Automates, mots et décision

Bruno Teheux

University of Luxembourg

What is the common point among…

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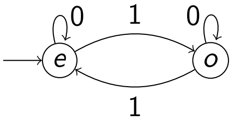

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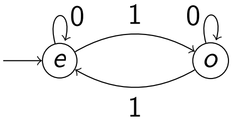
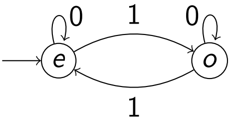


`sort()`

What is the common point among. . .



`sort()`

Notation

$$n\text{-tuples } \mathbf{x} \text{ in } X^n \quad \equiv \quad \textit{n-strings} \text{ over } X$$

0-string: ε,
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

$$|\mathbf{x}| \;=\; \text{length of } \mathbf{x}$$

$$X^* := \bigcup_{n \geq 0} X^n$$

We endow X^* with concatenation

Notation

Any $F : X^* \to Y$ is called a *variadic function*, and we set

$$F_n := F|_{X^n}.$$

Any $F : X^* \to X \cup \{\varepsilon\}$ is a *variadic operation*.

We assume

$$\boxed{F(\mathbf{x}) = \varepsilon \iff \mathbf{x} = \varepsilon}$$

Associativity for string functions

Definition. $F\colon X^* \to X^*$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

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Examples.

- sorting in alphabetical order
- letter removing, duplicate removing

Associativity entails 'distributivity'

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Example. $F = \mathtt{sort}()$

INPUT: $\mathbf{xzu}\cdots$ in blocks of unknown length given at unknown time intervals.

OUTPUT: $\mathtt{sort}(\mathbf{xzu}\cdots)$

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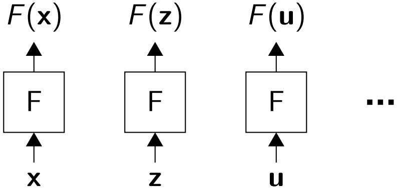

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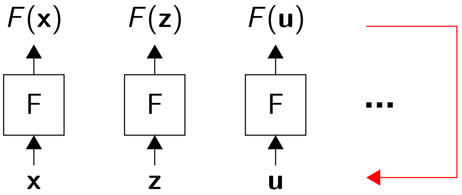

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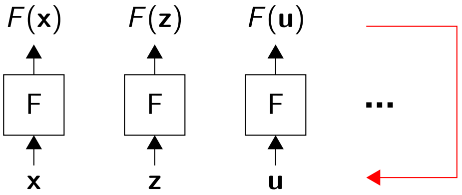

“Highly” distributed algorithms

Associativity for variadic functions?

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Quest: a notion of 'associativity' for variadic $F\colon X^* \to Y$

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Definition. We say that $F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{xyz}) \;=\; F(\mathbf{xy}'\mathbf{z})$$

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Examples. $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

$F_n(\mathbf{x}) = |\mathbf{x}| \quad (X \text{ arbitrary}, \; Y = \mathbb{N})$

Associativity and preassociativity

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{xyz}) \;=\; F(\mathbf{xy}'\mathbf{z})$$

Proposition. Let $F\colon X^* \to X^*$.

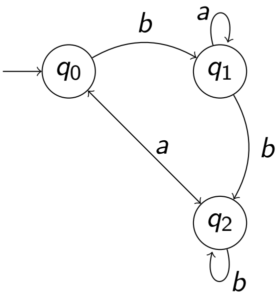
$$F \text{ is associative}$$

$$\Longleftrightarrow$$

$$F \text{ is preassociative} \qquad \text{and} \qquad F \circ F = F.$$

Slogan. Preassociativity is a *composition-free* version of associativity.

Semiautomata



A *semiautomaton* over X:

$$\mathcal{A} = (Q, q_0, \delta)$$

where $q_0 \in Q$ is the *initial state* and

$$\delta \colon Q \times X \to Q$$

is the *transition function.*

The map δ is extended to $Q \times X^*$ by

$$\delta(q, \varepsilon) := q,$$
$$\delta(q, \mathbf{x}y) := \delta(\delta(q, \mathbf{x}), y)$$

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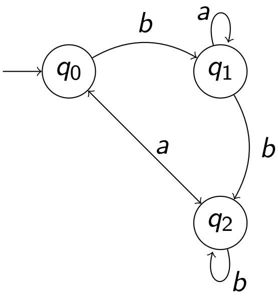

