Generalizing the isogeometric concept: weakening the tight coupling between geometry and simulation in IGA

Satyendra Tomar
University of Luxembourg

Joint work with
E. Atroshchenko, S. Bordas and G. Xu

Financial support from:
FP7-PEOPLE-2011-ITN (289361)
F1R-ING-PEU-14RLTC

There will be a lot of space for discussions, between mathematicians and engineers, between experts on finite element and on isogeometric analysis and between junior and senior scientists. Therefore we will not have parallel sessions, and we also strongly encourage oral presentations and posters which not only present latest results, but also raise open questions and identify new challenges.

There will be a lot of space for discussions, between mathematicians and engineers, between experts on finite element and on isogeometric analysis and between junior and senior scientists. Therefore we will not have parallel sessions, and we also strongly encourage oral presentations and posters which not only present latest results, but also raise open questions and identify new challenges.

- Focus on presenting latest results, raising some open questions, and identifying some new challenges

Outline

1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

Outline

1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

Outline

1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

Outline

1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

Outline

1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

Frequently encountered geometries in numerics

Frequently encountered geometries in numerics

- Straight-sided polygonal domains (including L-shaped)

Frequently encountered geometries in numerics

- Straight-sided polygonal domains (including L-shaped)
- Curved boundaries, typically, circular and elliptical shapes

Frequently encountered geometries in numerics

- Straight-sided polygonal domains (including L-shaped)
- Curved boundaries, typically, circular and elliptical shapes

Polynomial degree required to represent such a geometry?

Frequently encountered geometries in numerics

- Straight-sided polygonal domains (including L-shaped)
- Curved boundaries, typically, circular and elliptical shapes

Polynomial degree required to represent such a geometry?

Typically, $p_g = 1, 2, 3$!!

Frequently encountered geometries in numerics

- Straight-sided polygonal domains (including L-shaped)
- Curved boundaries, typically, circular and elliptical shapes

Polynomial degree required to represent such a geometry?

Typically, $p_g = 1, 2, \ldots 5 \ldots 20$!!

HOFEIM ☺

Polynomial degree for the numerical solution?

Polynomial degree for the numerical solution?

If the analytical solution is expected to be sufficiently regular, the *p*- or *hp*- method can be employed (with $p_u > p_g$) to obtain higher accuracy

Different basis for geometry and numerical solution?

Different basis for geometry and numerical solution?

Various splines basis in practice

B-Splines, NURBS, T-Splines,
LR-Splines, (truncated)Hierarchical B-Splines,
PHT-Splines, Generalized B-Splines, add your choice

Different basis for geometry and numerical solution?

Various splines basis in practice

B-Splines, NURBS, T-Splines,
LR-Splines, (truncated)Hierarchical B-Splines,
PHT-Splines, Generalized B-Splines, add your choice

Combining various basis

Different basis for geometry and numerical solution?

Various splines basis in practice

B-Splines, NURBS, T-Splines,
LR-Splines, (truncated)Hierarchical B-Splines,
PHT-Splines, Generalized B-Splines, add your choice

Combining various basis

Geo NURBS, T-Splines

Different basis for geometry and numerical solution?

Various splines basis in practice

B-Splines, NURBS, T-Splines,
LR-Splines, (truncated)Hierarchical B-Splines,
PHT-Splines, Generalized B-Splines, add your choice

Combining various basis

Geo NURBS, T-Splines

Solution B-Splines, LR-Splines, (truncated)Hierarchical B-Splines, PHT-Splines, Generalized B-Splines, add your choice

Different basis for geometry and numerical solution?

Various splines basis in practice

B-Splines, NURBS, T-Splines,
LR-Splines, (truncated)Hierarchical B-Splines,
PHT-Splines, Generalized B-Splines, add your choice

Combining various basis

Geo	NURBS, T-Splines
Solution	B-Splines, LR-Splines, (truncated)Hierarchical B-Splines, PHT-Splines, Generalized B-Splines, add your choice

R. Sevilla, S. Fernandez Mendez, and A. Huerta. NURBS-enhanced finite element method (NEFEM). *Int. J. Numer. Meth. Engrg.*, **76**, 56-83, 2008.

