AXIOMATIZATIONS OF QUASI-LOVÁSZ EXTENSIONS OF PSEUDO-BOOLEAN FUNCTIONS

MIGUEL COUCEIRO AND JEAN-LUC MARICHAL

Abstract. We introduce the concept of quasi-Lovász extension as being a mapping $f: I^n \to \mathbb{R}$ defined on a nonempty real interval I containing the origin and which can be factorized as $f(x_1,\ldots,x_n) = L(\varphi(x_1),\ldots,\varphi(x_n))$, where L is the Lovász extension of a pseudo-Boolean function $\psi: \{0,1\}^n \to \mathbb{R}$ (i.e., the function $L: \mathbb{R}^n \to \mathbb{R}$ whose restriction to each simplex of the standard triangulation of $[0,1]^n$ is the unique affine function which agrees with ψ at the vertices of this simplex) and $\varphi: I \to \mathbb{R}$ is a nondecreasing function vanishing at the origin. These functions appear naturally within the scope of decision making under uncertainty since they subsume overall preference functionals associated with discrete Choquet integrals whose variables are transformed by a given utility function. To axiomatize the class of quasi-Lovász extensions, we propose generalizations of properties used to characterize the Lovász extensions, including a comonotonic version of modularity and a natural relaxation of homogeneity. A variant of the latter property enables us to axiomatize also the class of symmetric quasi-Lovász extensions, which are compositions of symmetric Lovász extensions with 1-place nondecreasing odd functions.

1. Introduction

Aggregation functions arise wherever merging information is needed: applied and pure mathematics (probability, statistics, decision theory, functional equations), operations research, computer science, and many applied fields (economics and finance, pattern recognition and image processing, data fusion, etc.). For recent references, see Beliakov et al. [1] and Grabisch et al. [15].

The discrete Choquet integral has been widely investigated in aggregation theory due to its many applications, for instance, in decision making (see the edited book [16]). A convenient way to introduce the discrete Choquet integral is via the concept of Lovász extension. An n-place Lovász extension is a continuous function $L: \mathbb{R}^n \to \mathbb{R}$ whose restriction to each of the $n!$ subdomains

$$\mathbb{R}^n_\sigma = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}, \qquad \sigma \in S_n,$$

is an affine function, where S_n denotes the set of permutations on $[n] = \{1,\ldots,n\}$. An n-place Choquet integral is simply a nondecreasing (in each variable) n-place Lovász extension which vanishes at the origin. For general background, see [15, §5.4].

Date: June 22, 2011.

2010 *Mathematics Subject Classification.* Primary 39B22, 39B72; Secondary 26B35.

Key words and phrases. Aggregation function, discrete Choquet integral, Lovász extension, functional equation, comonotonic modularity, invariance under horizontal differences, axiomatization.

The class of n-place Lovász extensions has been axiomatized by the authors [10] by means of two noteworthy aggregation properties, namely comonotonic additivity and horizontal min-additivity (for earlier axiomatizations of the n-place Choquet integrals, see, e.g., [2, 13]). Recall that a function $f\colon\mathbb{R}^n\to\mathbb{R}$ is said to be *comonotonically additive* if, for every $\sigma\in S_n$, we have

$$f(\mathbf{x}+\mathbf{x}')=f(\mathbf{x})+f(\mathbf{x}'),\qquad \mathbf{x},\mathbf{x}'\in\mathbb{R}^n_\sigma.$$

The function f is said to be *horizontally min-additive* if

$$f(\mathbf{x})=f(\mathbf{x}\wedge c)+f(\mathbf{x}-(\mathbf{x}\wedge c)),\qquad \mathbf{x}\in\mathbb{R}^n,\ c\in\mathbb{R},$$

where $\mathbf{x}\wedge c$ denotes the n-tuple whose ith component is $x_i\wedge c=\min(x_i,c)$.

In this paper we consider a generalization of Lovász extensions, which we call quasi-Lovász extensions, and which are best described by the following equation

$$f(x_1,\ldots,x_n)=L(\varphi(x_1),\ldots,\varphi(x_n))$$

where L is a Lovász extension and φ a nondecreasing function such that $\varphi(0)=0$. Such an aggregation function is used in decision under uncertainty, where φ is a utility function and f an overall preference functional. It is also used in multi-criteria decision making where the criteria are commensurate (i.e., expressed in a common scale). For a recent reference, see Bouyssou et al. [3].

To axiomatize the class of quasi-Lovász extensions, we propose the following generalizations of comonotonic additivity and horizontal min-additivity, namely comonotonic modularity and invariance under horizontal min-differences (as well as its dual counterpart), which we now briefly describe. We say that a function $f\colon\mathbb{R}^n\to\mathbb{R}$ is *comonotonically modular* if, for every $\sigma\in S_n$, we have

$$f(\mathbf{x})+f(\mathbf{x}')=f(\mathbf{x}\wedge\mathbf{x}')+f(\mathbf{x}\vee\mathbf{x}'),\qquad \mathbf{x},\mathbf{x}'\in\mathbb{R}^n_\sigma,$$

where $\mathbf{x}\wedge\mathbf{x}'$ (resp. $\mathbf{x}\vee\mathbf{x}'$) denotes the n-tuple whose ith component is $x_i\wedge x_i'=\min(x_i,x_i')$ (resp. $x_i\vee x_i'=\max(x_i,x_i')$). We say that f is *invariant under horizontal min-differences* if

$$f(\mathbf{x})-f(\mathbf{x}\wedge c)=f([\mathbf{x}]_c)-f([\mathbf{x}]_c\wedge c),\qquad \mathbf{x}\in\mathbb{R}^n,\ c\in\mathbb{R},$$

where $[\mathbf{x}]_c$ denotes the n-tuple whose ith component is 0, if $x_i\leqslant c$, and x_i, otherwise.

The outline of this paper is as follows. In Section 2 we recall the definitions of Lovász extensions, discrete Choquet integrals, as well as their symmetric versions, and present representations for these functions. In Section 3 we define the concept of quasi-Lovász extension and its symmetric version, introduce natural relaxations of homogeneity, namely weak homogeneity and odd homogeneity, and characterize those quasi-Lovász extensions (resp. symmetric quasi-Lovász extensions) that are weakly homogeneous (resp. oddly homogeneous). In Section 4 we define the concepts of comonotonic modularity, invariance under horizontal min-differences and invariance under horizontal max-differences, and completely describe the function classes axiomatized by each of these properties. In Section 5 we give axiomatizations of the class of quasi-Lovász extensions by means of the properties above and describe all possible factorizations of quasi-Lovász extensions into compositions of Lovász extensions with 1-place functions. In Section 6 we present analogous results for the symmetric quasi-Lovász extensions. Finally, in Section 7 we show that the so-called quasi-polynomial functions [5] on closed intervals form a noteworthy subclass of comonotonically modular functions.

We employ the following notation throughout the paper. Let $\mathbb{B} = \{0,1\}$, $\mathbb{R}_+ = [0,+\infty[$, and $\mathbb{R}_- =]-\infty,0]$. The symbol I denotes a nonempty real interval, possibly unbounded, containing 0. We also introduce the notation $I_+ = I \cap \mathbb{R}_+$, $I_- = I \cap \mathbb{R}_-$, and $I^n_\sigma = I^n \cap \mathbb{R}^n_\sigma$. A function $f\colon I^n \to \mathbb{R}$, where I is centered at 0, is said to be *odd* if $f(-\mathbf{x}) = -f(\mathbf{x})$. For any function $f\colon I^n \to \mathbb{R}$, we define $f_0 = f - f(\mathbf{0})$. For every $A \subseteq [n]$, the symbol $\mathbf{1}_A$ denotes the n-tuple whose ith component is 1, if $i \in A$, and 0, otherwise. Let also $\mathbf{1} = \mathbf{1}_{[n]}$ and $\mathbf{0} = \mathbf{1}_\varnothing$. The symbols $\wedge$ and $\vee$ denote the minimum and maximum functions, respectively. For every $\mathbf{x} \in \mathbb{R}^n$, let $\mathbf{x}^+ = \mathbf{x} \vee 0$ and $\mathbf{x}^- = (-\mathbf{x})^+$. For every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}_+$ (resp. $c \in \mathbb{R}_-$) we denote by $[\mathbf{x}]_c$ (resp. $[\mathbf{x}]^c$) the n-tuple whose ith component is 0, if $x_i \leqslant c$ (resp. $x_i \geqslant c$), and x_i, otherwise.

