Axiomatic characterizations of generalized values

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Abstract

In the framework of cooperative game theory, the concept of generalized value, which is an extension of that of value, has been recently proposed to measure the overall influence of coalitions in games. Axiomatizations of two classes of generalized values, namely probabilistic generalized values and generalized semivalues, which extend probabilistic values and semivalues, respectively, are first proposed. The axioms we utilize are based on natural extensions of axioms involved in the axiomatizations of values. In the second half of the paper, special instances of generalized semivalues are also axiomatized.

Key words: Cooperative games, values (power indices), generalized values.

1 Introduction

It is well known that the notion of *value* (see e.g. [2, 5, 9, 17, 18]) has been introduced to measure the individual power of each player in a cooperative game. In some sense, this individual power can be regarded as the influence the player has in the game.

It is natural to extend this notion of power to coalitions of players. For example, suppose two or three players join together to form a partnership in a game. It is then much more relevant to measure the power or the strength of such a coalition in the game rather than the power of each of these players. This is exactly the role played by the *generalized values*, which have been recently introduced by Marichal [12] to measure the overall influence of every coalition in a game. More precisely, the concept of generalized value stems from the investigation of the influence of variables on Boolean and pseudo-Boolean functions [4, 10, 12], which was motivated by the problem of searching for robust voting schemes in game theory [3].

Let $v : 2^N \to \mathbb{R}$ be a cooperative game on a finite set of players N. For a coalition $S \subseteq N$, the main generalized values introduced in [12] are defined as

$$\begin{aligned} \Phi_{Sh}(v,S) &:= \sum_{T\subseteq N\setminus S} \frac{(n-s-t)!\,t!}{(n-s+1)!}\,[v(T\cup S)-v(T)], \\ \Phi_{B}(v,S) &:= \sum_{T\subseteq N\setminus S} \frac{1}{2^{n-s}}\,[v(T\cup S)-v(T)], \end{aligned}$$

where $n = |N|$, $s = |S|$, and $t = |T|$. As we can see, these expressions can be interpreted as weighted means of the marginal contributions $v(T \cup S) - v(T)$ of coalition S to outer coalitions $T \subseteq N\setminus S$. As they clearly coincide with Shapley and Banzhaf values on singletons (i.e., when $S = \{i\}$), we will naturally call them the *Shapley* and *Banzhaf generalized values*, respectively.

In this paper we introduce and axiomatize two families of generalized values, namely the broad class of *probabilistic generalized values* and the narrower subclass of *generalized semi-values* obtained by additionally imposing the symmetry axiom. Probabilistic generalized values can be seen as extensions of *probabilistic values* studied by Weber [18]. Generalized semivalues are extensions of *semivalues*, which were axiomatized by Dubey et al. [5]. We show that this latter subclass encompasses the Shapley and Banzhaf generalized values, but also the game itself and its dual. We also axiomatize these particular instances of generalized semivalues.

Besides the classical axiom of *linearity*, the axioms involved in the characterizations we present can be regarded as natural generalizations of those used in the axiomatizations of values. Two of the most important axioms in the proposed characterizations of probabilistic generalized values and generalized semivalues are the *dummy coalition* axiom, which is a natural extension of the *dummy player* axiom [18, §3], and the *positivity* axiom, which generalizes the one for values [11, §4] (called *monotonicity* in [18, §4]). The notion of *partnership* (see e.g. [11]) is also at the root of some of the axioms we additionally impose to characterize the Shapley and Banzhaf generalized values.

In addition to these characterization results we provide a representation theorem for generalized semivalues, extending that of semivalues by Dubey et al. [5]. We also provide a compact formula linking any generalized semivalue with the well-known Owen *multilinear extension* of a game [15].

This paper is organized as follows. In Section 2 we recall some basic definitions and results we use in this paper. In Section 3 we introduce the class of probabilistic generalized values and that of generalized semivalues. In the last section we present our characterization results. Probabilistic generalized values and generalized semivalues are first axiomatized. Then, we yield representation theorems for generalized semivalues. Finally, the Shapley and Banzhaf generalized values as well as three other instances of generalized semivalues are characterized by imposing additional axioms.

In order to avoid a heavy notation, we adopt that used in [8]. Thus, we will often omit braces for singletons, e.g., by writing $v(i)$, $U \setminus i$ instead of $v(\{i\})$, $U \setminus \{i\}$. Similarly, for pairs, we will write ij instead of $\{i,j\}$. Furthermore, cardinalities of subsets $S,T,\ldots$, will often be denoted by the corresponding lower case letters $s,t,\ldots$, otherwise by the standard notation $|S|,|T|,\ldots$

2 Preliminary definitions

We consider an infinite set U, the universe of players. As usual, a *game* on U is a set function $v : 2^U \to \mathbb{R}$ such that $v(\varnothing) = 0$, which assigns to each *coalition* $S \subseteq U$ its *worth* $v(S)$.

In this section we recall some concepts and results we will use throughout.

2.1 Carriers

A set $N \subseteq U$ is said to be a *carrier* (or *support*) of a game v when $v(S) = v(N \cap S)$ for all $S \subseteq U$. Thus, a game v with carrier $N \subseteq U$ is completely defined by the knowledge of the coefficients $\{v(S)\}_{S\subseteq N}$ and the players outside N have no influence on the game since they do not contribute to any coalition.

In this paper, we restrict our attention to finite games, that is, games that possess finite carriers. We denote by $\mathcal{G}$ the set of finite games on U and by $\mathcal{G}^N$ the set of games with finite carrier $N \subset U$.

2.2 Dividends and Möbius transform

Let us recall an equivalent representation of a game. Any game $v \in \mathcal{G}^N$ can be uniquely expressed in terms of its *dividends* $\{m(v,S)\}_{S\subseteq N}$ (see e.g. [9]) by

$$v(T) = \sum_{S\subseteq T} m(v,S), \qquad \forall\, T \subseteq N.$$

In combinatorics, the set function $m(v,\,\cdot\,) : 2^U \to \mathbb{R}$ is called the *Möbius transform* [16] of v and is given by

$$m(v,S) := \sum_{T\subseteq S} (-1)^{s-t} v(T), \qquad \forall\, S \subseteq U.$$

2.3 Unanimity games

Let us now recall two important simple games of $\mathcal{G}^N$.

The *unanimity game* for $T \subseteq N$, $T \neq \varnothing$, is defined as the game u_T such that, for all $S \subseteq N$, $u_T(S) := 1$ if and only if $S \supseteq T$ and 0 otherwise. It is easy to check that T is a carrier of u_T and that its Möbius transform is given, for all $S \subseteq N$, by $m(u_T, S) = 1$ if and only if $S = T$ and 0 otherwise.

Following Dubey et al. [5, §1], for any $T \subseteq N$, we also consider the game $\hat{u}_T \in \mathcal{G}^N$, defined for all $S \subseteq N$, by $\hat{u}_T(S) := 1$ if and only if $S \supsetneq T$ and 0 otherwise.

2.4 Permuted games

Following Shapley [17, §2], given a game $v \in \mathcal{G}^N$ and a permutation π on U (i.e., a one-to-one mapping from U onto itself), we denote by πv the game defined by

$$\pi v[\pi(S)] := v(S), \qquad \forall\, S \subseteq U,$$

where $\pi(S) := \{\pi(i) \mid i \in S\}$. Note that $\pi(N)$ is a carrier of πv.

2.5 Restricted and reduced games

Given a game $v \in \mathcal{G}^N$ and a coalition $A \subseteq N$, the *restriction* of v to A [8] is a game of $\mathcal{G}^A$ defined by

$$v^A(S) := v(S), \qquad \forall\, S \subseteq A.$$

This is equivalent to considering for v only coalitions containing players of A.

Given a coalition $B \subseteq N \setminus A$, the *restriction* of v to A *in the presence of* B [8] is a game of $\mathcal{G}^A$ defined by

$$v^A_{\cup B}(S) := v(S \cup B) - v(B), \qquad \forall\, S \subseteq A.$$

This is equivalent to considering for v only coalitions containing coalition B and some players of A.