B. Marussig, J. Zechner, G. Beer, T.P. Fries. Fast isogeometric boundary element method based on independent field approximation. *Comput. Methods Appl. Mech. Engrg.*, **284**, 458-488, 2015. (ECCOMAS 2014, arxiv/1406.3499)

Talk of S. Elgeti on Monday, 2016.05.30 (Spline-based FEM for fluid flow on deforming domains)

Recall

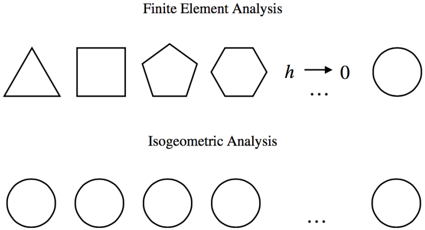


Fig. 2. "What is a circle?" In finite element analysis it is an idealization attained in the limit of mesh refinement but never for any finite mesh. In isogeometric analysis, the same exact geometry and parameterization are maintained for all meshes.

The main idea of isogeometric analysis (IGA)

J.A. Cottrell, A. Reali, Y. Bazilevs, T.J.R. Hughes. Isogeometric analysis of structural vibrations. *Comput. Methods Appl. Mech. Engrg.*, **195**, 5257-5296, 2006

Motivation summarized

Standard paradigm of IGA

Motivation summarized

Standard paradigm of IGA

- Geometry and simulation spaces are tightly integrated, i.e. same space for geometry and numerical solution

Motivation summarized

Standard paradigm of IGA

- Geometry and simulation spaces are tightly integrated, i.e. same space for geometry and numerical solution
- Situations where this tight integration can be *relaxed* for *improved solution quality*

Motivation summarized

Standard paradigm of IGA

- Geometry and simulation spaces are tightly integrated, i.e. same space for geometry and numerical solution
- Situations where this tight integration can be *relaxed* for *improved solution quality*
  - Geometry of the domain is simple enough to be represented by low order NURBS, but the solution is sufficiently regular. Higher order approximation delivers superior results.

Motivation summarized

Standard paradigm of IGA

- Geometry and simulation spaces are tightly integrated, i.e. same space for geometry and numerical solution
- Situations where this tight integration can be *relaxed* for *improved solution quality*
  - Geometry of the domain is simple enough to be represented by low order NURBS, but the solution is sufficiently regular. Higher order approximation delivers superior results.
  - Solution has low regularity (e.g. corner singularity) but the curved boundary can be represented by higher order NURBS.

Motivation summarized

Standard paradigm of IGA

- Geometry and simulation spaces are tightly integrated, i.e. same space for geometry and numerical solution
- Situations where this tight integration can be *relaxed* for *improved solution quality*
  - Geometry of the domain is simple enough to be represented by low order NURBS, but the solution is sufficiently regular. Higher order approximation delivers superior results.
  - Solution has low regularity (e.g. corner singularity) but the curved boundary can be represented by higher order NURBS.
  - In shape/topology optimization, the constraint of using the same space is particularly undesirable.

Motivation summarized

Standard paradigm of IGA

- Geometry and simulation spaces are tightly integrated, i.e. same space for geometry and numerical solution
- Situations where this tight integration can be *relaxed* for *improved solution quality*
  - Geometry of the domain is simple enough to be represented by low order NURBS, but the solution is sufficiently regular. Higher order approximation delivers superior results.
  - Solution has low regularity (e.g. corner singularity) but the curved boundary can be represented by higher order NURBS.
  - In shape/topology optimization, the constraint of using the same space is particularly undesirable.
  - Standard tools for the geometry/boundary but different (spline-)basis for solution (to exploit features like local refinement).