In order not to restrict our framework to functions defined on $\mathbb{R}$, we consider functions defined on intervals I containing 0, in particular of the forms I_+, I_-, and those centered at 0.

2. Lovász extensions and symmetric Lovász extensions

We now recall the concepts of Lovász extension and symmetric Lovász extension.

Consider an n-place *pseudo-Boolean function*, i.e. a function $\psi\colon \mathbb{B}^n \to \mathbb{R}$, and define the set function $v_\psi\colon 2^{[n]} \to \mathbb{R}$ by $v_\psi(A) = \psi(\mathbf{1}_A)$ for every $A \subseteq [n]$. Hammer and Rudeanu [18] showed that such a function has a unique representation as a multilinear polynomial of n variables

$$\psi(\mathbf{x}) = \sum_{A\subseteq[n]} a_\psi(A) \prod_{i\in A} x_i\,,$$

where the set function $a_\psi\colon 2^{[n]} \to \mathbb{R}$, called the *Möbius transform* of v_ψ, is defined by

$$a_\psi(A) = \sum_{B\subseteq A} (-1)^{|A|-|B|}\, v_\psi(B).$$

The *Lovász extension* of a pseudo-Boolean function $\psi\colon \mathbb{B}^n \to \mathbb{R}$ is the function $L_\psi\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each subdomain $\mathbb{R}^n_\sigma$ ($\sigma \in S_n$) is the unique affine function which agrees with ψ at the $n+1$ vertices of the n-simplex $[0,1]^n \cap \mathbb{R}^n_\sigma$ (see [19, 22]). We then have $L_\psi|_{\mathbb{B}^n} = \psi$.

It can be shown (see [15, §5.4.2]) that the Lovász extension of a pseudo-Boolean function $\psi\colon \mathbb{B}^n \to \mathbb{R}$ is the continuous function

$$L_\psi(\mathbf{x}) = \sum_{A\subseteq[n]} a_\psi(A) \bigwedge_{i\in A} x_i\,, \qquad \mathbf{x} \in \mathbb{R}^n. \tag{1}$$

Its restriction to $\mathbb{R}^n_\sigma$ is the affine function

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \sum_{i\in[n]} x_{\sigma(i)} \big(v_\psi(A^\uparrow_\sigma(i)) - v_\psi(A^\uparrow_\sigma(i+1))\big), \qquad \mathbf{x} \in \mathbb{R}^n_\sigma, \tag{2}$$

or equivalently,

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \sum_{i\in[n]} x_{\sigma(i)} \big(L_\psi(\mathbf{1}_{A^\uparrow_\sigma(i)}) - L_\psi(\mathbf{1}_{A^\uparrow_\sigma(i+1)})\big), \qquad \mathbf{x} \in \mathbb{R}^n_\sigma, \tag{3}$$

where $A^\uparrow_\sigma(i) = \{\sigma(i), \ldots, \sigma(n)\}$, with the convention that $A^\uparrow_\sigma(n+1) = \varnothing$. Indeed, for any $k \in [n+1]$, both sides of each of the equations (2) and (3) agree at $\mathbf{x} = \mathbf{1}_{A^\uparrow_\sigma(k)}$.

It is noteworthy that L_ψ can also be represented by

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \sum_{i\in[n]} x_{\sigma(i)} \big(L_\psi(-\mathbf{1}_{A^\downarrow_\sigma(i-1)}) - L_\psi(-\mathbf{1}_{A^\downarrow_\sigma(i)})\big), \qquad \mathbf{x} \in \mathbb{R}^n_\sigma, \tag{4}$$

where $A^{\downarrow}_{\sigma}(i) = \{\sigma(1),\ldots,\sigma(i)\}$, with the convention that $A^{\downarrow}_{\sigma}(0) = \varnothing$. Indeed, for any $k \in [n+1]$, by (3) we have

$$L_\psi(-\mathbf{1}_{A^{\downarrow}_{\sigma}(k-1)}) = \psi(\mathbf{0}) + L_\psi(\mathbf{1}_{A^{\uparrow}_{\sigma}(k)}) - L_\psi(\mathbf{1}_{A^{\uparrow}_{\sigma}(1)}).$$

Let ψ^d denotes the *dual* of ψ, that is the function $\psi^d\colon\mathbb{B}^n \to \mathbb{R}$ defined by $\psi^d(\mathbf{x}) = \psi(\mathbf{0}) + \psi(\mathbf{1}) - \psi(\mathbf{1}-\mathbf{x})$. The next result provides further representations for L_ψ.

Proposition 1. *The Lovász extension of a pseudo-Boolean function $\psi\colon\mathbb{B}^n \to \mathbb{R}$ is given by*

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \sum_{A\subseteq[n]} a_{\psi^d}(A) \bigvee_{i\in A} x_i\,, \tag{5}$$

and

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + L_\psi(\mathbf{x}^+) - L_{\psi^d}(\mathbf{x}^-). \tag{6}$$

Proof. Since the Lovász extension L_ψ is additive with respect to its restriction ψ (i.e., $L_{\psi+\psi'} = L_\psi + L_{\psi'}$), for every $\mathbf{x}\in\mathbb{R}^n$, we have

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \psi(\mathbf{1}) - L_{\psi^d}(\mathbf{1}-\mathbf{x}) = \psi(\mathbf{0}) + \psi^d(\mathbf{1}) - L_{\psi^d}(\mathbf{1}-\mathbf{x}),$$

that is, by using (1),

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \sum_{A\subseteq[n]} a_{\psi^d}(A) - \sum_{A\subseteq[n]} a_{\psi^d}(A)\Big(1 - \bigvee_{i\in A} x_i\Big),$$

which proves (5).

Now, for every $A\subseteq[n]$, we have $\bigwedge_{i\in A} x_i = \bigwedge_{i\in A} x_i^+ + \bigwedge_{i\in A}(-x_i^-)$ and hence by (1),

$$\begin{aligned} L_\psi(\mathbf{x}) &= \psi(\mathbf{0}) + \sum_{\varnothing\neq A\subseteq[n]} a_\psi(A)\Big(\bigwedge_{i\in A} x_i^+ + \bigwedge_{i\in A}(-x_i^-)\Big) \\ &= \psi(\mathbf{0}) + \sum_{\varnothing\neq A\subseteq[n]} a_\psi(A)\bigwedge_{i\in A} x_i^+ - \sum_{A\subseteq[n]} a_\psi(A)\bigvee_{i\in A} x_i^- \end{aligned}$$

which, using (5) and the identity $\psi^{dd} = \psi$, leads to (6). □

A function $f\colon\mathbb{R}^n \to \mathbb{R}$ is said to be a *Lovász extension* if there is a pseudo-Boolean function $\psi\colon\mathbb{B}^n \to \mathbb{R}$ such that $f = L_\psi$.

An n-place *Choquet integral* is a nondecreasing Lovász extension $L_\psi\colon\mathbb{R}^n \to \mathbb{R}$ such that $L_\psi(\mathbf{0}) = 0$. It is easy to see that a Lovász extension $L\colon\mathbb{R}^n \to \mathbb{R}$ is an n-place Choquet integral if and only if its underlying pseudo-Boolean function $\psi = L|_{\mathbb{B}^n}$ is nondecreasing and vanishes at the origin (see [15, §5.4]).

The *symmetric Lovász extension* of a pseudo-Boolean function $\psi\colon\mathbb{B}^n \to \mathbb{R}$ is the function $\check{L}\colon\mathbb{R}^n \to \mathbb{R}$ defined by (see [10])

$$\check{L}_\psi(\mathbf{x}) = \psi(\mathbf{0}) + L_\psi(\mathbf{x}^+) - L_\psi(\mathbf{x}^-).$$

In particular, we see that $\check{L}_\psi - \check{L}_\psi(\mathbf{0}) = \check{L}_\psi - \psi(\mathbf{0})$ is an odd function.