Given a game $v \in \mathcal{G}^N$ and a coalition $T \subseteq N$, $T \neq \varnothing$, the *reduced game* with respect to T [8, 14], denoted $v_{[T]}$, is a game of $\mathcal{G}^{(N\setminus T)\cup[T]}$ where $[T]$ indicates a single hypothetical player, which is the representative (or macro player) of the players in T. It is defined by

$$\begin{aligned} &v_{[T]}(S) := v(S),\\ &v_{[T]}(S \cup [T]) := v(S \cup T), \end{aligned}$$

for all $S \subseteq N \setminus T$.

2.6 Dummy coalitions and partnerships

A coalition $S \subseteq U$ is said to be *dummy* in a game $v \in \mathcal{G}^N$ if $v(T \cup S) = v(T) + v(S)$ for all $T \subseteq U \setminus S$. In other words, the marginal contribution of a dummy coalition S to any coalition T not containing elements of S is simply its worth $v(S)$.

A coalition $S \subseteq U$ in a game $v \in \mathcal{G}^N$ is said to be *null* if it is a dummy coalition in v such that $v(S) = 0$.

A *dummy* (resp. *null*) player is a *dummy* (resp. *null*) one-membered coalition.

A coalition $P \subseteq U$, $P \neq \varnothing$, is said to be a *partnership* [11, §4] in a game $v \in \mathcal{G}^N$ if $v(S \cup T) = v(T)$ for all $S \subsetneq P$ and all $T \subseteq U \setminus P$. In other words, as long as all the members of a partnership P are not all in coalition, the presence of some of them only leaves unchanged the worth of any coalition not containing elements of P. In particular $v(S) = 0$ for all $S \subsetneq P$.

Notice that, thus defined, a partnership behaves like a single hypothetical player, that is, the game v and its reduced version $v_{[P]}$ can be considered as equivalent.

Now, a *dummy partnership* is simply a partnership $P \subseteq U$ that is dummy. Thus, a dummy partnership can be regarded as a single hypothetical dummy player. It is easy to verify that any coalition $P \subseteq U$ is a dummy partnership in the corresponding unanimity game u_P.

3 The concept of generalized value

We now introduce the concepts of probabilistic generalized value and generalized semivalue and present five instances of them. Probabilistic values and semivalues are first recalled.

3.1 Probabilistic values and semivalues

As mentioned in the introduction, generalized values can be seen as extensions of values. In turn, a value can be seen as a function $\phi : \mathcal{G} \times U \to \mathbb{R}$ that assigns to every player $i \in U$ in a game $v \in \mathcal{G}$ his/her prospect $\phi(v, i)$ from playing the game. The exact form of a value depends on the axioms that are imposed on it. For instance, the well-known Shapley value can be defined as the sole value that satisfies the *linearity*, *dummy player*, *symmetry*, and *efficiency* axioms [18, Theorem 15].

Given a game $v \in \mathcal{G}^N$, the Shapley value of a player $i \in N$ is given by

$$\phi_{Sh}(v, i) := \sum_{T \subseteq N \setminus i} \frac{1}{n} \binom{n-1}{t}^{-1} [v(T \cup i) - v(T)].$$

If $i \notin N$, we set $\phi_{Sh}(v, i) := 0$; see also [17, Lemma 1].

Another frequently encountered value is the Banzhaf value [2, 6]. The Banzhaf value of a player $i \in N$ in a game $v \in \mathcal{G}^N$ is defined by

$$\phi_B(v, i) := \sum_{T \subseteq N \setminus i} \frac{1}{2^{n-1}} [v(T \cup i) - v(T)].$$

Here also, if $i \notin N$, we set $\phi_B(v, i) := 0$.

The Shapley and Banzhaf values are instances of *probabilistic values* [18] and, more precisely, of *semivalues* [5].

A *probabilistic value* ϕ_p of a player $i \in N$ in a game $v \in \mathcal{G}^N$ is a value of the form

$$\phi_p(v, i) := \sum_{T \subseteq N \setminus i} p_T^i(N) [v(T \cup i) - v(T)],$$

where the family of coefficients $\{p_T^i(N)\}_{T \subseteq N \setminus i}$ forms a probability distribution on $2^{N \setminus i}$. Again, if $i \notin N$, we naturally set $\phi_p(v, i) := 0$.

Thus defined, $\phi_p(v, i)$ can be interpreted as the mathematical expectation on $2^{N \setminus i}$ of the marginal contribution $v(T \cup i) - v(T)$ of player i to a coalition $T \subseteq N \setminus i$ with respect to the probability distribution $\{p_T^i(N)\}_{T \subseteq N \setminus i}$.

A *semivalue* is a probabilistic value such that, additionally, for all $i \in N$, the coefficients $p_T^i(N)$ $(T \subseteq N \setminus i)$ depend only on the cardinalities of the coalitions i, T, and N, i.e., there exist n nonnegative real numbers $\{p_t(n)\}_{t=0,\ldots,n-1}$ fulfilling

$$\sum_{t=0}^{n-1} \binom{n-1}{t} p_t(n) = 1$$

such that, for any $i \in N$ and any $T \subseteq N \setminus i$, we have $p_T^i(N) = p_t(n)$.

3.2 Probabilistic generalized values and generalized semivalues

By analogy with the works of Dubey et al. [5] and Weber [18] on values, we can define the class of probabilistic generalized values and the subclass of generalized semivalues.

A *probabilistic generalized value* of a coalition $S \subseteq N$ in a game $v \in \mathcal{G}^N$ is of the form

$$\Phi_p(v, S) := \sum_{T \subseteq N \setminus S} p_T^S(N) [v(T \cup S) - v(T)],$$

where, for any $S \subseteq N$, the family of coefficients $\{p_T^S(N)\}_{T \subseteq N \setminus S}$ forms a probability distribution on $2^{N \setminus S}$. If $S \nsubseteq N$, we naturally set $\Phi_p(v,S) := \Phi_p(v, S \cap N)$.

A *generalized semivalue* is a probabilistic generalized value such that, additionally, for any $S \subseteq N$, the coefficients $p_T^S(N)$ $(T \subseteq N \setminus S)$ depend only on the cardinalities of the coalitions S, T, and N, i.e., for any $s \in \{0, \dots, n\}$, there exists a family of nonnegative real numbers $\{p_t^s(n)\}_{t=0,\dots,n-s}$ fulfilling

$$\sum_{t=0}^{n-s} \binom{n-s}{t} p_t^s(n) = 1,$$

such that, for any $S \subseteq N$ and any $T \subseteq N \setminus S$, we have $p_T^S(N) = p_t^s(n)$.

As instances of generalized semivalues, we obviously have the Shapley and Banzhaf generalized values, already mentioned in the introduction. Recall that, for a game $v \in \mathcal{G}^N$ and a coalition $S \subseteq N$, they are respectively defined as

$$\begin{aligned}
\Phi_{Sh}(v,S) &:= \sum_{T \subseteq N \setminus S} \frac{1}{n-s+1} \binom{n-s}{t}^{-1} [v(T \cup S) - v(T)], \\
\Phi_B(v,S) &:= \sum_{T \subseteq N \setminus S} \frac{1}{2^{n-s}} [v(T \cup S) - v(T)].
\end{aligned}$$

These indices are extensions of the Shapley and Banzhaf values in the sense that $\Phi_{Sh}(v,i) = \phi_{Sh}(v,i)$ and $\Phi_B(v,i) = \phi_B(v,i)$ for all $i \in U$ and all $v \in \mathcal{G}$.

Another relevant generalized semivalue, which extends the Shapley value, is given by the formula

$$\Phi_{ch}(v,S) = \sum_{T \subseteq N \setminus S} \frac{s}{s+t} \binom{n}{s+t}^{-1} [v(T \cup S) - v(T)]. \tag{1}$$

This generalized value will be called the *chaining generalized value*, by analogy with the concept of chaining interaction index [13], which is another probabilistic linear expression constructed with the same coefficients as in formula (1). We will yield an axiomatization of it in the next section.

