Automates, mots et décision

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What is the common point among...
What is the common point among...
What is the common point among...
What is the common point among...
Notation

\[ n\text{-tuples } x \text{ in } X^n \equiv \text{n-strings over } X \]

0-string: \( \varepsilon \),
1-strings: \( x, y, z, \ldots \)
n-strings: \( x, y, z, \ldots \)

\[ |x| = \text{length of } x \]

\[ X^* := \bigcup_{n \geq 0} X^n \]

We endow \( X^* \) with concatenation
Notation

Any $F : X^* \to Y$ is called a *variadic function*, and we set

$$F_n := F|_{X^n}.$$ 

Any $F : X^* \to X \cup \{\varepsilon\}$ is a *variadic operation*.

We assume

$$F(x) = \varepsilon \iff x = \varepsilon.$$
Associativity for string functions

Definition. $F : X^* \rightarrow X^*$ is associative if

$$F(xyz) = F(xF(y)z) \quad \forall \ xyz \in X^*$$
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Examples.

- sorting in alphabetical order
- letter removing, duplicate removing
Associativity entails ‘distributivity’

\[ F(xyz) = F(xF(y)z) \quad \forall \ xyz \in X^* \]

**Example.** \( F = \text{sort}() \)

**Input:** \textit{xzu} \ldots \textit{in blocks of unknown length given at unknown time intervals.}

**Output:** \textit{sort(xzu} \ldots \textit{)}
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“Highly” distributed algorithms
Associativity for variadic functions?

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**Quest:** a notion of ‘associativity’ for variadic \( F : X^* \to Y \)
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**Definition.** We say that \( F : X^* \rightarrow Y \) is *preassociative* if

\[ F(y) = F(y') \Rightarrow F(xyz) = F(xy'z) \]
Associativity for variadic functions?

\[ F(\text{xyz}) = F(\text{x}F(\text{y})\text{z}) \quad \forall \text{xyz} \in X^* \]

**Quest:** a notion of ‘associativity’ for variadic \( F : X^* \rightarrow Y \)

**Definition.** We say that \( F : X^* \rightarrow Y \) is **preassociative** if

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**Examples.**
- \( F_n(x) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R}) \)
- \( F_n(x) = |x| \quad (X \text{ arbitrary, } Y = \mathbb{N}) \)
Associativity and preassociativity

\[ F(y) = F(y') \implies F(xyz) = F(xy'z) \]

**Proposition.** Let \( F : X^* \to X^* \).

\[
\begin{align*}
F \text{ is associative} & \iff \\ F \text{ is preassociative and } F \circ F &= F.
\end{align*}
\]

**Slogan.** Preassociativity is a *composition-free* version of associativity.
A semiautomaton over $X$:

$\mathcal{A} = (Q, q_0, \delta)$

where $q_0 \in Q$ is the initial state and

$\delta : Q \times X \to Q$

is the transition function.

The map $\delta$ is extended to $Q \times X^*$ by

$\delta(q, \varepsilon) := q,$

$\delta(q, xy) := \delta(\delta(q, x), y)$
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**Definition.** $F_\mathcal{A} : X^* \to Q$ is defined by

$$F_\mathcal{A}(x) := \delta(q_0, x)$$
Preassociativity and semiautomata

\[ F_\mathcal{A}(x) := \delta(q_0, x) \]

**Fact.** If \( \mathcal{A} \) is a semiautomaton,

- \( F_\mathcal{A} \) is “half”-preassociative:
  \[ F_\mathcal{A}(y) = F_\mathcal{A}(y') \implies F_\mathcal{A}(y'z) = F_\mathcal{A}(y'z) \]

- \( F_\mathcal{A} \) may not be preassociative:

\[
\begin{align*}
F_\mathcal{A}(b) &= q_1 = F_\mathcal{A}(ba) \\
F_\mathcal{A}(bb) &= q_2 \neq q_0 &= F_\mathcal{A}(bba)
\end{align*}
\]
Preassociativity and semiautomata

**Definition.** A semiautomaton is *preassociative* if it satisfies

\[ \delta(q_0, x) = \delta(q_0, y) \implies \delta(q_0, zx) = \delta(q_0, zy) \]
Preassociativity and semiautomata

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**Lemma.**

\[ A \text{ preassociative} \iff F_A \text{ preassociative} \]
Preassociativity and semiautomata

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**Lemma.**

\[ \mathcal{A} \text{ preassociative} \iff F_\mathcal{A} \text{ preassociative} \]

**Example.** \( X = \{0, 1\} \)

\[ F_\mathcal{A}(x) = e \iff \#\{ i \mid x_i = 1 \} \text{ is even,} \]

\[ F_\mathcal{A}(x) = o \iff \#\{ i \mid x_i = 1 \} \text{ is odd.} \]
Preassociativity and semiautomata

$X$, $Q$ finite.

**Definition.** For an onto $F : X^* \rightarrow Q$, set

$q_0 := F(\varepsilon)$,

$\delta(q, z) := \{ F(xz) \mid q = F(x) \}$,

$\mathcal{A}^F := (Q, q_0, \delta)$

Generally, $\mathcal{A}^F$ is a non-deterministic semiautomaton.
Preassociativity and semiautomata

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**Definition.** For an onto $F : X^* \to Q$, set

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Generally, $\mathcal{A}^F$ is a non-deterministic semiautomaton.

