Weakening the tight coupling between geometry and simulation in IGA

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Joint work with
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Recall

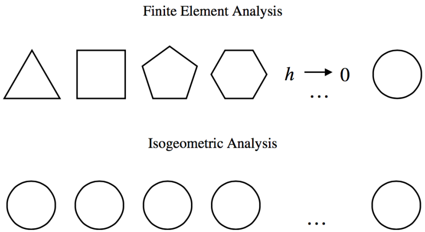

Fig. 2. "What is a circle?" In finite element analysis it is an idealization attained in the limit of mesh refinement but never for any finite mesh. In isogeometric analysis, the same exact geometry and parameterization are maintained for all meshes.

The main idea of isogeometric analysis (IGA)

J.A. Cottrell, A. Reali, Y. Bazilevs, T.J.R. Hughes. Isogeometric analysis of structural vibrations. *Comput. Methods Appl. Mech. Engrg.*, **195**, 5257-5296, 2006

Outline

1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

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Polynomial degree for the numerical solution?

If the analytical solution is expected to be sufficiently regular, the *p*- or *hp*- method can be employed (with $p_u > p_g$) to obtain higher accuracy

Different basis for geometry and numerical solution?

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Various splines basis in practice

B-Splines, NURBS, T-Splines,
LR-Splines, (truncated)Hierarchical B-Splines,
PHT-Splines, Generalized B-Splines, SubD, add your choice

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R. Sevilla, S. Fernandez Mendez, and A. Huerta. NURBS-enhanced finite element method (NEFEM). *Int. J. Numer. Meth. Engrg.*, **76**, 56-83, 2008.

B. Marussig, J. Zechner, G. Beer, T.P. Fries. Fast isogeometric boundary element method based on independent field approximation. *Comput. Methods Appl. Mech. Engrg.*, **284**, 458-488, 2015. (ECCOMAS 2014, arxiv/1406.3499)

Previous talk of S. Elgeti (Spline-based FEM for fluid flow)

Motivation summarized

Standard paradigm of IGA

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  - In shape/topology optimization, the constraint of using the same space is particularly undesirable.
  - Standard tools for the geometry/boundary but different (spline-)basis for solution (to exploit features like local refinement).

Outline

Some historical background of the patch test

I. Babuska and R. Narasimhan. The Babuska-Brezzi condition and the patch test: an example. *Comput. Methods Appl. Mech. Engrg.*, **140**, 183-199, 1997.

G.P. Bazeley, Y.K. Cheung, B.M. Irons, and O.C. Zienkiewicz. Triangular elements in plate bending - conforming and nonconforming solutions, in *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, Wright Patterson Air Force Base, Dayton, Ohio, 547-576, 1965.

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[redacted] Subsequently, the patch test has generated some mathematical controversy (see Stummel [65]) and undergone rumination (see Irons and Loikkanen [66] and Taylor et al. [67]). In addition, in the context of complicated theories, it is not always even clear how to pose patch tests. For these reasons faith in the patch test has eroded in some quarters. *This is unfortunate, for we firmly believe that, within the realm of problems dealt with so far in this book, the patch test is the most practically useful technique for assessing element behavior*. Thus we wish to avoid altogether the mathematically controversial facets of this subject and return to the spirit of Irons' original conception.

Original geometry parametrization of the domain

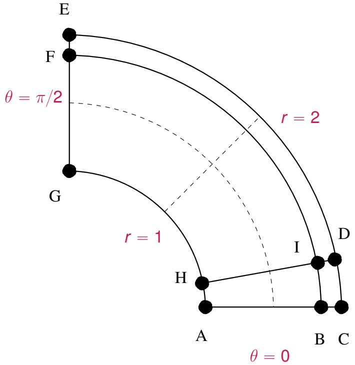
- The geometry is exactly represented by NURBS of degrees $1x2$
- Basic parametrization by one element, defined by 2 knot vectors

$$\Sigma = \{0, 0, 1, 1\}, \quad \Pi = \{0, 0, 0, 1, 1, 1\}.$$

- Together with NURBS basis, this is given by the following set of 6 control points, where the third value denotes the weight.

$$P[0,0] := \{1, 0, 1\}, \qquad P[1,0] := \{2, 0, 1\},$$
$$P[0,1] := \{1, 1, 1/\sqrt{2}\}, \qquad P[1,1] := \{2, 2, 1/\sqrt{2}\},$$
$$P[0,2] := \{0, 1, 1\}, \qquad P[1,2] := \{0, 2, 1\}.$$