Outline

Some historical background of the patch test

I. Babuska and R. Narasimhan. The Babuska-Brezzi condition and the patch test: an example. *Comput. Methods Appl. Mech. Engrg.*, **140**, 183-199, 1997.

G.P. Bazeley, Y.K. Cheung, B.M. Irons, and O.C. Zienkiewicz. Triangular elements in plate bending - conforming and nonconforming solutions, in *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, Wright Patterson Air Force Base, Dayton, Ohio, 547-576, 1965.

F. Stummel. The generalized patch test. *SIAM J. Numer. Anal.*, **16(3)**, 449-471, 1979.

M. Wang. On the necessity and sufficiency of the patch test for convergence of nonconforming finite elements. *SIAM J. Numer. Anal.*, **39(2)**, 363-384, 2001.

O.C. Zienkiewicz, and R.L. Taylor. The finite element patch test revisited: A computer test for convergence, validation and error estimates. *Comput. Methods Appl. Mech. Engrg.*, **149**, 223-254, 1997.

T.J.R. Hughes. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Prentice-Hall Inc., 1987.

Some historical background of the patch test

I. Babuska and R. Narasimhan. The Babuska-Brezzi condition and the patch test: an example. *Comput. Methods Appl. Mech. Engrg.*, **140**, 183-199, 1997.

G.P. Bazeley, Y.K. Cheung, B.M. Irons, and O.C. Zienkiewicz. Triangular elements in plate bending - conforming and nonconforming solutions, in *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, Wright Patterson Air Force Base, Dayton, Ohio, 547-576, 1965.

F. Stummel. The generalized patch test. *SIAM J. Numer. Anal.*, **16(3)**, 449-471, 1979.

M. Wang. On the necessity and sufficiency of the patch test for convergence of nonconforming finite elements. *SIAM J. Numer. Anal.*, **39(2)**, 363-384, 2001.

O.C. Zienkiewicz, and R.L. Taylor. The finite element patch test revisited: A computer test for convergence, validation and error estimates. *Comput. Methods Appl. Mech. Engrg.*, **149**, 223-254, 1997.

T.J.R. Hughes. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis.* Prentice-Hall Inc., 1987.

[redacted] Subsequently, the patch test has generated some mathematical controversy (see Stummel [65]) and undergone rumination (see Irons and Loikkanen [66] and Taylor et al. [67]). In addition, in the context of complicated theories, it is not always even clear how to pose patch tests. For these reasons faith in the patch test has eroded in some quarters. *This is unfortunate, for we firmly believe that, within the realm of problems dealt with so far in this book, the patch test is the most practically useful technique for assessing element behavior*. Thus we wish to avoid altogether the mathematically controversial facets of this subject and return to the spirit of Irons' original conception.

Original geometry parametrization of the domain

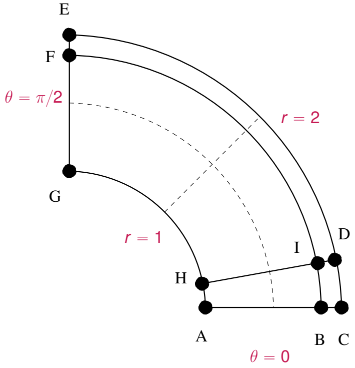
- The geometry is exactly represented by NURBS of degrees $1x2$
- Basic parametrization by one element, defined by 2 knot vectors

$$\Sigma = \{0, 0, 1, 1\}, \quad \Pi = \{0, 0, 0, 1, 1, 1\}.$$

- Together with NURBS basis, this is given by the following set of 6 control points, where the third value denotes the weight.

$$P[0,0] := \{1, 0, 1\}, \qquad P[1,0] := \{2, 0, 1\},$$
$$P[0,1] := \{1, 1, 1/\sqrt{2}\}, \qquad P[1,1] := \{2, 2, 1/\sqrt{2}\},$$
$$P[0,2] := \{0, 1, 1\}, \qquad P[1,2] := \{0, 2, 1\}.$$

