycles classes are geometric cocycles classes with respect to certain explicitely given one-forms
- this is done by showing that all cocycles are bounded cocycles with respect to the almost-grading induced by the standard splitting,
- now the classification result of bounded cocycle classes of the author is used which gives a complete classification and explicit expressions given by integrals over curves
- note that in genus zero the geometric cocycles can be obtained via integration over circles around the points in I, or alternatively around ∞
- and they can be calculate via residues
- In case that the Lie algebra is perfect the universal central extension can directly be given.

Function algebra – Heisenberg algebra

- γ is $\mathcal{L}$-invariant if $\gamma(e\,.\,f, g) + \gamma(f, e\,.\,g) = 0$, for all $f, g \in \mathcal{A}$, for all $e \in \mathcal{L}$,
- multiplicative if $\gamma(fg, h) + \gamma(gh, f) + \gamma(hf, g) = 0$, for all $f, g, h \in \mathcal{A}$
- Theorem: If one of the above properties is fulfilled then it is a geometric cocycle.
- basis

$$\gamma_i^{\mathcal{A}}(f, g) = \frac{1}{2\pi \mathrm{i}} \int_{C_i} f dg = \mathrm{res}_{a_i}(f dg), \quad i = 1, \dots, N-1.$$

- γ is bounded from above with respect to the almost-grading given by the standard splitting.

- Every $\mathcal{L}$-invariant cocycle is multiplicative and vice versa.
- Two point situation: $\gamma(A_n, A_m) = \alpha \cdot (-n) \cdot \delta_m^{-n}$
- Heisenberg algebra is such a central extension (the local one or the "full" one).
- for the full one the center is $(N-1)$-dimensional

Results: $g = 0$

- Every cocycle class is geometric and given by

$$\gamma^{\mathcal{L}}_{C,R}(e,f) = \frac{1}{2\pi\mathrm{i}}\int_C (\frac{1}{2}(ef''' - e'''f) - R(ef' - e'f)dz.$$

- R is a projective connection, with our coordinates we can take $R = 0$.
- after cohomological changes they are bounded
- $\mathrm{H}^2(\mathcal{L}, \mathbb{C})$ is $(N - 1)$-dimensional
- can be calculate by residues at the points
- these cocycles generate a universal central extension.
- By different techniques Skryabin has shown the existence of a universal central extension for arbitrary genus.

DIFFERENTIAL OPERATOR ALGEBRA

- Main result also here: all cocycle classes are geometric
- $\mathcal{L}$-invariant coycles for $\mathcal{A}$ and arbitrary cocycles for $\mathcal{L}$ define two cocycle types for $\mathcal{D}^1$.
- There is a another type: mixing cocycles

$$\gamma_{C,T}^{(m)}(e,g) := \frac{1}{2\pi \mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A},$$

- T is an affine connection. Can be taken to be zero on the affine part.
- also $\mathcal{D}^1$ is perfect and the universal central extension has $3 \cdot (N-1)$ dimensional center

OTHERS

Current algebra:

- $\mathfrak{g}$ a finite dimensional simple Lie algebra, β Cartan–Killing form

$$\gamma^{\overline{\mathfrak{g}}}_{\beta,C}(x\otimes f, y\otimes g) = \beta(x,y)\cdot\gamma^{\mathcal{A}}_{C}(f,g) = \beta(x,y)\cdot\frac{1}{2\pi\mathrm{i}}\int_C f dg$$

- all cocycles are cohomologous to such cocycles,
- $\widehat{\mathfrak{g}}$ is perfect, universal central extension has again $(N-1)$-dimensional center
- the multiplicativity of $\int_C f dg$ is crucial
- I have corresponding results for $\mathfrak{g}$ reductive.

Also results for Lie superalgebras: Each central extension of $\mathcal{L}$ gives a unique central extension of the superalgebra.

Short form

Every cocycle class is geometric and given by (for $\mathcal{A}$ we need either $\mathcal{L}$-invariance or multiplicativity)

$$\gamma_C^{\mathcal{A}}(f,g) = \frac{1}{2\pi \mathrm{i}} \int_C f dg$$

$$\gamma_{C,R}^{\mathcal{L}} = \frac{1}{2\pi \mathrm{i}} \int_C (\frac{1}{2}(ef''' - e'''f) - R(ef' - e'f)dz.$$

$$\gamma_{C,T}^{(m)}(e,g) := \frac{1}{2\pi \mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A},$$

$$\gamma_{\beta,C}^{\overline{\mathfrak{g}}}(x \otimes f, y \otimes g) = \beta(x,y) \cdot \gamma_C^{\mathcal{A}}(f,g) = \beta(x,y) \cdot \frac{1}{2\pi \mathrm{i}} \int_C f dg$$

Next use that C_i, $i = 1, \ldots, N-1$ is a basis of $\mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C})$ and that the integration over C_i can be done by calculating residues.