Besides the Shapley, Banzhaf, and chaining generalized values, the mappings

$$\Phi_{int} : (v,S) \mapsto v(S) \quad \text{and} \quad \Phi_{ext} : (v,S) \mapsto v^*(S),$$

where $v^* \in \mathcal{G}^N$ is the dual of $v \in \mathcal{G}^N$, defined by $v^*(S) = v(N) - v(N \setminus S)$, are also generalized semivalues. The coefficients of $\Phi_{int}(v,S)$ are defined by $p_t^s(n) = 1$ if $t = 0$ and 0 otherwise, while the coefficients of $\Phi_{ext}(v,S)$ are defined by $p_t^s(n) = 1$ if $t = n-s$ and 0 otherwise. These indices will be called *internal* and *external* generalized values, respectively. This terminology will be justified in Section 4.5.

3.3 Interpretation of probabilistic generalized values

Similarly to probabilistic values, an interpretation of probabilistic generalized values can be easily given.

Consider a game $v \in \mathcal{G}^N$ and suppose that any coalition $S \subseteq N$ joins an outer coalition $T \subseteq N \setminus S$ picked at random with probability $p_T^S(N)$. Then the generalized value $\Phi_p(v,S)$ can be immediately thought of as the mathematical expectation of the marginal contribution $[v(T \cup S) - v(T)]$ of S to the coalition T. Depending on the given randomization scheme, this generalized value takes a well defined form. For example,

- if the coalition S is equally likely to join any coalition $T \subseteq N \setminus S$, its probability to join is $p_T^S(N) = \frac{1}{2^{n-s}}$ and we get $\Phi_B(v, S)$;
- if the coalition S is equally likely to join any coalition $T \subseteq N \setminus S$ of size t ($0 \leq t \leq n - s$) and that all coalitions of size t are equally likely, its probability to join is $p_T^S(N) = \frac{1}{n-s+1}\binom{n-s}{t}^{-1}$ and we get $\Phi_{Sh}(v, S)$.

Such an interpretation of $\Phi_p(v, S)$, which naturally extends that of values (see e.g. [18]), is in full accordance with the idea of a generalized power index. The power of any coalition S in v should not be solely determined by its worth $v(S)$, but also by all $v(S \cup T)$ such that $T \subseteq N \setminus S$. Indeed, the worth $v(S)$ may be very low, suggesting that S has a rather weak power, while $v(S \cup T)$ may be much larger than $v(S)$ for many coalitions $T \subseteq N \setminus S$, suggesting that S actually has a great power.

As expected, we have $\Phi_p(v, \varnothing) = 0$ trivially, which means that the the empty coalition has no power.

4 Axiomatic characterizations

In this final section, we provide our axiomatization and representation results. We first axiomatize the classes of probabilistic generalized values and generalized semivalues and we yield representation theorems for the latter class. Then, by means of additional axioms, we characterize the Shapley, Banzhaf, chaining, internal, and external generalized values.

In the rest of the paper, a *generalized value* is regarded as a function $\Phi : \mathcal{G} \times 2^U \to \mathbb{R}$ such that, for any $v \in \mathcal{G}$ and any $S \subseteq U$, $\Phi(v, S)$ reflects the power of coalition S in the game v.

4.1 Characterizations of probabilistic generalized values and generalized semivalues

We shall now axiomatize the class of probabilistic generalized values and that of generalized semivalues. The proofs, mainly inspired from Weber [18], are given in Appendix A. The following axioms are first considered:

- *Carrier axiom (C)* : For any finite $N \subset U$, $n \geq 1$, and any $v \in \mathcal{G}^N$, we have $\Phi(v, S) = \Phi(v, S \cap N)$ for all $S \subseteq U$.
- *Linearity axiom (L)* : Φ is a linear function with respect to its first argument.
- *Additivity axiom (A)* : Φ is an additive function with respect to its first argument.
- *Positivity axiom (P)* : For any monotone $v \in \mathcal{G}$, we have $\Phi(v, S) \geq 0$ for all $S \subseteq U$.
- *First dummy coalition axiom (DC')* : If $S \subseteq U$ is a dummy coalition in a game $v \in \mathcal{G}$, then $\Phi(v, S) = v(S)$.
- *Symmetry axiom (S)* : For any permutation π on U, and any $v \in \mathcal{G}$, we have $\Phi(v, S) = \Phi(\pi v, \pi(S))$ for all $S \subseteq U$.

These axioms are very natural and have straightforward interpretations. Axiom (C) means that the players outside the carrier should not contribute to the power of any coalition. Axiom (L) (resp. (A)) indicates that generalized values should be decomposable linearly (resp. additively) whenever games are decomposable linearly (resp. additively). Axiom (P), used for one-membered coalitions by Weber [18, §4] to characterize probabilistic values, states that, since in a monotone game the marginal contributions of any coalition are necessarily nonnegative, its generalized value should be nonnegative, too. Axiom (DC'), which is a generalization of the classical dummy axiom [18, §3], states that a dummy coalition has a generalized value equal to its worth. Finally, axiom (S) indicates that the names of the players play no role in determining the generalized values.

We first present an immediate description of linear generalized values.

Proposition 4.1. *A function $\Phi : \mathcal{G} \times 2^U \to \mathbb{R}$ satisfies axiom (L) if and only if, for any finite set $N \subset U$, $n \geq 1$, and any $S \subseteq N$, there exists a family of real constants $\{\alpha_T^S(N)\}_{T \subseteq N}$ such that, for any $v \in \mathcal{G}^N$, we have*

$$\Phi(v, S) = \sum_{T \subseteq N} \alpha_T^S(N) v(T).$$

The following result, less trivial, shows that adding axiom (DC') makes the marginal contributions $[v(T \cup S) - v(T)]$ appear in the expression of $\Phi(v, S)$.

Proposition 4.2. *A function $\Phi : \mathcal{G} \times 2^U \to \mathbb{R}$ satisfies axioms (L) and (DC') if and only if, for any finite set $N \subset U$, $n \geq 1$, and any $S \subseteq N$, there exists a family of real constants $\{p_T^S(N)\}_{T \subseteq N \setminus S}$, satisfying $\sum_{T \subseteq N \setminus S} p_T^S(N) = 1$, such that, for any $v \in \mathcal{G}^N$, we have*

$$\Phi(v, S) = \sum_{T \subseteq N \setminus S} p_T^S(N) \, [v(T \cup S) - v(T)].$$

Now, by using axiom (P), we can easily characterize the classes of probabilistic generalized values and generalized semivalues. It is noteworthy that, under this axiom, axiom (L) can be weakened into (A), as the following lemma shows.

Lemma 4.1. *If $\Phi : \mathcal{G} \times 2^U \to \mathbb{R}$ satisfies axioms (A) and (P) then it also satisfies axiom (L).*

The axiomatizations of probabilistic generalized values and generalized semivalues can then be formulated as follows. Axiom (C) has been added only to define generalized values for coalitions not included in the carrier.

Theorem 4.1. *A function $\Phi : \mathcal{G} \times 2^U \to \mathbb{R}$ satisfies axioms (C), (A), (DC'), and (P) if and only if, for any finite set $N \subset U$, $n \geq 1$, and any $S \subseteq N$, there exists a family of nonnegative constants $\{p_T^S(N)\}_{T \subseteq N \setminus S}$, satisfying $\sum_{T \subseteq N \setminus S} p_T^S(N) = 1$, such that, for any $v \in \mathcal{G}^N$, we have*

$$\Phi(v, S) = \sum_{T \subseteq N \setminus S} p_T^S(N) \, [v(T \cup S) - v(T)],$$

and for any $S \nsubseteq N$ and any $v \in \mathcal{G}^N$, we have $\Phi(v, S) = \Phi(v, S \cap N)$.