Parametrization of the domain for the patch-test I



Quarter annulus region

- For patch-test in 2D, one-time h-refinement in both directions
- Consider the refined knot vectors

$$\Sigma = \{0, 0, s, 1, 1\}, \quad \Pi = \{0, 0, 0, t, 1, 1, 1\}.$$

Parametrization of the domain for the patch-test II

Shape A Uniform curvilinear elements $s = t = 1/2$

Shape B For non-uniform curvilinear elements, shift the points B, D, F, H and I. Set

$$t_1 := 1 - t + t/\sqrt{2}, \quad t_2 := t + \sqrt{2}t_1,$$

Updated set of control points in non-homogenized form

$$\{1, 0, 1\}, \quad \{1 + s, 0, 1\}, \quad \{2, 0, 1\}$$

$$\{1, \frac{t}{\sqrt{2}t_1}, t_1\}, \quad \{(1 + s), \frac{(1 + s)t}{\sqrt{2}t_1}, t_1\}, \quad \{2, \frac{\sqrt{2}t}{t_1}, t_1\}$$

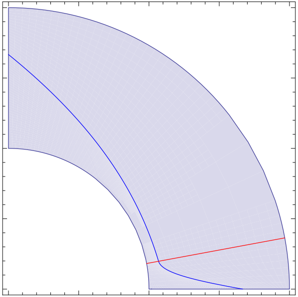
$$\{\frac{\sqrt{2}(1 - t)}{t_2}, 1, \frac{t_2}{2}\}, \{\frac{\sqrt{2}(1 + s)(1 - t)}{t_2}, (1 + s), \frac{t_2}{2}\}, \{\frac{2\sqrt{2}(1 - t)}{t_2}, 2, \frac{t_2}{2}\}$$

$$\{0, 1, 1\}, \quad \{0, 1 + s, 1\}, \quad \{0, 2, 1\}$$

Parametrization of the domain for the patch-test III

Shape C Add another parameter δ, two interior points changed as

$$\{\frac{(1+s)t_1}{t_1+\delta}, \frac{(1+s)t}{\sqrt{2}(t_1+\delta)}, t_1+\delta\}, \{\frac{\sqrt{2}(1+s)(1-t)}{t_2+2\delta}, \frac{(1+s)t_2}{t_2+2\delta}, \frac{t_2}{2}+\delta\}$$



Quarter annulus region with non-uniform elements,
$(s = 2/3, t = 1/8, \delta = 1/2)$

Various combinations of degrees and knots/weights

- $p_u = p_g$, and $\Sigma_u = \Sigma_g$ (isogeometric case)
- $p_u = p_g$, and $\Sigma_u \neq \Sigma_g$
- $p_u < p_g$, and $\Sigma_u = \Sigma_g$
- $p_u < p_g$, and $\Sigma_u \neq \Sigma_g$
- $p_u > p_g$, and $\Sigma_u = \Sigma_g$
- $p_u > p_g$, and $\Sigma_u \neq \Sigma_g$

Various combinations of degrees and knots/weights

- $p_u = p_g$, and $\Sigma_u = \Sigma_g$ (isogeometric case)
- $p_u = p_g$, and $\Sigma_u \neq \Sigma_g$
- $p_u < p_g$, and $\Sigma_u = \Sigma_g$
- $p_u < p_g$, and $\Sigma_u \neq \Sigma_g$
- $p_u > p_g$, and $\Sigma_u = \Sigma_g$
- $p_u > p_g$, and $\Sigma_u \neq \Sigma_g$

Total number of cases

- 3 choices of element shapes, and 6 choices of degrees/knots, total of 18 cases !!