Three-point algebras

- $A = I \cup O, \qquad I := \{0, 1\}, \quad \text{and} \quad O := \{\infty\}$
- basis elements ("symmetrized" and "anti-symmetrized")

$$A_n(z) = z^n(z-1)^n, \quad B_n(z) = z^n(z-1)^n(2z-1),$$

- structure equations:

$$\begin{aligned} A_n \cdot A_m &= A_{n+m}, \\ A_n \cdot B_m &= B_{n+m}, \\ B_n \cdot B_m &= A_{n+m} + 4A_{n+m+1}. \end{aligned}$$

- space of cocycles is two-dimensional, e.g. we take the residues around ∞ and around 0

- $$\gamma_\infty^{\mathcal{A}}(A_n, A_m) = 2n\,\delta_m^{-n},$$
  $$\gamma_\infty^{\mathcal{A}}(A_n, B_m) = 0,$$
  $$\gamma_\infty^{\mathcal{A}}(B_n, B_m) = 2n\delta_m^{-n} + 4(2n+1)\,\delta_m^{-n-1}\ .$$
- $$\gamma_0^{\mathcal{A}}(A_n, A_m) = -n\,\delta_m^{-n},$$
  $$\gamma_0^{\mathcal{A}}(A_n, B_m) = n\,\delta_m^{-n} + 2n\,\delta_m^{-n-1} + \sum_{k=2}^{\infty} n\,(-1)^{k-1} 2^k \frac{(2k-3)!!}{k!} \delta_m^{-n-k}\ ,$$
  $$\gamma_0^{\mathcal{A}}(B_n, B_m) = -n\delta_m^{-n} - 2(2n+1)\,\delta_m^{-n-1}\ .$$

- vector field algebra
- basis: $e_n := A_{n+1}\frac{d}{dz}, \qquad f_n := B_{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}$
- structure equation

$$
\begin{aligned}
[e_n, e_m] &= (m-n)\, f_{m+n}, \\
[e_n, f_m] &= (m-n)\, e_{m+n} + (4(m-n)+2)\, e_{n+m+1}, \\
[f_n, f_m] &= (m-n)\, f_{m+n} + 4(m-n)\, f_{n+m+1}.
\end{aligned}
$$

- the universal central extension is two -dimensional, as above obtained by calculating residues at ∞ and 0.
-

$$
\begin{aligned}
\gamma_0^{\mathcal{L}}(e,f) &= 1/2\,\mathrm{res}_0(e\cdot f''' - f\cdot e''')dz \\
\gamma_\infty^{\mathcal{L}}(e,f) &= 1/2\,\mathrm{res}_\infty(e\cdot f''' - f\cdot e''')
\end{aligned}
$$

$$\gamma_\infty^{\mathcal{L}}(e_n, e_m) = 2(n^3 - n)\,\delta_m^{-n} + 4n(n+1)(2n+1)\delta_m^{-n-1}$$

$$\gamma_\infty^{\mathcal{L}}(e_n, f_m) = 0,$$

$$\gamma_\infty^{\mathcal{L}}(f_n, f_m) = 2(n^3 - n)\,\delta_m^{-n} + 8n(n+1)(2n+1)\delta_m^{-n-1} + 8(n+1)(2n+1)(2n+3)\delta_m^{-n-2}$$

$$\gamma_0^{\mathcal{L}}(e_n, e_m) = -(n^3 - n)\,\delta_n^{-m} - 2n(n+1)(2n+1)\delta_m^{-n-1}$$

$$\gamma_0^{\mathcal{L}}(e_n, f_m) = (n^3 - n)\,\delta_m^{-n} + 6n^2(n+1)\delta_m^{-n-1} + 6n(n+1)^2\delta_m^{-n-2} + \sum_{k\geq 3} n(n+1)(n+k-1)(-1)^k 2^k \cdot 3 \cdot \frac{(2k-5)!!}{k!}\delta_m^{-n-k}$$

$$\gamma_0^{\mathcal{L}}(f_n, f_m) = -(n^3 - n)\,\delta_m^{-n} - 4n(n+1)(2n+1)\delta_m^{-n-1} - 4(n+1)(2n+1)(2n+3)\delta_m^{-n-2}\,.$$

Another basis

- our algebra $\mathcal{A}$ can be given as the algebra $\mathcal{A} = \mathbb{C}[(z-a_1),(z-a_1)^{-1},(z-a_2)^{-1},\dots,(z-a_{N-1})^{-1}]$, with the obvious relations.
- we set $A_n^{(i)} := (z-a_i)^n$
- $A_n^{(i)}, \quad n \in \mathbb{Z},\ i = 1,\dots,N-1$ is a generating set of $\mathcal{A}$
- A basis is given e.g. by $A_n^{(1)},\ n \in \mathbb{Z}, \quad A_{-n}^{(i)},\ n \in \mathbb{N},\ i = 2,\dots,N-1.$
- but this defines not an almost-graded structure