Theorem 4.2. *A function $\Phi : \mathcal{G} \times 2^U \to \mathbb{R}$ satisfies axioms (C), (A), (DC'), (P), and (S) if and only if, for any finite set $N \subset U$, $n \geq 1$, and any $S \subseteq N$, there exists a family of* *nonnegative constants* $\{p_t^s(n)\}_{t=0,\ldots,n-s}$ *satisfying* $\sum_{t=0}^{n-s}\binom{n-s}{t}p_t^s(n) = 1$*, such that, for any* $v \in \mathcal{G}^N$*, we have*

$$\Phi(v,S) = \sum_{T\subseteq N\setminus S} p_t^s(n)\,[v(T\cup S) - v(T)],$$

and for any $S \nsubseteq N$ *and any* $v \in \mathcal{G}^N$*, we have* $\Phi(v,S) = \Phi(v, S\cap N)$*.*

4.2 Representation theorems for generalized semivalues

We now present a generalization of the representation theorem given by Dubey et al. [5, Theorem 1(a)] for semivalues. The proof, given in Appendix B, is mainly based on the so-called power moment problem on $[0,1]$, also known as the Hausdorff's moment problem (see e.g. [1, Chapter 2, §6.4]).

Theorem 4.3. *If* Φ_p *is a generalized semivalue given in the form of Theorem 4.2, then, for any finite set* $N \subset U$*,* $n \geq 1$*, and any* $s \in \{1,\ldots,n\}$*, there exists a uniquely determined cumulative density function (CDF)* F_s *on* $[0,1]$ *such that*

$$p_t^s(n) = \int_0^1 x^t(1-x)^{n-s-t}dF_s(x), \qquad \forall t \in \{0,\ldots,n-s\}, \tag{2}$$

where the integral is to be understood in the sense of Riemann-Stieltjes.

Thus, with each generalized semivalue Φ_p is associated a unique denumerable family of CDFs $\mathcal{F} := \{F_s \mid s \geq 1\}$ and we will write $\Phi_{\mathcal{F}} := \Phi_p$. Note that, for $s = 0$, the coefficients $p_t^s(n)$ in Theorem 4.2 are undetermined and hence they can be defined also from Eq. (2).

Remark. It is easy to see that the CDFs corresponding to the Shapley, Banzhaf, and chaining generalized values as well as for the internal and external generalized values are given in the following table, where, for any $E \subseteq [0,1]$, $\mathbf{1}_E$ denotes the characteristic function of E.

	Φ_{Sh}	Φ_B	Φ_{ch}	Φ_{int}	Φ_{ext}
$F_s(x) =$	x	$\mathbf{1}_{[1/2,1]}$	$x^s\,\mathbf{1}_{]0,1]}$	$\mathbf{1}_{]0,1]}$	$\mathbf{1}_{\{1\}}$

The following result will be useful as we go on.

Proposition 4.3. *For any finite* $N \subset U$*,* $n \geq 1$*, and any* $S \subseteq N$*, any generalized semivalue can be rewritten in terms of the Möbius transform as*

$$\Phi_p(v,S) = \sum_{\substack{T\subseteq N\\ T\cap S\neq\varnothing}} q^s_{|T\setminus S|}(n)\, m(v,T), \qquad \forall\, v \in \mathcal{G}^N, \tag{3}$$

where, for all $T \subseteq N$*, with* $T\cap S \neq \varnothing$*,*

$$q^s_{|T\setminus S|}(n) = \sum_{k=0}^{n-s-|T\setminus S|}\binom{n-s-|T\setminus S|}{k} p^s_{k+|T\setminus S|}(n).$$

By combining Theorem 4.3 and Proposition 4.3, it is easy to see that, for any finite $N \subset U$, $n \geq 1$, we have

$$\Phi_{\mathcal{F}}(v,S) = \sum_{\substack{T\subseteq N\\ T\cap S\neq\varnothing}} \Big[\int_0^1 x^{|T\setminus S|}\, dF_s(x)\Big]\, m(v,T), \qquad \forall\, v\in\mathcal{G}^N, \forall\, S\subseteq N, \tag{4}$$

which shows that the coefficients $q^s_{|T\setminus S|}(n)$ of Proposition 4.3 do not depend on n. More precisely, we have

$$q^s_{|T\setminus S|}(n) \;=\; q^s_{|T\setminus S|} \;=\; p^s_{|T\setminus S|}(s+|T\setminus S|)$$

for all $S,T \subseteq N$, with $T\cap S \neq \varnothing$.

The coefficients $q^s_{|T\setminus S|}$ for the particular generalized values introduced thus far are given in the following table.

	Φ_{Sh}	Φ_B	Φ_{ch}	Φ_{int}	Φ_{ext}
$q^s_{\lvert T\setminus S\rvert} =$	$\frac{1}{\lvert T\setminus S\rvert+1}$	$\frac{1}{2^{\lvert T\setminus S\rvert}}$	$\frac{s}{\lvert S\cup T\rvert}$	$\begin{cases}1, & \text{if } T\subseteq S\\ 0, & \text{else}\end{cases}$	1

Remark. Eq. (3) shows that any generalized semivalue Φ_p is a weighted sum over $T \subseteq N$, with $T \cap S \neq \varnothing$, of the dividends $m(v,T)$. In this sum, each coefficient $q^s_{|T\setminus S|}$ represents the extent to which $m(v,T)$ contributes in the computation of Φ_p. For example, for the chaining and Banzhaf interaction indices, these coefficients are respectively given by

$$\frac{s}{|S\cup T|} = \frac{|S|}{|S\cup(T\setminus S)|} \quad\text{and}\quad \frac{1}{2^{|T\setminus S|}} = \frac{|2^S|}{|2^{S\cup(T\setminus S)}|},$$

which shows that $m(v,T)$ is weighted by the contribution of S as a subset of $S\cup T$, where we reason on the elements in the first case and on the subsets in the second case. Similarly, for the Shapley and Banzhaf interaction indices, the coefficients are respectively given by

$$\frac{1}{|T\setminus S|+1} = \frac{|[S]|}{|[S]\cup(T\setminus S)|} \quad\text{and}\quad \frac{1}{2^{|T\setminus S|}} = \frac{|2^{[S]}|}{|2^{[S]\cup(T\setminus S)}|},$$

again showing that $m(v,T)$ is weighted by the contribution of S in $S\cup T$ except that, this time, S is regarded as a single representative $[S]$.

Now, let $N \subset U$ finite, $n \geq 1$, and $v \in \mathcal{G}^N$. Consider the so-called *Owen multilinear extension* $g : [0,1]^n \to \mathbb{R}$ of v [15], namely

$$g(x) := \sum_{T\subseteq N} v(T) \prod_{i\in T} x_i \prod_{i\notin T}(1-x_i), \qquad \forall\, x\in[0,1]^n$$

and, for any $S \subseteq N$, let $\sigma_S g$ denote its *S-slack*, i.e., the difference

$$\sigma_S g(x) := g(1_S x_{-S}) - g(0_S x_{-S}), \qquad \forall x \in \mathbb{R},$$

where, for any $x, y \in \mathbb{R}$, $z = x_S y_{-S}$ denotes the n-dimensional vector defined by $z_i = x$, if $i \in S$, and $z_i = y$, else.

It can be proved [12, §3] (see Owen [15] for the case $s = 1$) that

$$\Phi_{Sh}(v,S) = \int_0^1 \sigma_S g(x)\, dx, \qquad \forall\, S\subseteq N, \tag{5}$$

which means that the Shapley generalized value related to S can be obtained by integrating the S-slack of g along the main diagonal of the unit hypercube.

By combining Eq. (4) and the formula

$$\sigma_S g(x) = \sum_{\substack{T\subseteq N \\ T\cap S\neq\varnothing}} x^{|T\setminus S|}\, m(v,T), \qquad \forall x\in\mathbb{R},$$

(see [12, §3]), we can immediately extend formula (5) to any generalized semivalue. The result can be stated as follows.

Theorem 4.4. *Let $N\subset U$ finite, $n\geq 1$, and $v\in\mathcal{G}^N$. For any generalized semivalue $\Phi_{\mathcal{F}}$, associated with the family of CDFs $\mathcal{F}:=\{F_s \mid s\geq 1\}$, we have*

$$\Phi_{\mathcal{F}}(v,S) = \int_0^1 \sigma_S g(x)\, dF_s(x) \qquad \forall\, S\subseteq N,$$

where $g:[0,1]^n\to\mathbb{R}$ is the multilinear extension of v.