It is easy to see that the restriction of $\check{L}_\psi$ to $\mathbb{R}^n_\sigma$ is the function

$$\begin{aligned} \check{L}_\psi(\mathbf{x}) &= \psi(\mathbf{0}) + \sum_{1\leqslant i\leqslant p} x_{\sigma(i)}\big(L_\psi(\mathbf{1}_{A^{\downarrow}_{\sigma}(i)}) - L_\psi(\mathbf{1}_{A^{\downarrow}_{\sigma}(i-1)})\big) \\ &\quad + \sum_{p+1\leqslant i\leqslant n} x_{\sigma(i)}\big(L_\psi(\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) - L_\psi(\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big), \qquad \mathbf{x}\in\mathbb{R}^n_\sigma, \end{aligned} \tag{7}$$

where the integer $p\in\{0,\ldots,n\}$ is such that $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$.

A function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be a *symmetric Lovász extension* if there is a pseudo-Boolean function $\psi:\mathbb{B}^n \to \mathbb{R}$ such that $f = \check{L}_\psi$.

Nondecreasing symmetric Lovász extensions vanishing at the origin, also called *discrete symmetric Choquet integrals*, were introduced by Šipoš [23] (see also [15, §5.4]).

3. Quasi-Lovász extensions and symmetric quasi-Lovász extensions

In this section we introduce the concepts of quasi-Lovász extension and symmetric quasi-Lovász extension. We also introduce natural relaxations of homogeneity, namely weak homogeneity and odd homogeneity, and characterize those quasi-Lovász extensions (resp. symmetric quasi-Lovász extensions) that are weakly homogeneous (resp. oddly homogeneous). Recall that I is a real interval containing 0.

A *quasi-Lovász extension* is a function $f:I^n \to \mathbb{R}$ defined by

$$f = L \circ (\varphi, \dots, \varphi),$$

also written $f = L \circ \varphi$, where $L:\mathbb{R}^n \to \mathbb{R}$ is a Lovász extension and $\varphi: I \to \mathbb{R}$ is a nondecreasing function satisfying $\varphi(0) = 0$. Observe that a function $f:I^n \to \mathbb{R}$ is a quasi-Lovász extension if and only if $f_0 = L_0 \circ \varphi$.

Lemma 2. *Assume $I \subseteq \mathbb{R}_+$. For every quasi-Lovász extension $f:I^n \to \mathbb{R}$, $f = L \circ \varphi$, we have*

$$f_0(x\mathbf{1}_A) = \varphi(x) L_0(\mathbf{1}_A), \qquad x \in I,\ A \subseteq [n]. \tag{8}$$

Proof. For every $x \in I$ and every $A \subseteq [n]$, there exists $\sigma \in S_n$ such that $x\mathbf{1}_A \in I^n_\sigma$ and, using (3), we then obtain

$$f_0(x\mathbf{1}_A) = \sum_{n-|A|+1 \leqslant i \leqslant n} \varphi(x) \left(L(\mathbf{1}_{A^\uparrow_\sigma(i)}) - L(\mathbf{1}_{A^\uparrow_\sigma(i+1)}) \right) = \varphi(x) L_0(\mathbf{1}_A). \quad \square$$

Observe that if $[0,1] \subseteq I \subseteq \mathbb{R}_+$ and $\varphi(1) = 1$, then the equation in (8) becomes $f_0(x\mathbf{1}_A) = \varphi(x) f_0(\mathbf{1}_A)$. This motivates the following definition. We say that a function $f:I^n \to \mathbb{R}$, where $I \subseteq \mathbb{R}_+$, is *weakly homogeneous* if there exists a nondecreasing function $\varphi: I \to \mathbb{R}$ satisfying $\varphi(0) = 0$ such that $f(x\mathbf{1}_A) = \varphi(x) f(\mathbf{1}_A)$ for every $x \in I$ and every $A \subseteq [n]$.

Clearly, every weakly homogeneous function f satisfies $f(\mathbf{0}) = 0$ (take $x = 0$ in the definition).

The following proposition provides necessary and sufficient conditions on a nonconstant quasi-Lovász extension $f:I^n \to \mathbb{R}$ for the function f_0 to be weakly homogeneous.

Proposition 3. *Assume $[0,1] \subseteq I \subseteq \mathbb{R}_+$. Let $f:I^n \to \mathbb{R}$ be a nonconstant quasi-Lovász extension, $f = L \circ \varphi$. Then the following conditions are equivalent.*

- (*i*) *f_0 is weakly homogeneous.*
- (*ii*) *There exists $A \subseteq [n]$ such that $f_0(\mathbf{1}_A) \neq 0$.*
- (*iii*) *$\varphi(1) \neq 0$.*

In this case we have $f_0(x\mathbf{1}_A) = \frac{\varphi(x)}{\varphi(1)} f_0(\mathbf{1}_A)$ for every $x \in I$ and every $A \subseteq [n]$.

Proof. Let us prove that $(i) \Rightarrow (ii)$ by contradiction. Assume that $f_0(\mathbf{1}_A) = 0$ for every $A \subseteq [n]$. Since f_0 is weakly homogeneous, we must have $f_0(x\mathbf{1}_A) = 0$ for every $x \in I$ and every $A \subseteq [n]$. By (8), we then have $\varphi \equiv 0$ or $L_0(\mathbf{1}_A) = 0$ for every $A \subseteq [n]$. In either case, by (3), we have $f_0 \equiv 0$, i.e. f is constant, a contradiction.

Let us prove that $(ii) \Rightarrow (iii)$ by contradiction. If we had $\varphi(1) = 0$, then by (8) we would have $f_0(\mathbf{1}_A) = 0$ for every $A \subseteq [n]$, a contradiction.

Let us prove that $(iii) \Rightarrow (i)$. By (8), we have $f_0(x\mathbf{1}_A) = \frac{\varphi(x)}{\varphi(1)} f_0(\mathbf{1}_A)$, which shows that f_0 is weakly homogeneous. □

Remark 1. (*a*) If $[0,1] \subsetneq I \subseteq \mathbb{R}_+$, then the quasi-Lovász extension $f\colon I^n \to \mathbb{R}$ defined by $f(\mathbf{x}) = \bigwedge_{i\in[n]} \varphi(x_i)$, where $\varphi(x) = 0 \vee (x-1)$, is not weakly homogeneous.

(*b*) When $I = [0,1]$, the assumption that f is nonconstant implies immediately that $\varphi(1) \neq 0$. We then see by Proposition 3 that f_0 is weakly homogeneous. Note also that, if f is constant, then $f_0 \equiv 0$ is clearly weakly homogeneous. Thus, for any quasi-Lovász extension $f\colon [0,1]^n \to \mathbb{R}$, the function f_0 is weakly homogeneous.

Dually, we say that a function $f\colon I^n \to \mathbb{R}$, where $I \subseteq \mathbb{R}_-$, is *weakly homogeneous* if there exists a nondecreasing function $\varphi\colon I \to \mathbb{R}$ satisfying $\varphi(0) = 0$ such that $f(x\mathbf{1}_A) = -\varphi(x)f(-\mathbf{1}_A)$ for every $x \in I$ and every $A \subseteq [n]$.

Using (4) (instead of (3)), we can easily obtain the following negative counterparts of Lemma 2 and Proposition 3.

Lemma 4. *Assume $I \subseteq \mathbb{R}_-$. For every quasi-Lovász extension $f\colon I^n \to \mathbb{R}$, $f = L \circ \varphi$, we have*

$$f_0(x\mathbf{1}_A) = -\varphi(x)L_0(-\mathbf{1}_A), \qquad x \in I,\ A \subseteq [n].$$

Proposition 5. *Assume $[-1,0] \subseteq I \subseteq \mathbb{R}_-$. Let $f\colon I^n \to \mathbb{R}$ be a nonconstant quasi-Lovász extension, $f = L \circ \varphi$. Then the following conditions are equivalent.*

(*i*) *f_0 is weakly homogeneous.*

(*ii*) *There exists $A \subseteq [n]$ such that $f_0(-\mathbf{1}_A) \neq 0$.*

(*iii*) *$\varphi(-1) \neq 0$.*

In this case we have $f_0(x\mathbf{1}_A) = \frac{\varphi(x)}{\varphi(-1)} f_0(-\mathbf{1}_A)$ for every $x \in I$ and every $A \subseteq [n]$.