Outline

1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

p and Σ	Shape A	Shape B	Shape C
$p_u = p_g$, $\Sigma_u = \Sigma_g$	8.2469e-016	8.5695e-016	1.5017e-015

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

p and Σ	Shape A	Shape B	Shape C
$p_u = p_g$, $\Sigma_u = \Sigma_g$	8.2469e-016	8.5695e-016	1.5017e-015
$p_u < p_g$, $\Sigma_u = \Sigma_g$	8.5489e-016	8.8080e-016	1.5172e-015

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

p and Σ	Shape A	Shape B	Shape C
$p_u = p_g$, $\Sigma_u = \Sigma_g$	8.2469e-016	8.5695e-016	1.5017e-015
$p_u < p_g$, $\Sigma_u = \Sigma_g$	8.5489e-016	8.8080e-016	1.5172e-015
$p_u > p_g$, $\Sigma_u = \Sigma_g$	1.1979e-015	1.8185e-015	3.6019e-015

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

p and Σ	Shape A	Shape B	Shape C
$p_u = p_g$, $\Sigma_u = \Sigma_g$	8.2469e-016	8.5695e-016	1.5017e-015
$p_u < p_g$, $\Sigma_u = \Sigma_g$	8.5489e-016	8.8080e-016	1.5172e-015
$p_u > p_g$, $\Sigma_u = \Sigma_g$	1.1979e-015	1.8185e-015	3.6019e-015
$p_u < p_g$, $\Sigma_u \neq \Sigma_g$	1.1401e-015	1.0907e-015	0.20155

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

p and Σ	Shape A	Shape B	Shape C
$p_u = p_g$, $\Sigma_u = \Sigma_g$	8.2469e-016	8.5695e-016	1.5017e-015
$p_u < p_g$, $\Sigma_u = \Sigma_g$	8.5489e-016	8.8080e-016	1.5172e-015
$p_u > p_g$, $\Sigma_u = \Sigma_g$	1.1979e-015	1.8185e-015	3.6019e-015
$p_u < p_g$, $\Sigma_u \neq \Sigma_g$	1.1401e-015	1.0907e-015	0.20155
$p_u > p_g$, $\Sigma_u \neq \Sigma_g$	1.5010e-015	1.4193e-015	0.03019

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

p and Σ	Shape A	Shape B	Shape C
$p_u = p_g$, $\Sigma_u = \Sigma_g$	8.2469e-016	8.5695e-016	1.5017e-015
$p_u < p_g$, $\Sigma_u = \Sigma_g$	8.5489e-016	8.8080e-016	1.5172e-015
$p_u > p_g$, $\Sigma_u = \Sigma_g$	1.1979e-015	1.8185e-015	3.6019e-015
$p_u < p_g$, $\Sigma_u \neq \Sigma_g$	1.1401e-015	1.0907e-015	0.20155
$p_u > p_g$, $\Sigma_u \neq \Sigma_g$	1.5010e-015	1.4193e-015	0.03019
$p_u = p_g$, $\Sigma_u \neq \Sigma_g$	7.4477e-016	1.0611e-015	0.20155

Results for various choices of $p_u, p_g, \Sigma_u, \Sigma_g$

- L^2 error in the solution $u = 1 + x + y$
- $p_u = p_g$, i.e. $p_u = p_g = 1x2$
- $p_u < p_g$, i.e. $p_u = 1x2$, $p_g = 2x3$
- $p_u > p_g$, i.e. $p_u = 2x3$, $p_g = 1x2$

p and Σ	Shape A	Shape B	Shape C
$p_u = p_g, \Sigma_u = \Sigma_g$	8.2469e-016	8.5695e-016	1.5017e-015
$p_u < p_g, \Sigma_u = \Sigma_g$	8.5489e-016	8.8080e-016	1.5172e-015
$p_u > p_g, \Sigma_u = \Sigma_g$	1.1979e-015	1.8185e-015	3.6019e-015
$p_u < p_g, \Sigma_u \neq \Sigma_g$	1.1401e-015	1.0907e-015	0.20155
$p_u > p_g, \Sigma_u \neq \Sigma_g$	1.5010e-015	1.4193e-015	0.03019
$p_u = p_g, \Sigma_u \neq \Sigma_g$	7.4477e-016	1.0611e-015	0.20155

- What is expected for $u = 1 + x^2 + y^2$?