We shall now proceed with the characterizations of the Shapley, Banzhaf, and chaining generalized values as well as the internal and external generalized values, which all are instances of generalized semivalues.

4.3 Characterizations of the Shapley and Banzhaf generalized values

The following axioms are first additionally considered:

- *Recursivity axiom (R)* : For any finite $N\subset U$, $n\geq 1$, and any disjoint coalitions $S,T\subseteq N$ in a game $v\in\mathcal{G}^N$,

$$\Phi(v,S\cup T) = \Phi(v^{N\setminus T},S) + \Phi(v^{N\setminus S}_{\cup S},T).$$

- *Second dummy coalition axiom (DC")* : For any finite $N\subset U$, $n\geq 1$, if $T\subseteq N$ is a dummy coalition in a game $v\in\mathcal{G}^N$ then

$$\Phi(v,S\cup T) = \Phi(v^{N\setminus T},S) + \Phi(v,T), \qquad \forall\, S\subseteq N\setminus T.$$

- *Reduced partnership consistency axiom (RPC)* : For any finite $N\subset U$, $n\geq 1$, if $P\subseteq N$ is a partnership in a game $v\in\mathcal{G}^N$ then

$$\Phi(v,P) = \Phi(v_{[P]},[P]).$$

Axiom (R) means that when a coalition $T\subseteq N$ joins a coalition $S\subseteq N\setminus T$, the resulting power equals the power of S in the absence of T plus the power of T in the presence of S. If, in addition, the joining coalition T is dummy then axiom (DC") says that the power of T is not influenced by the presence of S.

Now, recall that a partnership P can be considered as behaving like a single hypothetical player. Axiom (RPC) identifies the power of this partnership with that of its representative $[P]$ in the corresponding reduced game $v_{[P]}$.

We now prove that, for any generalized semivalue Φ_p, the three axioms above are equivalent. Moreover, in this case, Φ_p is completely determined by its corresponding value on singletons. The proof of this result is given in Appendix C.

Lemma 4.2. *Let Φ_p be a generalized semivalue given in the form of Theorem 4.2. Then the following assertions are equivalent:*

i) *Φ_p satisfies axiom (R).*

ii) *Φ_p satisfies axiom (DC").*

iii) *Φ_p satisfies axiom (RPC).*

iv) *For any $n \geq 1$, any $s \in \{1, \ldots, n\}$, and any $t \in \{0, \ldots, n-s\}$, we have*

$$p_t^s(n) = p_t^1(n-s+1).$$

Actually, it is easy to see that, under conditions of Lemma 4.2, the identity $\Phi_p(v, S) = \Phi_p(v_{[S]}, [S])$ also holds for any generalized semivalue Φ_p, any finite $N \subset U$, $n \geq 1$, any coalition $S \subseteq N$, and any $v \in \mathcal{G}^N$.

Lemma 4.2 enables us to characterize Shapley and Banzhaf generalized values from axiomatizations of Shapley and Banzhaf values. For this purpose, we consider the following two axioms:

- *Efficiency (E)* : For any $N \subset U$ finite, $n \geq 1$, and any $v \in \mathcal{G}^N$, we have

$$\sum_{i \in N} \Phi(v, i) = v(N).$$

- *2-efficiency (2-E)* : For any $N \subset U$ finite, $n \geq 2$, and any $v \in \mathcal{G}^N$, we have

$$\Phi(v, i) + \Phi(v, j) = \Phi(v_{[ij]}, [ij]), \qquad \forall\, ij \subseteq N.$$

Axiom (E), initially considered by Shapley [17], is dedicated to values and ensures that the players of N in a game $v \in \mathcal{G}^N$ share the total amount $v(N)$ among them in terms of their respective values. Axiom (2-E), initially considered by Nowak [14], expresses the fact that the sum of the values of two players should be equal to the value of these players considered as twins in the corresponding reduced game.

The following lemmas can be immediately deduced from [18, Theorem 15] and [8, Theorem 2].

Lemma 4.3. *If Φ_p is a generalized semivalue additionally satisfying axiom (E), then, for any $v \in \mathcal{G}$ and any $i \in U$, $\Phi_p(v, i)$ is the Shapley value of i in the game v.*

Lemma 4.4. *If Φ_p is a generalized semivalue additionally satisfying axiom (2-E), then, for any $v \in \mathcal{G}$ and any $i \in U$, $\Phi_p(v, i)$ is the Banzhaf value of i in the game v.*

We are now ready to state characterizations of the Shapley and Banzhaf generalized values. They immediately follow from Lemmas 4.2, 4.3, and 4.4.

Theorem 4.5. *The Shapley generalized value is the only generalized semivalue additionally satisfying axioms (R or DC" or RPC) and (E). As a consequence, the Shapley generalized value is the only generalized value satisfying axioms (C), (A), (DC'), (P), (S), (R or DC" or RPC), and (E).*

Theorem 4.6. *The Banzhaf generalized value is the only generalized semivalue additionally satisfying axioms (R or DC" or RPC) and (2-E). As a consequence, the Banzhaf interaction index is the only generalized value satisfying axioms (C), (A), (DC'), (P), (S), (R or DC" or RPC), and (2-E).*

4.4 Characterizations of the Banzhaf and chaining generalized values by means of the partnership-allocation axiom

We consider the following additional axiom :

- *Partnership-allocation axiom (PA)* : For any $N \subset U$ finite, $n \geq 1$, and any partnership $P \subseteq N$ in $v \in \mathcal{G}^N$, there exists $\alpha_{|P|} \in \mathbb{R}$ such that

$$\Phi(v, P) = \alpha_{|P|}\, \Phi(v, i), \qquad \forall\, i \in P. \tag{6}$$

Let $N \subset U$ finite, $n \geq 1$, Φ_p be a generalized semivalue, $P \subseteq N$ be a partnership in a game $v \in \mathcal{G}^N$, and i be a member of P. Axiom (PA) is based on the following intuitive reasoning.

1. We know that $\Phi_p(v, P)$ is a weighted arithmetic mean of the marginal contributions $v(T \cup P) - v(T)$ $(T \subseteq N \setminus P)$ and it is easy to verify that $\Phi_p(v, i)$ is a weighted sum of these same marginal contributions. In other words, both $\Phi_p(v, P)$ and $\Phi_p(v, i)$ can be considered as measuring the value in the game v of the hypothetical macro player corresponding to P.

2. Let α be a real number such that $\Phi_p(v, P) = \alpha\, \Phi_p(v, i)$. Notice that this equality still holds if i is replaced with any other player $j \in P$, since all players in a partnership play symmetric roles. Hence, the coefficient α depends only on v and P and can then be seen as determining the way $\Phi_p(v, P)$ is calculated from the value of any of the players of the partnership, quantity that contains all the "relevant information" as discussed in Point 1.

3. It could then be required that the way the value of P is determined from the value of any player of the partnership does not depend on the underlying game. Therefore, it depends only on $|P|$, which justifies axiom (PA).

For generalized semivalues Φ_p, the expression of $\alpha_{|P|}$ in Eq. (6) can be easily obtained. We simply have

$$1 = \alpha_{|P|} \Phi_p(u_P, i), \qquad \forall\, i \in P,$$

since coalition P is always a dummy partnership in the unanimity game u_P.

We now show that, under axiom (PA), any generalized semivalue Φ_p is completely determined by its corresponding value on singletons. The proof is given in Appendix D.

Lemma 4.5. *Let Φ_p be a generalized semivalue given in the form of Proposition 4.3. Then, Φ_p satisfies axiom (PA) if and only if*

$$q^s_{t-s}\, q^1_{s-1} = q^1_{t-1}, \qquad \forall\, t \geq s \geq 1.$$

We can now state another characterization of the Banzhaf generalized value and a characterization of the chaining generalized value. These characterizations immediately follow from Lemmas 4.3, 4.4, and 4.5.