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Definition. $F_{\mathcal{A}} \colon X^* \to Q$ is defined by

$$F_{\mathcal{A}}(\mathbf{x}) := \delta(q_0, \mathbf{x})$$

Preassociativity and semiautomata

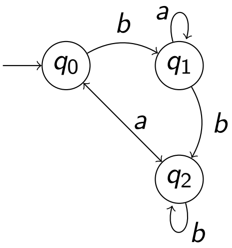
$$\boxed{F_{\mathcal{A}}(\mathbf{x}) := \delta(q_0, \mathbf{x})}$$

Fact. If $\mathcal{A}$ is a semiautomaton,

- $F_{\mathcal{A}}$ is "half"-preassociative:

$$F_{\mathcal{A}}(\mathbf{y}) \;=\; F_{\mathcal{A}}(\mathbf{y}') \quad\Longrightarrow\quad F_{\mathcal{A}}(\mathbf{y}'\mathbf{z}) \;=\; F_{\mathcal{A}}(\mathbf{y}'\mathbf{z})$$

- $F_{\mathcal{A}}$ may not be preassociative:

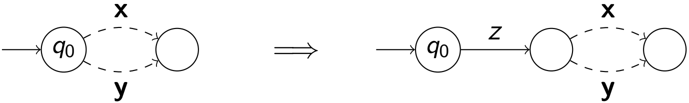


$$F_{\mathcal{A}}(b) \;=\; q_1 \;=\; F_{\mathcal{A}}(ba)$$
$$F_{\mathcal{A}}(bb) \;=\; q_2 \;\neq\; q_0 \;=\; F_{\mathcal{A}}(bba)$$

Preassociativity and semiautomata

Definition. A semiautomaton is *preassociative* if it satisfies

$$\delta(q_0, \mathbf{x}) = \delta(q_0, \mathbf{y}) \quad \Longrightarrow \quad \delta(q_0, z\mathbf{x}) = \delta(q_0, z\mathbf{y})$$



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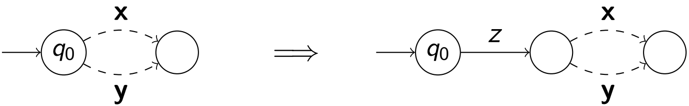

Lemma.

$$\mathcal{A} \text{ preassociative} \quad \Longleftrightarrow \quad F_{\mathcal{A}} \text{ preassociative}$$

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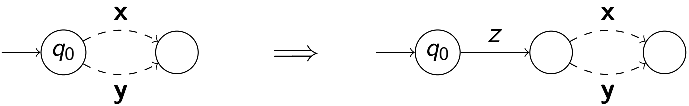

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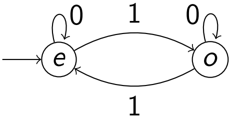
Example. $X = \{0, 1\}$



$$F_{\mathcal{A}}(\mathbf{x}) = e \iff \#\{i \mid x_i = 1\} \text{ is even},$$
$$F_{\mathcal{A}}(\mathbf{x}) = o \iff \#\{i \mid x_i = 1\} \text{ is odd}.$$

Preassociativity and semiautomata

X, Q finite.

Definition. For an onto $F\colon X^* \to Q$, set

$$q_0 := F(\varepsilon),$$
$$\delta(q, z) := \{F(\mathbf{x}z) \mid q = F(\mathbf{x})\},$$
$$\boxed{\mathcal{A}^F := (Q, q_0, \delta)}$$

Generally, $\mathcal{A}^F$ is a non-deterministic semiautomaton.

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Lemma.

F is preassociative $\iff$ $\mathcal{A}^F$ is deterministic and preassociative

A criterion for preassociativity

F is preassociative $\iff$ $\mathcal{A}^F$ is deterministic and preassociative

For any state q of $\mathcal{A} = (Q, q_0, \delta)$, any $L \subseteq 2^{X^*}$ and $z \in X$, set

$$L^{\mathcal{A}}(q) := \{\mathbf{x} \in X^* \mid \delta(q_0, \mathbf{x}) = q\}$$
$$z.L := \{z\mathbf{x} \mid \mathbf{x} \in L\}$$

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Proposition. Let $\mathcal{A} = (Q, q_0, \delta)$ be a semiautomaton. The following conditions are equivalent.