Assume now that $-x \in I$ whenever $x \in I$, that is, I is centered at 0. A *symmetric quasi-Lovász extension* is a function $f\colon I^n \to \mathbb{R}$ defined by

$$f = \check{L} \circ \varphi,$$

where $\check{L}\colon \mathbb{R}^n \to \mathbb{R}$ is a symmetric Lovász extension and $\varphi\colon I \to \mathbb{R}$ is a nondecreasing odd function.

Combining Lemmas 2 and 4 with the fact that $\check{L}_0$ and φ are odd functions, we obtain immediately the following result.

Lemma 6. *Assume that I is centered at 0. For every symmetric quasi-Lovász extension $f\colon I^n \to \mathbb{R}$, $f = \check{L} \circ \varphi$, we have*

$$f_0(x\mathbf{1}_A) = \varphi(x)\check{L}_0(\mathbf{1}_A), \qquad x \in I,\ A \subseteq [n]. \tag{9}$$

We say that a function $f\colon I^n \to \mathbb{R}$, where I centered at 0, is *oddly homogeneous* if there exists a nondecreasing odd function $\varphi\colon I \to \mathbb{R}$ such that $f(x\mathbf{1}_A) = \varphi(x)f(\mathbf{1}_A)$ for every $x \in I$ and every $A \subseteq [n]$.

Clearly, for every oddly homogeneous function f, the functions $f|_{I_+^n}$ and $f|_{I_-^n}$ are weakly homogeneous.

The following proposition provides necessary and sufficient conditions on a nonconstant symmetric quasi-Lovász extension $f: I^n \to \mathbb{R}$ for the function f_0 to be oddly homogeneous.

Proposition 7. *Assume that I is centered at 0 with $[-1,1] \subseteq I$. Let $f: I^n \to \mathbb{R}$ be a symmetric quasi-Lovász extension, $f = \check{L} \circ \varphi$, such that $f|_{I_+^n}$ or $f|_{I_-^n}$ is nonconstant. Then the following conditions are equivalent.*

(*i*) *f_0 is oddly homogeneous.*
(*ii*) *There exists $A \subseteq [n]$ such that $f_0(\mathbf{1}_A) \neq 0$.*
(*iii*) *$\varphi(1) \neq 0$.*

In this case we have $f_0(x\mathbf{1}_A) = \frac{\varphi(x)}{\varphi(1)} f_0(\mathbf{1}_A)$ for every $x \in I$ and every $A \subseteq [n]$.

Proof. Since $f|_{I_+^n}$ or $f|_{I_-^n}$ is nonconstant and f_0 is odd, we have $f_0|_{I_+^n} \not\equiv 0$ and $f|_{I_-^n} \not\equiv 0$.

The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow from Proposition 3 and the fact that, if f_0 is oddly homogeneous, then $f_0|_{I_+^n}$ is weakly homogeneous.

Now, assume that (iii) holds. Since $\check{L}_0$ and φ are odd, by Propositions 3 and 5 we clearly have $f_0(x\mathbf{1}_A) = \frac{\varphi(x)}{\varphi(1)} f_0(\mathbf{1}_A)$ for every $x \in I$ and every $A \subseteq [n]$, which shows that (i) also holds. □

Remark 2. Similarly to Remark 1(*b*), we see that, for any symmetric quasi-Lovász extension $f: [-1,1]^n \to \mathbb{R}$, the function f_0 is oddly homogeneous.

4. Comonotonic modularity

Recall that a function $f: I^n \to \mathbb{R}$ is said to be *modular* (or a *valuation*) if

$$f(\mathbf{x}) + f(\mathbf{x}') = f(\mathbf{x} \wedge \mathbf{x}') + f(\mathbf{x} \vee \mathbf{x}') \tag{10}$$

for every $\mathbf{x}, \mathbf{x}' \in I^n$. It was proved (see Topkis [24, Thm 3.3]) that a function $f: I^n \to \mathbb{R}$ is modular if and only if it is *separable*, that is, there exist n functions $f_i: I \to \mathbb{R}$, $i \in [n]$, such that $f = \sum_{i\in[n]} f_i$.[1] In particular, any 1-place function $f: I \to \mathbb{R}$ is modular.

Two n-tuples $\mathbf{x}, \mathbf{x}' \in I^n$ are said to be *comonotonic* if there exists $\sigma \in S_n$ such that $\mathbf{x}, \mathbf{x}' \in I_\sigma^n$. A function $f: I^n \to \mathbb{R}$ is said to be *comonotonically modular* (or a *comonotonic valuation*) if (10) holds for every comonotonic n-tuples $\mathbf{x}, \mathbf{x}' \in I^n$. This notion was considered in the special case when $I = [0,1]$ in [20]. We observe that, for any function $f: I^n \to \mathbb{R}$, condition (10) holds for every $\mathbf{x}, \mathbf{x}' \in I^n$ of the forms $\mathbf{x} = x\mathbf{1}_A$ and $\mathbf{x}' = x'\mathbf{1}_A$, where $x, x' \in I$ and $A \subseteq [n]$.

Observe also that, for every $\mathbf{x} \in \mathbb{R}_+^n$ and every $c \in \mathbb{R}_+$, we have

$$\mathbf{x} - \mathbf{x} \wedge c = [\mathbf{x}]_c - [\mathbf{x}]_c \wedge c.$$

This motivates the following definition. We say that a function $f: I^n \to \mathbb{R}$, where $I \subseteq \mathbb{R}_+$, is *invariant under horizontal min-differences* if, for every $\mathbf{x} \in I^n$ and every $c \in I$, we have

$$f(\mathbf{x}) - f(\mathbf{x} \wedge c) = f([\mathbf{x}]_c) - f([\mathbf{x}]_c \wedge c). \tag{11}$$

Dually, we say that a function $f: I^n \to \mathbb{R}$, where $I \subseteq \mathbb{R}_-$, is *invariant under horizontal max-differences* if, for every $\mathbf{x} \in I^n$ and every $c \in I$, we have

$$f(\mathbf{x}) - f(\mathbf{x} \vee c) = f([\mathbf{x}]^c) - f([\mathbf{x}]^c \vee c). \tag{12}$$

[1] This result still holds in the more general framework where f is defined on a product of chains.

Fact 8. *Assume $I \subseteq \mathbb{R}_+$. A function $f\colon (-I)^n \to \mathbb{R}$, where $-I = \{-x : x \in I\}$, is invariant under horizontal max-differences if and only if the function $f'\colon I^n \to \mathbb{R}$, defined by $f'(\mathbf{x}) = f(-\mathbf{x})$ for every $\mathbf{x} \in I^n$, is invariant under horizontal min-differences.*

We observe that, for any function $f\colon I^n \to \mathbb{R}$, where $I \subseteq \mathbb{R}_+$, condition (11) holds for every $\mathbf{x} \in I^n$ of the form $\mathbf{x} = x\mathbf{1}_A$, where $x \in I$ and $A \subseteq [n]$. Dually, for any function $f\colon I^n \to \mathbb{R}$, where $I \subseteq \mathbb{R}_-$, condition (12) holds for every tuple $\mathbf{x} \in I^n$ of the form $\mathbf{x} = x\mathbf{1}_A$, where $x \in I$ and $A \subseteq [n]$.

We also observe that a function f is comonotonically modular (resp. invariant under horizontal min-differences, invariant under horizontal max-differences) if and only if so is the function f_0.

Theorem 9. *Assume $I \subseteq \mathbb{R}_+$ and let $f\colon I^n \to \mathbb{R}$ be a function. Then the following assertions are equivalent.*

(i) f is comonotonically modular.

(ii) f is invariant under horizontal min-differences.

(iii) There exists a function $g\colon I^n \to \mathbb{R}$ such that, for every $\sigma \in S_n$ and every $\mathbf{x} \in I^n_\sigma$, we have

$$f(\mathbf{x}) = g(\mathbf{0}) + \sum_{i\in[n]} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) - g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big). \tag{13}$$

In this case, we can choose $g = f$.

