A characterization of associative idempotent nondecreasing functions with neutral elements

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Elementary properties of binary functions

Let $I = [a, b]$ be a closed real interval and let $F : I^2 \to I$ be a binary function (operation).
Elementary properties of binary functions

Let $I = [a, b]$ be a closed real interval and let $F : I^2 \rightarrow I$ be a binary function (operation). We may define natural algebraic and analytic assumptions.

1. $F$ is idempotent, iff $F(x, x) = x$ holds for every $x \in I$.
2. $F$ has a neutral element, iff there exists an $e \in X$ such that $F(e, x) = x$ and $F(x, e) = x$ for every $x \in I$.
3. $F$ is associative, iff $F(F(x, y), z) = F(x, F(y, z))$ for every $x, y, z \in I$.
4. $F$ is symmetric or commutative, iff $F(x, y) = F(y, x)$ if $\forall x, y \in I$.

Notation: If $F : I^2 \rightarrow I$ is associative, then we also say that the pair $(I, F)$ is a (2-ary) semigroup.
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**Notation:** If \( F : I^2 \to I \) is associative, then we also say that the pair \( (I, F) \) is a (2-ary) semigroup.
Analytic:

1. $F$ is monotone increasing
   1.1 in each variable iff $x_1 \leq x_2, y_1 \leq y_2 \Rightarrow F(x_1, y_1) \leq F(x_2, y_2)$ $(\forall x_i, y_i \in I, i = 1, 2)$
   1.2 in the first variable iff $x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$ $(\forall x_i, y \in I, i = 1, 2)$
   1.3 in the second variable.

2. $F$ is monotone decreasing.

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Our first aim is to characterize idempotent, monotone increasing (in each variable), 2-ary semigroups which have neutral element.
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Main tool:

Theorem (Czogala, Drewniak, 1984)

Let $I = [a, b]$ be a closed real interval. If a function $F: I^2 \rightarrow I$ is associative, idempotent, monotone which has a neutral element $e \in I$, then there exists a monotone decreasing function $g: I \rightarrow I$, with $g(e) = e$, such that

$$F(x, y) = \begin{cases} 
\min (x, y), & \text{if } y < g(x) \\
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\end{cases} \tag{1}$$

Lemma

If $F$ is associative, idempotent and monotone (in each variable) then it is monotone increasing (in each variable).
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**Theorem (Czogala,Drewniak, 1984)**

Let \( I = [a, b] \) be a closed real interval. If a function \( F : I^2 \rightarrow I \) is associative, idempotent, **monotone** which has a neutral element \( e \in I \), then there exits a monotone decreasing function \( g : I \rightarrow I \), with \( g(e) = e \), such that

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**Lemma**

*If* \( F \) *is associative, idempotent and monotone (in each variable) then it is monotone increasing (in each variable).*
The 'extended' graph of $g$

Further analysis shows that $g$ which arise in precious theorem also satisfies the following equations:

$x < y (x, y \in I) \Rightarrow x \geq g(y)$ or $y \leq g(x)$

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The set $\Gamma_g$ denotes the 'extended' graph of $g$ which is the graph of $g$ with vertical line segments in the discontinuity points of $g$.

Lemma

If $g$ satisfies (2) then

1. $g$ is monotone decreasing.
2. The 'extended' graph $\Gamma_g = \{(x, y) : g(x^-) \geq y \geq g(x^+)\}$ is symmetric with respect to the line $x = y$. 
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is symmetric with respect to the line $x = y$. 

Characterization of associative, idempotent, monotone increasing functions with neutral element

Theorem (Martín-Mayor-Torrens,'03; K-Marichal-Teheux,'16)

Let $I \subseteq \mathbb{R}$ be a closed interval. The function $F : I^2 \rightarrow I$ is associative, monotone increasing, idempotent and has a neutral element $e \in X$
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Let $I \subseteq \mathbb{R}$ be a closed interval. The function $F : I^2 \rightarrow I$ is associative, monotone increasing, idempotent and has a neutral element $e \in X$ if and only if there exists a decreasing function $g : X \rightarrow X$ with $g(e) = e$ such that extension of $\Gamma_g$ is symmetric
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Moreover, in this case $F$ must be commutative except perhaps on the set of points $(x, y)$ such that $y = g(x)$ and $x = g(y)$.
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The $n$-ary semigroups are generalizations of semigroups.
\( n \)-ary semigroups and basic properties

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\( F_n : I^n \to I \) is \( n \)-associative if for every \( x_1, \ldots, x_{2n-1} \in I \) and for every \( 1 \leq i \leq n-1 \) we have

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- \( F_n \) has neutral element \( e \) if for every \( x \in I \) and \( 1 \leq i \leq n \) we have \( F(e, \ldots, e, x, e, \ldots, e) = x \), where \( x \) is substituted into the \( i \)'th coordinate.
\textbf{\textit{n}-ary semigroups and basic properties}

The \textit{n}-ary semigroups are generalizations of semigroups.

\begin{itemize}
  \item $F_n : I^n \rightarrow I$ is \textit{n}-associative if for every $x_1, \ldots, x_{2n-1} \in I$ and for every $1 \leq i \leq n - 1$ we have
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    \end{equation}

  \item $F_n$ is \textit{idempotent} if $F_n(a, \ldots, a) = a$ for all $a \in I$.