(i) $\mathcal{A}$ is preassociative,

(ii) for all $z \in X$ and $q \in Q$,

$$z.L^{\mathcal{A}}(q) \subseteq L^{\mathcal{A}}(q'), \qquad \text{for some } q' \in Q.$$

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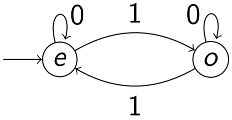

$$L^{\mathcal{A}}(e) = \{\mathbf{x} \mid \mathbf{x} \text{ contains an even number of } 1\}$$
$$L^{\mathcal{A}}(o) = \{\mathbf{x} \mid \mathbf{x} \text{ contains an odd number of } 1\}$$

$0.L^{\mathcal{A}}(o) \subseteq L^{\mathcal{A}}(o)$ $\qquad$ $0.L^{\mathcal{A}}(e) \subseteq L^{\mathcal{A}}(e)$

$1.L^{\mathcal{A}}(o) \subseteq L^{\mathcal{A}}(e)$ $\qquad$ $1.L^{\mathcal{A}}(e) \subseteq L^{\mathcal{A}}(o)$

An example of characterization

Definition. $F\colon X^* \to X^*$ is *length-based* if

$$F \;=\; \phi \circ |\cdot| \qquad \text{for some } \phi\colon \mathbb{N} \to X^*.$$

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Proposition. Let $F\colon X^* \to X^*$ be a length-based function. The following conditions are equivalent.

(i) F is associative

(ii)

$$|F(\mathbf{x})| \;=\; \alpha(|\mathbf{x}|)$$

where $\alpha\colon \mathbb{N} \to \mathbb{N}$ satisfies

$$\alpha(n+k) \;=\; \alpha(\alpha(n)+k), \qquad \forall n,k \in \mathbb{N}$$

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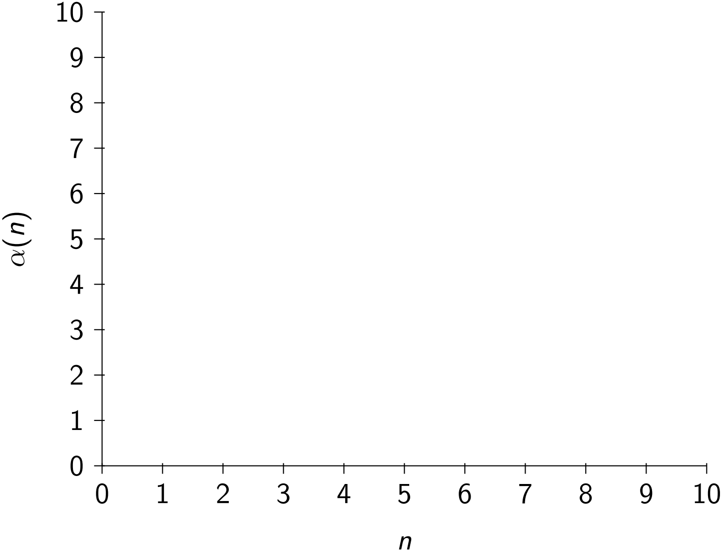

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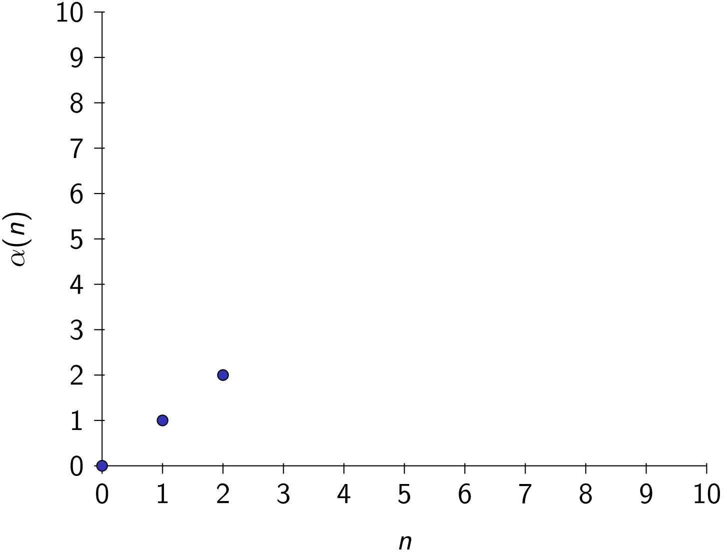