Numerical setup

Analytic solution with Dirichlet BC

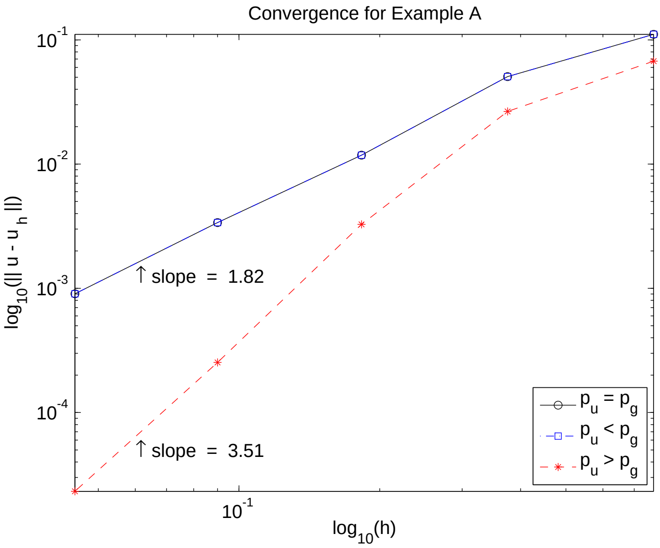
$u = \log(\sqrt{((x-0.1)*(x-0.1)+(y-0.1)*(y-0.1))})$

Example A Same knot vectors, same starting approximation for solution as geo (using NURBS)

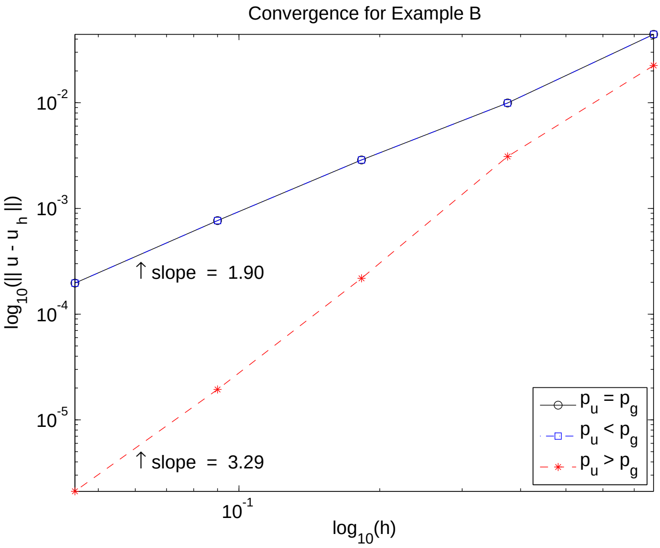
Example B Same geo, but different knot vectors for geo and solution (still using NURBS), which also represents geo

Example C Same knot vectors for geo and field, geo using NURBS, and solution using B-Splines