**Lemma.**

$F$ is preassociative $\iff \mathcal{A}^F$ is deterministic and preassociative
A criterion for preassociativity

\[ F \text{ is preassociative } \iff \mathcal{A}^F \text{ is deterministic and preassociative} \]

For any state \( q \) of \( \mathcal{A} = (Q, q_0, \delta) \), any \( L \subseteq 2^{X^*} \) and \( z \in X \), set

\[
L^A(q) := \{ x \in X^* | \delta(q_0, x) = q \}
\]

\[
z.L := \{zx | x \in L\}
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A criterion for preassociativity

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L^A(q) := \{ x \in X^* \mid \delta(q_0, x) = q \}
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z.L := \{zx \mid x \in L\}
\]

**Proposition.** Let \( \mathcal{A} = (Q, q_0, \delta) \) be a semiautomaton. The following conditions are equivalent.

(i) \( \mathcal{A} \) is preassociative,

(ii) for all \( z \in X \) and \( q \in Q \),

\[
z.L^A(q) \subseteq L^A(q')
\]

for some \( q' \in Q \).
$z.L^A(q) \subseteq L^A(q')$, for some $q' \in Q$.

Example. $X = \{0, 1\}$

$L^A(e) = \{x \mid x \text{ contains an even number of } 1\}$
$L^A(o) = \{x \mid x \text{ contains an odd number of } 1\}$

$0.L^A(o) \subseteq L^A(o)$
$0.L^A(e) \subseteq L^A(e)$
$1.L^A(o) \subseteq L^A(e)$
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An example of characterization

**Definition.** $F : X^* \rightarrow X^*$ is *length-based* if

$$F = \phi \circ |\cdot|$$

for some $\phi : \mathbb{N} \rightarrow X^*$. 
An example of characterization

**Definition.** $F : X^* \rightarrow X^*$ is *length-based* if

$$F = \phi \circ |\cdot|$$

for some $\phi : \mathbb{N} \rightarrow X^*$.

**Proposition.** Let $F : X^* \rightarrow X^*$ be a length-based function. The following conditions are equivalent.

1. **(i)** $F$ is associative
2. **(ii)** $|F(x)| = \alpha(|x|)$

where $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ satisfies

$$\alpha(n + k) = \alpha(\alpha(n) + k), \quad \forall n, k \in \mathbb{N}$$
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Relaxing the associativity property

\[ X := \mathbb{L} \cup \mathbb{N} \text{ where } \mathbb{L} = \{a, b, c, \ldots, z\} \]

\[ |x|_\mathbb{L} = \text{number of letters of } x \text{ that are in } \mathbb{L}. \]

The functions \( F, G \) defined by

\[
F(x) = \begin{cases} 
  x, & \text{if } |x| < m \\
  x_1 \cdots x_{m-1} |x|, & \text{if } |x| \geq m 
\end{cases}
\]

\[
G(x) = \begin{cases} 
  x, & \text{if } |x| < m \\
  x_1 \cdots x_m |x|_\mathbb{L}, & \text{if } |x| \geq m 
\end{cases}
\]

are not associative,
Relaxing the associativity property

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\end{cases}
\]

are not associative, but they satisfy

\[
F(xyz) = F(xF(y)z) \quad \forall \, xz \in X^* \text{ such that } |y| \leq m
\]
The origin of the terminology

\[ f : X \times X \to X \] is associative if

\[ f(x, f(y, z)) = f(f(x, y), z) \]

Associativity enables us to define expressions like

\[ f(x, y, z, t) = f(f(f(x, y), z), t) = f(x, f(f(y, z), t)) = \cdots \]

Define \( F : X^* \to X \cup \{ \varepsilon \} \) by

\[ F(\varepsilon) = \varepsilon, \quad F(x) = x, \quad F(x) = f(x_1, \ldots, x_n) \]

Then \( F \) is an associative variadic operation.
What about...
What about...
Let

- $H : X^* \to X^*$ be associative and length preserving
- $f_n : \text{ran}(H_n) \to X$ be one-to-one for every $n \geq 1$

Set

$$F_n = f_n \circ H_n, \quad n \geq 1$$
Let
\[ H: X^* \rightarrow X^* \] be associative and length preserving
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Set
\[ F_n = f_n \circ H_n, \quad n \geq 1 \]
If \( F(F(y)|y|) = F(y) \) for all \( y \in X^* \), then
\[ F(xF(y)|y|z) = F(xyz), \quad xyz \in X^* \]
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Set
\[ F_n = f_n \circ H_n, \quad n \geq 1 \]

If \( F(F(y) |y|) = F(y) \) for all \( y \in X^* \), then
\[ F(xF(y) |y| z) = F(xyz), \quad xyz \in X^* \]

This property is called \textit{barycentric associativity} and is satisfied by a wide class of \textit{means}. 
Conclusion
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The ubiquity of the associativity property

http://math.uni.lu/~teheux
And now for something completely different
An invitation

The first International Symposium on Aggregation and Structures

Luxembourg, July 5 – 8, 2016

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The first
International Symposium on Aggregation and Structures

Scientific Committee:

Miguel Couceiro, Bernard De Baets, Radko Mesiar.

Invited speakers:

Marek Gagolewski, Michel Grabisch, Carlos Lopez-Molina, Gabriella Pigozzi.

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