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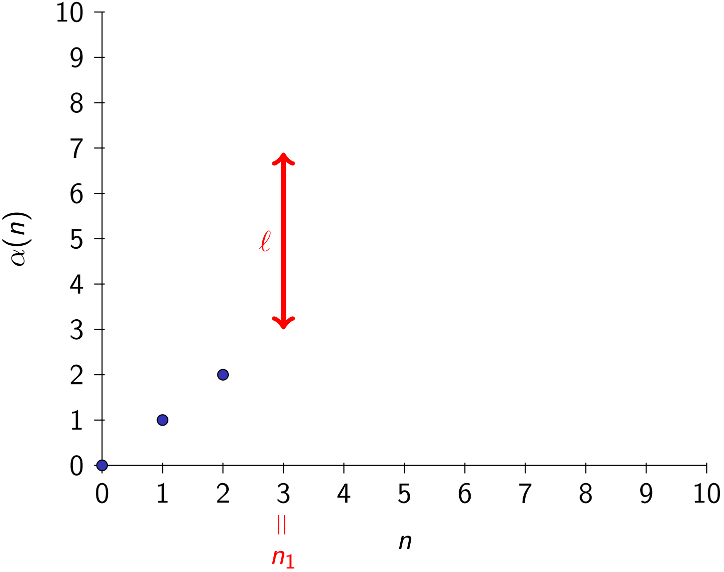

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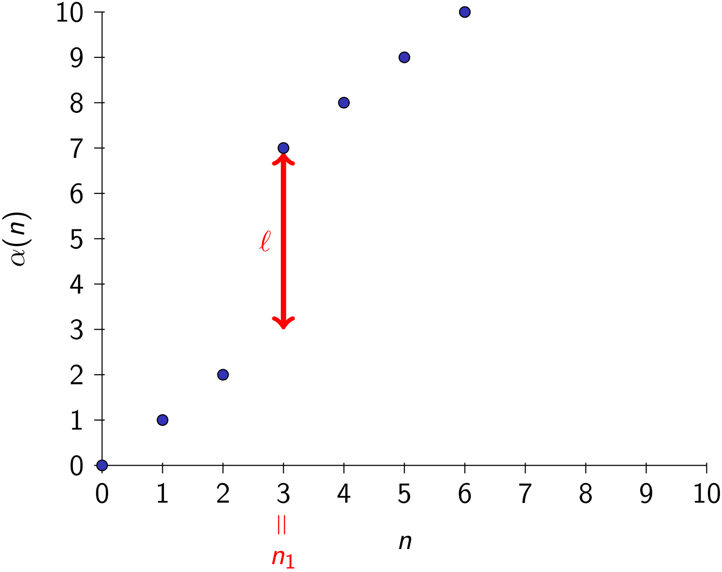

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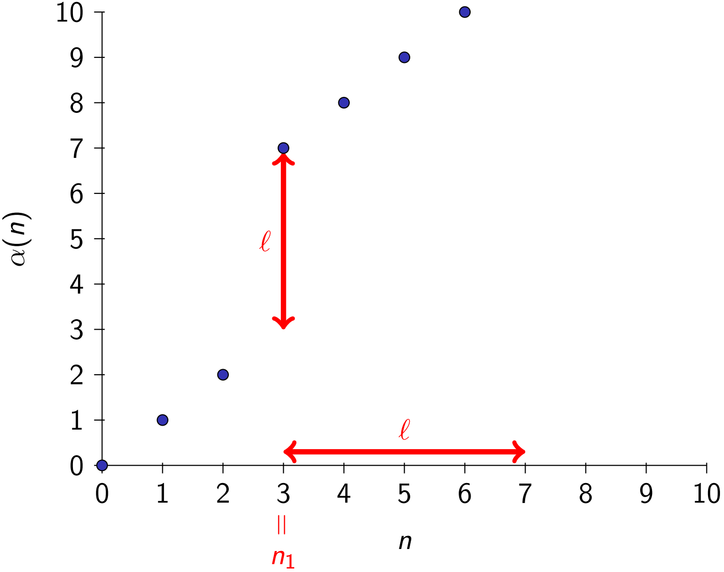