Proof. $(i) \Rightarrow (iii)$ Let $\sigma \in S_n$ and $\mathbf{x} \in I^n_\sigma$. By comonotonic modularity, for every $i \in [n-1]$ we have

$$f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) + f(\mathbf{x}^0_{A^{\downarrow}_{\sigma}(i)}) = f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)}) + f(\mathbf{x}^0_{A^{\downarrow}_{\sigma}(i-1)}),$$

that is,

$$f(\mathbf{x}^0_{A^{\downarrow}_{\sigma}(i-1)}) = \big(f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) - f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big) + f(\mathbf{x}^0_{A^{\downarrow}_{\sigma}(i)}). \tag{14}$$

By using (14) for $i = 1, \dots, n-1$, we obtain (13) with $g = f$.

$(iii) \Rightarrow (i)$ For every $\sigma \in S_n$ and every $\mathbf{x}, \mathbf{x}' \in I^n_\sigma$, we have

$$\begin{aligned} f_0(\mathbf{x}) + f_0(\mathbf{x}') &= \sum_{i\in[n]} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) + g(x'_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)})\big) \\ &\quad - \sum_{i\in[n]} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)}) + g(x'_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big) \end{aligned}$$

and, since g satisfies property (10) for every $\mathbf{x}, \mathbf{x}' \in I^n$ of the forms $\mathbf{x} = x\mathbf{1}_A$ and $\mathbf{x}' = x'\mathbf{1}_A$, where $x, x' \in I$ and $A \subseteq [n]$, we have that (i) holds.

$(ii) \Rightarrow (iii)$ Let $\sigma \in S_n$ and $\mathbf{x} \in I^n_\sigma$. There exists $p \in [n]$ such that $x_{\sigma(1)} = \cdots = x_{\sigma(p)} < x_{\sigma(p+1)}$.[2] Then, using (11) with $c = x_{\sigma(1)}$, we get

$$f(\mathbf{x}) - f(x_{\sigma(1)}\mathbf{1}_{A^{\uparrow}_{\sigma}(1)}) = f(\mathbf{x}^0_{A^{\downarrow}_{\sigma}(p)}) - f(x_{\sigma(1)}\mathbf{1}_{A^{\uparrow}_{\sigma}(p+1)}).$$

Using a telescoping sum and the fact that $x_{\sigma(1)} = \cdots = x_{\sigma(p)}$, we obtain

$$\begin{aligned} f(\mathbf{x}) &= \big(f(x_{\sigma(1)}\mathbf{1}_{A^{\uparrow}_{\sigma}(1)}) - f(x_{\sigma(1)}\mathbf{1}_{A^{\uparrow}_{\sigma}(p+1)})\big) + f(\mathbf{x}^0_{A^{\downarrow}_{\sigma}(p)}) \\ &= \sum_{i=1}^{p} \big(f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) - f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big) + f(\mathbf{x}^0_{A^{\downarrow}_{\sigma}(p)}). \end{aligned} \tag{15}$$

[2] Here $x_{\sigma(n+1)} = +\infty$.

If $p = n-1$ or $p = n$, then (13) holds with $g = f$. Otherwise, there exists $q \in [n-p]$ such that $x_{\sigma(p+1)} = \cdots = x_{\sigma(p+q)} < x_{\sigma(p+q+1)}$ and we expand the last term in (15) similarly by using (11) with $c = x_{\sigma(p+1)}$. We then repeat this procedure until the last term is $f(\mathbf{0})$, thus obtaining (13) with $g = f$.

To illustrate, suppose $x_1 < x_2 = x_3 < x_4$. Then

$$f(x_1,x_2,x_3,x_4) = \big(f(x_1,x_1,x_1,x_1) - f(0,x_1,x_1,x_1)\big) + f(0,x_2,x_3,x_4)$$

with

$$\begin{aligned} f(0,x_2,x_3,x_4) &= \big(f(0,x_2,x_2,x_2) - f(0,0,x_2,x_2)\big) \\ &\quad + \big(f(0,0,x_3,x_3) - f(0,0,0,x_3)\big) + f(0,0,0,x_4) \end{aligned}$$

and

$$f(0,0,0,x_4) = \big(f(0,0,0,x_4) - f(0,0,0,0)\big) + f(0,0,0,0).$$

$(iii) \Rightarrow (ii)$ For every $\sigma \in S_n$, every $\mathbf{x} \in I^n_\sigma$, and every $c \in I$, we have

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{x} \wedge c) &= \sum_{i\in[n]} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) - g((x_{\sigma(i)} \wedge c)\mathbf{1}_{A^{\uparrow}_{\sigma}(i)})\big) \\ &\quad - \sum_{i\in[n]} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)}) - g((x_{\sigma(i)} \wedge c)\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big) \end{aligned}$$

and, since g satisfies property (11) for every $\mathbf{x} \in I^n$ of the form $\mathbf{x} = x\mathbf{1}_A$, where $x \in I$ and $A \subseteq [n]$, we have that (ii) holds. □

Remark 3. The equivalence between (i) and (iii) in Theorem 9 generalizes Theorem 1 in [20], which describes the class of comonotonically modular functions $f\colon [0,1]^n \to [0,1]$ under the additional conditions of symmetry and idempotence.

The following theorem is the negative counterpart of Theorem 9 and its proof follows dually by taking into account Fact 8.

Theorem 10. *Assume $I \subseteq \mathbb{R}_-$ and let $f\colon I^n \to \mathbb{R}$ be a function. Then the following assertions are equivalent.*

(i) *f is comonotonically modular.*
(ii) *f is invariant under horizontal max-differences.*
(iii) *There exists a function $g\colon I^n \to \mathbb{R}$ such that, for every $\sigma \in S_n$ and every $\mathbf{x} \in I^n_\sigma$, we have*

$$f(\mathbf{x}) = g(\mathbf{0}) + \sum_{i\in[n]} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_{\sigma}(i)}) - g(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_{\sigma}(i-1)})\big).$$

In this case, we can choose $g = f$.

We observe that if $f\colon I^n \to \mathbb{R}$ is comonotonically modular then necessarily

$$f_0(\mathbf{x}) = f_0(\mathbf{x}^+) + f_0(-\mathbf{x}^-) \tag{16}$$

(take $\mathbf{x}' = \mathbf{0}$ in (10)).

We may now characterize the class of comonotonically modular functions on an arbitrary interval I containing 0.

Theorem 11. *For any function $f\colon I^n \to \mathbb{R}$, the following assertions are equivalent.*

(i) *f is comonotonically modular.*

(*ii*) *There exist* $g\colon I_+^n \to \mathbb{R}$ *comonotonically modular (or invariant under horizontal min-differences) and* $h\colon I_-^n \to \mathbb{R}$ *comonotonically modular (or invariant under horizontal max-differences) such that* $f_0(\mathbf{x}) = g_0(\mathbf{x}^+) + h_0(-\mathbf{x}^-)$ *for every* $\mathbf{x} \in I^n$. *In this case, we can choose* $g = f|_{I_+^n}$ *and* $h = f|_{I_-^n}$.

(*iii*) *There exist* $g\colon I_+^n \to \mathbb{R}$ *and* $h\colon I_-^n \to \mathbb{R}$ *such that, for every* $\sigma \in S_n$ *and every* $\mathbf{x} \in I_\sigma^n$,

$$\begin{aligned} f_0(\mathbf{x}) &= \sum_{1\leqslant i\leqslant p} \big(h(x_{\sigma(i)}\mathbf{1}_{A_\sigma^\downarrow(i)}) - h(x_{\sigma(i)}\mathbf{1}_{A_\sigma^\downarrow(i-1)})\big) \\ &\quad + \sum_{p+1\leqslant i\leqslant n} \big(g(x_{\sigma(i)}\mathbf{1}_{A_\sigma^\uparrow(i)}) - g(x_{\sigma(i)}\mathbf{1}_{A_\sigma^\uparrow(i+1)})\big), \end{aligned}$$

where $p \in \{0,\ldots,n\}$ *is such that* $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$. *In this case, we can choose* $g = f|_{I_+^n}$ *and* $h = f|_{I_-^n}$.

Proof. (*i*) ⇒ (*ii*) Follows from (16) and Theorems 9 and 10.

(*ii*) ⇒ (*iii*) Follows from Theorems 9 and 10.