Theorem 4.7. *The Banzhaf generalized value is the only generalized semivalue additionally satisfying axioms (PA) and (2-E). As a consequence, the Banzhaf generalized value is the only generalized value satisfying axioms (C), (A), (DC'), (P), (S), (PA), and (2-E).*

Theorem 4.8. *The chaining generalized value is the only generalized semivalue additionally satisfying axioms (PA) and (E). As a consequence, the chaining interaction index is the only generalized value satisfying axioms (C), (A), (DC'), (P), (S), (PA), and (E).*

4.5 Characterizations of the internal and external generalized values

Finally, we consider two last axioms in order to characterize the internal and external generalized values (i.e., the game itself and its dual) :

- *Internal generalized value axiom (IGV)* : For any $S \subseteq U$, $s \geq 1$, and any $v \in \mathcal{G}$, we have $\Phi(v, S) = \Phi(v^S, S)$.
- *External generalized value axiom (EGV)* : For any $N \subset U$ finite, $n \geq 1$, any $S \subseteq N$, $s \geq 1$, and any $v \in \mathcal{G}^N$, we have $\Phi(v, S) = \Phi(v^S_{\cup N \setminus S}, S)$.

Axiom (IGV) simply states that the generalized value of a coalition S in a game v should be independent of players that are outside S. At the opposite, axiom (EGV) states the generalized value of a coalition S in a game v should be measured in the presence of all outside players.

We can then state the following two characterizations. They are immediate since, for any probabilistic generalized value Φ_p, we have $\Phi_p(v^S, S) = v(S)$ and $\Phi_p(v^S_{\cup N \setminus S}, S) = v^*(S)$.

Theorem 4.9. *The internal generalized value is the only generalized semivalue additionally satisfying axiom (IGV). As a consequence, the internal generalized value is the only generalized value satisfying axioms (C), (A), (DC'), (P), and (IGV).*

Theorem 4.10. *The external generalized value is the only generalized semivalue additionally satisfying axiom (EGV). As a consequence, the external generalized value is the only generalized value satisfying axioms (C), (A), (DC'), (P), and (EGV).*

5 Conclusion

Axiomatic characterizations of the broad class of probabilistic generalized values and of the narrower subclass of generalized semivalues have been proposed. The presented characterizations are mainly based on a natural generalization of the dummy player axiom, namely the dummy coalition axiom. Then, by further imposing classical axioms such as efficiency, 2-efficiency, as well as some other natural axioms, we have characterized the Shapley, Banzhaf, and chaining generalized values.

In a companion paper [7] we provide axiomatizations of probabilistic and cardinal-probabilistic *interaction indices*, that is, probabilistic linear indices measuring the interaction phenomena among players [8]. Although the approach is formally similar, the axioms proposed in [7] are more focused on the concept of partnership and dummy partnership.

A Proofs of results from Section 4.1

Proof of Proposition 4.2

(Sufficiency) Trivial.

(Necessity) The proof is inspired from that of Weber [18, Theorem 2] for values.

Consider a finite set $N \subset U$, $n \geq 1$. The result holds trivially when $S = N$ since N is dummy in any $v \in \mathcal{G}^N$.

Let $S \subsetneq N$. From Proposition 4.1, for any $v \in \mathcal{G}^N$, we have

$$\Phi(v, S) = \sum_{T \subseteq N \setminus S} \sum_{L \subseteq S} \alpha^S_{T \cup L}(N)\, v(T \cup L). \tag{7}$$

Let us prove that

$$\sum_{L \subseteq S} \alpha^S_{T \cup L}(N) = 0, \qquad \forall T \subseteq N \setminus S,\ T \neq \varnothing. \tag{8}$$

This is true when $T = N \setminus S$. Indeed, since S is dummy in $u_{N \setminus S}$, we have

$$\sum_{L \subseteq S} \alpha^S_{(N \setminus S) \cup L}(N) = \Phi(u_{N \setminus S}, S) = u_{N \setminus S}(S) = 0.$$

Now, assume the result holds for $|T| \geq k+1$, with $1 \leq k \leq n-s-1$, and show it still holds for $|T| = k$. Considering $K \subseteq N \setminus S$, with $|K| = k \geq 1$, we have, since S is dummy in u_K,

$$\begin{aligned}
0 = u_K(S) = \Phi(u_K, S) &= \sum_{\substack{T \subseteq N \setminus S \\ T \supseteq K}} \sum_{L \subseteq S} \alpha^S_{T \cup L}(N) \\
&= \sum_{\substack{T \subseteq N \setminus S \\ T \supsetneq K}} \sum_{L \subseteq S} \alpha^S_{T \cup L}(N) + \sum_{L \subseteq S} \alpha^S_{K \cup L}(N) \\
&= \sum_{L \subseteq S} \alpha^S_{K \cup L}(N)
\end{aligned}$$

by induction hypothesis and axiom (DC').

Now, it follows from Eqs. (7) and (8) that

$$\Phi(v, S) = \sum_{T \subseteq N \setminus S} \sum_{L \subseteq S} \alpha^S_{T \cup L}(N)[v(T \cup L) - v(T)]$$

for all $v \in \mathcal{G}^N$. Clearly, we can now assume that $S \neq \varnothing$.

Coalition S being dummy in u_S, we immediately have that $\sum_{T \subseteq N \setminus S} \alpha^S_{T \cup S}(N) = 1$ from axiom (DC'). It follows that, for any $w \in \mathcal{G}^N$ in which S is a dummy coalition, we have

$$\sum_{T \subseteq N \setminus S} \sum_{L \subsetneq S} \alpha^S_{T \cup L}(N)[w(T \cup L) - w(T)] = 0.$$

By choosing an appropriate such game w, we can easily prove that the coefficients $\alpha^S_{T \cup L}(N)$ of this latter equation all are zero. Just fix $T^* \subseteq N \setminus S$ and $L^* \subsetneq S$ and consider any game $w \in \mathcal{G}^N$ fulfilling the conditions

$$\begin{aligned}
w(T \cup S) &= w(T) + w(S), \qquad \forall T \subseteq N \setminus S, \\
w(T \cup L) &= w(T), \qquad \forall T \subseteq N \setminus S,\ \forall L \subsetneq S,\ L \neq L^*, \\
w(T \cup L^*) &= w(T), \qquad \forall T \subseteq N \setminus S,\ T \neq T^*, \\
w(T^* \cup L^*) &\neq w(T^*).
\end{aligned}$$

Consequently, for any $v \in \mathcal{G}^N$, we have

$$\Phi(v, S) = \sum_{T \subseteq N \setminus S} p^S_T(N)[v(T \cup S) - v(T)],$$

where $p_T^S(N) := \alpha_{T\cup S}^S(N)$ and $\sum_{T\subseteq N\setminus S} p_T^S(N) = 1$. $\square$

Proof of Lemma 4.1

We only need to show that, for any $v \in \mathcal{G}$, any $\lambda \in \mathbb{R}$, and any $S \subseteq U$, we have $\Phi(\lambda v, S) = \lambda\Phi(v, S)$. Since the family of unanimity games $\{u_T\}_{T\subseteq U, T\neq\varnothing}$ is a basis of $\mathcal{G}$, it suffices to prove the equality when v is an arbitrary unanimity game.

Let $S, T \subseteq U$, with $T \neq \varnothing$, and let $\lambda \in \mathbb{R}$. Consider sequences r_k and s_k of rational numbers converging to λ and such that $s_k \leq \lambda \leq r_k$ for all k. By (A), we have $\Phi(r_k u_T, S) = r_k\Phi(u_T, S)$ and $\Phi(s_k u_T, S) = s_k\Phi(u_T, S)$ for all k.

Now, by (P) the following real sequences

$$\Phi((r_k - \lambda)u_T, S) \quad \text{and} \quad \Phi((\lambda - s_k)u_T, S)$$

are clearly nonnegative. On the other hand, by (A), the first one converges to $l := \lambda\Phi(u_T, S) - \Phi(\lambda u_T, S)$ and the second one converges to $-l$. It follows that $l = 0$, which completes the proof. $\square$

Proof of Theorem 4.1

(Sufficiency) Trivial.