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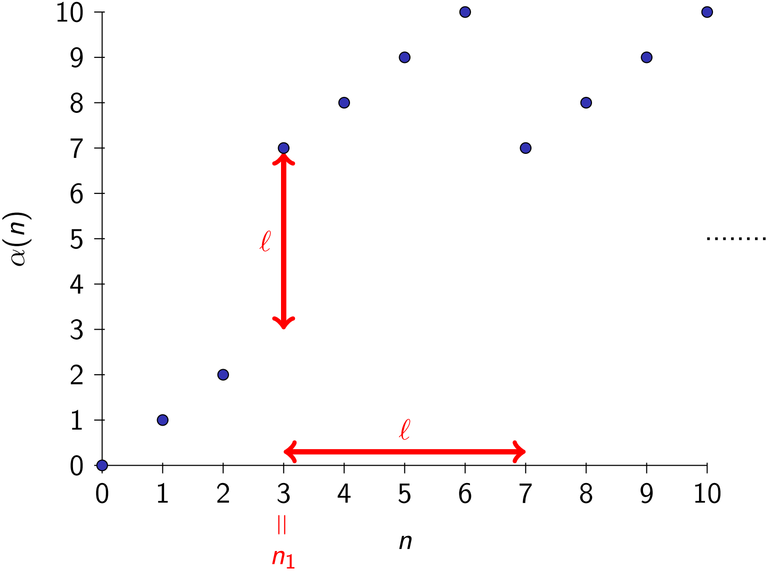

Relaxing the associativity property

$X := \mathbb{L} \cup \mathbb{N}$ where $\mathbb{L} = \{a, b, c, \ldots, z\}$

$|\mathbf{x}|_{\mathbb{L}} =$ number of letters of $\mathbf{x}$ that are in $\mathbb{L}$.

The functions F, G defined by

$$F(\mathbf{x}) = \begin{cases} \mathbf{x}, & \text{if } |\mathbf{x}| < m \\ x_1 \cdots x_{m-1} |\mathbf{x}|, & \text{if } |\mathbf{x}| \geq m \end{cases}$$

$$G(\mathbf{x}) = \begin{cases} \mathbf{x}, & \text{if } |\mathbf{x}| < m \\ x_1 \cdots x_m |\mathbf{x}|_{\mathbb{L}}, & \text{if } |\mathbf{x}| \geq m \end{cases}$$

are not associative,

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are not associative, but they satisfy

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xz} \in X^* \text{ such that } |\mathbf{y}| \leq m$$

The origin of the terminology

$f\colon X \times X \to X$ is *associative* if

$$f(x, f(y,z)) = f(f(x,y),z)$$

Associativity enables us to define expressions like

$$\begin{aligned} f(x,y,z,t) &= f(f(f(x,y),z),t) \\ &= f(x, f(f(y,z),t)) \ = \ \cdots \end{aligned}$$

Define $F\colon X^* \to X \cup \{\varepsilon\}$ by

$$F(\varepsilon) = \varepsilon, \quad F(x) = x, \quad F(\mathbf{x}) = f(x_1, \ldots, x_n)$$

Then F is an associative variadic operation.

What about...

What about. . .

Let

- $H\colon X^* \to X^*$ be associative and length preserving
- $f_n\colon \mathrm{ran}(H_n) \to X$ be one-to-one for every $n \geq 1$

Set

$$F_n = f_n \circ H_n, \quad n \geq 1$$

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If $F(F(\mathbf{y})^{|\mathbf{y}|}) = F(\mathbf{y})$ for all $\mathbf{y} \in X^*$, then

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This property is called *barycentric associativity* and is satisfied by a wide class of means.

Conclusion

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The ubiquity of the associativity property

http://math.uni.lu/~teheux

And now for something completely different

An invitation

An invitation

The first
International Symposium on Aggregation and Structures

Luxembourg, July 5 – 8, 2016
http://math.uni.lu/isas/

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The first

International Symposium on Aggregation and Structures

Scientific Committee:

Miguel Couceiro,
Bernard De Baets,
Radko Mesiar.

Invited speakers:

Marek Gagolewski,
Michel Grabisch,
Carlos Lopez-Molina,
Gabriella Pigozzi.

Luxembourg, July 5 – 8, 2016
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