(*iii*) ⇒ (*i*) Clearly, f_0 satisfies (16). Let $\sigma \in S_n$ and $\mathbf{x}, \mathbf{x}' \in I_\sigma^n$. By (16) we have

$$f_0(\mathbf{x}) + f_0(\mathbf{x}') = f_0(\mathbf{x}^+) + f_0(\mathbf{x}'^+) + f_0(-\mathbf{x}^-) + f(-\mathbf{x}'^-)$$

Using Theorems 9 and 10, we see that this identity can be rewritten as

$$\begin{aligned} f_0(\mathbf{x}) + f_0(\mathbf{x}') &= f_0(\mathbf{x}^+ \wedge \mathbf{x}'^+) + f_0(\mathbf{x}^+ \vee \mathbf{x}'^+) + f_0(-\mathbf{x}^- \wedge -\mathbf{x}'^-) + f_0(-\mathbf{x}^- \vee -\mathbf{x}'^-) \\ &= f_0\big((\mathbf{x}\wedge\mathbf{x}')^+\big) + f_0\big((\mathbf{x}\vee\mathbf{x}')^+\big) + f_0\big(-(\mathbf{x}\wedge\mathbf{x}')^-\big) + f_0\big(-(\mathbf{x}\vee\mathbf{x}')^-\big), \end{aligned}$$

which, by (16), becomes $f_0(\mathbf{x}) + f_0(\mathbf{x}') = f_0(\mathbf{x}\wedge\mathbf{x}') + f_0(\mathbf{x}\vee\mathbf{x}')$. Therefore, f_0 is comonotonically modular and, hence, so is f. □

From Theorem 11 we obtain the "comonotonic" analogue of Topkis' characterization [24] of modular functions as separable functions, and which provides an alternative description of comonotonically modular functions. We make use of the following fact.

Fact 12. *Let* J *be any nonempty real interval, possibly unbounded, and let* $c \in J$. *A function* $g\colon J^n \to \mathbb{R}$ *is modular (resp. comonotonically modular) if and only if the function* $f\colon I^n \to \mathbb{R}$, *defined by* $f(\mathbf{x}) = g(\mathbf{x} + c\mathbf{1})$, *where* $I = J - c = \{z - c : z \in J\}$, *is modular (resp. comonotonically modular).*

Corollary 13. *Let* J *be any nonempty real interval, possibly unbounded. A function* $f\colon J^n \to \mathbb{R}$ *is comonotonically modular if and only if it is comonotonically separable, that is, for every* $\sigma \in S_n$, *there exist functions* $f_i^\sigma\colon J \to \mathbb{R}$, $i \in [n]$, *such that*

$$f(\mathbf{x}) = \sum_{i=1}^n f_i^\sigma(x_{\sigma(i)}) = \sum_{i=1}^n f_{\sigma^{-1}(i)}^\sigma(x_i), \qquad \mathbf{x} \in J^n \cap \mathbb{R}_\sigma^n.$$

Proof. (Necessity) By Fact 12 we can assume that J contains the origin. The result then follows from the equivalence (*i*) ⇔ (*iii*) stated in Theorem 11.

(Sufficiency) For every $\sigma \in S_n$ and every $i \in [n]$, the function $f_{\sigma^{-1}(i)}^\sigma$ is clearly modular and hence comonotonically modular. Since the class of comonotonically modular functions is closed under addition, the proof is now complete. □

5. Axiomatization and representation of quasi-Lovász extensions

We now provide axiomatizations of the class of quasi-Lovász extensions and describe all possible factorizations of quasi-Lovász extensions into compositions of Lovász extensions with 1-place nondecreasing functions.

Theorem 14. *Assume $[0,1]\subseteq I\subseteq \mathbb{R}_+$ and let $f\colon I^n\to\mathbb{R}$ be a nonconstant function. Then the following assertions are equivalent.*

- (i) *f is a quasi-Lovász extension and there exists $A\subseteq[n]$ such that $f_0(\mathbf{1}_A)\neq 0$.*
- (ii) *f is comonotonically modular (or invariant under horizontal min-differences) and f_0 is weakly homogeneous.*
- (iii) *There is a nondecreasing function $\varphi_f\colon I\to\mathbb{R}$ satisfying $\varphi_f(0)=0$ and $\varphi_f(1)=1$ such that $f=L_{f|_{\mathbb{B}^n}}\circ\varphi_f$.*

Proof. Let us prove that $(i)\Rightarrow(ii)$. By definition, we have $f=L\circ\varphi$, where $L\colon\mathbb{R}^n\to\mathbb{R}$ is a Lovász extension and $\varphi\colon I\to\mathbb{R}$ is a nondecreasing function satisfying $\varphi(0)=0$. By Proposition 3, f_0 is weakly homogeneous. Moreover, by (3) and (8) we have that, for every $\sigma\in S_n$ and every $\mathbf{x}\in I^n_\sigma$,

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{0})+\sum_{i\in[n]}\varphi(x_{\sigma(i)})\big(L_0(\mathbf{1}_{A^{\uparrow}_\sigma(i)})-L_0(\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big)\\ &= f(\mathbf{0})+\sum_{i\in[n]}\big(f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i)})-f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big).\end{aligned}$$

Theorem 9 then shows that f is comonotonically modular.

Let us prove that $(ii)\Rightarrow(iii)$. Since f is comonotonically modular, by Theorem 9 it follows that, for every $\sigma\in S_n$ and every $\mathbf{x}\in I^n_\sigma$,

$$f(\mathbf{x})=f(\mathbf{0})+\sum_{i\in[n]}\big(f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i)})-f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big),$$

and, since f_0 is weakly homogeneous,

$$f(\mathbf{x})=f(\mathbf{0})+\sum_{i\in[n]}\varphi_f(x_{\sigma(i)})\big(f(\mathbf{1}_{A^{\uparrow}_\sigma(i)})-f(\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big)\tag{17}$$

for some nondecreasing function $\varphi_f\colon I\to\mathbb{R}$ satisfying $\varphi_f(0)=0$. By (3), we then obtain $f=L_{f|_{\mathbb{B}^n}}\circ\varphi_f$. Finally, by (17) we have that, for every $A\subseteq[n]$,

$$f_0(\mathbf{1}_A)=\varphi_f(1)f_0(\mathbf{1}_A).$$

Since there exists $A\subseteq[n]$ such that $f_0(\mathbf{1}_A)\neq 0$ (for otherwise, we would have $f_0\equiv 0$ by (17)), we obtain $\varphi_f(1)=1$.

The implication $(iii)\Rightarrow(i)$ follows from Proposition 3. □

Let $f\colon I^n\to\mathbb{R}$ be a quasi-Lovász extension, where $[0,1]\subseteq I\subseteq\mathbb{R}_+$, for which there exists $A^*\subseteq[n]$ such that $f_0(\mathbf{1}_{A^*})\neq 0$. Then the inner function φ_f introduced in Theorem 14 is unique. Indeed, by Proposition 3, we have $f_0(x\mathbf{1}_A)=\varphi_f(x)f_0(\mathbf{1}_A)$ for every $x\in I$ and every $A\subseteq[n]$. The function φ_f is then defined by

$$\varphi_f(x)=\frac{f_0(x\mathbf{1}_{A^*})}{f_0(\mathbf{1}_{A^*})}\,,\qquad x\in I.$$

We can now describe the possible factorizations of f into compositions of Lovász extensions with nondecreasing functions.

Theorem 15. *Assume $[0,1] \subseteq I \subseteq \mathbb{R}_+$ and let $f\colon I^n \to \mathbb{R}$ be a quasi-Lovász extension, $f = L \circ \varphi$. Then there exists $A^* \subseteq [n]$ such that $f_0(\mathbf{1}_{A^*}) \neq 0$ if and only if there exists $a > 0$ such that $\varphi = a\,\varphi_f$ and $L_0 = \frac{1}{a}(L_{f|_{\mathbb{B}^n}})_0$.*

Proof. (Sufficiency) We have $f_0 = L_0 \circ \varphi = (L_{f|_{\mathbb{B}^n}})_0 \circ \varphi_f$, and by Theorem 14 we see that the conditions are sufficient.