(Necessity) The proof is similar to that of Weber [18, Theorem 4].

By lemma 4.1, Φ fulfills axiom (L) and Proposition 4.2 applies. Consider a finite set $N \subset U$, $n \geq 1$, let $S \subseteq N$, $S \neq \varnothing$, and fix $T^* \subseteq N \setminus S$. The simple game $\hat{u}_{T^*}$ is monotone and we have $\Phi(\hat{u}_{T^*}, S) = p_{T^*}^S(N)$. From axiom (P), this coefficient is nonnegative.

When $S = \varnothing$, the coefficients can be chosen arbitrarily, and hence nonnegative.

Finally, axiom (C) enables us to define Φ for coalitions $S \nsubseteq N$. $\square$

Proof of Theorem 4.2

(Sufficiency) Trivial.

(Necessity) The proof is similar to that of Weber [18, Theorem 10].

We first observe that Theorem 4.1 applies. Hence, consider a finite set $N \subset U$, $n \geq 1$, and a subset $S \subseteq N$. Again, we can assume that $S \neq \varnothing$.

Consider $T_1, T_2 \subseteq N \setminus S$ and a permutation π on N such that $\pi(T_1) = T_2$ while leaving S fixed. Then, using axiom (S) and the games $\hat{u}_{T_1}, \hat{u}_{T_2} \in \mathcal{G}^N$, we can write

$$p_{T_1}^S(N) = \Phi(\hat{u}_{T_1}, S) = \Phi(\pi\hat{u}_{T_1}, \pi(S)) = \Phi(\hat{u}_{T_2}, S) = p_{T_2}^S(N).$$

Next, consider nonempty sets $S_1, S_2 \subseteq N$ of the same cardinality, a set $T \subseteq N\setminus(S_1\cup S_2)$, and a permutation π on N such that $\pi(S_1) = S_2$ while leaving T invariant. Then, using axiom (S) and the game $\hat{u}_T \in \mathcal{G}^N$, we have $\pi\hat{u}_T = \hat{u}_T$ and

$$p_T^{S_1}(N) = \Phi(\hat{u}_T, S_1) = \Phi(\pi\hat{u}_T, \pi(S_1)) = \Phi(\hat{u}_T, S_2) = p_T^{S_2}(N).$$

Finally, consider a nonempty finite set $S \subset U$, a finite set $T \subset U \setminus S$, two finite sets $N_1, N_2 \subset U$ such that $n_1 = n_2 \geq 1$ and $N_1 \cap N_2 \supseteq S \cup T$, and a permutation π on U such that $\pi(N_1) = N_2$ leaving S and T invariant. Then,

$$p_T^S(N_1) = \Phi(\hat{u}_T^{N_1}, S) = \Phi(\pi\hat{u}_T^{N_1}, \pi(S)) = \Phi(\hat{u}_T^{N_2}, S) = p_T^S(N_2).$$

It follows that, for any finite $N \subset U$, $n \geq 1$, and any $S \subseteq N$, the coefficients $p_T^S(N)$ $(T \subseteq N \setminus S)$ depend only on the cardinalities of the coalitions S, T, and N. Equivalently, for any integers $n \geq 1$ and $s \in \{0, \dots, n\}$, there exists a family of nonnegative real numbers $\{p_t^s(n)\}_{t=0,\dots,n-s}$ fulfilling

$$\sum_{t=0}^{n-s} \binom{n-s}{t} p_t^s(n) = 1,$$

such that, for any finite $N \subset U$, $n \geq 1$, any $S \subseteq N$, and any $T \subseteq N \setminus S$, we have $p_T^S(N) = p_t^s(n)$.

Finally, axiom (C) enables us to define Φ for coalitions $S \nsubseteq N$. $\square$

B Proofs of results from Section 4.2

Proof of Theorem 4.3

We proceed nearly as in [5, Theorem 1(a)].

Let $N \subset U$ finite, with $n \geq 1$, let $T \subseteq N$, and consider the simple game $\hat{u}_T \in \mathcal{G}^N$.

We know from Theorem 4.2 that, for each $K \subseteq N$, $k \geq 1$, there exists a family of nonnegative numbers $\{p_l^k(n)\}_{l=0,\dots,n-k}$ such that, for any $v \in \mathcal{G}^N$,

$$\Phi_p(v, K) = \sum_{L \subseteq N \setminus K} p_l^k(n) \, [v(L \cup K) - v(L)].$$

Let $S \subseteq N \setminus T$, $s \geq 1$. For the game $\hat{u}_T$, it is easy to verify that

$$\Phi_p(\hat{u}_T, S) = p_t^s(n). \tag{9}$$

Let $i \in U \setminus N$. N being a carrier of $\hat{u}_T$, $N \cup i$ is also a carrier of $\hat{u}_T$ in which i is a null player. We know that, for each $K \subseteq N \cup i$, $k \geq 1$, there exists a family of nonnegative numbers $\{p_l^k(n+1)\}_{l=0,\dots,n+1-k}$ such that, for any $v \in \mathcal{G}^{N \cup i}$,

$$\Phi_p(v, K) = \sum_{L \subseteq (N \cup i) \setminus K} p_l^k(n+1) \, [v(L \cup K) - v(L)].$$

For the game $\hat{u}_T$ seen as an element of $\mathcal{G}^{N \cup i}$ and the coalition S, it is easy to verify that

$$\Phi_p(\hat{u}_T, S) = p_t^s(n+1) + p_{t+1}^s(n+1).$$

From Eq. (9), it follows that the coefficients $p_t^s(n)$ obey the recurrence relation

$$p_t^s(n) = p_t^s(n+1) + p_{t+1}^s(n+1).$$

Setting $\alpha_l^s := p_l^s(s+l)$ for all $l \in \mathbb{N}$, we can prove by induction that

$$\begin{aligned} p_t^s(n) &= (-1)^{n-s-t} \sum_{i=0}^{n-s-t} (-1)^i \binom{n-s-t}{i} \alpha_{n-s-i}^s \\ &= (-1)^{n-s-t} \nabla^{n-s-t} \alpha_{n-s}^s, \end{aligned} \tag{10}$$

for all $n \geq 1$, all $s \in \{1, \dots, n\}$, and all $t \in \{0, \dots, n-s\}$, where ∇^k denotes the kth iterate of the standard backward difference operator ∇.

Clearly, the sequence $(\alpha_m^s)_{m \geq 0}$ is nonnegative and $\alpha_0^s = p_0^s(s) = 1$. Moreover, we have $(-1)^k \nabla^k \alpha_m^s \geq 0$ for all $k \leq m$. Then, according to Hausdorff's moment problem (see e.g.