  \item $F_n$ has \textit{neutral element} $e$ if for every $x \in I$ and $1 \leq i \leq n$ we have $F(e, \ldots, e, x, e, \ldots, e) = x$, where $x$ is substituted into the $i$'th coordinate.
\end{itemize}

An important construction:
Let $(X, F_2)$ be a binary semigroup and $F_n := F_2 \circ F_2 \circ \ldots \circ F_2$.\footnote{$n-1$ times}

Then $F_n$ is \textit{n}-associative.
Theorem (Dudek-Mukhin, 2006)

If an $n$-associative $F_n$ has a neutral element $e$, then $F_n$ is derived from an associative function $F_2 : I^2 \to I$ where $F_2(a, b) = F_n(a, e, \ldots, e, b)$. (i.e: $F_n = F_2 \circ \cdots \circ F_2$.)
Theorem (Dudek-Mukhin, 2006)

If an $n$-associative $F_n$ has a neutral element $e$, then $F_n$ is derived from an associative function $F_2 : I^2 \rightarrow I$ where $F_2(a, b) = F_n(a, e, \ldots, e, b)$. (i.e: $F_n = F_2 \circ \cdots \circ F_2$.)

By the definition of $F_2$, the element $e$ is also a neutral element of $F_2$. 
Main lemmas

Lemma

Let $F_n$ be $n$-associative, idempotent, monotone in at least two variables and derived from $F_2$. Then $F_2$ is also monotone.
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Let $F_n = F_2 \circ \cdots \circ F_2$ be idempotent and monotone increasing, $n$-associative. Then $F_2$ is idempotent as well.
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Let $F_n = F_2 \circ \cdots \circ F_2$ be idempotent and monotone increasing, $n$-associative. Then $F_2$ is idempotent as well.

By a previous lemma, if $F_2$ is monotone, idempotent, associative, then $F_2$ is monotone increasing in each variable. Easily, $F_n$ is also monotone increasing in each variable.
Generalization of Czogala-Drewniak theorem

We denote $\min(a_1, \ldots, a_n)$ and $\max(a_1, \ldots, a_n)$ by $\min(a_1, \ldots, n)$ and $\max(a_1, \ldots, n)$, respectively.
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**Theorem**

*Let \( I \subseteq \mathbb{R} \) be an interval. Let \( F_n : I^n \to I \) be idempotent, \( n \)-associative, monotone in at least two variable and has a neutral element.*
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**Theorem**

Let $I \subseteq \mathbb{R}$ be an interval. Let $F_n : I^n \rightarrow I$ be idempotent, $n$-associative, monotone in at least two variable and has a neutral element. Then there exists monotone decreasing function $g$ such that $\Gamma_g$ is symmetric and for every $a_1, \ldots, a_n$ for which $g(a_i) \neq a_j$ ($\forall i \neq j$)
Generalization of Czogala-Drewniak theorem

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\[
F_n(a_1, \ldots, a_n) = \begin{cases} 
\min(a_1, \ldots, n), & \text{if } g(\max(a_1, \ldots, n)) > \min(a_1, \ldots, n) \\
\max(a_1, \ldots, n), & \text{if } g(\max(a_1, \ldots, n)) < \min(a_1, \ldots, n)
\end{cases}
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Characterization of idempotent, monotone increasing, $n$-ary semigroups with neutral elements
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Theorem

Let $I$ be as above. Let $F_n : I^n \rightarrow I$ be an idempotent $n$-ary semigroup, which is monotone increasing in each variable and has a neutral element iff

\[
\begin{align*}
\Gamma g &\text{ is symmetric and } \\
F_n(a_1, \ldots, a_n) &= \begin{cases} \\
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\max(a_1, \ldots, a_n), & \text{if } g(\max(a_1, \ldots, a_n)) < \min(a_1, \ldots, a_n) \\
\text{max or min}, & \text{if } g(\max(a_1, \ldots, a_n)) = \min(a_1, \ldots, a_n) \\
g(\min(a_1, \ldots, a_n)) &= \max(a_1, \ldots, a_n) \\
\end{cases}
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& \text{or } g(\min(a_1, \ldots, n)) > \max(a_1, \ldots, n) \\
\max \text{ or } \min, & \text{if } g(\max(a_1, \ldots, n)) = \min(a_1, \ldots, n) \\
& \text{and } g(\min(a_1, \ldots, n)) = \max(a_1, \ldots, n)
\end{cases}
\]
Proof of Idempotency: Backward induction

Lemma

Let $F_n = F_2 \circ \cdots \circ F_2$ be idempotent and monotone increasing, $n$-associative. Then $F_2$ is idempotent as well.
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$$
\begin{align*}
F_k(a, \ldots, a, a, b) & \quad F_k(a, \ldots, a, b, b) & \quad \cdots & \quad F_k(a, b, \ldots, b, b) \\
F_k(a, \ldots, a, a, a) & \quad F_k(a, \ldots, a, b, a) & \quad \cdots & \quad F_k(a, b, \ldots, b, a)
\end{align*}
$$
Lemma
Let $a$ and $b$ be as above. Further let $x_1 = \ldots = x_l = a$ and $x_{l+1} = \ldots = x_k = b$. Then for every $\pi \in \text{Sym}(k)$ we have

$$F_k(x_1, \ldots, x_k) = F_k(x_{\pi(1)}, \ldots, x_{\pi(k)}).$$

Lemma
Let $l$ and $m$ be fixed and $l + m = k$. Then for any $1 \leq m \leq k - 2$

$$F_k(a, \ldots, a, b, \ldots, b) = F_l(a, \ldots, a),$$

and $F_k(a, b, \ldots, b) = a$. 
Lemma
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and $F_k(a, b, \ldots, b) = a$. 

\[
\begin{array}{c|c|c|c|c}
  b & F_{k-1}(a, \ldots, a) & F_{k-2}(a, \ldots, a) & \ldots & a \\
  a & b & F_{k-1}(a, \ldots, a) & \ldots & F_2(a, a)
\end{array}
\]
Thank you for your kind attention!