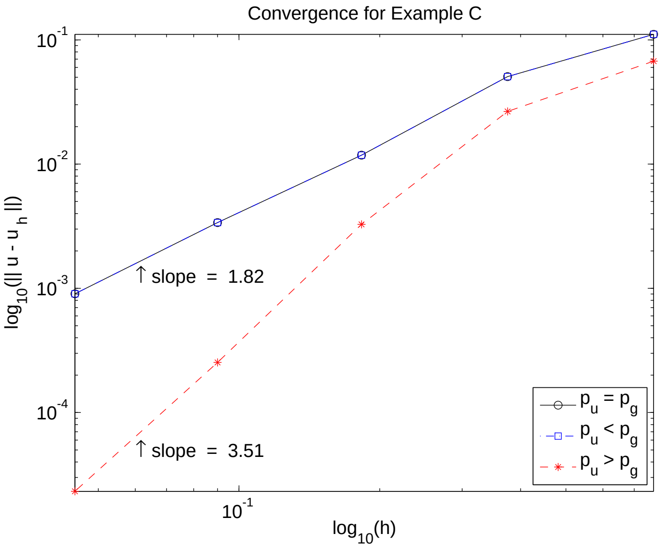
Convergence



Convergence



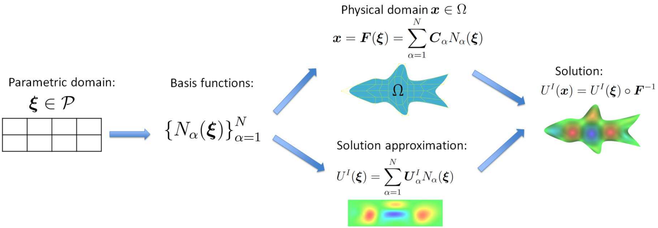
Convergence



Outline

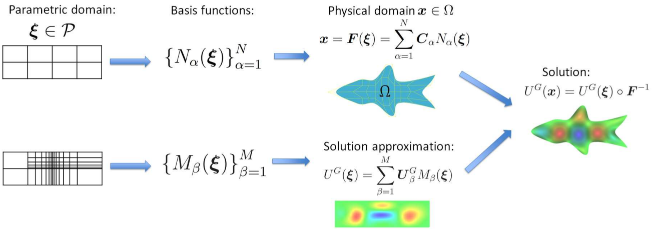
1. Motivation
   - Different degrees for geometry and solution
   - Different basis for geometry and solution
2. Patch tests
   - Various partitioning of the domain
   - Various combinations of degrees and knots/weights
3. Some numerical results
   - Patch test results
   - Convergence results
4. Conclusions

Approaches of IGA and GIA



The main idea of isogeometric analysis (IGA)

Approaches of IGA and GIA



The main idea of Geometry-Independent Field approximaTion (GIFT)

G. Beer, B. Marussig, J. Zechner, C. Dünser, T.P. Fries. Boundary Element Analysis with trimmed NURBS and a generalized IGA approach. *http://arxiv.org/abs/1406.3499*, 2014.

B. Marussig, J. Zechner, G. Beer, T.P. Fries. Fast isogeometric boundary element method based on independent field approximation. *Comput. Methods Appl. Mech. Engrg.*, **284**, 458-488, 2015.

Approaches of IGA and GIA

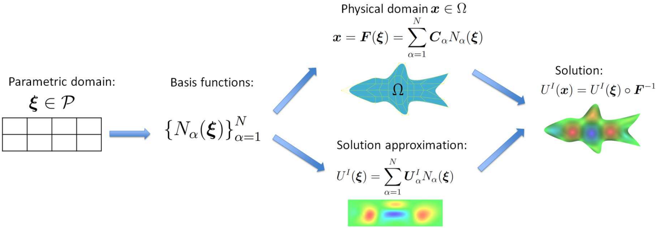
School of Athens, from the Stanza della Segnatura, 1510-11 (fresco), Raphael (Raffaello Sanzio of Urbino) (1483-1520)/Vatican Museums and Galleries, Vatican City, Italy/Giraudon/The Bridgeman Art Library. Legend has it that over the door to Plato's Academy in Athens there was an inscription "Let no man ignorant of geometry enter here." Words to live by, in antiquity and today.

Approaches of IGA and GIA

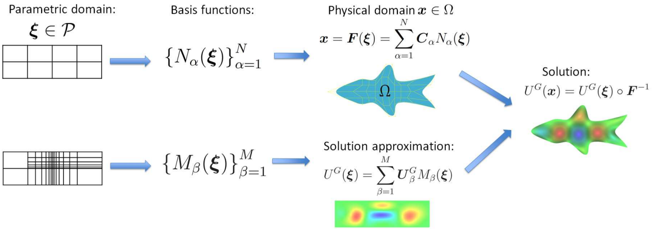
School of Athens, from the Stanza della Segnatura, 1510-11 (fresco), Raphael (Raffaello Sanzio of Urbino) (1483-1520)/Vatican Museums and Galleries, Vatican City, Italy/Giraudon/The Bridgeman Art Library. Legend has it that over the door to Plato's Academy in Athens there was an inscription "Let no man ignorant of geometry enter here." Words to live by, in antiquity and today.