[1, Theorem 2.6.4]), we know that $\alpha_0^s, \alpha_1^s, \dots$ are the moments of a uniquely determined CDF F_s on $[0,1]$, that is, we have

$$\alpha_m^s = \int_0^1 x^m dF_s(x), \qquad \forall\, m \geq 0.$$

Therefore, for each $t \in \{0, \dots, n-s\}$, we have, by Eq. (10),

$$\begin{aligned} p_t^s(n) &= \int_0^1 x^t \sum_{i=0}^{n-s-t} \binom{n-s-t}{i} (-x)^{n-s-t-i} dF_s(x) \\ &= \int_0^1 x^t (1-x)^{n-s-t} dF_s(x), \end{aligned}$$

which completes the proof. $\square$

Proof of Proposition 4.3

Let $N \subset U$ finite, $n \geq 1$, let $S \subseteq N$ and let $v \in \mathcal{G}^N$. Then, by using the linear expression of the game in terms of its dividends, we have successively

$$\begin{aligned} \Phi_p(v,S) &= \sum_{T \subseteq N \setminus S} p_t^s(n)\, [v(T \cup S) - v(T)] \\ &= \sum_{T \subseteq N \setminus S} p_t^s(n) \sum_{\substack{K \subseteq T \cup S \\ K \cap S \neq \varnothing}} m(v,K) \\ &= \sum_{\substack{K \subseteq N \\ K \cap S \neq \varnothing}} m(v,K) \sum_{\substack{T \subseteq N \setminus S \\ T \supseteq K \setminus S}} p_t^s(n), \end{aligned}$$

and the second sum reads

$$\sum_{t=|K \setminus S|}^{n-s} \binom{n-s-|K\setminus S|}{t-|K\setminus S|} p_t^s(n) = \sum_{t=0}^{n-s-|K\setminus S|} \binom{n-s-|K\setminus S|}{t} p_{t+|K\setminus S|}^s(n),$$

which is sufficient. $\square$

C Proofs of results from Section 4.3

Proof of Lemma 4.2

i) $\Rightarrow$ *ii)* Let $N \subset U$ finite, $n \geq 1$ and let $v \in \mathcal{G}^N$. Consider a dummy coalition $K \subseteq N$ and any coalition $S \subseteq N \setminus K$. For any $T \subseteq N \setminus K$, we have $v(T \cup K) - v(T) = v(K)$ and hence $\Phi_p(v_{\cup S}^{N \setminus S}, K) = v(K) = \Phi_p(v, K)$.

ii) $\Rightarrow$ *iv)* Let $N \subset U$ finite, $n \geq 1$ and let $v \in \mathcal{G}^N$. Consider a dummy coalition $K \subseteq N$ and any nonempty coalition $S \subseteq N \setminus K$. Then we have $\Phi_p(v,K) = v(K)$ and

$$\begin{aligned} \Phi_p(v, S \cup K) &= \sum_{T \subseteq N \setminus (S \cup K)} p_t^{s+k}(n)\, [v(T \cup S) - v(T)] + v(K), \\ \Phi_p(v^{N \setminus K}, S) &= \sum_{T \subseteq N \setminus (S \cup K)} p_t^s(n-k)\, [v(T \cup S) - v(T)]. \end{aligned}$$

It follows from axiom (DC”) that

$$\sum_{T \subseteq N \setminus (S \cup K)} [p_t^{s+k}(n) - p_t^s(n-k)][v(T \cup S) - v(T)] = 0.$$

Now, fix $T^* \subseteq N \setminus (S \cup K)$ and consider a game in which K is a dummy coalition and whose restriction on $N \setminus K$ is $\hat{u}_{T^*}$. Then, considering this game in the previous equality yields

$$p_t^{s+k}(n) = p_t^s(n-k) \tag{11}$$

for all $n \geq 1$, all $s \in \{1, \ldots, n\}$, all $k \in \{0, \ldots, n\}$ such that $s + k \leq n$, and all $t \in \{0, \ldots, n - s - k\}$. Finally, for any $s' \in \{1, \ldots, n\}$, setting $s := 1$ and $k := s' - 1$ in Eq. (11), we get

$$p_t^1(n - s' + 1) = p_t^1(n-k) = p_t^{s'}(n).$$

iv) $\Rightarrow$ *i)* Let $N \subset U$ finite, $n \geq 1$. For any disjoint coalitions $S, T \subseteq N$ in a game $v \in \mathcal{G}^N$, we have

$$\begin{aligned}
\Phi_p(v, S \cup T) &= \sum_{K \subseteq N \setminus (S \cup T)} p_k^{s+t}(n)\,[v(K \cup S \cup T) - v(K)], \\
\Phi_p(v^{N \setminus T}, S) &= \sum_{K \subseteq N \setminus (S \cup T)} p_k^{s}(n-t)\,[v(K \cup S) - v(K)], \\
\Phi_p(v_{\cup S}^{N \setminus S}, T) &= \sum_{K \subseteq N \setminus (S \cup T)} p_k^{t}(n-s)\,[v(K \cup S \cup T) - v(K \cup S)].
\end{aligned}$$

But we have

$$p_k^{s+t}(n) = p_k^s(n-t) = p_k^t(n-s) = p_k^1(n-s-t+1)$$

and hence Φ_p satisfies axiom (R).

iii) $\Rightarrow$ *iv)* Let $N \subset U$ finite, $n \geq 1$, and let $S \subseteq N$ be a nonempty partnership in $v \in \mathcal{G}^N$. The function Φ_p being a generalized semivalue, we can write

$$\Phi_p(v, S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\,[v(T \cup S) - v(T)]. \tag{12}$$

Furthermore, we have

$$\Phi_p(v_{[S]}, [S]) = \sum_{T \subseteq ((N \setminus S) \cup [S]) \setminus [S]} p_t^1(n-s+1)[v(T \cup S) - v(T)].$$

Since Φ_p also satisfies axiom (RPC), we obtain

$$\sum_{T \subseteq N \setminus S} [p_t^s(n) - p_t^1(n-s+1)][v(T \cup S) - v(T)] = 0. \tag{13}$$

Now, it is easy to verify that, for each $T \subseteq N \setminus S$, S is a partnership in the unanimity game $u_{T \cup S}$ and that $u_{T \cup S}(K \cup S) - u_{T \cup S}(K) = 0$ for $K \subsetneq T$ and 1 for $K \subseteq N \setminus S$, $K \supseteq T$. Thus, using the family of games $\{u_{T \cup S}\}_{T \subseteq N \setminus S}$ (starting with u_N) in Eq. (13), we obtain that

$$p_t^s(n) = p_t^1(n-s+1), \qquad \forall\, t \in \{0, \ldots, n-s\}.$$

iv) $\Rightarrow$ *iii)* Starting from Eq. (12) and the fact that $p_t^s(n) = p_t^1(n-s+1)$ for all $t \in \{0, \ldots, n-s\}$, we immediately obtain that Φ_p satisfies (RPC). $\square$

D Proofs of results from Section 4.4

Proof of Lemma 4.5

(Necessity) Let $N \subset U$ finite, $n \geq 1$, let $S \subseteq N$, $s \geq 1$, and let $T \supseteq S$. Clearly, S is a partnership in the game u_T. Then, since Φ_p satisfies axiom (PA), we have

$$\Phi_p(u_T, S)\Phi_p(u_S, i) = \Phi_p(u_T, i), \qquad \forall\, i \in S.$$

Let $i \in S$. On the one hand, we have

$$\Phi_p(u_T, S)\Phi_p(u_S, i) = q^s_{t-s}\, q^1_{s-1},$$

from the result given in Section 2.3 on the Möbius transform of a unanimity game. On the other hand,

$$\Phi_p(u_T, i) = q^1_{t-1},$$

which implies the necessity.

(Sufficiency) Let $N \subset U$ finite, $n \geq 1$, let $S \subseteq N$, $s \geq 1$, and let $v \in \mathcal{G}^N$ be a game in which S is a partnership. Let $i \in S$, $T \ni i$, and $K = T \setminus S$. Then,

$$m(v, T) = \sum_{R \subseteq K} \sum_{L \subseteq T \cap S} (-1)^{t-l-r} v(R \cup L).$$

If $T \not\supseteq S$, then, for all $L \subseteq T \cap S$, $v(R \cup L) = v(R)$, since S is a partnership in v. It follows that

$$m(v, T) = \sum_{R \subseteq K} (-1)^{t-r} v(R) \sum_{L \subseteq T \cap S} (-1)^l = 0.$$

Hence, $m(v, L) = 0$ for all $L \ni i$ such that $L \not\supseteq S$. Then, using Proposition 4.3 and the results given in Section 2.3, we have

$$\begin{aligned}
\Phi_p(v, S)\, \Phi_p(u_S, i) &= \sum_{T \supseteq S} q^s_{t-s} m(v, T) \sum_{L \supseteq S} q^1_{l-1} m(u_S, L) \\
&= \sum_{T \supseteq S} q^s_{t-s} q^1_{s-1} m(v, T) \\
&= \sum_{T \supseteq S} q^1_{t-1} m(v, T) \\
&= \Phi_p(v, i),
\end{aligned}$$

and thus the sufficiency. □

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