(Necessity) By Proposition 3, we have

$$\frac{\varphi(x)}{\varphi(1)} = \frac{f_0(x\mathbf{1}_{A^*})}{f_0(\mathbf{1}_{A^*})} = \varphi_f(x).$$

We then have $\varphi = a\,\varphi_f$ for some $a > 0$. Moreover, for every $\mathbf{x} \in \mathbb{B}^n$, we have

$$\begin{aligned}(L_{f|_{\mathbb{B}^n}})_0(\mathbf{x}) &= \big((L_{f|_{\mathbb{B}^n}})_0 \circ \varphi_f\big)(\mathbf{x}) = f_0(\mathbf{x}) = (L_0 \circ \varphi)(\mathbf{x}) \\ &= a(L_0 \circ \varphi_f)(\mathbf{x}) = a\,L_0(\mathbf{x}).\end{aligned}$$

Since a Lovász extension is uniquely determined by its values on $\mathbb{B}^n$, we have $(L_{f|_{\mathbb{B}^n}})_0 = a\,L_0$. □

The following two theorems are the negative counterparts of Theorems 14 and 15 and their proofs follow dually.

Theorem 16. *Assume $[-1,0] \subseteq I \subseteq \mathbb{R}_-$ and let $f\colon I^n \to \mathbb{R}$ be a nonconstant function. Then the following assertions are equivalent.*

- (*i*) *f is a quasi-Lovász extension and there exists $A \subseteq [n]$ such that $f_0(-\mathbf{1}_A) \neq 0$.*
- (*ii*) *f is comonotonically modular (or invariant under horizontal max-differences) and f_0 is weakly homogeneous.*
- (*iii*) *There is a nondecreasing function $\varphi_f\colon I \to \mathbb{R}$ satisfying $\varphi_f(0) = 0$ and $\varphi_f(-1) = -1$ such that $f = L_{f|_{-\mathbb{B}^n}} \circ \varphi_f$.*

Theorem 17. *Assume $[-1,0] \subseteq I \subseteq \mathbb{R}_-$ and let $f\colon I^n \to \mathbb{R}$ be a quasi-Lovász extension, $f = L \circ \varphi$. Then there exists $A^* \subseteq [n]$ such that $f_0(-\mathbf{1}_{A^*}) \neq 0$ if and only if there exists $a > 0$ such that $\varphi = a\,\varphi_f$ and $L_0 = \frac{1}{a}(L_{f|_{-\mathbb{B}^n}})_0$.*

Remark 4. If $I = [0,1]$ (resp. $I = [-1,0]$), then the "nonconstant" assumption and the second condition in assertion (i) of Theorem 14 (resp. Theorem 16) can be dropped off.

6. Axiomatization and representation of symmetric quasi-Lovász extensions

We now provide an axiomatization of the class of symmetric quasi-Lovász extensions and describe all possible factorizations of symmetric quasi-Lovász extensions into compositions of symmetric Lovász extensions with 1-place nondecreasing odd functions. We proceed in complete analogy as in the previous section.

Theorem 18. *Assume that I is centered at 0 with $[-1,1] \subseteq I$ and let $f\colon I^n \to \mathbb{R}$ be a function such that $f|_{I_+^n}$ or $f|_{I_-^n}$ is nonconstant. Then the following assertions are equivalent.*

- (*i*) *f is a symmetric quasi-Lovász extension and there exists $A \subseteq [n]$ such that $f_0(\mathbf{1}_A) \neq 0$.*
- (*ii*) *f is comonotonically modular and f_0 is oddly homogeneous.*

(*iii*) *There is a nondecreasing odd function* $\varphi_f\colon I\to\mathbb{R}$ *satisfying* $\varphi_f(1)=1$ *such that* $f=\check{L}_{f|_{\mathbb{B}^n}}\circ\varphi_f$.

Proof. Let us prove that $(i)\Rightarrow(ii)$. By definition, we have $f=\check{L}\circ\varphi$, where $\check{L}\colon\mathbb{R}^n\to\mathbb{R}$ is a symmetric Lovász extension and $\varphi\colon I\to\mathbb{R}$ is a nondecreasing odd function. By Proposition 7, f_0 is oddly homogeneous. Moreover, for every $\sigma\in S_n$ and every $\mathbf{x}\in I^n_\sigma$, by (7) and (9) we have

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{0})+\sum_{1\leqslant i\leqslant p}\varphi(x_{\sigma(i)})\big(L_0(\mathbf{1}_{A^{\downarrow}_\sigma(i)})-L_0(\mathbf{1}_{A^{\downarrow}_\sigma(i-1)})\big)\\ &\quad+\sum_{p+1\leqslant i\leqslant n}\varphi(x_{\sigma(i)})\big(L_0(\mathbf{1}_{A^{\uparrow}_\sigma(i)})-L_0(\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big)\\ &= f(\mathbf{0})+\sum_{1\leqslant i\leqslant p}\big(f(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_\sigma(i)})-f(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_\sigma(i-1)})\big)\\ &\quad+\sum_{p+1\leqslant i\leqslant n}\big(f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i)})-f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big),\end{aligned}$$

where $p\in\{0,\dots,n\}$ is such that $x_{\sigma(p)}<0\leqslant x_{\sigma(p+1)}$. By Theorem 11 it then follows that f is comonotonically modular.

Let us prove that $(ii)\Rightarrow(iii)$. Since f is comonotonically modular and f_0 is oddly homogeneous, by Theorem 11 we have that, for every $\sigma\in S_n$ and every $\mathbf{x}\in I^n_\sigma$,

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{0})+\sum_{1\leqslant i\leqslant p}\big(f(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_\sigma(i)})-f(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_\sigma(i-1)})\big)\\ &\quad+\sum_{p+1\leqslant i\leqslant n}\big(f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i)})-f(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big)\\ &= f(\mathbf{0})+\sum_{1\leqslant i\leqslant p}\varphi_f(x_{\sigma(i)})\big(f(\mathbf{1}_{A^{\downarrow}_\sigma(i)})-f(\mathbf{1}_{A^{\downarrow}_\sigma(i-1)})\big)\\ &\quad+\sum_{p+1\leqslant i\leqslant n}\varphi_f(x_{\sigma(i)})\big(f(\mathbf{1}_{A^{\uparrow}_\sigma(i)})-f(\mathbf{1}_{A^{\uparrow}_\sigma(i+1)})\big) \qquad (18)\end{aligned}$$

for some nondecreasing odd function $\varphi_f\colon I\to\mathbb{R}$, where $p\in\{0,\dots,n\}$ is such that $x_{\sigma(p)}<0\leqslant x_{\sigma(p+1)}$. By (7), we then obtain $f=\check{L}_{f|_{\mathbb{B}^n}}\circ\varphi_f$. Finally, by (18) we then have that, for every $A\subseteq[n]$,

$$f_0(\mathbf{1}_A)=\varphi_f(1)f_0(\mathbf{1}_A).$$

Since there exists $A\subseteq[n]$ such that $f_0(\mathbf{1}_A)\neq 0$ (for otherwise we would have $f_0\equiv 0$ by (18)), we obtain $\varphi_f(1)=1$.

The implication $(iii)\Rightarrow(i)$ follows from Proposition 7. □

Assume again that I is centered at 0 with $[-1,1]\subseteq I$ and let $f\colon I^n\to\mathbb{R}$ be a symmetric quasi-Lovász extension for which there exists $A^*\subseteq[n]$ such that $f_0(\mathbf{1}_{A^*})\neq 0$. Then the inner function φ_f introduced in Theorem 18 is unique. Indeed, by Proposition 7, we have $f_0(x\mathbf{1}_A)=\varphi_f(x)f_0(\mathbf{1}_A)$ for every $x\in I$ and every $A\subseteq[n]$. The function φ_f is then defined by

$$\varphi_f(x)=\frac{f_0(x\mathbf{1}_{A^*})}{f_0(\mathbf{1}_{A^*})}\,,\qquad x\in I.$$

We can now describe the possible factorizations of f into compositions of symmetric Lovász extensions with nondecreasing odd functions. The proof is similar to that of Theorem 15 and thus it is omitted.

Theorem 19. *Assume that I is centered at 0 with $[-1,1]\subseteq I$ and let $f\colon I^n\to\mathbb{R}$ be a symmetric quasi-Lovász extension, $f=\check{L}\circ\varphi$. Then there exists $A^*\subseteq[n]$ such that $f_0(\mathbf{1}_{A^*})\neq 0$ if and only if there exists $a>0$ such that $\varphi=a\,\varphi_f$ and $\check{L}_0=\frac{1}{a}(\check{L}_{f|_{\mathbb{B}^n}})_0$.*

Remark 5. If $I=[-1,1]$, then the "nonconstant" assumption and the second condition in assertion (i) of Theorem 18 can be dropped off.