J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA*, Wiley, 2009.

Approaches of IGA and GIA



The main idea of isogeometric analysis (IGA)



The main idea of geometry-induced analysis (GIA)

Naming convention

Sub-parametric interpolation: The order of the interpolation for $\mathbf{x}$ is lower than that for ϕ.

Isoparametric interpolation: The order of the interpolation for $\mathbf{x}$ is the same as that for ϕ.

Super-parametric interpolation: The order of the interpolation for $\mathbf{x}$ is higher than that for ϕ.

In developing solutions to C_0 problems one may use either "sub-parametric" or "isoparametric" interpolations since either ensures that the polynomials 1, x, y and for three dimensions z are always available, thus ensuring that constant derivatives can be computed. On the other hand use of "super-parametric" interpolation should generally be avoided.

O.C. Zienkiewicz, R.L. Taylor, and J.Z. Zhu. *The Finite Element Method: Its Basis and Fundamentals*. Elsevier, 2013.

Naming convention

Sub-parametric interpolation:	The order of the interpolation for $\mathbf{x}$ is lower than that for ϕ.
Isoparametric interpolation:	The order of the interpolation for $\mathbf{x}$ is the same as that for ϕ.
Super-parametric interpolation:	The order of the interpolation for $\mathbf{x}$ is higher than that for ϕ.

In developing solutions to C_0 problems one may use either "sub-parametric" or "isoparametric" interpolations since either ensures that the polynomials $1, x, y$ and for three dimensions z are always available, thus ensuring that constant derivatives can be computed. On the other hand use of "super-parametric" interpolation should generally be avoided.

O.C. Zienkiewicz, R.L. Taylor, and J.Z. Zhu. *The Finite Element Method: Its Basis and Fundamentals*. Elsevier, 2013.

$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
Iso-parametric	Super-parametric	Sub-parametric

Naming convention

Sub-parametric interpolation: The order of the interpolation for **x** is lower than that for ϕ.

Isoparametric interpolation: The order of the interpolation for **x** is the same as that for ϕ.

Super-parametric interpolation: The order of the interpolation for **x** is higher than that for ϕ.

In developing solutions to C_0 problems one may use either "sub-parametric" or "isoparametric" interpolations since either ensures that the polynomials 1, x, y and for three dimensions z are always available, thus ensuring that constant derivatives can be computed. On the other hand use of "super-parametric" interpolation should generally be avoided.

O.C. Zienkiewicz, R.L. Taylor, and J.Z. Zhu. *The Finite Element Method: Its Basis and Fundamentals*. Elsevier, 2013.

$p_u = p_g$	$p_u < p_g$	$p_u > p_g$
Iso-parametric Iso-geometric	Super-parametric Sub-geometric	Sub-parametric Super-geometric

Concluding remarks

- When knot data are same (same representation/basis)

Concluding remarks

- When knot data are same (same representation/basis)
  - various combinations of polynomial degrees pass the test for all kind of parametric elements

Concluding remarks

- When knot data are same (same representation/basis)
  - various combinations of polynomial degrees pass the test for all kind of parametric elements
- When knot data are different (allowing different basis)

Concluding remarks

- When knot data are same (same representation/basis)
  - various combinations of polynomial degrees pass the test for all kind of parametric elements
- When knot data are different (allowing different basis)
  - various combinations of polynomial degrees pass the test on rectangular elements in parametric domain

Concluding remarks

- When knot data are same (same representation/basis)
  - various combinations of polynomial degrees pass the test for all kind of parametric elements
- When knot data are different (allowing different basis)
  - various combinations of polynomial degrees pass the test on rectangular elements in parametric domain

One message

- Without any fancy/weird parametric elements, various combinations of different basis and polynomial degrees pass the test, and can be used in practice