Parametrization of the domain for the patch-test I



Quarter annulus region

- For patch-test in 2D, one-time *h*-refinement in both directions
- Consider the refined knot vectors

$$\Sigma = \{0, 0, s, 1, 1\}, \quad \Pi = \{0, 0, 0, t, 1, 1, 1\}.$$

Parametrization of the domain for the patch-test II

Shape A For non-uniform curvilinear elements, shift the points B, D, F, H and I. Set

$$t_1 := 1 - t + t/\sqrt{2}, \quad t_2 := t + \sqrt{2}t_1,$$

Updated set of control points in non-homogenized form

$$\{1, 0, 1\}, \quad \{1 + s, 0, 1\}, \quad \{2, 0, 1\}$$

$$\{1, \frac{t}{\sqrt{2}t_1}, t_1\}, \quad \{(1 + s), \frac{(1 + s)t}{\sqrt{2}t_1}, t_1\}, \quad \{2, \frac{\sqrt{2}t}{t_1}, t_1\}$$

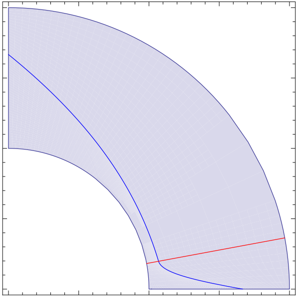
$$\{\frac{\sqrt{2}(1 - t)}{t_2}, 1, \frac{t_2}{2}\}, \{\frac{\sqrt{2}(1 + s)(1 - t)}{t_2}, (1 + s), \frac{t_2}{2}\}, \{\frac{2\sqrt{2}(1 - t)}{t_2}, 2, \frac{t_2}{2}\}$$

$$\{0, 1, 1\}, \quad \{0, 1 + s, 1\}, \quad \{0, 2, 1\}$$

Parametrization of the domain for the patch-test III

Shape B Add another parameter δ, two interior points changed as

$$\{\frac{(1+s)t_1}{t_1+\delta}, \frac{(1+s)t}{\sqrt{2}(t_1+\delta)}, t_1+\delta\}, \{\frac{\sqrt{2}(1+s)(1-t)}{t_2+2\delta}, \frac{(1+s)t_2}{t_2+2\delta}, \frac{t_2}{2}+\delta\}$$



Quarter annulus region with non-uniform elements,
$(s = 2/3, t = 1/8, \delta = 1/2)$

Various combinations of degrees and knots/weights

- $p_u = p_g$, and $\Sigma_u = \Sigma_g$ (isogeometric case)
- $p_u < p_g$, and $\Sigma_u = \Sigma_g$ (different end knots)
- $p_u > p_g$, and $\Sigma_u = \Sigma_g$ (different end knots)
- $p_u = p_g$, and $\Sigma_u \neq \Sigma_g$
- $p_u < p_g$, and $\Sigma_u \neq \Sigma_g$
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Outline

Problem setup

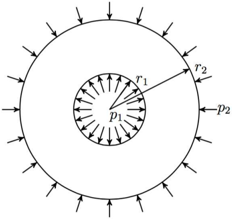

Figure 1: Pressurized cylinder.

Problem domain

- L^2 error in the solution

$$u_r = \frac{1+\nu}{E}\left(-\frac{r_1^2 r_2^2 (p_2 - p_1)}{r(r_2^2 - r_1^2)} + (1-2\nu) r \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}\right)$$

Patch tests

- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

PT1 Shape A, $\Sigma_u = \Sigma_g$, 0.17 and 0.25 interior knots pt

PT2 Shape A, $\Sigma_u \neq \Sigma_g$. Σ_g has 0.17 and 0.25 interior knot pt, and Σ_u has 0.35 and 0.81 interior knot pt

PT3 Same as PT1, NURBS for geo, and B-Splines for solution

PT4 Shape B, $\Sigma_u \neq \Sigma_g$

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

Test	$p_u = p_g$	$p_u < p_g$	$p_u > p_g$

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Test	$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
PT1	2.34666e-15	4.46571e-15	1.59281e-13

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Test	$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
PT1	2.34666e-15	4.46571e-15	1.59281e-13
PT2	4.04412e-14	1.64516e-15	2.14010e-15

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PT3	2.47975e-15	1.14036e-14	8.12925e-15

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PT4	0.00212	$\star$	$\star$

Numerical setup

Example 1: Quarter annulus domain.