7. Application: Quasi-polynomial functions on chains

In this section we show that prominent classes of lattice functions on closed real intervals are comonotonically modular. To this extent we need to introduce some basic concepts and terminology.

Let L be a bounded distributive lattice. Recall that a *lattice polynomial function on L* is a mapping $p\colon L^n\to L$ which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. As it is well known, the notion of lattice polynomial function generalizes that of the discrete Sugeno integral. For further background on lattice polynomial functions and discrete Sugeno integrals see, e.g., [7, 8, 9, 14]; see also [4, 17, 21] for general background on lattice theory.

In [5] the authors introduced the notion of "quasi-polynomial function" as being a mapping $f\colon X^n\to X$ defined and valued on a bounded chain X and which can be factorized into a composition of a lattice polynomial function with a nondecreasing function.

In the current paper we restrict ourselves to such mappings on closed intervals $J\subseteq\overline{\mathbb{R}}=[-\infty,+\infty]$. More precisely, by a *quasi-polynomial function on J* we mean a mapping $f\colon J^n\to\mathbb{R}$ which can be factorized as

$$f=p\circ\varphi,$$

where $\varphi\colon J\to\overline{\mathbb{R}}$ is an order-preserving map and $p\colon\overline{\mathbb{R}}^n\to\overline{\mathbb{R}}$ is a lattice polynomial function on $\overline{\mathbb{R}}$. For further extensions and generalizations, see [6, 11, 12].

The class of quasi-polynomial functions was axiomatized in [5] in terms of two well-known conditions in aggregation theory, which we now briefly describe.

A function $f\colon J^n\to\overline{\mathbb{R}}$ is said to be *comonotonically maxitive* if, for any two comonotonic tuples $\mathbf{x},\mathbf{x}'\in J^n$,

$$f(\mathbf{x}\vee\mathbf{x}')=f(\mathbf{x})\vee f(\mathbf{x}').$$

Dually, $f\colon J^n\to\overline{\mathbb{R}}$ is said to be *comonotonically minitive* if, for any two comonotonic tuples $\mathbf{x},\mathbf{x}'\in J^n$,

$$f(\mathbf{x}\wedge\mathbf{x}')=f(\mathbf{x})\wedge f(\mathbf{x}').$$

Theorem 20 ([5, 6]). *A function $f\colon J^n\to\overline{\mathbb{R}}$ is a quasi-polynomial function if and only if it is is comonotonically maxitive and comonotonically minitive.*

Immediately from Theorem 20 it follows that every quasi-polynomial function $f\colon J^n\to\mathbb{R}$ is comonotonically modular. Indeed, by comonotonic maxitivity and comonotonic minitivity, we have that, for any two comonotonic tuples $\mathbf{x},\mathbf{x}'\in J^n$,

$$\begin{aligned} f(\mathbf{x}\wedge\mathbf{x}')+f(\mathbf{x}\vee\mathbf{x}') &= \big(f(\mathbf{x})\wedge f(\mathbf{x}')\big)+\big(f(\mathbf{x})\vee f(\mathbf{x}')\big)\\ &= f(\mathbf{x})+f(\mathbf{x}'). \end{aligned}$$

In fact, from Corollary 13, we obtain the following factorization of quasi-polynomial functions into a sum of unary mappings.

Corollary 21. *Every quasi-polynomial function* $f\colon J^n \to \overline{\mathbb{R}}$ *is comonotonically modular. Moreover, for every* $\sigma \in S_n$*, there exist functions* $f_i^\sigma\colon J \to \overline{\mathbb{R}}$*,* $i \in [n]$*, such that*

$$f(\mathbf{x}) = \sum_{i=1}^{n} f_i^\sigma(x_{\sigma(i)}) = \sum_{i=1}^{n} f_{\sigma^{-1}(i)}^\sigma(x_i), \qquad \mathbf{x} \in J^n \cap \overline{\mathbb{R}}_\sigma^n.$$

Acknowledgments

This research is supported by the internal research project F1R-MTH-PUL-09MRDO of the University of Luxembourg.

References

[1] G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners.* Studies in Fuziness and Soft Computing. Springer, Berlin, 2007.

[2] P. Benvenuti, R. Mesiar, and D. Vivona. Monotone set functions-based integrals. In *Handbook of measure theory, Vol. II*, pages 1329–1379. North-Holland, Amsterdam, 2002.

[3] D. Bouyssou, D. Dubois, H. Prade, and M. Pirlot, editors. *Decision-Making Process - Concepts and Methods.* ISTE/John Wiley, London, 2009.

[4] S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1981.

[5] M. Couceiro and J.-L. Marichal. Axiomatizations of quasi-polynomial functions on bounded chains. *Aeq. Math.*, 78(1-2):195–213, 2009.

[6] M. Couceiro and J.-L. Marichal. Quasi-polynomial functions over bounded distributive lattices. *Aeq. Math.*, 80(3):319–334, 2010.

[7] M. Couceiro and J.-L. Marichal. Characterizations of discrete Sugeno integrals as polynomial functions over distributive lattices. *Fuzzy Sets and Systems*, 161(5):694–707, 2010.

[8] M. Couceiro and J.-L. Marichal. Representations and characterizations of polynomial functions on chains. *J. Mult.-Valued Logic Soft Comput.*, 16(1-2):65–86, 2010.

[9] M. Couceiro and J.-L. Marichal. Polynomial functions over bounded distributive lattices. *J. Mult.-Valued Logic Soft Comput.* In press.

[10] M. Couceiro and J.-L. Marichal. Axiomatizations of Lovász extensions and symmetric Lovász extensions of pseudo-Boolean functions. *Fuzzy Sets and Systems.* In press.

[11] M. Couceiro and T. Waldhauser. Sugeno Utility Functions I: Axiomatizations. In *Lecture Notes in Artificial Intelligence, vol. 6408*, pp. 79–90. Springer, 2010.

[12] M. Couceiro and T. Waldhauser. Sugeno Utility Functions II: Factorizations. In *Lecture Notes in Artificial Intelligence, vol. 6408*, pp. 91–103. Springer, 2010.

[13] L. M. de Campos and M. J. Bolaños. Characterization and comparison of Sugeno and Choquet integrals. *Fuzzy Sets and Systems*, 52(1):61–67, 1992.

[14] R. L. Goodstein. The solution of equations in a lattice. *Proc. Roy. Soc. Edinburgh Sect. A*, 67:231–242, 1965/1967.

[15] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions.* Encyclopedia of Mathematics and its Applications 127. Cambridge University Press, Cambridge, UK, 2009.

[16] M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy measures and integrals - Theory and applications*, volume 40 of *Studies in Fuzziness and Soft Computing.* Physica-Verlag, Heidelberg, 2000.

[17] G. Grätzer. *General lattice theory.* Birkhäuser Verlag, Berlin, 2003. Second edition.

[18] P. Hammer and S. Rudeanu. *Boolean methods in operations research and related areas.* Berlin-Heidelberg-New York: Springer-Verlag, 1968.

[19] L. Lovász. Submodular functions and convexity. In *Mathematical programming, 11th int. Symp., Bonn 1982, 235–257.* 1983.

[20] R. Mesiar and A. Mesiarová-Zemánková. The ordered modular averages. *IEEE Trans. Fuzzy Syst.*, 19(1):42–50, 2011.

[21] S. Rudeanu. *Lattice functions and equations.* Springer Series in Discrete Mathematics and Theoretical Computer Science. Springer-Verlag London Ltd., London, 2001.

[22] I. Singer. Extensions of functions of 0-1 variables and applications to combinatorial optimization. *Numer. Funct. Anal. Optimization*, 7:23–62, 1984.

[23] J. Šipoš. Integral with respect to a pre-measure. *Mathematica Slovaca*, 29(2):141–155, 1979.

[24] D. M. Topkis. Minimizing a submodular function on a lattice. *Operations Research*, 26(2):305–321, 1978.

Mathematics Research Unit, FSTC, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg
E-mail address: miguel.couceiro[at]uni.lu

Mathematics Research Unit, FSTC, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg
E-mail address: jean-luc.marichal[at]uni.lu