- **Case A1** Similar elements, both geo and solution using NURBS, $\Sigma_u = \Sigma_g$ (except end knots for $p_u \neq p_g$)
- **Case A2** Same as A1 except $\Sigma_u \neq \Sigma_g$
- **Case B1** Similar elements, geo using NURBS, and solution using B-Splines, $\Sigma_u = \Sigma_g$ (except weights, and end knots for $p_u \neq p_g$)
- **Case B2** Same as B1 except $\Sigma_u \neq \Sigma_g$
- **Case C1** Nonuniform elements (with parameter δ), both geo and solution using NURBS, $\Sigma_u = \Sigma_g$, $\delta_u = \delta_g$ (except end knots for $p_u \neq p_g$)
- **Case C2** Same as C1 except $\delta_u \neq \delta_g$
- **Case C3** Same as C1 except $\Sigma_u \neq \Sigma_g$, and $\delta_u \neq \delta_g$

Convergence: Example 1

L^2 error in the solution

Case A1			Case A2		
$p_u = p_g$	$p_u < p_g$	$p_u > p_g$	$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
0.004332	0.004332	0.000440	0.002128	0.002128	0.000145
3.768	3.768	8.664	3.877	3.877	8.807
3.936	3.936	8.704	3.968	3.968	8.670
3.983	3.983	8.436	3.992	3.992	8.425
3.996	3.996	8.231	3.998	3.998	8.237

Convergence: Example 1

L^2 error in the solution

Case B1			Case B2		
$p_u = p_g$	$p_u < p_g$	$p_u > p_g$	$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
0.004601	0.004601	0.000472	0.002701	0.002701	0.000249
3.953	3.953	8.666	4.674	4.674	8.700
3.974	3.974	9.163	4.129	4.129	12.689
3.992	3.992	8.546	4.029	4.029	9.587
3.998	3.998	8.257	4.007	4.007	8.527

Convergence: Example 1

L^2 error in the solution

Case C1			Case C2		
$p_u = p_g$	$p_u < p_g$	$p_u > p_g$	$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
0.004584	0.004584	0.000274	0.003676	0.003676	0.000241
3.782	3.782	10.254	3.810	3.810	9.885
3.931	3.931	8.659	3.935	3.935	8.548
3.984	3.984	8.118	3.984	3.984	8.100
3.996	3.996	8.028	3.996	3.996	8.024

Convergence: Example 1

L^2 error in the solution

Case C3		
$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
0.008824	0.008824	0.003984
3.779	3.779	1.891
3.944	3.944	2.018
3.983	3.983	2.013
3.995	3.995	2.005
4.000	4.000	2.002

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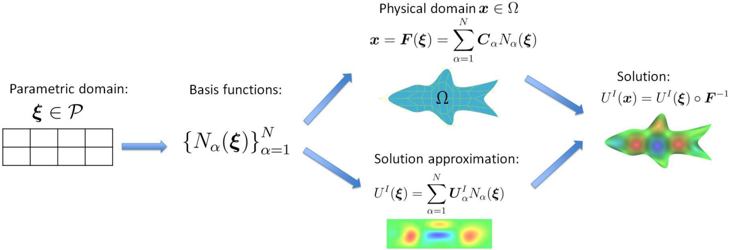
Approaches of IGA and GIA



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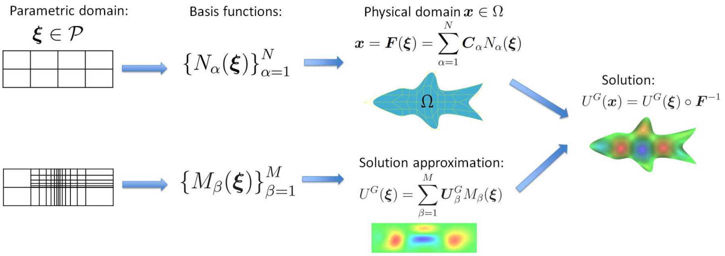

The main idea of geometry-induced analysis (GIA)

G. Beer, B. Marussig, J. Zechner, C. Dünser, T.P. Fries. Boundary Element Analysis with trimmed NURBS and a generalized IGA approach. *http://arxiv.org/abs/1406.3499*, 2014.

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Naming convention

Sub-parametric interpolation: The order of the interpolation for $\mathbf{x}$ is lower than that for ϕ.

Isoparametric interpolation: The order of the interpolation for $\mathbf{x}$ is the same as that for ϕ.

Super-parametric interpolation: The order of the interpolation for $\mathbf{x}$ is higher than that for ϕ.

In developing solutions to C_0 problems one may use either "sub-parametric" or "isoparametric" interpolations since either ensures that the polynomials 1, x, y and for three dimensions z are always available, thus ensuring that constant derivatives can be computed. On the other hand use of "super-parametric" interpolation should generally be avoided.

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$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
Iso-parametric	Super-parametric	Sub-parametric
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One message

- Without any fancy/weird elements, various combinations of different basis and polynomial degrees pass the test, and can be used in practice